

The Iordanskii Force in Helium II

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Thesis submitted for the Degree of
Candidatus Scientiarum

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May 2001

Acknowledgments

First of all thanks to my supervisor professor Jon Magne Leinaas for guidance and support through the whole work with this thesis, and for selecting the fascinating topic of quantized vortices and forces in helium II.

Also thanks to my father, Jørgen Vestgården, for correcting the language.

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Oslo, May 2001

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Introduction

Among the most controversial topics in superfluid helium II, is the Iordanskii force. This is the phonon contribution to the force perpendicular to the normal fluid velocity. It is an apparent conflict between direct calculations, which get a Iordanskii force, and an indirect determination, which gets none. The goal of this thesis is to offer a careful presentation of the topic and the importance is attached to the direct calculations.

We will study phonons scattered on a vortex. The theory used is the quantum mechanical Ginzburg-Landau theory, and the equations are solved by both Born approximation and partial wave analysis. To complete the understanding of the problem, the analogy to Aharonov-Bohm scattering is discussed. The vortex will be seen to impose a change of the incoming waves, in addition to the scattered wave. The region right behind the vortex is of special interest and will be discussed separately.

The transverse force from one phonon is found, by considering the momentum balance on a contour surrounding the vortex. Analytical answers are found in the limits when the circuit is far from the vortex and close to the vortex. Numerical studies will be done in the intermediate regions. The contribution from the region right behind the vortex will prove to be essential to the force. The Iordanskii force is found as the force from a thermal assembly of phonons.

The indirect determination of the Iordanskii force comes through the effective Magnus force, which is the force perpendicular to a moving vortex, in a stationary liquid. The effective Magnus force is found from a many-particle quantum mechanics expression, and is associated with the geometrical Berry phase. Applied on helium II, the result implies no Iordanskii force, opposed to the direct calculations.

Most of the topics discussed in this thesis are built on the tension between these conflicting results.

The contents of the chapters are:

- *Chapter 1:* Presentation of Ginzburg-Landau theory, which is the formalism used in most of the thesis. Quantized vortices and plane wave solutions are introduced. The chapter also includes two applications which have nothing to do with the Iordanskii force: the motion of a free vortex, and

the analogy between Ginzburg-Landau theory and two-dimensional electromagnetism.

- *Chapter 2:* Calculations of a phonon scattered on a vortex. The methods used are the Born approximation, and the partial wave analysis. The analogy to Aharonov-Bohm scattering is discussed.
- *Chapter 3:* The results from chapter 2 is used to calculate the transverse force from one phonon, by considering the momentum balance on a contour enclosing the vortex. The force is found both analytically and numerically. In addition the background fluid force is found.
- *Chapter 4:* A presentation of the two-fluid model and how the model can be constructed from Ginzburg-Landau theory in the low temperature regime. The normal fluid circulations is calculated from the scattering results from chapter 2.
- *Chapter 5:* Discussion of the different forces on a vortex in helium II, at finite temperature. Presentation of the Iordanskii force, the superfluid Magnus force and the effective Magnus force. Derivation of the effective Magnus force and discussion of the disagreement with the direct calculations.

Notation

In Ginzburg-Landau theory, dimensionless units are used. This means that the speed of sound is $c = 1$, and the constant density of the background fluid is $\rho_0 = 1$. The circulation of a q fold vortex is $\kappa = 2\pi q$.

Some of the symbols used are not standard in condensed matter physics, but we have tried to be consequent in our notation throughout the thesis.

$\theta(\mathbf{r}, t)$	Phase of Ginzburg-landau wavefunction
$\rho(\mathbf{r}, t)$	Density of Ginzburg-landau wavefunction
$\chi(\mathbf{r}, t)$	Phase fluctuation
$\phi(\mathbf{r})$	Time-independent phase fluctuation
$\eta(\mathbf{r}, t)$	Density fluctuation
$\nu(\mathbf{r})$	Time-independent density fluctuation
$q = \pm 1$	Vortex winding number
$\alpha = qk$	Aharonov-Bohm flux & scattering parameter
$f(r)$	Density profile of stationary vortex
$\Omega_l = \sqrt{l^2 + 2\alpha l}$	Phase shift in acoustic Aharonov-Bohm scattering

Chapter 1

Background

1.1 Classical Hydrodynamics

The language used in most of this thesis is the quantum mechanical Ginzburg-Landau theory, but since many of the quantities used there have classical analogies, it is useful first to write down the equations ruling a classical fluid. A more careful treatment of classical fluids can be found in a textbook, for example [LL87].

The first equation to mention is the continuity equation. This is a statement of *mass conservation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1.1)$$

The quantities used here is ρ which is the fluid mass density and \mathbf{v} which is the velocity. Conservation of momentum gives the important *Navier-Stoke's equation*

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P - \frac{1}{m} \nabla V_{eq} + \nu \nabla^2 \mathbf{v} \quad (1.2)$$

where P is the pressure, ν is the dynamical viscosity and V_{eq} an external potential. The Navier-Stoke's equation is difficult to solve and in many cases it is enough to consider a case without viscosity. Then the simpler Euler's equation can be used instead

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P - \frac{1}{m} \nabla V_{eq} \quad (1.3)$$

In many fluids Euler's equation gives an adequate description away from boundaries and turbulent regions. Most practical solutions in hydrodynamics are done by fitting solutions of Navier-Stoke's equation near boundaries with solutions of Euler's equation in the rest of the fluid.

1.1.1 The Circulation

In classical hydrodynamics the circulation is defined as a closed contour integral of the velocity. As the name implies, the circulation tells us how much the fluid is rotating in the area enclosed by the contour. It can be shown ([LL87]) that in case of incompressible classical fluid, the circulation is conserved when the contour follows the flow of the fluid. Classically the circulation can take any value

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l} = \text{constant} \quad (1.4)$$

Another classical quantity, is the vorticity which is defined as the curl of the velocity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. If Stoke's theorem is applied to the definition of the circulation (1.4), the line integral of the velocity can be transformed to a surface integral of the vorticity

$$\Gamma = \int_S d\mathbf{S} \cdot \boldsymbol{\omega} = \text{constant} \quad (1.5)$$

which shows that vorticity is circulation per. unit area. A point vortex situated at the origin has vorticity $\boldsymbol{\omega} = \Gamma \delta(\mathbf{r})$.

The circulation of ordinary fluids can take any value. In superfluids this is no longer true, and the the circulation is *quantized* in integer steps of

$$\kappa = \frac{h}{m} \quad (1.6)$$

where h is Plank's constant and m is the physical atom/molecule mass. The circulation of a superfluid cannot be continuously changed and circulation of the whole fluid, κN , is thus a topological quantity; quantized vortex solution can not be found by perturbation of non-vortex solutions. The quantized vortices, as they appear in Ginzburg-Landau theory, are discussed in section 1.2.3.

The idea of quantization of circulation in superfluids can be ascribed to Onsager [Ons49]. In a footnote(!) he writes: "Vortices in a superfluid are presumably quantized; the quantum of circulation is h/m , where m is the mass of a single molecule."

1.1.2 The Magnus Force

The main attention of this thesis will be devoted to the transverse force on a vortex in superfluid helium II. This force will prove to have similarities to the force on a body in a conventional fluid, so it can be useful first to study the classical situation. A classical fluid passing a solid body will in many cases exert a force on the body. The force can be split in one parallel and one transverse to the initial fluid velocity. These components are of different natures. In the

cases when the longitudinal force is non-zero it is found to be strongly dependent on the shape and the surface of the body. In case of potential flow there is no longitudinal force on the body, which is a statement of d'Alembert's paradox. The transverse force, when it appears, is on the contrary not dependent on the details of the body, but only on the circulation in the fluid. The transverse force on a body from a fluid is called the Magnus force, after the German physicist Magnus who discovered it in 1852, on spinning bodies in a liquid.

In the following we will work in two dimensions. The result can easily be generalized to three dimensions by letting the force be the force per unit length. To find the force, let us suppose that the body is at rest and the fluid has the velocity \mathbf{v} . The force on the body can be found by considering the momentum-flux tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j \quad (1.7)$$

The momentum-flux tensor is interpreted as the density of j -momentum in the direction $+i$. The force on the fluid from the body is found by integrating the tensor around a surface enclosing the body. From Newton's third law the force on the body from the fluid is the same, but in opposite direction ($-i$ -direction). The components of the force are then

$$F_i = - \oint dS_j \Pi_{ij} \quad (1.8)$$

Let us suppose that far from the body solutions of the hydrodynamical equations with constant pressure P and density ρ can be found. Besides that, the velocity must be very near a constant velocity which we put along the x -axis, $\mathbf{v} = v_\infty \mathbf{e}_x + \delta\mathbf{v}$. The constant terms with P and $v_{\infty i} v_{\infty j}$ do not contribute to the force. If the second order term in the velocity difference $\delta\mathbf{v}$ is ignored, we get for the transverse force

$$F_\perp = - \oint dS_j \rho v_\infty \delta v_j \quad (1.9)$$

With constant ρ , the integral is nothing but the circulation (1.4). In vector notation, the Magnus force is

$$\mathbf{F}_M = \rho \mathbf{v}_\infty \times \mathbf{\Gamma} \quad (1.10)$$

The direction of the $\mathbf{\Gamma}$ vector is out of the plane. In this case with constant ρ and P , the longitudinal force is zero. With the above choice of solutions for P and \mathbf{v} , only the $v_i v_j$ term in the momentum-flux tensor is non-vanishing, but in general both terms will contribute. On an airplane wing, for example, the transverse force is caused by the pressure difference on the two sides of the wing. With other solutions a longitudinal force component could have been experienced, in addition.

If the body moves with velocity \mathbf{u} , $\mathbf{v}_\infty - \mathbf{u}$ can be substituted for \mathbf{v}_∞ in the Magnus force (1.10). For a body moving along with the velocity of the fluid, there is no Magnus force.

1.2 Ginzburg-Landau Theory

The understanding of superfluid helium has developed in many steps and there are many contributors. The theory which is presented here is called Ginzburg-Landau theory. The name is often used on the more general theory of superconductors as well, but here it will be applied to superfluids only (also referred to as Ginzburg-Pitaevskii theory). The historical development of the theories on liquid helium will not be presented here, but can be found in many text-books, e.g. [Gly94], [NP90] and [WB87]. This presentation will just serve as a brief introduction and the main point is to establish the formalism which will be used for most of the calculations in this thesis.

In nature there are two species of helium atoms ${}^4\text{He}$, which are Bosons and ${}^3\text{He}$ which are Fermions. For low temperatures these isotopes have very different behavior so that each of them must be discussed separately. This thesis will be restricted to *bosonic helium* only. With normal pressure we talk about three different phases of helium. For high temperatures it is of course a gas and below the boiling point it is an ordinary viscid liquid called helium I. In bosonic helium we can in addition experience a second order phase transition at a critical temperature T_λ . Below this temperature we speak of helium II, which is partly superfluid. At the absolute zero, helium is a pure superfluid. Helium II is more carefully described in chapter 4. bosonic helium is liquid down to the absolute zero and solid helium is only found at high pressure.

The theory describing liquid helium can start with the grand canonical Hamiltonian for a D -dimensional interacting system

$$\begin{aligned} \hat{K} &= \hat{H} - \mu \hat{N} \\ &= \int d^D r \hat{\Psi}^\dagger(\mathbf{r}) \left[-\frac{\hbar^2 \nabla^2}{2m} - \mu \right] \hat{\Psi}(\mathbf{r}) \\ &\quad + \frac{1}{2} \int d^D r \int d^D r' \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') V(\mathbf{r} - \mathbf{r}') \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}) \end{aligned} \quad (1.11)$$

where $\hat{\Psi}^\dagger(\mathbf{r})$ and $\hat{\Psi}(\mathbf{r})$ are the helium atom creation and annihilation operators and $V(\mathbf{r} - \mathbf{r}')$ is the interaction potential. The chemical potential is μ . The fields satisfy the Bose commutation rule

$$\left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}') \right] = \delta(\mathbf{r} - \mathbf{r}') \quad (1.12)$$

To get a reliable theory of the liquid, an appropriate interaction must be found. Interactions which are realistic at a microscopic level, do often distort the mathematical handling and hide the macroscopic behavior. The Ginzburg-Landau

theory is, on the contrary, a simple model where the fluid is treated as consisting of weakly repelling Bosons, i.e. the only interaction entering into our calculations is the repulsion when two atoms happen to be at the same point. The potential is then

$$V(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}') \quad (1.13)$$

Such a theory is not valid for small length-scales, but it can serve a good description at a larger level. The mathematical simplification of the model is overwhelming and makes it possible to carry out more detailed calculations. The equation of motion is

$$\begin{aligned} i\hbar \frac{\partial \hat{\Psi}(\mathbf{r})}{\partial t} &= [\hat{\Psi}(\mathbf{r}), \hat{K}] \\ &= \left[-\frac{\hbar^2}{2m} \nabla^2 + g\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}) - \mu \right] \hat{\Psi}(\mathbf{r}) \end{aligned} \quad (1.14)$$

Here m is the helium mass, g is the interaction strength and μ is the chemical potential. The interpretation of the chemical potential is that the system has a density $\rho_0 = m\mu/g$ which is energetically favorable. At temperatures near the absolute zero, the quantized fields can be replaced by their mean fields, $\langle \hat{\Psi} \rangle = \Psi$. Since Ψ is a classical function, it is much easier to handle than the operator $\hat{\Psi}$. Mean field in this context means the off-diagonal matrix element $\langle \Psi(N-1) | \hat{\Psi} | \Psi(N) \rangle$, where $\Psi(N)$ is the N particle wavefunction [NP90]. This is called the Gross-Pitaevskii approximation, which will be used in most of the calculations done later. The equation of motion for the classical field is

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + g|\Psi|^2 - \mu \right] \Psi \quad (1.15)$$

which is often in literature referred to as the nonlinear Schrödinger equation. A Lagrangian formalism can also be used instead of the Hamiltonian formalism. The Ginzburg-Landau Lagrangian density is

$$\mathcal{L} = \frac{i\hbar}{2} \left(\dot{\Psi}\Psi^* - \dot{\Psi}^*\Psi \right) - \frac{\hbar^2}{2m} |\nabla\Psi|^2 - \frac{g}{2m} (m|\Psi|^2 - \rho_0)^2 \quad (1.16)$$

The theory as described above contains many parameters. These are important when the thermodynamical properties of the system is considered, but are just distorting the equations when working at zero temperature. The fields and coordinates can be rescaled to *dimensionless form*, denoted with primes

$$\Psi' = \sqrt{\frac{\rho_0}{m}} \Psi \quad t' = \frac{g\rho_0}{\hbar m} t \quad x' = \frac{\sqrt{g\rho_0}}{\hbar} x \quad (1.17)$$

This transformations correspond to setting all the constants \hbar, g, m and ρ_0 to unity in our equation. Most calculations done in the rest of this chapter and in chapter 2 and 3 are in dimensionless units. The speed of sound is (when the velocity is small) equal to unity in dimensionless units. In dimensional units it is $c = \sqrt{g\rho_0/m}$.

