Growth - lecture note for ECON1910

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This lecture note is meant as a supplement to the curriculum, in particular to Ray (1998). In some of the lectures I will use slightly different figures and calculations from Ray, and the main purpose of this note is to give you that in written form.

The main role is to act as a supplement on the Harrod-Domar and Solow models. I have explicitly used *a lot of* intermediate calculations to reach the results, to make the derivations easy to follow.

It also acts as an overview of the reading list, grouping the chapters in Ray and Banerjee into what I think is relevant headings.

The information in this text corresponds to the three lectures labelled "growth" on the lecture plan.

Text in Ray should be read even if it is not referred here - **important** things such as section 3.5 in Ray is not covered here.

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Contents

1 What you should read and what you do not need to read

The curriculum is

- Ray (1998), chapter 3, 4 and 5
- Banerjee et al. (2006), chapter 2, 3 and 5
- Hand-out on the Big Push Model (from a textbook by Todaro and Smith), which I will talk about in the third lecture on growth, related to Ray ch. 5. Available from the course webpage.
- You will probably have some use of reading this note

However, you can skip advanced algebra:

- Section 3.5.6 in Ray is not requred reading (but you will probably have good use from reading page 87 anyway)
- The appendices to the chapters in Ray are not required reading

As I will say in the lectures, if you have serious problems following algebra (like it is laid out in this note), it is not strictly mandatory. But I do not think it necessarily is any easier to understand the models using only other forms of intuition. You should at the very least be able to interpret equations like equation (32) below, and understand the figures.

2 Introduction

Facts, history and statistics - see Ray (1998), especially chapter 3. Also treated at the lecture.

3 The Harrod-Domar model

Ray ch. 3.3.1

3.1 How to derive it

This follows Ray chapter 3.3.1, I just added some intermediate calculation. See Ray for the full details.

$$Y_t = C_t + S_t \tag{1}$$
$$Y_t = C_t + I_t \tag{2}$$
$$S_t = I_t \tag{3}$$
$$\tag{4}$$

Capital next period:

$$K_{t+1} = (1-\delta)K_t + I_t \tag{5}$$
$$K_{t+1} = (1-\delta)K_t + S_t \tag{6}$$

Now introduce savings rates and capital-output ratios to get

$$s = sY \tag{7}$$
$$K = \theta Y \tag{8}$$
$$Y = \frac{1}{\theta}K \tag{9}$$
$$\theta = \frac{K}{Y} \tag{10}$$

We then derive the Harrod-Domar results as (see Ray for more discussion)

$$\begin{aligned}
\theta Y_{t+1} &= (1-\delta)\theta Y_t + sY_t && (11)\\
\theta Y_{t+1} &= \theta Y_t - \delta\theta Y_t + sY_t && (12)\\
\theta Y_{t+1} - \theta Y_t &= sY_t - \delta\theta Y_t && (13)\\
\frac{\theta Y_{t+1} - \theta Y_t}{Y_t} &= s - \delta\theta && (14)\\
\frac{Y_{t+1} - Y_t}{Y_t} &= \frac{s}{\theta} - \delta && (15)\\
g &= \frac{s}{\theta} - \delta && (16)\\
\frac{s}{\theta} &= g + \delta && (17)
\end{aligned}$$

3.2 Policy implications

Capital-output ratio is seen as exogenous, but technology-driven.

Savings rates can be affected by policy.

Implicitly, there is a production function with fixed factor shares[1] with a labor reserve in the "non-developed" sector. The central issue for the planner is "how can we get enough capital to get everyone into the industrialized sector"? The Harrod-Domar model was written after the Great Depression, with great surplus of labor. More talk about this at the lecture

3.3 Harrod-Domar with population growth

The growth equation becomes

$$\frac{s}{\theta} \approx g^* + n + \theta \qquad (18)$$

where g^* is per capita growth. The result is quite straightforward: as there are more people each period, we must invest more for a given level of per capita growth.