1.2.1 Hydrodynamical form

The Ginzburg Landau theory is a quantum theory. The application of the theory is however on a liquid which in many cases can be treated like a classical liquid. The analogy between the classical equations in section 1.1 and the Ginzburg-Landau theory is best seen by writing the wavefunction on polar form

$$\Psi = \sqrt{\frac{\rho}{m}} e^{i\theta} \quad (1.18)$$

where ρ and θ are real functions. The wavefunction is a solution of the Schrödinger equation

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m}\nabla^2\Psi + \frac{g}{m}(m|\Psi|^2 - \rho_0)\Psi \quad (1.19)$$

The standard interpretation of quantum mechanics is that $|\Psi|^2$ is the probability density. The quantity $\rho = m|\Psi|^2$ is then *mass density*. The phase of the wavefunction will be a potential for the velocity, $\mathbf{v} = \frac{\hbar}{m}\nabla\theta$. By separating out the real and imaginary part of (1.18), we get two equations

$$\dot{\rho} + \frac{\hbar}{m}\nabla \cdot (\rho\nabla\theta) = 0 \quad (1.20)$$

$$\dot{\theta} - \frac{\hbar}{m}\frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}} + \frac{\hbar}{2m}(\nabla\theta)^2 + \frac{1}{\hbar}\frac{g}{m}(m|\Psi|^2 - \rho_0) = 0 \quad (1.21)$$

The first equation is the continuity equation. If we take the gradient of the second equation it begins to look like the Euler equation. Supposing that θ is non-singular, the order of time derivative and space derivative can be interchanged. This is however not true in the case of vortices as we will see later. Expressing the equations by the density and velocity, we get

$$\dot{\rho} + \nabla \cdot (\rho\mathbf{v}) = 0 \quad (1.22)$$

$$\dot{\mathbf{v}} + \mathbf{v}(\nabla \cdot \mathbf{v}) = -\frac{1}{m}\nabla \left[\frac{g}{m}\rho - \frac{\hbar^2\nabla^2\rho}{2m\sqrt{\rho}} \right] \quad (1.23)$$

where we have used the mathematical identity $\frac{1}{2}\nabla(\mathbf{v}^2) = \mathbf{v}(\nabla \cdot \mathbf{v})$. The last equation is a variant of Euler's equation (1.3), where the quantum effects are hidden in the peculiar looking potential on the right hand side.

1.2.2 Plane Waves

The physical ground-state of the Ginzburg-Landau liquid is a state minimizing the interaction energy, thus having density $\rho = 1$ in dimensionless units. Considering small fluctuations around this density, plane wave solutions can be found. In

addition it is supposed that the background fluid has the constant velocity $\mathbf{v}_0 = \nabla\theta_0$. If the velocity is small, the correction to the density ρ of order \mathbf{v}_0^2 , can be ignored. Seeking solutions of the Ginzburg-Landau equations near the ground state, the field equations can be linearized in the small quantities $\eta = \rho - 1$ and $\chi = \theta - \theta_0$.

$$\dot{\chi} + \eta - \frac{1}{4}\nabla^2\eta + \mathbf{v}_0 \cdot \nabla\chi = 0 \quad \dot{\eta} + \nabla^2\chi + \mathbf{v}_0 \cdot \nabla\eta = 0 \quad (1.24)$$

Plane wave solutions for η and χ are specially important, since other non-vortex solutions can be expressed as superpositions of such waves. Plane waves are of the form

$$\chi_{\mathbf{k}}(\mathbf{x}, t) = \chi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \quad \eta_{\mathbf{k}}(\mathbf{x}, t) = \eta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \quad (1.25)$$

where ω is the frequency and \mathbf{k} is the wave-number of the wave. In dimensionless units ω is also equal to the wave energy. It must be pointed out that all physical solutions of η and χ must be real, so we must be ready taking the real part of our solutions at any time. Inserting (1.25) into one of the linearized field equations leads to the following relations between the constants $\chi_{\mathbf{k}}$ and $\eta_{\mathbf{k}}$

$$\chi_{\mathbf{k}} = -i \frac{\omega_0}{k^2} \eta_{\mathbf{k}} \quad (1.26)$$

where $\omega_0 = \omega - \mathbf{k} \cdot \mathbf{v}_0$ is the energy in the rest frame where condensate velocity is zero. From the other field equation we get

$$\omega_0 = k^2 \left(1 + \frac{k^2}{4}\right) \quad (1.27)$$

with $k = |\mathbf{k}|$. For low k the ordinary linear phonon energy $\omega \approx k$ is obtained, but for higher momenta the energy deviates from this. For $k \gg 1$ it gives the peculiar phonon energy $\omega \approx \frac{k^2}{2} + 1$ and the speed of sound is also frequency dependent $c_k = \omega_0/k \approx k/2$. We must be careful to use this energy spectrum for higher energies in quantum liquids, since it does not fit the experimental curve, figure 4.1. In this thesis all calculations are in the low energy regime, with a linear phonon energy $\omega_0 \approx k$ and a constant speed of sound, $c = 1$ (In full units $c = \sqrt{g\rho_0/m}$).

An important quantity is the time average mass flux¹ of a plane wave. The definition is

$$\langle \mathbf{j} \rangle = \langle (1 + \eta)(\mathbf{v}_0 + \nabla\chi) \rangle \quad (1.28)$$

The plane waves η and χ are real, and only the real part of (1.25) count. The product of the real parts are $\Re(\eta)\Re(\nabla\chi) = \frac{1}{2}\Re(\eta^*\nabla\chi + \eta\nabla\chi)$. The first term,

¹In our dimensionless units this is also equal to the probability flux and the energy flux.

$\Re(\eta^* \nabla \chi)$, is time-independent and survives the time averaging, while the latter has time dependency $e^{2i\omega}$, and vanishes.

$$\langle \mathbf{j}_{\mathbf{k}} \rangle = \frac{1}{2} \frac{k^2}{\omega_k} |\chi_{\mathbf{k}}|^2 \mathbf{k} \equiv n_k \mathbf{k} \quad (1.29)$$

where n_k , as the the number density of the plane wave, is defined. In a thermal assembly of phonons this will appear as the distribution function.

1.2.3 Stationary Vortex

In Ginzburg-Landau theory the quantized vortices appear as multi-valued phases of the wavefunction. Since the wavefunction itself must be single-valued, the only transformations allowed for the phase are $\theta \rightarrow \theta + 2\pi q$, with q as an integer. There is no way to change it continuously from one value to another, and q can then be seen as a *topological quantity* of the fluid; it can only change in discrete steps.

If the vortex is *at rest at the origin* the q -fold vortex-solution is $\theta = q\varphi$ where $\varphi = \arctan(y/x)$ is the angle between y and x . The wave function on polar form is now

$$\Psi = f e^{iq\varphi} \quad (1.30)$$

where f is the square root of the density profile (or just the density profile for short) of the vortex. The symbol $f = f(r)$ will in the rest of the thesis be used to denote the density profile of a stationary vortex. The velocity field is $\mathbf{v} = \frac{q\hbar}{rm} \mathbf{e}_\varphi$. The vorticity is

$$q \frac{\hbar}{m} \nabla \times \nabla \varphi = q \frac{h}{m} \delta(\mathbf{r}) \quad (1.31)$$

where the delta function is recognized from the formula in appendix B. This tells us that the circulation is an integer multiple q of the circulation quantum $h/m = \kappa$ if the integral is enclosing the vortex, in all other cases it is zero.

The density profile is found by substituting $f = \sqrt{\rho}$ in the equations (1.20) and (1.21), which here are presented on dimensionless form. The first equation is now simply $\nabla \varphi \cdot \nabla f$, which means that f is a function of the radius alone, $f = f(r)$. From the second we obtain the differential equation

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \left(2 - \frac{q^2}{r^2}\right) f - 2f^3 = 0 \quad (1.32)$$

The boundary condition far from the vortex is that the density must reach the density of the rest of the fluid, which in dimensionless units means that $f \rightarrow 1$. Near the origin we must have $f \rightarrow 0$ to avoid a singular wave-function. An analytical solution of (1.32) with these boundary conditions is not possible

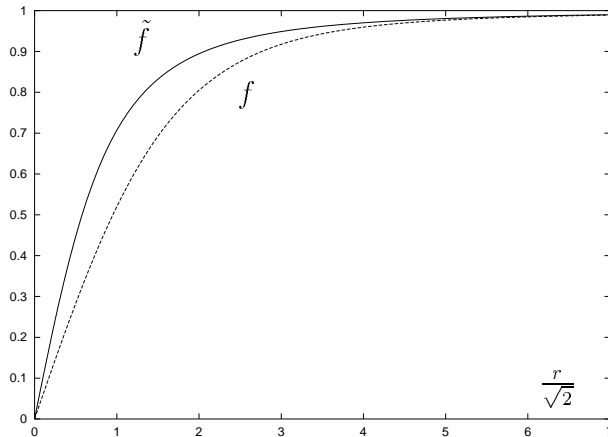


Figure 1.1: Numerical solution of the density profile f , and the analytical approximation \tilde{f} , as functions of $r/\sqrt{2}$.

to achieve, so the best we can do is to find the asymptotic behavior and plot numerical solutions.

By doing series expansions near the origin we easily find $f \sim r^{|q|}$, but the proportionality factor is still undetermined. It must be found by fitting solutions near the vortex to those far from it. Far from the vortex the profile can be expanded in powers of r^{-1} , giving $f \sim 1 - \frac{1}{4r^2}$. Since the gradient of the phase goes as r^{-1} the tail of the vortex profile far from the vortex can in many cases be ignored. This is called the point vortex approximation. These two asymptotic expressions can, when $q = \pm 1$, be brought together in an analytical approximation for f . The function

$$\tilde{f} = \frac{r}{\sqrt{r^2 + \frac{1}{2}}} \quad (1.33)$$

which was found by Fetter [Fet65] will serve as an adequate approximation to the full function f . The figure 1.1 shows a plot comparing the analytical approximation (1.33) and the numerical solution of Kawatra and Pathria [KP66].

For the rest of the thesis we will suppose that vortices have winding-numbers $|q| = 1$, where a solution with $q = 1$ will be called a vortex, and a solution with $q = -1$ will be called an anti-vortex.

1.2.4 Moving Vortex

From stationary vortex solutions, it is possible to find approximate solutions for a slowly moving vortex, by assuming that the solutions of the moving case can be seen a small perturbation of the stationary case. The phase of the wave function is thought to be $\theta = q\varphi_0 = q \arctan\left(\frac{y-y_0}{x-x_0}\right)$, where $\mathbf{r}_0 = \mathbf{r}_0(t)$ is the vortex

position. For the phase we write $\rho = f^2(1 + \delta\rho)$, where $f = f(|\mathbf{r} - \mathbf{r}_0|)$ is the density profile of a stationary vortex, defined in the last section. By inserting this into the Ginzburg-Landau equation of hydrodynamical form, (1.20) and (1.21), the definition of f , (1.32), can be used to get rid of some terms. By ignoring second order terms in $\delta\rho$, we get the following two equations

$$2f\dot{f} + f^2\delta\dot{\rho} + qf^2\nabla\delta\rho \cdot \nabla\varphi_0 = 0 \quad (1.34)$$

$$q\dot{\varphi}_0 + f^2\delta\rho - \frac{1}{f}\nabla f \cdot \nabla\delta\rho - \frac{1}{2}\nabla\delta\rho \approx 0 \quad (1.35)$$

The time derivatives of f and φ_0 are entirely through the vortex position \mathbf{r}_0 , so that $\dot{f} = -\nabla f \cdot \dot{\mathbf{r}}_0$ and $\dot{\varphi}_0 = -\nabla\varphi_0 \cdot \dot{\mathbf{r}}_0$. When making a rough estimate of $\delta\rho$ far from the vortex, it is enough to work to order $\sim |\mathbf{r} - \mathbf{r}_0|^{-1}$, thus putting $f = 1$. Seeking slowly changing solutions, the $\nabla^2\delta\rho$ term can be ignored giving a simple equation for the density correction

$$\delta\rho \approx q\nabla\varphi_0 \cdot \dot{\mathbf{r}}_0 \quad (1.36)$$

The correction is of order $\sim |\mathbf{r} - \mathbf{r}_0|^{-1}$ and thus dominating compared to the correction in the constant density profile $1 - f \sim |\mathbf{r} - \mathbf{r}_0|^{-2}$. If the vortex moves with constant velocity along the x -axis, so that $x_0 = vt$ and $y_0 = 0$, the density correction is $\delta\rho = -qv\frac{y}{(x-vt)^2+y^2}$. The correction is anti-symmetric about the x -axis, and proportional to the vortex winding number q .

This result can also be applied to the situation when the background fluid is not at rest, but moving with some velocity \mathbf{v}_s , by substituting $\dot{\mathbf{r}}_0 \rightarrow \dot{\mathbf{r}}_0 - \mathbf{v}_s$. The force on the vortex in this situation will be calculated in section 3.4.