4 The Solow model

Ray ch. 3.3.3

4.1 Production function

In the Solow model, production is explicitly a result of two production factors: Labor and Capital. Denoting total output as Y, and using P for labor and K for capital:[2]

$$Y = F(K, P) \qquad (19)$$

We will assume that the production function has constant returns to scale. By this I mean that if we increase **both factors** by the same fraction, total output will increase by the same. For example, if we double the number of production factors from above, we get twice the output:

$$2Y = F(2K, 2P) \qquad (20)$$

[1] This can be referred to as a Leontief production function. I might talk more about this at the lecture

[2] It is more common (and logical, perhaps) to use L for labor. However, I will follow the letters used by Ray (1998).

or, more generally

$$aY = F(aK, aP) \tag{21}$$

where a is any constant.

Note, however (and this is important) that if you increase only one of the factors, production increases by less. Think of a factory consisting of ten machines and one hundred workers that produces one thousand cars a year. Now, if you double both the number of machines and workers (to 20 and 200), it is not unreasonable to think that you get twice as many cars. However, if we only double the number of machines (so we have 20 machines and 200 workers), we will no doubt get more than one thousand cars, but as the workers have to be spread more thinly across the machines output will not double.

For our application: If we **keep the number of people constant**, adding capital will increase production, but with smaller and smaller increases for a given amount of capital.

Because of the constant returns to scale assumption above, we can show this by multiplying both factors by $\frac{1}{P}$ (setting $a = \frac{1}{P}$ in equation (21)) to get

$$\frac{1}{P} \cdot Y = F(\frac{1}{P} \cdot K, \frac{1}{P} \cdot P) \tag{22}$$

$$\frac{Y}{P} = F(\frac{K}{P}, \frac{P}{P} \tag{23}$$

$$y = F(k, 1) = f(k) \tag{24}$$

4.2 Baseline Solow model

Introducing dot notation, I now declare $\dot{k}$ to mean "the change in k from one period to the next period". In other words, $\dot{k}_t \equiv k_{t+1} - k_t$.

We first declare that change in total capital comes from saving (which adds to capital) and depreciation (which "destroys" capital), and saying that both saving (s) and depreciation (δ, the Greek letter delta, you can write d in your own notes if you like):

$$\dot{K}_t = sY_t - \delta K_t \tag{25}$$

$$\tag{26}$$

We then introduce the variable "capital per person", and as in Ray, use lower case letter for per-capita variables. The definition of k is then, of course, $\frac{K}{L}$.

Now how does k change over time? First assume that there is constant population. Then the change in capital per person will be the same as the relative change in total capital. However, if population grows, there has to be higher total capital growth for a given capital per person growth. It is really quite logical. This is expressed in the following equation (where we divide by K and L and k because we want relative growth):

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \tag{27}$$

We then use the equations (25) and (37) to derive the following:

$$\frac{\dot{k}}{k} = \frac{sY - \delta K}{K} - \frac{\dot{L}}{L} \tag{28}$$

$$\frac{\dot{k}}{k} = \frac{sY}{K} - \frac{\delta K}{K} - \frac{\dot{L}}{L} \tag{29}$$

$$\frac{\dot{k}}{k} = \frac{sy}{k} - \delta - n \tag{30}$$

$$\dot{k} = sy - \delta k - nk \tag{31}$$

$$\dot{k} = sy - (\delta + n)k \tag{32}$$

Equation (32) tells us how capital per worker grows (given what we proposed in equations (25) and (37)). Increased savings increse capital, while high depreciation rates and high population growth slows capital growth.

Also note that there is a level

$$sy = (\delta + n)k \tag{33}$$

where capital growth is zero.

Remember from 4.1 that if we increase capital while holding population constant, output increases (but less as we reach higher levels). This can be written as

$$y = f(k) \tag{34}$$

where $f(\cdot)$ is a function having these properties.

We can then rewrite (32) as

$$\dot{k} = s \cdot f(k) - (\delta + n)k \tag{35}$$

This relationship can be illustrated graphically. Figure 1 shows k on the horizontal axis and y on the vertical axis.

- The green line is $y = f(k)$. Note how, as we said, it always increases, but less so as we reach higher levels of k.
- The blue line is $s \cdot y$. Now, as we have said that s is constant, this line will always be a constant fraction of the green line.
- The red line is $(n + \delta)k$. This relationship is linear.

Now, for the level of capital k shown in Figure 1 we can see the level of production, the level of saving, and how much capital per person is reduced because of depreciation and population growth. We see that for this particular situation capital will increase, we will move towards the right in the figure over time.

These "moves" will depend on the distance between the savings curve and the population-depreciation curve, and we will move in the directions given in figure 2

As discussed above, there is a level where $f(k) = (\delta + n)k$, meaning capital growth is zero. This is referred to as the "steady state" level, and is shown in Figure 3. For further discussion of the steady state, see the textbook.

4.3 Solow with technological growth

Ray ch. 3.4

Obviously, the baseline Solow implication that there is no growth does not fit reality. We therefore postulate some exogenous ("outside-the-model") growth, which we do not try to explain at this point - we just state that "technology" grows by some constant fraction each year. The "technology-augmented" production function is then written

$$Y = F(K, EP) \tag{36}$$

that is, we multiply the labor force by the technology.[3]

In short, it means that over time, labor becomes more and more productive, so that the labor of one person "matters more" in the production. We now scale all variables by this technology growth (Ray denotes

[3] Three possible ways of doing this would be to modify equation (19), adding an A to represent technology: (a) $Y = A \cdot F(K, P)$, (b) $Y = F(A \cdot K, P)$, (c) $Y = F(K, A \cdot P)$ Ray uses the third alternative, but you should not be very concerned about the differences here.