1.3 Vortex Motion

In section 1.2.4 the change of the fields far from the vortex, due to the vortex motion, was studied. Now we want to examine the equations inside the vortex core, in order to find the vortex equations of motion. The derivation will follow that of E. Schröder and O. Törnkvist [ST97], except that their derivation is done for a vortex string in three dimensions while we only will consider a vortex point in two dimensions. We start with the Ginzburg-Landau field equations on hydrodynamical form

$$\dot{\rho} + \nabla \cdot (\rho\nabla\theta) = 0 \quad (1.37)$$

$$\dot{\theta} + \frac{1}{2}(\nabla\theta)^2 - \frac{1}{2}\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho} + \rho - 1 = 0 \quad (1.38)$$

where ρ and θ is the density and the phase of the wave function. The contributions originating from the vortex can be separated out

$$\theta \rightarrow q\varphi + \tilde{\theta} \quad \rho \rightarrow f^2\tilde{\rho}$$

where the density profile of the vortex is $f = f(R)$. We use $R = |\mathbf{r} - \mathbf{r}_0|$ and $\varphi = \arctan(\frac{y-y_0}{x-x_0})$ which is the polar coordinate system centered at the moving vortex point $\mathbf{r}_0 = \mathbf{r}_0(t)$. The remaining functions $\tilde{\rho}$ and $\tilde{\theta}$ are supposed to be non-singular and well behaved near the vortex point. The field equations can now be written as

$$2\frac{\dot{f}}{f} + \frac{\dot{\tilde{\rho}}}{\tilde{\rho}} + \frac{2}{f}\nabla f \cdot \nabla \tilde{\theta} + \frac{1}{\tilde{\rho}}\nabla \tilde{\rho} \cdot \nabla (q\varphi + \tilde{\theta}) + \tilde{\rho}\nabla^2 \tilde{\theta} = 0 \quad (1.39)$$

$$q\dot{\varphi} + \dot{\tilde{\theta}} + \frac{1}{2}\left(q\varphi + \nabla \tilde{\theta}\right)^2 - \frac{1}{2}\frac{1}{f\sqrt{\tilde{\rho}}}\nabla^2(f\sqrt{\tilde{\rho}}) = 0 \quad (1.40)$$

where the identities $\nabla f \cdot \nabla \varphi = 0$ and $\nabla^2 \varphi = 0$ have been used. All the time dependency in f and φ is through \mathbf{r}_0 , so that $\dot{\varphi} = -\nabla \varphi \cdot \dot{\mathbf{r}}_0$ and $\dot{f} = -\nabla f \cdot \dot{\mathbf{r}}_0$. We now do expansions in powers of R^{-1} . The gradient of the phase is $\nabla \varphi = \frac{1}{R}\mathbf{e}_\varphi$ and the density profile is proportional to R , $f \sim R$. All the terms of power R^{-2} fall out and the terms proportional to R^{-1} are

$$\left(\nabla \tilde{\theta} - \dot{\mathbf{r}}_0 + \frac{q}{2}\nabla \ln(\tilde{\rho}) \times \mathbf{e}_z\right) \cdot \mathbf{e}_R = 0 \quad (1.41)$$

$$\left(\nabla \tilde{\theta} - \dot{\mathbf{r}}_0 + \frac{q}{2}\nabla \ln(\tilde{\rho}) \times \mathbf{e}_z\right) \cdot \mathbf{e}_\varphi = 0 \quad (1.42)$$

In the limit where $R \rightarrow 0$ these solutions are exact. The following equation relates the motion of the vortex to the fields $\tilde{\rho}$ and $\tilde{\theta}$

$$\dot{\mathbf{r}}_0 = \left[\nabla \tilde{\theta} + \frac{q}{2}\nabla \ln(\tilde{\rho}) \times \mathbf{e}_z\right]_{\mathbf{r}=\mathbf{r}_0} \quad (1.43)$$

The exact vortex motion is found as self-consistent solutions for the vortex equation of motion (1.43) and the fields. And approximation of the vortex motion can be found by inserting the vortex into solutions of the field equation without bothering about the influence of the vortex on the solutions. The only application which will be considered in this thesis, is the vortex in a plane wave, in section 1.3.1.

1.3.1 A Vortex in a Plane Wave

One application of the vortex motion formula, (1.43), is to see how a vortex behaves in a plane wave. This means that the modification of the wave because of the vortex (which is so carefully discussed in chapter 2) is ignored.

The linearized field equations have plane wave solutions, which were studied in section 1.2.2. To avoid confusion, let us suppose that η_k is real. The plane wave solutions of the phase χ and density $\eta = \rho - 1$ are then

$$\chi = \frac{1}{k}\eta_k \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad \eta = \eta_k \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad (1.44)$$

where the frequency, in the low energy limit is, $\omega = k$. If the plane wave is propagating along the x -axes the equations of motion (1.43) for the vortex are

$$\dot{x}_0 = \eta_{\mathbf{k}} \cos(kx_0 - \omega t) \quad (1.45)$$

$$\dot{y}_0 = -\frac{q}{2}\eta_{\mathbf{k}} \sin(kx_0 - \omega t) \quad (1.46)$$

This is in qualitative agreement with what is earlier done by L. K. Myklebust [Myk96] in his thesis for Cand. Scient. He uses $\tilde{f} = r/\sqrt{r^2 + 1/2}$ as an approximation for the vortex profile, and in the small k limit, he gets the same k -dependency as above. But his result differs from ours in that he gets $\frac{q}{4}$ instead of $\frac{q}{2}$ as the factor in the \dot{y}_0 equation.

A very rough estimate of the solution of the above equations can be found if $k\mathbf{r}_0$ is small. If the kx_0 terms are ignored on the right hand side, we get

$$\dot{x}_0 \approx \eta_{\mathbf{k}} \cos(\omega t) \quad (1.47)$$

$$\dot{y}_0 \approx -\frac{q}{2}k\eta_{\mathbf{k}} \sin(\omega t) \quad (1.48)$$

The boundary conditions can be chosen so that $y_0(0) = -\frac{q}{2}\eta_{\mathbf{k}}$ and $x_0(0) = 0$, which gives

$$x_0(t) = \frac{1}{k}\eta_{\mathbf{k}} \sin(\omega t) \quad (1.49)$$

$$y_0(t) = -\frac{q}{2}\eta_{\mathbf{k}} \cos(\omega t) \quad (1.50)$$

which describes an ellipse. The relation between x_0 and y_0 is then

$$\left(\frac{x_0}{a}\right)^2 + \left(\frac{y_0}{b}\right)^2 = 1 \quad (1.51)$$

with $a = \eta_{\mathbf{k}}/k$ and $b = \frac{q}{2}\eta_{\mathbf{k}}$. When k gets smaller, the motion gets more and more in the x -direction. The area of the ellipse is $\pi ab = \frac{\pi q}{2k}\eta_{\mathbf{k}}^2$. The area is independent of k if the normalization $\eta_{\mathbf{k}} \sim \sqrt{k}$ is chosen.

In this estimate of the motion, a lot of effects have been ignored. The scattering of the wave on the vortex will, as seen in chapter 2 and 3, exert a force on the vortex. If the vortex has a mass² it would give a drift of the vortex in the direction of the force. If the analogy to electro-magnetism is applied (section 1.4), where vortices are interpreted as charges, one could expect a radiation of energy from the accelerating vortex. The vortex losing energy will spiral inwards.

1.4 Analogy to Electro-Magnetism

For a long time there has been known analogies between two-dimensional electro-magnetism and hydrodynamics on a thin film. Before Maxwell's equations were

²The question of the vortex mass is still unsolved.

well established people could gain insight on electro-magnetism from such analogies by experimenting on fluids. An introduction to the analogies between a classical fluid and electro-magnetism can be found in [Myk96]. Today the situation is the other way round: electro-magnetism is very well understood and studied, while there still exist phenomena in fluid mechanics which are still not entirely understood. One such phenomenon is quantized vortices in superfluid films, which can be shown to have some of the same behavior as (quantized) point charges in an electro-magnetic theory. A reference starting with Ginzburg-Landau theory is an article by D.F. Arovas and J.A. Freire [AF96]. They work directly with the Ginzburg-Landau Lagrangian density. By requiring that the partition function shall be invariant, they can integrate out some of the variables and obtain a Lagrangian which is to second order in the fields equal to the QED-Lagrangian. We will do a more simple and direct approach instead, by starting with the field equations on polar form, (1.20) and (1.21), and try to transform them into Maxwell's equations.

There also exists an analogy to the potential formulation of electro-dynamics, which is described in [Myk96]. In chapter 2 the analogy between phonon scattering on a vortex and Aharonov-Bohm scattering is discussed. This analogy comes from terms quadratic in the fields and will not be transparent in the linearized equations considered here.

1.4.1 2-D Electro-Magnetism

First we will take brief look at two-dimensional electro-magnetism, where the \mathbf{E} -field is lying in the plane while the B -field is perpendicular to the plane. This B -vector can hence be treated as a scalar, since it has a fixed direction. In two dimensions there are only three Maxwell's equations, since $\nabla \cdot \mathbf{B} = 0$ always holds.

$$\nabla \cdot \mathbf{E} = \rho_c \quad (1.52)$$

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla B \times \mathbf{e}_z = \mathbf{J}_c \quad (1.53)$$

$$\frac{\partial B}{\partial t} + \nabla \times \mathbf{E} = 0 \quad (1.54)$$

The equations are Gauss' law, Ampere's law and Faraday's law. The density ρ_c is the charge density and \mathbf{J}_c is the current density. Instead of using the fields B and \mathbf{E} , we could use a potential formulation and introduce a Lagrangian density as a function of the potential ϕ and \mathbf{A} . Then the first two equations would be the field equations, while Faraday's law would be a consequence of the definitions $B = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}$.

Charged particles moving in an electric and magnetic field will also experience a force; the Lorentz force. If the particle position is \mathbf{r}_0 and it has mass m and

charge Q , the force is simply

$$m\ddot{\mathbf{r}}_0 = Q(\mathbf{E} + B\dot{\mathbf{r}}_0 \times \mathbf{e}_z) \quad (1.55)$$

where \mathbf{E} and B is the electric and magnetic field evaluated at the particle position. These are all the basic equations needed in classical electro-magnetism in two dimensions.

1.4.2 The Analogy to Electro-Magnetism

In dimensionless units, the Ginzburg-Landau field equations on hydrodynamical form, (1.20) and (1.21) are

$$\dot{\rho} + \nabla \cdot (\rho \nabla \theta) = 0 \quad (1.56)$$

$$\dot{\theta} + \frac{1}{2}(\nabla \theta)^2 - \frac{1}{2} \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} + \rho - 1 = 0 \quad (1.57)$$

We want to do the identification

$$Q = 2\pi q \quad B = 1 - \rho \quad \mathbf{E} = \nabla \theta \times \mathbf{e}_z \quad (1.58)$$

where Q will be identified as charge, B as the magnetic field and \mathbf{E} as the electric field. The most common way is to define the electric field as proportional to $\rho \nabla \theta \times \mathbf{e}_z$. The continuity equation, (1.56), is then Faraday's law exactly. The reason why we choose to do it differently, is that our choice makes the \mathbf{E} -field singular. This makes it possible to draw the analogy to point charges and still keep information from the density profile³. If there are N vortices present with positions $\mathbf{r}_k(t)$, Gauss' law is directly found from the definition of \mathbf{E}

$$\nabla \cdot \mathbf{E} = \nabla \cdot (\nabla \theta \times \mathbf{e}_z) = \sum_k 2\pi q_k \delta(\mathbf{r} - \mathbf{r}_k) \equiv \rho_v \quad (1.59)$$

where the delta functions are identified from $\nabla \times \nabla \varphi = 2\pi \delta(\mathbf{r})$ found in appendix B. The quantized vortices appear as point charges in the electro-magnetic formulations! The charges are given by $Q = 2\pi q$. In full units this would have been proportional to the quantized circulation of the vortex.

In the first equation (1.56) the substitutions can be done without much trouble. In the second (1.57) we have to take the gradient of the whole equations and take the cross product with \mathbf{e}_z to be able to do the identifications. The gradient and the time derivative do not commute, since the \mathbf{E} -field is singular. From appendix B the commutator is found $[\nabla, \frac{\partial}{\partial t}] \varphi_0 \times \mathbf{e}_z = 2\pi \dot{\mathbf{r}}_0 \delta(\mathbf{r} - \mathbf{r}_0)$. The vortex current can be identified

$$\left[\nabla, \frac{\partial}{\partial t} \right] \theta \times \mathbf{e}_z = \sum_k 2\pi q_k \dot{\mathbf{r}}_k \delta(\mathbf{r} - \mathbf{r}_k) \equiv \mathbf{J}_v$$

³In most analogies only point vortices are considered.

The two hydrodynamical field equations are now

$$\dot{B} + \nabla \times \mathbf{E} = B \nabla \times \mathbf{E} - \mathbf{E} \cdot (\nabla B \times \mathbf{e}_z) \quad (1.60)$$

$$-\dot{\mathbf{E}} + \nabla B \times \mathbf{e}_z = \mathbf{J}_v + \frac{1}{2} \nabla \left[\mathbf{E}^2 + \frac{\nabla^2 \sqrt{1-B}}{\sqrt{1-B}} \right] \times \mathbf{e}_z \quad (1.61)$$

To compare this with the last two of Maxwell's equations we must linearize in B and \mathbf{E} . It must be noted that linearization is only valid far from the vortex cores, since $B \rightarrow 1$ near the vortex. After the linearization the equations are still not exact Maxwell's equations, since (1.61) has an additional term $\sim \nabla^2 B$. In the low energy limit, where the B field is slowly varying, this term can be neglected. The analogy to electro-magnetism is hence valid just in the low energy limit.

1.4.3 The Lorentz Force

The last equation needed is the Lorentz force, or the force on a moving charge in an electro-magnetic field. In hydrodynamics this must be analogous to the force on a moving vortex in a Bose-liquid. In section 1.3 an equation for the motion of a vortex (1.43) was derived

$$\dot{\mathbf{r}}_0 = \left[\nabla \theta + \frac{q}{2} \nabla \ln \rho \times \mathbf{e}_z \right]_{\mathbf{r}=\mathbf{r}_0} \quad (1.62)$$

where \mathbf{r}_0 is the vortex position. Applying the analogy (1.58), this can be written

$$Q [\mathbf{E} + \dot{\mathbf{r}}_0 \times \mathbf{B}_{ex}] = -\pi \nabla B \quad (1.63)$$

where a large constant external magnetic field is defined to be $\mathbf{B}_{ex} = -\mathbf{e}_z$. The evaluations of \mathbf{E} and ∇B are still at the vortex position \mathbf{r}_0 . The left hand side of (1.63) looks pretty much as a Lorentz force. The term $\dot{\mathbf{r}}_0 \times \mathbf{B}_{ex}$ is also equal to the background fluid Magnus force, which is discussed in section 3.4.

The right hand side is not proportional to the vortex acceleration as expected for a massive vortex, nor zero, as for a massless vortex. From the form of the equation (1.63) we can thus not draw any conclusion about the vortex mass. In [AF96] the right hand side of the equation was zero, implying a massless vortex. The reason why we arrived on different looking result, is that the derivation done in section 1.3 kept information from the density profile of the vortex, while most analogies are done within the point vortex approximation.

Chapter 2

Phonon Scattering

A well-known phenomenon in quantum physics is the Aharonov-Bohm effect, which says that an electron beam is affected by a magnetic flux-string, even if the electron itself is not in direct contact with the flux. The whole effect is due to the vector potential around the flux-string and hence a pure quantum phenomenon. In this chapter, we will see that this is to some extent analogous to the effect of a stationary vortex string on a phonon wave, which will be called *the acoustic Aharonov-Bohm effect*. We will exploit this analogy by using the well known solutions of the Aharonov-Bohm effect to discuss our acoustic scattering problem. In recent time there has been a lot of interest in this subject, since it is still not entirely understood and it is important when discussing the forces acting in helium II. A more complete discussion of the forces in helium II is held in chapter 5. In this chapter *only* the effect of a single plane wave on a stationary vortex is considered. The discussion will mainly follow the arguments of Sonin [Son96] and Stone [Sto99]. The original paper by Aharonov and Bohm [AB59] will also prove a useful source. The summation in the partial wave analysis will be done according to Sommerfield and Minakata [SM00].