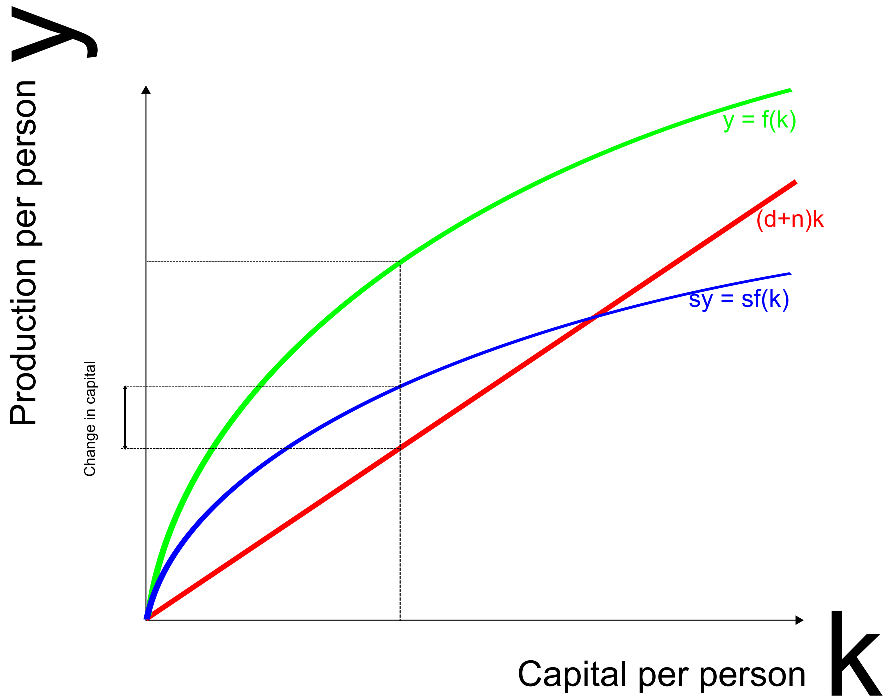

Figure 1: The Solow model

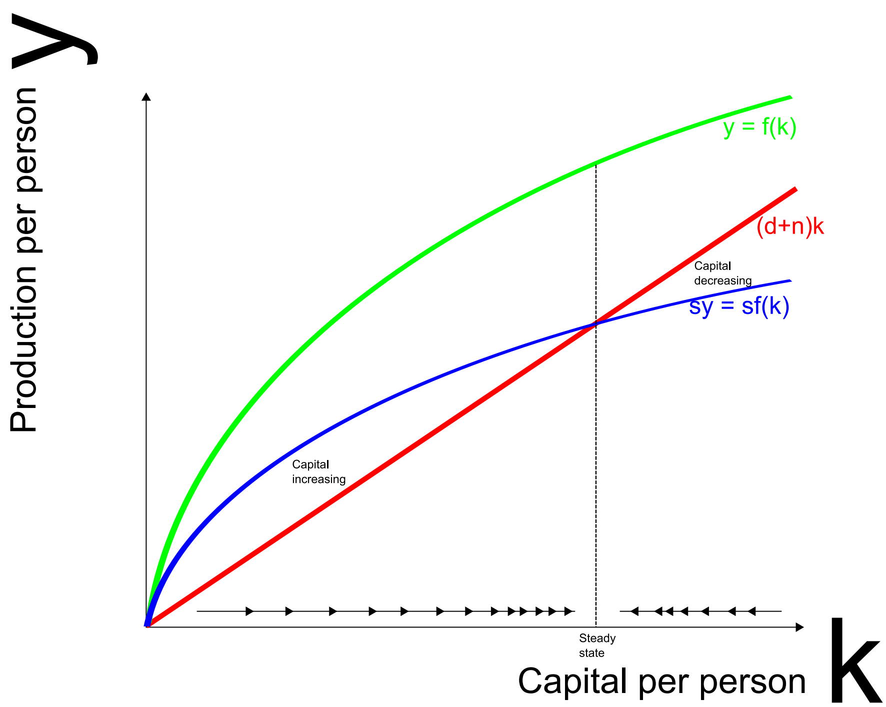

Figure 2: Paths of movement in the Solow model

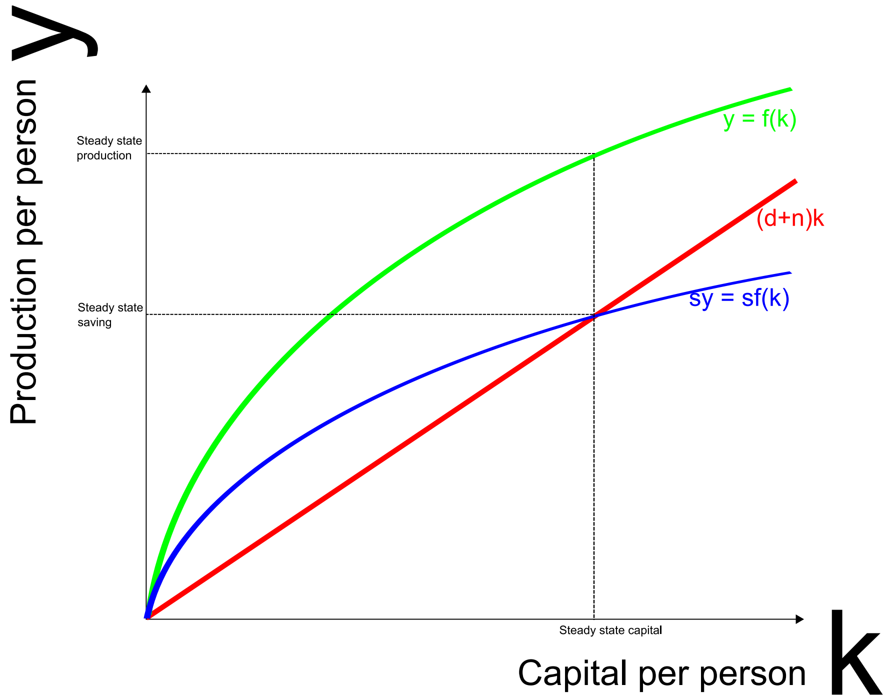

Figure 3: Steady state

this by adding a "hat" on the variables). Then, a "steady state" $\hat{k}$ - that does not change over time - means that k actually increases over time at the same rate as technology growth.

To get the $\hat{k}$, we divide the "big K" by "effective labor force" - that is, dividing by EP. This means that the variables we use no longer directly relate to the number of people - a given level of $\hat{k}$ or $\hat{y}$ may correspond to different levels of per-capita income depending on what the technology level is.

We declare that technology grows by some pre-determined value π.[4]

By the definition of $\hat{k}$, we can show that

$$\frac{\dot{\hat{k}}}{\hat{k}} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} - \frac{\dot{E}}{E} \tag{37}$$

where $\frac{\dot{E}}{E} = \pi$ as stated above.

Using the same algebra as in the previous section, we get the equation for capital change

$$\dot{\hat{k}} = s \cdot f(\hat{k}) - (\delta + n + \pi)\hat{k} \tag{38}$$

and the steady state equation

$$sf(\hat{k}) = (\delta + n + \pi)\hat{k} \tag{39}$$

The "steady state" level of $\hat{k}$ now denotes a situation where total production grows by π. Why? Observe that in the steady state $\dot{\hat{y}}$ is zero. That means that the growth of production measured per "effective

[4]Note for people new to economics: this has nothing to do with the mathematical constant 3.14. π is simply the greek letter pi, here used to denote a number that may be, for example, 2 or 3 percent.

worker" is zero. But the "effective worker" becomes more and more productive. Therefore, output *per person* is steadily increasing. With some algebra:

$$\hat{y} = \frac{Y}{EP} \tag{40}$$

$$\hat{y} \cdot E = \frac{Y}{P} \tag{41}$$

We then see that even though $\hat{y}$ is constant, $\frac{Y}{P}$, per capita income, is growing at the same rate as E.

We can draw diagrams similar to Figures 1- 3 by having $\hat{k}$ and $\hat{y}$ on the axes.

First lecture ends about here

5 Growth model extensions

To some extent, all of these can be seen as extensions of the Solow model.

This section corresponds to Ray (1998) chapter 4.

5.1 Other production factors

In the previous chapter, the factors of production were capital and labor. We can add more, or use something else instead, based on what we want to model. In reality, there is an infinite number of production factors (and often the differences will not be clear-cut), such as cars, computers, the labor people with master's degrees in development economics, the labor of hairdressers, steel, land, etc). We have to do some grouping of these. In the following we will look at human capital, an "extended" view on labor.

5.2 Human capital

Ray ch. 4.2 and 4.3

High and low skill - complementarities or substitutes?

In the following we will assume they are substitutes (it takes X persons to do the work of one person with high human capital, where $X > 1$).

If H is total human capital, h is human capital per person - you can think of this of the average years of schooling.

$$H = h \cdot P \tag{42}$$

$$\frac{H}{P} = h \tag{43}$$

(Note that there is some "depreciation" here as well - obviously one has to keep a certain amount of "investment" just to keep the human capital level constant - people die and new people are born, so you have to school people all the time. We abstract from this here)

Modify the Solow equation to read

$$y = f(k, h) \tag{44}$$

We can now "invest" both in capital, k, and human capital, h. Income is divided into

- Investment in physical capital $s \cdot y$
- Investment in human capital $q \cdot y$
- Consumption $(1 - s - q) \cdot y$

The detailed exposition is shown in section 8.1 below for those especially interested. It is not very complicated, but some calculation is involved.

This gives several implications: (Ray p. 102-104)

- There can be decreasing returns to physical capital yet constant returns in total
- Policy can influence total growth rates and not only levels
- Convergence not necessarily the case
- Return to capital may be higher in rich countries because of high levels of human capital

Note that this is possible because we *assumed* such a production function. If we add more factors of production (such as "unskilled labor"), this might not be the case.

5.3 Endogenous growth

Ray ch. 4.4

Baseline: We can use labor to "discover" new things, in addition to producing things.

Ray 4.4.2: The difference between "inventions" and diffusions of knowledge.

Ray 4.4.3: A model of technical process, shown in equation (4.5) to (4.7): A share u of human capital is used in production of goods (for consumption and investment), while a share $(1-u)$ is used for research (improving technology). This gives a trade-off between production today and better technology tomorrow.

How is such "production of technology" financed? In a planned economy, the state could just dictate that some people should do research. In a mixed economy, like Norway, a large part is financed via taxes. But if you want private initiatives for research, you need some kind of intellectual property rights - in essence, a (limited) monopoly for those who discover things.