2.1 The Scattering Equation

There are at least two possible starting points when discussing the scattering of a plane wave on a vortex. The first one is the classical hydrodynamical equation for a superfluid; the second is the quantum mechanical Ginzburg-Landau theory. We will use the latter in the semi-classical Gross-Pitaevskii approximation. If the vortex line is straight, we have cylindrical symmetry and effectively a two-dimensional problem.

We start with the Schrödinger equation on polar form. With one vortex situated at the origin, $\theta = \chi + q\varphi$ and $\rho = f(1+\eta)$ are substituted. If the deviation from a stationary vortex solution is small, the equations (1.20) and (1.21) can be

linearized in η and χ , to obtain

$$\dot{\chi} + \eta - \frac{1}{4}\nabla^2\eta = \lambda \left\{ -q\nabla\varphi \cdot \nabla\chi + (1 - f^2)\eta + \frac{1}{2}\nabla\ln(f) \cdot \nabla\eta \right\} \quad (2.1)$$

and

$$\dot{\eta} + \nabla^2\chi = \lambda \left\{ -q\nabla\varphi \cdot \nabla\eta - 2\nabla\ln(f) \cdot \nabla\chi \right\} \quad (2.2)$$

where the parameter $\lambda = 1$ is introduced to ease the book-keeping. Notice that $\lambda = 0$ correspond to no vortex present, giving the wave equations studied in (1.2.2). In the case of plane waves, without disturbing vortices, $\nabla^2\eta = -k^2\eta$. In the long wavelength limit, where $k \ll 1$, we thus have

$$|\nabla^2\eta| \ll |\eta| \quad (2.3)$$

We will now suppose that this condition holds with a vortex in the background too. The $\nabla^2\eta$ term in (2.1) will thus be ignored. It is not obvious that small k always ensures that $\nabla^2\eta$ is small when the vortex is present, but this can be explicitly checked later when the solutions are found (see section 2.4.5).

The equations (2.1) and (2.2) will now be solved by treating the appearance of the vortex as a perturbation. We want the solutions to be as close as possible to plane waves, since such waves later enable us to introduce temperature in the fluid. In the point vortex approximation $f = 1$, and q could have been used as the perturbation parameter. But since we will try to extract some information from the density profile, the new parameter $\lambda = 1$ is introduced. Both λ and q have the disadvantage that they are not small numbers themselves. The perturbation must then in some sense reflect that the operators appearing with the vortex shall not be too influential. This can happen in at least two ways. First: since the operators contain derivatives, their effect can be small in the low energy limit. Second: The operators can be small far from the vortex. Typically the phase contribution goes as $\sim r^{-1}$ and the density as $\sim r^{-2}$. In full units we would in addition have noticed that the vortex contributions are proportional to the flux quantum $\frac{h}{m}$, and the perturbation could also be seen as a semi-classical approximation in powers of h . Formally all this reduces to series expansions in λ . From the first equation (2.1) the density can be expressed by the phase exclusively

$$\eta = -\dot{\chi} + \lambda \left[-q\nabla\varphi \cdot \nabla - \left(1 - f^2 + \frac{1}{2}\nabla\ln(f) \cdot \nabla \right) \frac{\partial}{\partial t} \right] \chi + \mathcal{O}(\lambda^2) \quad (2.4)$$

Inserting this into the equation (2.2), the scattering equation is found

$$\begin{aligned} \left[\nabla^2 - \frac{\partial^2}{\partial t^2} \right] \chi = \lambda \left[2q\nabla\varphi \cdot \nabla \frac{\partial}{\partial t} \right] \chi \\ \lambda \left[(1 - f^2) \frac{\partial^2}{\partial t^2} - 2 \left(1 - \frac{1}{4} \frac{\partial^2}{\partial t^2} \right) \nabla\ln(f) \cdot \nabla \right] \chi + \mathcal{O}(\lambda^2) \end{aligned} \quad (2.5)$$

Further we assume a harmonic time dependency

$$\chi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-i\omega t} \quad \eta(\mathbf{r}, t) = \nu(\mathbf{r})e^{-i\omega t} \quad (2.6)$$

In the low energy regime the frequency is $\omega \approx |\mathbf{k}|$. After dropping a higher order term in k and putting λ back to unity, the scattering equation is

$$[\nabla^2 + k^2] \phi = [-2ikq \nabla \varphi \cdot \nabla - (1 - f^2)k^2 - 2\nabla \ln(f) \cdot \nabla] \phi \quad (2.7)$$

In most cases only the phase contributions to the equations will be considered, since it is probably dominant. The function f will then be put to unity.

The contents of the rest of this chapter are discussions and solutions of the scattering equation (2.7). In section 2.2 the analogy between the scattering equation and Aharonov-Bohm scattering is considered. In section 2.3 the equation is solved by using the Born approximation, while in 2.4 the tools of partial wave analysis is used. The density corrections are studied in section 2.5.

When the phase ϕ is known, the density fluctuation η can be found from (2.4).

2.2 Analogy to Aharonov-Bohm Scattering

Consider a magnetic flux-string piercing a plane while the magnetic field is zero anywhere else. The vector potential generating the magnetic field is non-zero however, and it will interact with a passing electron beam, even if the beam is not in any direct contact with the flux itself. The effect is then totally due to the vector potential, and this is what is called the Aharonov-Bohm effect. The time-independent Schrödinger equation for an electron interacting with a vector potential is

$$\left[\nabla + i \frac{e}{c} \mathbf{A} \right]^2 \psi = -2m E \psi \quad (2.8)$$

where $E = \frac{p^2}{2m}$ is the electron energy and $\hbar = 1$. A common example of such a potential is one giving a magnetic field proportional to a delta function

$$\mathbf{A} = \frac{1}{2\pi} \Phi_m \nabla \varphi \quad (2.9)$$

where Φ_m is the total magnetic flux and $\nabla \varphi = \frac{1}{r} \mathbf{e}_\varphi$. It is convenient to define $\alpha = \frac{e}{2\pi c} \Phi_m$. In their article [AB59], Aharonov and Bohm solved this equation by partial wave analysis and from that solution they extracted, in the limit of large r , the famous expression¹

$$\psi \sim e^{ikr \cos(\varphi) - i\alpha(\varphi - \pi)} + \frac{a^{AB}(\varphi)}{\sqrt{r}} e^{ikr} \quad (2.10)$$

¹In their paper, the incoming wave was from the right, not left as in our case.

where the Aharonov-Bohm scattering amplitude is

$$a^{AB}(\varphi) = \frac{\sin(\pi\alpha)}{\sqrt{2\pi ik}} \frac{e^{-i\frac{\varphi}{2}}}{\sin\left(\frac{\varphi}{2}\right)} \quad (2.11)$$

Notice that the first term in (2.10) solves the Aharonov-Bohm equation (2.8) but this solution does not satisfy the right boundary conditions. Aharonov and Bohm solved the equation (2.8) by partial wave analysis, and this method is presented in section 2.4. One problem with partial wave analysis is to determine the right boundary conditions. In the original paper [AB59] the constraint was that the current density should be along the x -axis asymptotically far from the vortex. Exactly the same solution was found by M.V. Berry [Ber80] by requiring that the wavefunction should be single-valued. The single-valuedness of the solution is not transparent in the asymptotic form (2.10) but is obvious in the partial wave expression. The partial wave expression for the wavefunction is finite and well behaved at all distances r from the origin and at all angles φ . The asymptotic expression for the scattering amplitude (2.11) is on the contrary singular as $\varphi \rightarrow 0$. This singularity appears because the attempt to extract a scattering amplitude fails near the forward direction! A different expression is needed. A good treatment of the near forward direction is found in an article by C.M. Sommerfield and H. Minakata [SM00]. E.B. Sonin has also derived a version for the acoustic problem [Son96]. It turns out that the forward direction will be extremely important when the force on a vortex is found in chapter 3.

An analogy to the magnetic Aharonov-Bohm scattering can be found in superfluid helium, where the time independent phase variation of the Ginzburg-Landau wavefunction ϕ in (2.7) plays the role of the electron wavefunction in (2.8). The static field around the point vortex plays the role of the vector potential. In the point vortex approximation, the acoustic scattering equation (2.7) is

$$(\nabla^2 + k^2) \phi = -2i\alpha \nabla\varphi \cdot \nabla\phi \quad (2.12)$$

where the vortex flux is $\alpha = qk$, and q is quantized, $q = \pm 1$, with sign dependent whether we have the vortex, or the anti-vortex solution. Working to first order in the vortex contribution, a second order term in α might as well be added, giving

$$[\nabla + i\alpha \nabla\varphi]^2 \phi \approx -k^2 \phi \quad (2.13)$$

which is similar to the original Aharonov-Bohm equation. The rewriting can be done since $\nabla^2\varphi = 0$. In contrast to the original Aharonov-Bohm problem, where the magnetic flux is an external parameter which can be changed independently of the electron energy, the total flux in the acoustic case is *dependent of the frequency* ω of the incoming wave. For a situation with a fixed wave number this will not cause any problems, but as soon as we have to manipulate the boundary conditions in a specified situation this difference will be visible.

2.3 Solution by Born Approximation

A standard method of solving scattering equations is the Born approximation. It gives the solution of the equation to first order in the scattering parameter. The scattering equation (2.7) can be written as

$$(k^2 + \nabla^2)\phi = \lambda O\phi \quad (2.14)$$

where the operator O is

$$O = [-2ikq\nabla\varphi \cdot \nabla - (1 - f^2)k^2 - 2\nabla \ln(f) \cdot \nabla] \quad (2.15)$$

The solution of the equation to first order in λ is

$$\phi = \phi_k (e^{i\mathbf{k}\cdot\mathbf{r}} + \lambda\delta\phi) + \mathcal{O}(\lambda^2) \quad (2.16)$$

where ϕ_k is a constant. The scattered wave can be expressed by the Green's function, which is a solution of the equation

$$(\nabla^2 + k^2)D(\mathbf{r}) = -\delta^2(\mathbf{r}) \quad (2.17)$$

The two-dimensional time-independent Green's function is

$$D(\mathbf{r}) = \frac{i}{4}H_0^{(1)}(kr) \quad (2.18)$$

found in appendix C. The function $H_0^{(1)}(kr)$ is the zeroth order Hankel function of the first kind. The Born solution for the scattered wave is

$$\delta\phi = \frac{i}{4} \int d^2r' N(\mathbf{r}') H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|) \quad (2.19)$$

The function $N(\mathbf{r})$ is

$$N(\mathbf{r}) = Oe^{i\mathbf{k}\cdot\mathbf{r}} = \left[2\alpha k \frac{\sin(\varphi)}{r} - k^2(1 - f^2) + 2\frac{f'}{f}ik \cos(\varphi) \right] e^{i\mathbf{k}\cdot\mathbf{r}} \quad (2.20)$$

where $\alpha = qk$ and $f = f(r)$ is the density profile of the vortex. Notice that in a point vortex description, with $f = 1$, the last two terms disappear, leaving only the first term, which we will call the phase contribution. Most likely this is the most important term, but we are interested in seeing if the other two terms give any considerable contribution, so they will be kept until a good reason to throw them away has been found. The last two terms will be called the density contributions since they are the contributions from the density profile of the vortex.

2.3.1 The Scattering Amplitude

In scattering problems it is often possible to separate the r dependency and the angle dependency in the scattered wave. The factor which is a function of φ is then the scattering amplitude. The scattering amplitude for a phonon scattered on a vortex will now be found.

The phase contribution to the scattering equation is usually considered as most important, so in the following we will only use the first term in (2.20), giving the scattered wave as

$$\delta\phi = \frac{i\alpha k}{2} \int d^2r' \frac{\sin(\varphi')}{r'} H_0^{(1)}(k|\mathbf{r}' - \mathbf{r}|) e^{ikr' \cos(\varphi')} \quad (2.21)$$

The effect of the density profile of the vortex will be studied in section 2.5. The Born integral is difficult to carry out in general and we need to do different approximations. The ordinary scattering amplitude is found by assuming that the main contribution to the integral comes from the region where $k|\mathbf{r} - \mathbf{r}'| \gg 1$. Doing series expansions in r^{-1} , we get

$$|\mathbf{r} - \mathbf{r}'| = r - r' \cos(\varphi' - \varphi) + \mathcal{O}(r^{-1}) \quad (2.22)$$

The expansion of the Hankel function for large arguments is

$$H_0^{(1)}(z) \xrightarrow{z \rightarrow \infty} \sqrt{\frac{2}{\pi iz}} e^{iz} \quad (2.23)$$

A trigonometric identity is also needed

$$\cos(\varphi') - \cos(\varphi' - \varphi) = -2 \sin\left(\frac{\varphi}{2}\right) \sin\left(\varphi' - \frac{\varphi}{2}\right)$$

Notice that if $\varphi \approx 0$ the φ' dependency in the exponent in the integrand disappears and therefore more terms are needed in the expansion to get a correct result for small angles (section 2.3.3). If the integration is performed in polar angles (r', φ') , the integral is

$$\delta\phi_a = \alpha \frac{ik}{2} \sqrt{\frac{2}{\pi ikr}} e^{ikr} \int_0^\infty dr' \int_0^{2\pi} d\varphi' \sin(\varphi') e^{-2ikr' \sin(\varphi/2) \sin(\varphi' - \varphi/2)}$$

First the integration with respect to φ' is done. Since the integration is over a whole period, the substitution $\varphi' \rightarrow \varphi' + \frac{\varphi}{2}$ can be made without changing the integration limits

$$\begin{aligned} & \int_0^{2\pi} d\varphi' \sin(\varphi' + \varphi/2) e^{-i2kr' \sin(\varphi/2) \sin(\varphi')} \\ &= \int_0^{2\pi} d\varphi' [\sin(\varphi') \cos(\varphi/2) + \cos(\varphi') \sin(\varphi/2)] e^{-i2kr' \sin(\varphi/2) \sin(\varphi')} \end{aligned}$$

By substituting $u = \sin(\varphi')$ it is easily seen that the second integral is zero. The first integral can be rewritten to a form recognizable as a Bessel function (see appendix A)

$$\int_0^{2\pi} d\varphi' \cos(\varphi') e^{-i2kr' \sin(\varphi/2) \sin(\varphi')} = 2\pi i J_1(-2kr' \sin(\varphi/2))$$

The r' integration is now only the integral over a Bessel function. The condition that φ shall be non-zero is now clearly seen to be essential to the r' integration

$$\int_0^\infty dr' J_1(-2kr' \sin(\varphi/2)) = \frac{1}{-2k \sin(\varphi/2)}$$

The finite angle scattered wave can far from the vortex be separated in a φ part and a r part. When all constants are gathered, the phase contribution to the scattered wave can be written as

$$\delta\phi_a = \frac{a(\varphi)}{\sqrt{r}} e^{ikr} \quad (2.24)$$

where the scattering amplitude is

$$a(\varphi) = \frac{\pi\alpha}{\sqrt{2\pi ik}} \cot\left(\frac{\varphi}{2}\right) \quad (2.25)$$

This is the most common form seen for the scattered wave and it is valid for finite angles. The index a , at the scattered wave, is to remind us that this is the term which can be expressed by the scattering amplitude. The result deviates from the original Aharonov-Bohm scattering amplitude (2.11) expanded to first order in α , by an additional s-wave. The partial wave analysis (see section 2.4.2) will show that this s-wave comes from the second order term in α which marks the difference between the acoustic and magnetic Aharonov-Bohm scattering.