5.4 Externalities

Ray 4.4.4

An externality is an unintended consequence of an action. If you invent the steam engine because it is profitable for your factory, and you don't have any legal system to protect intellectual property, a likely externality is that everyone else can also run their factory by copying your steam engine. In fact, even if you have intellectual property rights it is likely that *some* of what you have done cannot be protected, and thus gives ideas to others. (Note that there may also be negative externalities: pollution in the neighbouring village, for example).

Positive externalities between economic activities may lead to increasing returns to scale on the "macro" level.

Complementarities: Your actions depend on what everyone else does. See paragraph on complementarities and Figure 4.3 in Ray.

Aghion and de Aghion (2006) has an important discussion related to externalities and complementarities (see the paragraph "New Growth Theories in a nutshell"). The authors argue that the following factors are important to growth:

- A legal environment that allows inventors a share of the revenues generated by their innovation
- Expectations of macroeconomic stability increases expected profits (and thereby would lead more people to innovate?)

There are also several other important points in the article related to other topics we have discussed (or will discuss in the third lecture), such as human capital accumulation, financial markets and competition environment.

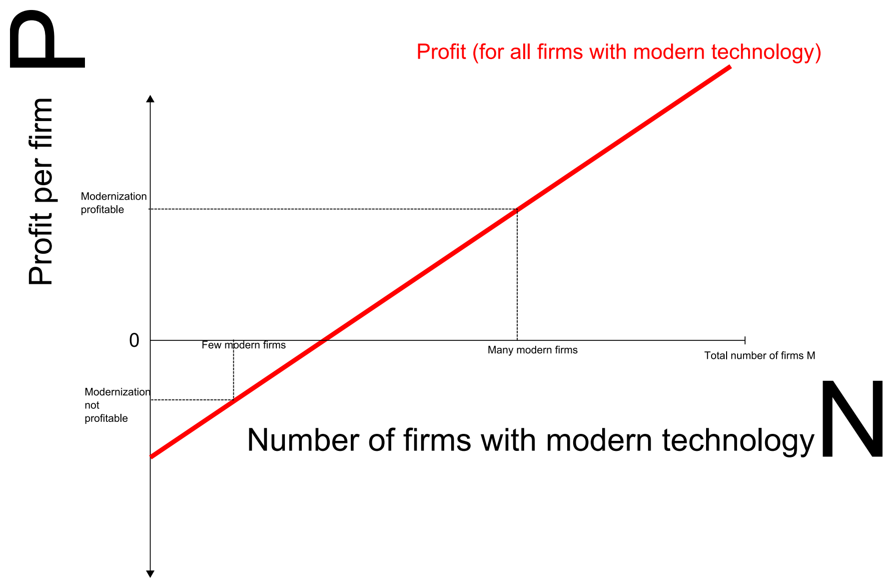

Figure 4: Big push

5.5 Total factor productivity growth

I will use TFP to mean Total Factor Productivity.

TFP growth is "growth that is not due to any change in factors of production". If you have higher GDP because you save more, that is not TFP growth. TFP is similar to the "technology parameter" in Section 4.3. See Ray 4.4.5

TFP and the East Asian Miracle: See Ray ch. 4.5.

Second lecture ends about here

6 The Big Push model

See handout from Smith/Todaro available at the course page here:

http://www.uio.no/studier/emner/sv/oekonomi/ECON1910/v08/undervisningsmateriale/todarosmith.html (UiO username and password required)

I will use Figure 4 at the lecture. It shows that profits from a (potentially) industrialized sector depends on the number of sectors that already are (or are expected to be) industrialized.

6.1 Growth coordination

Chapter 5.2 in Ray. The Big Push model is an example of a model where coordination plays an important role.

Complementarities and coordination (QWERTY/DVORAK as example)
Linkages: Ray 5.2.3
Increasing returns: Ray 5.3.2

7 Institutions and the role of history

- Ray ch. 5.5 ("Other roles for history")

- Acemoglu et al. (2006) - discussed at the lecture
- Engerman and Sokoloff (2006) - discussed at the lecture

This has some overlap with Section 5.4 above

Third lecture ends here

References

Acemoglu, D., S. Johnson, and J. Robinson (2006). Understanding prosperity and poverty: Geography, institutions, and the reversal of fortune. In A. V. Banerjee, R. Bénabou, and D. Mookherjee (Eds.), *Understanding Poverty*, Chapter 2. Oxford University Press.