The Aharonov-Bohm scattering amplitude is divergent and becomes infinitely large as the angle reaches the forward direction. In the section about small angle scattering (section 2.3.3), we will see that this apparent singularity is not physical, but a consequence of how approximations are dealt with.

2.3.2 The Hidden Terms

The asymptotic wave in the original Aharonov-Bohm result (2.10) is, opposed to the Born solution, not a plane wave but of the form $\sim e^{ikx - i\alpha(\varphi - \pi)}$. The missing factor is the *twist* of the incoming wave, but since it is proportional to the perturbation parameter α , it is still hope that it is not gone for good, but has been lost in the approximations done when extracting the scattering amplitude (section 2.3.1). The scattered wave in the Born approximation was

$$\alpha \frac{ik}{2} \int d^2r' \frac{y'}{r'^2} e^{ikx'} H_0^{(1)}(k|\mathbf{r}' - \mathbf{r}|) \quad (2.26)$$

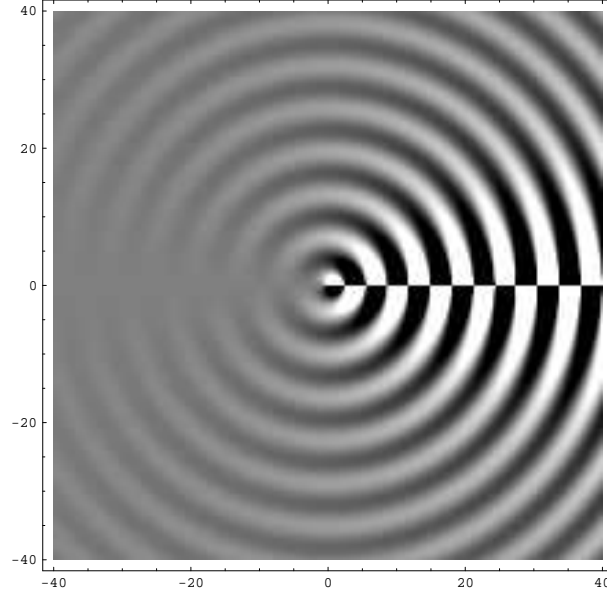


Figure 2.1: The real part of the scattered wave (2.24) expressed by the Born scattering amplitude (2.25). The plot clearly shows the phase difference at the positive x -axis. What is not clear from the plot, however, is that the scattered wave becomes infinitely large when $\varphi \rightarrow 0$.

The region in \mathbf{r}' space giving the scattering amplitude was $r' \ll r$. The hidden terms must then be found by some other approximation. We will now search in the region where $|y - y'| \ll |x - x'|$ and $|y| \gg |y - y'|$. Since these contributions are centered around the point $r' \approx r$, it is convenient to do a change of variables

$$\alpha \frac{ik}{2} e^{ikx} \int d^2 r'' \frac{y'' + y}{|\mathbf{r}'' + \mathbf{r}|^2} e^{ikx''} H_0^{(1)}(kr'') \quad (2.27)$$

where we have substituted $\mathbf{r}'' = \mathbf{r}' - \mathbf{r}$. In last section the asymptotic form of the Hankel function was used on the basis of $r \gg r'$. Now the same approximation is used, with the assumption that $|y''| \ll |x''|$. The Hankel function is then

$$H_0^{(1)}(kr'') \approx \sqrt{\frac{2}{\pi ik|x''|}} e^{ik|x''| + \frac{ik}{2|x''|}(y'')^2} \quad (2.28)$$

Supposing that $|y| \gg |y''|$, gives us a Gaussian integral in y'' , which by the ordinary formula (appendix A) gives

$$\sqrt{\frac{2}{\pi ik|x''|}} \int_{-\infty}^{\infty} dy'' e^{-\frac{k}{2|x''|}(y'')^2} = \frac{2}{k}$$

The condition that $|y| \gg |y''|$ makes the result just valid for finite angles. Introducing the step function $|x''| + x'' = 2\Theta(x'')$ gives us the hidden terms as

$$\delta\phi_{hidden} = i\alpha e^{ikx} \int_{-\infty}^{\infty} dx'' \frac{y}{y^2 + (x'' + x)^2} e^{2ikx''\Theta(x'')}$$

This integral can be solved by doing a trick. We have

$\frac{y}{y^2+x^2} = -\partial_x \left[\arctan\left(\frac{y}{x}\right) - \pi \text{sign}(y)\Theta(x) \right]$. The step function $\Theta(x)$ does not contribute to the derivative, but it makes the whole function continuous since it cancels the singularity in the $\text{Arctan}\left(\frac{y}{x}\right)$ function.

$$\begin{aligned} & \int_{-\infty}^{\infty} dx'' \frac{y}{y^2 + (x'' + x)^2} e^{2ikx''\Theta(x'')} \\ &= \left[- \left(\arctan\left(\frac{y}{x'' + x}\right) - \pi \text{sign}(y)\Theta(x'' + x) \right) e^{2ikx''\Theta(x'')} \right]_{x''=-\infty}^{x''=\infty} \\ &+ \int_0^{\infty} dx'' \left[\arctan\left(\frac{y}{x + x''}\right) - \pi \text{sign}(y)\Theta(x'' + x) \right] \frac{\partial}{\partial x''} \left(e^{2ikx''} \right) \end{aligned}$$

where the integration limit in the last integral has changed because of the step function in the exponent. The first term vanishes. The $\arctan\left(\frac{y}{x''+x}\right)$ is zero in both limits and in the $-\infty$ limit is the step function zero, while the $e^{2ikx''\Theta(x'')}$ term disappears in the ∞ limit because of rapid oscillations. The function $\arctan\left(\frac{y}{x+x''}\right) - \pi \text{sign}(y)\Theta(x'' + x)$ is continuous and slowly varying. Opposed to this, the last term is rapidly oscillating, and most contribution to the integral comes from small x'' . An approximation is to put the slowly varying term outside the integral, evaluated in $x'' = 0$. The remaining integral just gives -1 , so the hidden terms are

$$\delta\phi_{hidden} = -i\alpha e^{ikx} \left[\arctan\left(\frac{y}{x}\right) - \pi \text{sign}(y)\Theta(x) \right] \quad (2.29)$$

The angle $\varphi \in (0, 2\pi)$ is $\varphi = \arctan\left(\frac{y}{x}\right) + \pi\Theta(-x) + 2\pi\Theta(x)\Theta(-y)$. This gives the hidden terms as

$$\delta\phi_{hidden} = -i\alpha e^{kx} (\varphi - \pi) \quad (2.30)$$

To first order in α , this is the lost factor in $e^{ikx - i\alpha(\varphi - \pi)}$ from the original Aharonov-Bohm solution. In the Born approximation, this term appears as a part of the scattered wave since it is proportional to α . But it is proportional to $\exp(ikx)$, not $\exp(ikr)$, and it does not disappear far from the vortex, which makes it more natural to interpret it, in accordance with Aharonov and Bohm, as a modification of the incoming wave. The Born approximation is usually applied on scattering problems where the scattering center is well localized so that the incoming waves are just plane waves. In the scattering of a phonon on a vortex this is impossible,

which is a manifestation of the long range of the vortex; there is no localized scattering center. Despite this, the Born approximation gives the correct answer, and the reason why these non-scattering terms are usually not found in the Born approximation, is just because of the mathematical approximation done in the evaluation of the integral.

The scattered wave found in section 2.3.1 together with these hidden terms, form the solution for finite angles. Near the forward direction this solution is not valid, which leads us on a search for a specific small angle result in section 2.3.3.

2.3.3 Small Angle Scattering

In section 2.3.1 an expression for the scattered wave was found, but the solution diverged for small angles. By doing different approximations an expression which can describe the important forward direction will now be looked for. Since our incoming wave is propagating in direction of positive x , the small angle region is the area centered at the positive x -axis. The integration will here be performed in Cartesian coordinates x' and y' . The main contributions to the integral is thought to come from the terms where \mathbf{r}' is small, so that we can do series expansion of $|\mathbf{r} - \mathbf{r}'|$ with $x \gg y, x', y'$ as a condition. This derivation relies on one done by Sonin [Son96]. The only difference is that he uses polar coordinates describing \mathbf{r} . We use the following expansion

$$|\mathbf{r} - \mathbf{r}'| = x - x' + \frac{1}{2x}(y - y')^2 + \mathcal{O}(x^{-2})$$

Compared with the expansion done in section 2.3.1 one extra term must be kept, since the $-x'$ term cancels against the $+x'$ term from the incoming wave. The extra terms also saves us from a breakdown when $x' \approx x$, so in the small angle scattering there is no need to search for hidden terms as in section 2.3.2. The asymptotic expansion of the Hankel function for large arguments (2.23) gives

$$H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|)e^{ikx'} \approx \sqrt{\frac{2}{\pi kx}} e^{ikx + \frac{ik}{2x}(y-y')^2}$$

The small angle scattered wave can hence be written

$$\delta\phi_s \approx \alpha \frac{ik}{2} \sqrt{\frac{2}{\pi ikx}} e^{ikx} \int d^2r' \frac{y'}{r'^2} e^{\frac{ik}{2x}(y-y')^2} \quad (2.31)$$

There is no x' dependency in the exponent, so the x' integration can simply be done

$$\int_{-\infty}^{\infty} \frac{dx'}{x'^2 + y'^2} = \frac{\pi}{|y'|}$$

The y' integration is a bit more bothersome. It can not be carried out completely, but by some manipulations it can be transformed to an integral where all y -dependency is in the integration limits.

$$\begin{aligned}
& \int_{-\infty}^{\infty} dy' \frac{y'}{|y'|} e^{\frac{ik}{2x}(y'-y)^2} \\
&= \int_{-\infty}^{\infty} du \frac{u+y}{|u+y|} e^{\frac{ik}{2x}u^2} \\
&= \left(- \int_{-\infty}^{-y} + \int_{-y}^{\infty} \right) du e^{\frac{ik}{2x}u^2} \\
&= \left(- \int_y^{\infty} + \int_{-y}^{\infty} \right) du e^{\frac{ik}{2x}u^2} \\
&= 2 \int_0^y du e^{\frac{ik}{2x}u^2}
\end{aligned}$$

The small angle result can then be expressed as the integral

$$\delta\phi_s = \alpha \sqrt{\frac{2\pi ik}{x}} e^{ikx} \int_0^y du e^{\frac{ik}{2x}u^2} \quad (2.32)$$

It is not a problem that the final result can not be expressed by elementary functions, since usually only the derivative with respect to y is needed. The answer can be written more compact with the error function

$$\delta\phi_s = i\pi\alpha e^{ikx} \operatorname{erf} \left(y \sqrt{\frac{k}{2xi}} \right) \quad (2.33)$$

with the error function defined as $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dt e^{-t^2}$. This integral is well defined in the limit $y \rightarrow 0$, so that there is no trouble with singularities in contrast to the scattering amplitude expression (2.25).

2.3.4 Discussion of the Born Approximation

From the Born approximation three different results have been found: two being valid for finite angles and one which is valid just for small angles. All the terms originate in the same integral and they appear as a result of the different approximations. If one is clever enough it should be possible to extract *one* asymptotic expression for all angles, but for so long the different cases must be treated separately. The Born approximation result can be summarized as

$$\phi_s^{Born} = \phi_k \left[e^{ikx} + i\alpha\pi \operatorname{erf} \left(y \sqrt{\frac{k}{2xi}} \right) \right] \quad (2.34)$$

for small angles and

$$\phi_a^{Born} = \phi_k \left[e^{ikx} (1 - i\alpha(\varphi - \pi)) + \frac{a(\varphi)}{\sqrt{r}} e^{ikr} \right] \quad (2.35)$$

for finite angles. The scattering amplitude is

$$a(\varphi) = \frac{\pi\alpha}{\sqrt{2\pi ik}} \cot\left(\frac{\varphi}{2}\right) \quad (2.36)$$

Because of the long range of the vortex, it is a twist on incoming waves. Both solutions are continuous in their domains. The error function is well defined for small argument and actually it approaches zero at the positive x -axis, leaving just a plane wave there. The argument of the error function goes to infinity for $x \gg y \gg \sqrt{x}$, which are the largest angles where the small angle approximation is still valid. The asymptotic expansion of the error function for large arguments is

$$\text{erf}(z) \xrightarrow{z \rightarrow \infty} \text{sign}(\Re(z)) - \frac{1}{\sqrt{\pi}} \frac{1}{z} e^{-z^2} \quad (2.37)$$

For small angles, the approximations $\varphi \approx y/x$ and $r \approx x(1 + \frac{y^2}{2x^2})$ are valid. In the region $\varphi \ll 1$ and $\sqrt{r}\varphi \gg 1$, the small angle expression (2.34) can thus be written

$$\phi_s^{Born} \approx \phi_k \left\{ e^{ikx} [1 + i\pi\alpha \text{sign}(y)] + \frac{\pi\alpha}{\sqrt{2\pi ik}} \frac{e^{ikr}}{\sqrt{r}} \frac{2}{\varphi} \right\} \quad (2.38)$$

The first term is the plane wave, while the second comes from the twist of the incoming wave. The last is the scattered wave. The expression is exactly the same as the finite angle expression (2.35), expanded for small angles. *The Born solution is thus continuous.*

The conclusion of the Born approximation is that it gives a continuous solution to the scattering problem. The solution seems to be in agreement with the original Aharonov-Bohm result, to first order in α . In addition a special small angle result is developed, giving a continuous solution at all angles. The hidden term found in 2.3.2, giving the twist of the incoming wave, was essential to get a continuous solution. The solution presented here is the match of different solutions in different regions, but since all these solutions come from the same integral it should be theoretically possible to get one solution valid at all angles, just as done in the partial wave expansion (section 2.4.3).