Aghion, P. and B. A. de Aghion (2006). New growth approach to poverty alleviation. In A. V. Banerjee, R. Benabou, and D. Mookherjee (Eds.), *Understanding Poverty*, Chapter 5. Oxford University Press.

Banerjee, A. V., R. Benabou, and D. Mookherjee (2006, April). *Understanding Poverty*. Oxford University Press, USA.

Engerman, S. and K. Sokoloff (2006). Colonialism, inequality, and long-run paths of development. In A. V. Banerjee, R. Benabou, and D. Mookherjee (Eds.), *Understanding Poverty*, Chapter 3. Oxford University Press.

Ray, D. (1998, January). *Development Economics*. Princeton University Press.

8 Appendix: Some calculations

8.1 Human capital

This refers to section 5.2.

Capital grows as shown in (32), except that we (for simplicity) assume that $\delta = n = 0$

$$\dot{k} = sy \tag{45}$$

while human capital growth depends on the "schooling investment" q:

$$\dot{h} = qy \tag{46}$$

which, just like the "regular" savings, is assumed to be a constant fraction of total production. This says, for example, that a country uses ten percent of total GDP on education (we do not distinguish between public and private organization here).

We now introduce a new variable - the ratio of human capital to physical capital - and call it r. Then note that all these three equations hold:

$$r = \frac{h}{k} \tag{47}$$

$$k = \frac{h}{r} \tag{48}$$

$$h = \frac{r}{k} \tag{49}$$

Using this, we can manipulate the production function a bit. Total production is given by

$$Y = f(K, hP) \tag{50}$$
$$y = f(k, h) \tag{51}$$
$$\tag{52}$$

We then multiply by the constant $\frac{1}{r}$, as we showed in equation (21)

$$y = \frac{1}{r} f(rk, rh) \tag{53}$$
$$y = \frac{1}{r} f(\frac{h}{k} k, rh) \tag{54}$$
$$y = \frac{1}{r} f(h, rh) \tag{55}$$
$$y = \frac{h}{r} f(1, r) \tag{56}$$

to get the production function in terms of h and r.

It can be shown that in steady state all types of capital and total output grow at the same rate (see Appendix to Ray ch. 4). (You should just accept this and not look at the appendix). This means that

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} = \frac{\dot{h}}{h} \tag{57}$$

This also means that in the long run, r is constant.

We now start again at (45), inserting for production from (56).

$$\dot{k} = s \frac{h}{r} f(1, r) \tag{58}$$
$$\dot{k} = skf(1, r) \tag{59}$$
$$\frac{\dot{k}}{k} = sf(1, r) \tag{60}$$

Similarly for h from (46):

$$\dot{h} = q \frac{h}{r} f(1, r) \tag{61}$$
$$\frac{\dot{h}}{h} = q \frac{f(1, r)}{r} \tag{62}$$

Remember that these must be equal in the long run:

$$\frac{\dot{h}}{h} = \frac{\dot{k}}{k} \tag{63}$$
$$q \frac{f(1, r)}{r} = sf(1, r) \tag{64}$$
$$\frac{q}{r} = s \tag{65}$$
$$r = \frac{q}{s} \tag{66}$$

Inserting in (45) or (46), we then get the long-run growth rate:

$$\frac{\dot{k}}{k} = sf(1, r) \tag{67}$$

$$\frac{\dot{k}}{k} = sf(1, \frac{q}{s}) \tag{68}$$

$$\frac{\dot{k}}{k} = s\frac{1}{s}f(s, \frac{q}{s}s) \tag{69}$$

$$\frac{\dot{k}}{k} = f(s, q) \tag{70}$$

(To get the same expressions as in Ray, use the functional form $f(k, h) = k^{\alpha}h^{1-\alpha}$.)