2.4 Solution by Partial Wave Analysis

In the original article by Aharonov and Bohm [AB59], the Schrödinger equation was solved by partial wave analysis. This means that the solution is an expansion

in eigenstates of the angular momentum operator. To write down a general expression for the wavefunction is very easy, so the worry is how to treat the boundary conditions right, and later how to sum up. We will see that there are differences between the acoustic and magnetic Aharonov-Bohm scattering in deciding the boundary conditions, but that the result is mainly the same. Aharonov and Bohm also provided a method for doing the summation when kr is large, but their result is not satisfactory right behind the vortex, and we will instead present a solution found by Sommerfield and Minakata [SM00]. This is done in section 2.4.3.

In contrast to the Born approximation, the partial wave analysis is non-perturbative by nature and the calculations are best done by keeping all orders of α . But the scattering equation was derived by perturbation, which means that the answer obtained is just valid to lowest order. The scattering equation for a point vortex is

$$(\nabla^2 + k^2) \phi = -2i\alpha \nabla \varphi \cdot \nabla \phi \quad (2.39)$$

where $\alpha = qk$ is still supposed small compared to unity. The partial wave expansion is

$$\phi(\mathbf{r}) = \sum_{l=-\infty}^{\infty} \phi_l(r) e^{il\varphi}$$

Since l is just running over integers we have no problems with multivaluedness. Substituting this into our equation all derivatives of φ just give the eigenvalues il , which means that $\nabla \varphi \cdot \nabla \rightarrow \frac{il}{r^2}$ and $\nabla^2 \rightarrow \nabla_l^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{l^2}{r^2}$. Demanding that the equation holds for each l , we get the Bessel equations

$$\left[k^2 + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{l^2 + 2\alpha l}{r^2} \right] \phi_l = 0 \quad (2.40)$$

where the flux $\alpha = qk$. Note that in the magnetic Aharonov-Bohm problem, we would have got an additional term α^2/r^2 . The total solution can be written as superpositions of Bessel and Neumann functions

$$\phi_l = \alpha_l J_{\Omega_l}(kr) + \beta_l N_{\Omega_l}(kr) \quad (2.41)$$

where $\Omega_k = \sqrt{l^2 + 2\alpha l}$, and the constants α_l and β_l are dependent of k . The Bessel functions $J_{-\Omega_l}(kr)$ with negative subscripts could have been used instead of the Neumann functions as well, but since this, in the acoustic case, cause a degeneration for $l = 0$, the Neumann functions are chosen.

2.4.1 Boundary Conditions Near the Vortex

In the magnetic Aharonov-Bohm scattering, all the constants β_l in the general partial wave analysis solution (2.41) are zero. This can be found by examining a

flux tube of finite core size, with the constraint that the total wavefunction shall be normalizable. Since Neumann functions get arbitrarily large for small arguments, a wave-function containing the Neumann functions is not normalizable when the size of the flux tube is shrunk to zero.

The solution of the acoustic Aharonov-Bohm problem presented in this section, is a solution within the point vortex approximation. Opposed to the magnetic problem, it is in the acoustic case not possible to make an arbitrary solution with finite core size, since the density profile of the vortex is determined of the field equations. Instead we will try to derive advantage from the known behavior of the vortex profile. The scattering equation (2.7) was derived by requiring that presence of the vortex could be treated as a perturbation of a wave equation. Near, or inside, the vortex core this condition fails, since the free wave contribution in the equations are vanishing compared to those terms containing contributions from the vortex. However, the original linearized field equations (2.1) and (2.2) can be used inside the vortex core, where the vortex profile can be approximated with $f \sim r$. If the fields ν and ϕ and their derivatives are non-singular, the only singular terms are $\nabla\varphi \sim r^{-1}$ and $\nabla \ln(f) \sim r^{-1}$. So the terms ruling inside the vortex core are

$$\begin{aligned} -q\nabla\varphi \cdot \nabla\phi + \frac{1}{2}\nabla \ln(f) \cdot \nabla\nu &\approx 0 \\ -q\nabla\varphi \cdot \nabla\nu - 2\nabla \ln(f) \cdot \nabla\phi &\approx 0 \end{aligned}$$

Actually these are the same equations that were found in section 1.3, about vortex motion, applied on a stationary vortex. In this section the main concern, thought, is the boundary conditions in the partial wave analysis. Let us then use the partial wave expression for the fields, $\nabla\varphi \rightarrow \frac{il}{r^2}\mathbf{e}_\varphi$ and $\nabla \ln(r) \rightarrow \frac{1}{r}\mathbf{e}_r$. The above equations then look like

$$\begin{aligned} -\frac{iq}{r^2}\phi_l + \frac{1}{2r}\frac{\partial\nu_l}{\partial r} &= 0 \\ -\frac{iq}{r^2}\nu_l - \frac{2}{r}\frac{\partial\phi_l}{\partial r} &= 0 \end{aligned}$$

If ν_l is isolated from the second equations and put into the first, the result is a second order ordinary differential equation in ϕ_l alone.

$$r^2\frac{\partial^2\phi_l}{\partial r^2} + r\frac{\partial\phi_l}{\partial r} - l^2\phi_l = 0 \quad (2.42)$$

As all second order partial differential equations, the general solution can be written as a superposition of two functions. In this case the functions are $\phi_l \sim r^{\pm|l|}$. Because of degeneracy there is for $l = 0$ an additional solution $\phi_0 \sim \ln(r)$. The total solution for ϕ must be nonsingular when $r \rightarrow 0$, which means that

$$\phi_l \sim r^{+|l|} \quad (2.43)$$

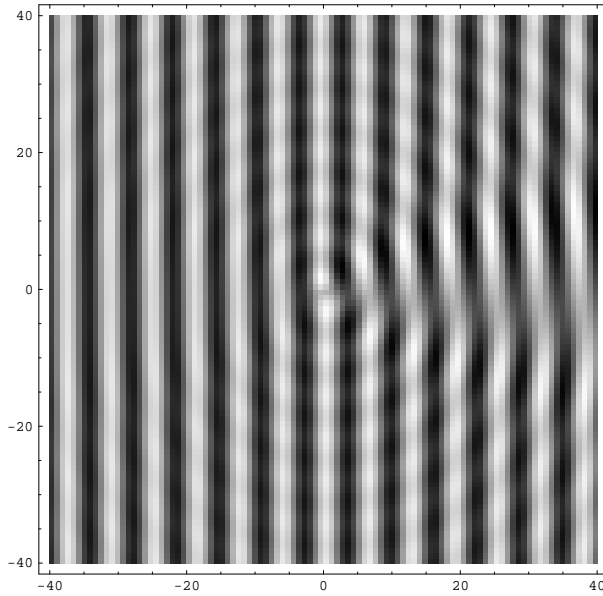


Figure 2.2: The real part of the partial wave analysis solution to the scattering problem (2.48). The plot is generated with $|l| \leq 60$ and $\alpha = 0.25$.

The partial wave expression (2.41) should be fit to the constraint (2.43). Taking the limit $r \rightarrow 0$ of (2.41) is not allowed, since it is a point vortex result. But kr can be taken to zero, by keeping r fixed and letting $k \rightarrow 0$. For small angles the Bessel function goes as $J_\nu(z) \sim z^\nu$ while the behavior of the Neumann functions are $N_\nu(z) \sim z^{-\nu}$ when $\nu \neq 0$ and $N_0(z) \sim \ln(z)$. We thus have the condition

$$\beta_l \xrightarrow{k \rightarrow 0} 0 \quad (2.44)$$

In the low energy limit the constants β_l are vanishing, just as in the magnetic case. The fact that our point vortex solution just satisfies the constraints for small k , can be seen as a clear indication that the point vortex approximation is just valid in the long wavelength limit. This is natural since waves with shorter wavelengths will have more interference with the vortex core. Practically this means that the point vortex approximation works fine together with long wavelength solutions, but if the intention is to keep higher orders in k , it is more dubious if the point vortex approximation holds. This would for example have consequences for the calculation of the longitudinal force on a vortex, since it is higher order in k .

2.4.2 Boundary Conditions Far From the Vortex

In last section restrictions were put on the general partial wave solution (2.41) from conditions on the behavior close to the vortex point. Now the behavior at a large distance from the vortex is considered. The normal fluid in helium II

(chapter 4) is constructed as a weighted integral over plane waves. To identify the normal fluid it is then a need for solutions which are as close as possible to plane waves, far from the vortex. The ideal situation must then be a solution which far from the vortex can be separated in an incoming plane wave, and an outgoing scattered wave. We will thus apply the constraint that each partial wave far from the vortex can be divided in one term belonging to the incoming plane wave and one term proportional to $\exp(ikr)$, demanding that the terms proportional to $\exp(-ikr)$ are zero. But this does not mean that the total solution can be divided in a plane wave and a term proportional to $\exp(ikr)$. Because of the summing of l to infinity the total wave looks different, as we will see.

For simplicity, the x -axis is put along the direction of the plane wave, which in partial wave expansion looks like

$$e^{ikr \cos \varphi} = \sum_l i^l J_l(kr) e^{il\varphi} \quad (2.45)$$

The asymptotic expansions of the Bessel functions for large kr are also needed

$$J_\nu(kr) \xrightarrow{kr \rightarrow \infty} \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \quad (2.46)$$

An expansion like this proves to be problematic when summing up, since it is just valid for $\nu \ll kr$. It is, however, working fine when applied at one partial wave at a fixed l . The partial waves of the scattered phonons are $\phi_l = \alpha_l J_{\Omega_l}(kr)$ where the constants α_l are to be determined. Subtracting the partial waves of $\exp(ikx)$, writing out the asymptotic expressions and decoupling the cosine terms, leads to

$$\begin{aligned} \{\phi_l(kr) - i^l J_l(kr)\} &= \sqrt{\frac{2}{\pi kr}} \frac{1}{2} e^{i(kr - \frac{\pi}{4})} \{\alpha_l e^{-i\frac{\pi}{2}\Omega_l} - 1\} \\ &+ \sqrt{\frac{2}{\pi kr}} \frac{1}{2} e^{-i(kr - \frac{\pi}{4})} \{\alpha_l e^{i\frac{\pi}{2}\Omega_l} - e^{i\pi l}\} \end{aligned}$$

In the partial wave analysis the overall constant ϕ_k will be put to unity. It can easily be put back in the final answers. Our boundary conditions are fulfilled if we let the coefficient before $\exp(-ikr)$ be zero, which leads to

$$\alpha_l = e^{i\pi l - i\frac{\pi}{2}\Omega_l} \quad (2.47)$$

The constants are then determined so that the total wave can be written as

$$\phi = \sum_l (-1)^l e^{-i\frac{\pi}{2}\Omega_l} J_{\Omega_l}(kr) e^{il\varphi} \quad (2.48)$$

The solution is plotted in figure 2.2. The anti-symmetry between the upper and lower half plane is clearly seen. We also notice that even though the tear right behind the vortex gets broader far from the vortex, the angle it occupies gets

smaller. This leads to the apparent singularity in polar coordinates at an infinite distance behind the vortex. The plot can be compared to the photographs of water waves scattered on a classical vortex in [BCL⁺80].

The subscripts of the Bessel functions are in the acoustic case $\Omega_l = \sqrt{l^2 + 2\alpha l}$. For small values of α this is approximately the same as the magnetic version, namely $|l + \alpha|$. An exception is for the s-wave, or the $l = 0$ term. To first order in α , the difference between acoustic and magnetic Aharonov-Bohm scattering is

$$\Delta = e^{-i\frac{\pi}{2}\alpha} J_\alpha(kr) - J_0(kr) \quad (2.49)$$

For large arguments the expansion of the Bessel function (2.46) can be safely used, so that to first order in α

$$\Delta \approx -i\alpha \sqrt{\frac{\pi}{2ikr}} e^{ikr} \quad (2.50)$$

which is exactly the difference between the found solutions of the magnetic (2.11) and acoustic (2.25) scattering problem.

The sum (2.48) is convergent for all kr ², since for the index ν much greater than kr the Bessel functions behave as

$$J_\nu(kr) \sim \frac{1}{\sqrt{2\pi kr}} \left(\frac{ekr}{2\nu} \right)^\nu$$

This means that when doing numerical calculations there is just need to keep terms with l just a bit larger than kr .

When expanding the Bessel functions in terms of cosine functions this convergence is lost, so that our expression for the wave becomes divergent for all finite r . This means we have to be extremely careful when using these asymptotic expressions. To avoid problems with singularities, will follow the example of Sommerfield and Minakata [SM00] and instead express the Bessel functions as contour integrals.

2.4.3 The Aharonov-Bohm Sum

Aharonov and Bohm found the wavefunction of an electron scattered on a magnetic flux string as a series of partial waves. Their solution was

$$\phi^{AB} = \sum_l (-1)^l e^{-i\frac{\pi}{2}|l+\alpha|} J_{|l+\alpha|}(kr) e^{il\varphi} \quad (2.51)$$

Most of the article [AB59] is about how to sum up for large r , and they end up with a solution as a sum of a twisted plane wave $\exp(ikx - i\alpha(\varphi - \pi))$,

²If there exist terms proportional to the Neumann functions for all l , the expression for the wave would be divergent.

and a scattered wave on the form $\frac{a(\varphi)}{\sqrt{r}} \exp(ikr)$. The unfortunate thing about their solution is that the scattering amplitude $a(\varphi)$ becomes infinite as the angle approaches the forward direction $\varphi \approx 0, 2\pi$. This is in sharp contrast to the partial wave expression they started with (2.51), which is convergent, continuous, single-valued and well behaved for all finite r and φ . The reason for the apparent singularity, is the use of the asymptotic expansion of the Bessel functions for large arguments (2.46), which is used for all values of $\nu = |l + \alpha|$, while the approximation is just valid for $kr \gg \nu$. This is the reason why they end up with a divergent series.

Even though the expanded series is strictly speaking divergent, Aharonov and Bohm end up with an expression which turns out to be right for all large angles, since they are careful in analyzing the asymptotic expression before summing up. Many references ([Son96],[WT98],[She98a]) use these expansions of the Bessel function when discussing Aharonov-Bohm scattering.

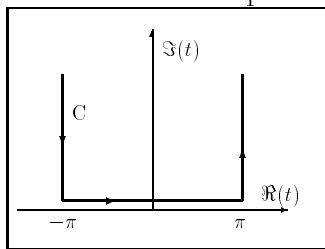


Figure 2.3: The integration path C in the complex t -plane.

The most important step is the first, expressing the Bessel functions as contour integrals

$$J_\nu(z) = \frac{1}{2\pi} e^{i\frac{\pi}{2}\nu} \int_C dt e^{-iz \cos(t) + i\nu t} \quad (2.52)$$

The integration contour C is in the upper half of the complex plane. The nice thing about this contour is that $it < 0$ for $t \in C$. It is this quality which makes our series convergent. The sum is now

$$\phi^{AB} = \frac{1}{2\pi} \int_C dt e^{-ikr \cos(t)} \sum_l (-1)^l e^{i|l+\alpha|t} e^{il\varphi}$$

To ease the notation we will now restrict ourselves to *positive values of α* . Negative values of α (anti-vortices) can easily be obtained from the final answer by letting $\alpha \rightarrow -\alpha$ and $\varphi \rightarrow -\varphi + 2\pi$. The above sum can be divided into one sum for $l \geq 0$ and one for $l < 0$. Both of these series are geometric and by using the ordinary formula for such series we arrive at

$$\phi^{AB} = \frac{1}{2\pi} \int_C dt e^{-ikr \cos(t)} \left[\frac{e^{i\alpha t}}{1 + e^{i(\varphi+t)}} - \frac{e^{-i\alpha t}}{1 + e^{i(\varphi-t)}} \right]$$

which put on one fraction line is

$$\phi^{AB} = \frac{1}{4\pi} \int_C dt e^{-ikr \cos(t)} \left[\frac{e^{i\alpha t}(e^{-i\varphi} + e^{-it}) - e^{-i\alpha t}(e^{-i\varphi} + e^{it})}{\cos(\varphi) + \cos(t)} \right]$$

The question now is how to remove the denominator so that (2.52) can be used to get the answer expressed by Bessel functions. The trick is to use

$$\int_0^{kr} dz e^{-iz(\cot(t)+\cos(\phi))} = \frac{e^{-ikr(\cos(t)+\cos(\phi))} - 1}{-i(\cos(t) + \cos(\phi))}$$

The term $1/(-i(\cot(t) + \cos(\phi)))$ will disappear from the final answer. This can be seen by noting that the original Aharonov-Bohm sum (2.51) is zero at $kr = 0$, which also applies to the integral $\int_0^{kr} dz$.

$$\begin{aligned} \phi^{AB} &= \frac{-i}{4\pi} e^{ikr \cos(\varphi)} \int_0^{kr} dz e^{-iz \cos(\varphi)} \\ &\times \int_C e^{-iz \cos(t)} [e^{-i\varphi}(e^{i\alpha t} - e^{-i\alpha t}) + e^{-i(1-\alpha)t} - e^{i(1-\alpha)t}] \end{aligned}$$

By using the formula (2.52) we can transform back to Bessel functions and hence get a sum of four Bessel functions of the first kind. These can in turn be expressed by two Hankel functions of the first kind

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-i\pi\nu} J_\nu(z)}{i \sin(\pi\nu)} \quad (2.53)$$

The whole solution can now be written as an integral over two Hankel functions. By making use of $\sin(\pi(1-\alpha)) = \sin(\alpha)$ the solution can be expressed as

$$\phi^{AB} = e^{ikr \cos(\varphi)} A(kr) \quad (2.54)$$

where we, to ease notation, have defined

$$\begin{aligned} A(kr) &\equiv \frac{\sin(\pi\alpha)}{2} \int_0^{kr} dz e^{-iz \cos(\varphi)} \\ &\times \left[-e^{-i\varphi + i\frac{\pi}{2}\alpha} H_\alpha^{(1)}(z) + e^{i\frac{\pi}{2}(1-\alpha)} H_{1-\alpha}^{(1)}(z) \right] \end{aligned} \quad (2.55)$$

What we have managed to do so far is to transform the original Aharonov-Bohm sum to an integral. This transcription is exact. The expression in itself is not very useful, but it turns out to be possible to do a very nice expansion of it far from the flux point.

2.4.4 The Solution Far From the Flux-Point

As promised, the expression from the last section will be simplified in the large kr limit, and we will start by carrying the integral in (2.55) to infinity. The integration can then be performed exactly with the formula

$$e^{i\frac{\pi}{2}\nu} \int_0^\infty dz e^{-iz \cos(\varphi)} H_\nu^{(1)}(z) = -2 \frac{\sin \nu(\varphi - \pi)}{\sin(\pi\nu) \sin(\varphi)} \quad (2.56)$$

From now on it is important that the angle φ is running between 0 and 2π to get the correct result. By doing some trigonometric manipulations we get

$$A(\infty) = e^{-i\varphi} \frac{\sin[\alpha(\varphi - \pi)]}{\sin(\varphi)} - \frac{\sin[(1 - \alpha)(\varphi - \pi)]}{\sin(\varphi)} = e^{-i\alpha(\varphi - \pi)}$$

This is the asymptotic behavior far from the vortex. The deviation from this expression can now be expressed as an integral from kr to infinity. Now the asymptotic expansion of the Hankel functions can be used

$$H_\nu^{(1)}(z) \xrightarrow{z \rightarrow \infty} \sqrt{\frac{2}{\pi iz}} e^{iz - i\frac{\pi}{2}\nu} \quad (2.57)$$

Opposed to the sum (2.51), the subscripts of the Hankel functions are now fixed and small. This means that the rewriting is safe, and the the approximation is actually very good.

$$\begin{aligned} A(\infty) - A(kr) &\approx \frac{\sin(\pi\alpha)}{2\pi i} (-e^{-i\varphi} + 1) \int_{kr}^{\infty} \frac{dz}{\sqrt{z}} e^{i(1 - \cos(\varphi))z} \\ &= i \sin(\pi\alpha) e^{-i\frac{\varphi}{2}} \left[1 - \operatorname{erf} \left(\sqrt{-2ikr} \sin \left(\frac{\varphi}{2} \right) \right) \right] \end{aligned}$$

This was obtained by substituting $1 - \cos(\varphi) = 2 \sin^2(\varphi/2)$ and $u = \sin(\varphi/2)\sqrt{-2iz}$. The error function is $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z du \exp(-u^2)$. Since $\sin(\varphi/2)$ is positive for angles between 0 and 2π we have no problems with multivaluedness. The final solution of the Aharonov-Bohm problem far from the flux point and all angles is

$$\phi^{AB} = e^{ikr \cos(\varphi)} \left\{ e^{-i\alpha(\varphi - \pi)} - i \sin(\pi\alpha) e^{-i\frac{\varphi}{2}} \left[1 - \operatorname{erf} \left(\sqrt{-2ikr} \sin \left(\frac{\varphi}{2} \right) \right) \right] \right\} \quad (2.58)$$

This is the solution obtained by Sommerfield and Minakata. We notice that the first term is discontinuous at forward angles, but the singularity cancel against a similar discontinuity in the scattered wave, leaving the full solution continuous and single valued.

2.4.5 Discussion of the Partial Wave Analysis

Far from the flux-point the sum of partial waves (2.48) has been transformed to the function (2.58) which is valid for all angles. The Born solution was divided in one expression for finite angles and one for small angles. The partial wave result can be expanded in each of these regions to see if the solutions are identical. For finite angles, which means that $\sin(\frac{\varphi}{2})\sqrt{r} \gg 1$, the error function can be expanded by the use of (2.37). The total wave (2.58) can hence be written

$$\phi^{AB} \approx \left[e^{ikr \cos(\varphi) - i\alpha(\varphi - \pi)} + \frac{\sin(\pi\alpha)}{\sqrt{2\pi i}} \frac{e^{-i\frac{\varphi}{2}}}{\sin(\frac{\varphi}{2})} \right] \quad (2.59)$$

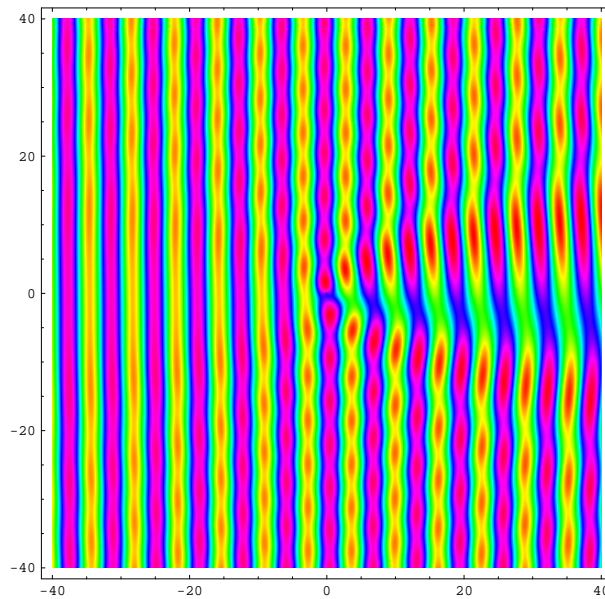


Figure 2.4: The real part of Sommerfield and Minakata's summation of the Aharonov-Bohm sum (2.58), for $\alpha = 0.25$. It must be compared with the direct plot of the partial wave expression (figure 2.2). The phase shift of the wave is clearly seen. The scattered wave is seen as the “dots” distributed over the plot.

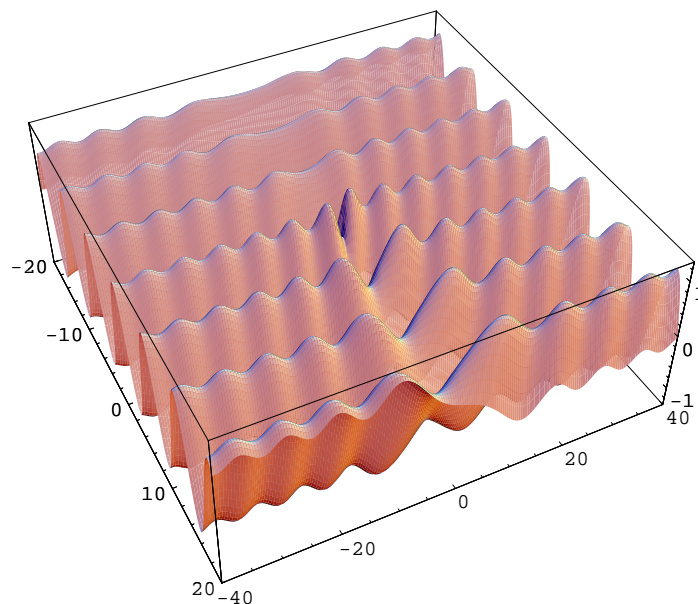


Figure 2.5: The same plot as figure 2.4, but seen from another angle, and with $|kx| < 20$ and $|ky| < 40$. This plot clearly shows the damping of the wave right behind the vortex.

as expected. This is the ordinary expression found by Aharonov and Bohm. It differs from the Born solution in an additional s-wave, which marks the difference between acoustic and magnetic Aharonov-Bohm scattering.

The other expression to compare it with, is Sonin's small angle expression, (2.33). This result is only to first order in α , so we must expand (2.58) in the same way. In the small angle limit, the angle is $\varphi \sim 0, 2\pi$, so that $\sin(\varphi/2) \approx y/x$, $\varphi - \pi \approx -\pi \text{sign}(y)$ and $\exp(-i\varphi/2) \approx \text{sign}(y)$. Then the discontinuous terms cancel and

$$\phi^{AB} \approx e^{ikx} \left[1 + i\pi\alpha \operatorname{erf} \left(y \sqrt{\frac{k}{2ix}} \right) \right] \quad (2.60)$$

This is the same as the small angle result found by Born approximation (2.33). The cancellation of the discontinuous terms is just shown to first order in α , but the full solution is continuous at arbitrary order.

It is important that the vortex is not a mass sink or source. There must be no net mass flux passing through a contour surrounding the vortex. We will now use the original partial wave expression for the mass flux

$$\langle \mathbf{j}_k \rangle = \frac{1}{2} \sum_{ml} \nu_m^* \left[\frac{\partial \phi_l}{\partial r} \mathbf{e}_r + \frac{il}{r} \mathbf{e}_\varphi \right] e^{i(l-m)\varphi} \quad (2.61)$$

Integrating the mass transport through a circle surrounding the vortex, means an integral with line segment $d\mathbf{l} = r d\varphi \mathbf{e}_r$. We need the density found from (2.4)

$$\nu_m = ik \left(1 - \frac{\alpha m}{(kr)^2} \right) \alpha_m J_{\Omega_m}(kr) \quad (2.62)$$

and the component in r -direction of gradient of the phase

$$\frac{\partial \phi_l}{\partial r} = \alpha_l \frac{\partial J_{\Omega_l}(kr)}{\partial r} \quad (2.63)$$

where $\Omega_l = \sqrt{l^2 + 2\alpha l}$ and $\alpha_l = (-1)^l \exp(-i\frac{\pi}{2}\Omega_l)$. The integral just gives contributions for diagonal terms, $\int_0^{2\pi} d\varphi \exp(i(l-m)\varphi) = 2\pi\delta_{ml}$. The net flux through the contour is then

$$r \int_0^{2\pi} d\varphi \langle j_k \rangle_r = \Re \left[-ikr\pi \sum_l \left(1 - \frac{\alpha l}{(kr)^2} \right) J_{\Omega_l} \frac{\partial J_{\Omega_l}}{\partial r} \right] = 0 \quad (2.64)$$

The expression is purely imaginary. The real part, which is the physical component, is zero, and the mass is conserved.

In the derivation of the scattering equations, derivatives of the phase was neglected to ease the mathematical handling. The condition $|\nabla^2 \nu| \ll |\nu|$ will now

be explicitly checked by inserting the solution (2.62). There are two assumptions: the wave-number is small, $k \ll 1$, and the distance from the vortex is large, $r \gg 1$. The derivatives of $\left(1 - \frac{\alpha l}{(kr)^2}\right)$ with respect to r will thus be neglected as higher order in r^{-1} . The Bessel functions satisfies the Bessel equation, which makes the derivations easy to perform

$$\nabla_l^2 J_{\Omega_l}(kr) = \left[-k^2 + \frac{2\alpha l}{r^2}\right] J_{\Omega_l} \quad (2.65)$$

where $\nabla_l = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{l^2}{r^2}$. The first terms is second order in k , and can be neglected. The second term is small when r is large. In addition is proportional to $\alpha = qk$, which is small. The conclusion is that the assumption $|\nabla^2 \nu| \ll |\nu|$ is justified.

2.5 Density Corrections

So far only the point vortex approximation has been studied, both for the Born approximation 2.3 and for the partial wave analysis 2.4. Now we will again use the Born approximation and try to extract information from the terms containing the density profile f in (2.20). Of special interest is the long wavelength limit; to see how the k dependency is compared to the phase contributions. The first density term in (2.20) is $\sim k^2(1 - f^2) \exp(ikr' \cos(\varphi'))$. If the approximation of the density profile (1.33) and the asymptotic form of the Hankel function (2.23) is used, we get

$$\delta\phi_{d1} \approx \frac{ik^2}{4} \sqrt{\frac{2}{\pi ikr}} e^{ikr} \int \frac{d^2r'}{1 + 2r'^2} e^{2ikr' \sin(\frac{\varphi}{2}) \sin(\varphi' - \frac{\varphi}{2})} \quad (2.66)$$

After substituting $\varphi' \rightarrow \varphi' + \frac{\varphi}{2}$, the angle part of the integral gives a Bessel function ³

$$\int_0^{2\pi} d\varphi' e^{2ikr' \sin(\frac{\varphi}{2}) \sin(\varphi')} = 2\pi J_0\left(2kr' \sin(\frac{\varphi}{2})\right)$$

The r' integration gives a modified Bessel function as an answer, so that

$$\delta\phi_{d1} \approx \frac{1}{4} \sqrt{\frac{2\pi i}{r}} k^{3/2} e^{ikr} K_0\left(\sqrt{2}k \sin(\frac{\varphi}{2})\right) \quad (2.67)$$

The second density correction in (2.20) is $\sim ik \frac{f'}{f} \cos(\varphi') \exp(ik \cos(\varphi'))$, where f' here denotes derivative with respect to r' . This gives the contribution

$$\delta\phi_{d2} \approx -\frac{k}{4} \sqrt{\frac{2}{\pi ikr}} e^{ikr} \int d^2r' \cos(\varphi') \frac{f'}{f} e^{2ikr' \sin(\frac{\varphi}{2}) \sin(\varphi' + \frac{\varphi}{2})} \quad (2.68)$$

³The formulae used can be found in appendix A.

As for the other integral, the angle integration here also gives a Bessel function

$$\int_0^{2\pi} d\varphi' \cos(\varphi' + \frac{\varphi}{2}) e^{2ikr' \sin(\frac{\varphi}{2}) \sin(\varphi')} = -\sin(\frac{\varphi}{2}) 2\pi i J_0\left(2kr' \sin(\frac{\varphi}{2})\right)$$

Using the same approximation for the density profile as above, we have $r' f'(r')/f(r') \approx \frac{1}{1+2r'^2}$. The r' integration is then an integral of the kind

$$\begin{aligned} \int_0^\infty \frac{dx}{2x^2+1} J_1(ax) &= \int_0^\infty dx \left[1 - \frac{x^2}{x^2 + \frac{1}{2}}\right] J_1(ax) \\ &= \frac{1}{a} - \frac{1}{\sqrt{2}} K_1\left(\frac{1}{\sqrt{2}}a\right) \end{aligned}$$

This gives that the second density correction is

$$\delta\phi_{d2} \approx -\frac{1}{2} \sqrt{\frac{2\pi i}{r}} k^{1/2} \sin(\frac{\varphi}{2}) \left[\frac{1}{2k \sin(\frac{\varphi}{2})} - \frac{1}{\sqrt{2}} K_1\left(\sqrt{2}k \sin(\frac{\varphi}{2})\right) \right] \quad (2.69)$$

The leading order of the modified Bessel functions for small arguments are $K_0(z) \sim -\ln(z)$ and $K_1(z) \sim \frac{1}{z} + \frac{z}{2} \ln z$. The first density correction (2.67) is thus logarithmically divergent near the forward direction. As in the case of the phase contributions we must think that this is just a consequence of the approximations being made, and not a physical property. If we forget this term as higher order in k , and take the low k limit, we see that both the density corrections have leading order terms of order $k^{3/2} \ln(k)$. The density correction is then in the low k limit

$$\delta\phi_d \approx -\frac{1}{8} \sqrt{2\pi i} (3 - \cos(\varphi)) k^{3/2} \ln(k) \frac{e^{ikr}}{\sqrt{r}} \quad (2.70)$$

The ordinary scattering amplitude goes for small wavenumbers as $k^{1/2}$. The density corrections can thus in the small wavelength limit be ignored. But the difference is only of order $k \ln(k)$, so that if we shall go up one order from the leading one, we must also take the density profile into consideration.

Chapter 3

The Phonon Force

The goal of this chapter is to provide an expression for the force on a vortex from a single scattered wave. Or more accurately: to find the transverse force component to order $\alpha = qk$. The answer to this question is of vital importance for the discussion of the Iordanskii force in helium II, as discussed in chapter 5. There are several articles available calculating the force from a single wave. Sonin [Son96][Son01] and Shelankov [She98a][She98b] end up with the conclusion that the force is proportional to α , while Wexler and Thouless [WT98] and Demircan, Ao and Niu [DAN95] conclude that it must be higher order in α . Stone's article [Sto99] is also important.

Central in the interpretation of the result is that the force originates in the asymmetry of the phonon wave in the presence of a vortex. This seems to be an acoustic analogy to the Aharonov-Bohm effect, discussed in section 2.2. Central in the discussion is also the region right behind the vortex, where we remember from chapter 2 that the formalism with the scattering amplitude failed. Stone [Sto99] interprets the force as the creation of transverse momentum in this region. The criticism of the results [WT98] and [DAN95] also points to the fact that they do not include the effects of small scattering angles.

All calculations in this chapter will use the solutions of the scattering problem from chapter 2, where a plane wave scattered on a stationary vortex was examined. The fact that the vortex was not allowed to move, was in the scattering problem presented as a boundary condition introduced to ease the mathematical handling, but in this chapter it will be of great physical significance, since the vortex at rest can also be thought of as being held still by *an external force*. There must be a momentum transfer between the vortex and the wave which is possible to extract by examining the *momentum-flux tensor*. The momentum-flux tensor Π_{ij} is the density of i -momentum in the j direction, so the force on the wave is $\oint dS_i \Pi_{ij}$. To get the correct result the momentum transfer must take place inside the integration contour. The region over which the force is applied, corresponds to region which is kept at rest. If a point vortex is considered, it does not matter where the contour is, as long as it encloses the vortex. If a larger

part of the vortex is kept at rest, for example the core, or the whole profile, we must be more careful in our choice of contour. A safe choice in all cases is to put it infinitely away from the vortex.

The force will fluctuate a lot, but we are only interested in the time-average. When from now on speaking about the force on the vortex, the meaning is the time-average of the force. The force *on the vortex* is then

$$F_j = - \oint dS_i \langle \Pi_{ij} \rangle \quad (3.1)$$

with summation over $i = x, y$. The dS_i is the components of the vector describing an infinitesimal surfaces segment (line in two dimensions).

The calculations will be done in several ways: In 3.2.1 the scattering result from the Born approximation will be used to find the force by letting the integration contour be infinitely away from the vortex. In the spirit of the Born approximation only terms to first order in the scattering parameter will be brought into consideration. If problems with divergences appear near $\varphi = 0$, the small angle result for the scattered wave must be used instead.

The limit when the integration contour is placed close to the vortex is studied in section 3.2.2. This limit requires that a point vortex is considered, since all interaction between the force and the wave has to be inside the integration contour. The calculation is amazingly simple, and it is not even necessary to solve the scattering problem to get the result.

Instead of trying to extract an analytical expression from the partial wave analysis, the calculation is done numerically in section 3.2.3 at a finite distance from the vortex and by keeping a finite number of terms. This result can be compared with the two analytical results. The numerical treatment is straightforward, even though the expressions themselves are long. It seems to be consensus of what the partial wave expression should look like, and since much of the problem is how to take the limits infinitely from the vortex, a numerical treatment at finite distance is a way to get a reliable result, and complement the analytical calculations.

3.1 The Momentum-Flux Tensor

In the calculation of the force the momentum-flux tensor is needed. By Noethers theorem the momentum-flux tensor can be found from the Lagrangian density. If the Lagrangian density \mathcal{L} is a function of the fields ρ and θ and their derivatives, the definition of the tensor is

$$\Pi_{ij} = \mathcal{L} \delta_{ij} - \frac{\partial \mathcal{L}}{\partial(\partial_i \theta)} \partial_j \theta - \frac{\partial \mathcal{L}}{\partial(\partial_i \rho)} \partial_j \rho \quad (3.2)$$

where ∂_i denotes derivation with respect to coordinate i . The symbol \mathcal{L} is in this context no longer a function of the fields, but the function of the spatial

coordinates and time that we get when a special solution for the fields is inserted. The Ginzburg-Landau Lagrangian on polar form is

$$\mathcal{L} = -\rho\dot{\theta} - \frac{1}{2}\rho(\nabla\theta)^2 - \frac{1}{8\rho}(\nabla\rho)^2 - \frac{1}{2}(\rho - 1)^2 \quad (3.3)$$

which, by the definition, gives the energy momentum tensor

$$\Pi_{ij} = \mathcal{L}\delta_{ij} + \rho\partial_i\theta\partial_j\theta + \frac{1}{4\rho}\partial_i\rho\partial_j\rho \quad (3.4)$$

The solutions which we are going to insert are plane waves scattered on a stationary vortex. In the long wavelength limit it will be assumed that the derivatives of the density can be ignored, so that such terms can be dropped both in the Lagrangian and the momentum-flux tensor. Suppose that the integration contour is at a large radius r . The density profile of the vortex goes as $1 - \frac{1}{4r^2}$. Since the length of the integrations contour is of dimension r , the density profile can be completely ignored in the momentum-flux tensor, though it can not necessarily be ignored when solving the field equations. In the rest of this derivation only a point vortex will be considered, so that the following substitutions are done

$$\rho = 1 + \eta \qquad \theta = \chi + q\varphi \quad (3.5)$$

The fields η and χ are thought to be small, so that the Lagrangian can be expanded in these fields. In chapter 2 the linearized field equations were solved, and we want to use these solutions to get an expression for the force. Upon thermal averaging all first order terms in the momentum-flux tensor vanish and we are left with an expression to second order in η and χ . There are at least two ways of finding the right expressions for the momentum-flux tensor. The first is to expand the tensor to second order in the fields, and then insert the linear fields. The other way is to expand the Lagrangian to third order so that the field equations are quadratic in χ and η . Then we must be careful since the field equations are no longer linear, and terms that appear to be linear in the Lagrangian density can still contain hidden quadratic terms that will survive the time averaging.

Those two procedures sketched above do anyway give the same result, so we are free to use the easiest and only consider the quadratic terms in the Lagrangian density

$$\mathcal{L}^{(2)} = -\eta\dot{\chi} - \frac{1}{2}(\nabla\chi)^2 - \eta\nabla\chi \cdot q\nabla\varphi - \frac{1}{2}\eta^2 \quad (3.6)$$

The diagonal term $\mathcal{L}\delta_{ij}$ can be rewritten by using the field equation

$$-\frac{\partial\mathcal{L}}{\partial\eta} = 0 = \dot{\chi} + \nabla\chi \cdot q\nabla\varphi - \eta \quad (3.7)$$

So that

$$\mathcal{L} = \frac{1}{2}\eta^2 - \frac{1}{2}(\nabla\chi)^2 \quad (3.8)$$

The time average of the excitation part of the momentum-flux tensor is then

$$\langle \Pi_{ij} \rangle = P\delta_{ij} + \langle \partial_i\chi\partial_j\chi \rangle + \langle \eta\partial_i\chi \rangle q\partial_j\varphi + \langle \eta\partial_j\chi \rangle q\partial_i\varphi \quad (3.9)$$

The coefficient before the diagonal part of the tensor is what we define as the average pressure $P = \langle \frac{1}{2}\eta^2 - \frac{1}{2}(\nabla\chi)^2 \rangle$. According to the solutions in chapter 2, it is useful to assume the solutions to have a harmonic time dependency

$$\eta(\mathbf{r}, t) = e^{-i\omega t}\nu(\mathbf{r}) \quad \chi(\mathbf{r}, t) = e^{-i\omega t}\phi(\mathbf{r})$$

Since both the phase χ and density η are *real* functions, we have to be a bit careful. In most calculations it is easiest to think of them as the real parts of complex functions, but for products of functions, the products of the real parts are not the same as the real part of the product. The formula to use in our multiplications is

$$\Re(ae^{-i\omega t})\Re(be^{-i\omega t}) = \frac{1}{2}\Re(a^*b + (a+b)e^{-2i\omega t})$$

Taking the time average, the $e^{-2i\omega t}$ term vanishes and

$$\langle \Re(ae^{-i\omega t})\Re(be^{-i\omega t}) \rangle = \frac{1}{2}\Re(a^*b)$$

For convenience we forget to write \Re before all terms from now on. The final expression for the time average of the momentum-flux tensor is then

$$\langle \Pi_{ij} \rangle = P\delta_{ij} + \frac{1}{2}\partial_i\phi^*\partial_j\phi + \frac{q}{2}\nu^*\partial_i\phi\partial_j\varphi + \frac{q}{2}\nu^*\partial_j\phi\partial_i\varphi \quad (3.10)$$

where the pressure is

$$P = \frac{1}{4}(|\nu|^2 - |\nabla\phi|^2) \quad (3.11)$$

The recipes of how to find the force is: plug in solutions of ϕ and ν into the energy momentum tensor (3.10), insert this in the expression for the force (3.1), and finally do the contour integration.

3.2 Calculation of the Force

In chapter 2 the scattering problem of a phonon on a vortex was considered both with the Born approximation and by partial wave analysis. These two solutions

will be inserted in the force expression, and the answers will be compared. But before this is done, let us do some more rewriting. With vector notation the force is

$$\mathbf{F} = - \oint \left\{ P \mathbf{dS} + \frac{1}{2} \nabla \phi^* (\nabla \phi \cdot \mathbf{dS}) + \frac{q}{2} \nu^* [\nabla \phi (\nabla \varphi \cdot \mathbf{dS}) + \nabla \varphi (\nabla \phi \cdot \mathbf{dS})] \right\} \quad (3.12)$$

where the pressure is $P = \frac{1}{4}(|\nu|^2 - |\nabla \phi|^2)$. If the integration contour is a circle with radius r , the surface segment (a curve in two dimensions) is $\mathbf{dS} = \mathbf{r} d\varphi$. The force then becomes

$$\mathbf{F} = - \int_0^{2\pi} d\varphi \left[P \mathbf{r} + \frac{r}{2} \frac{\partial \phi}{\partial r} (\nabla \phi^* + q \nu^* \nabla \varphi) \right] \quad (3.13)$$

This expression should be valid both for the transverse and the longitudinal force component. In our thesis only the transverse force is of interest. If we were really pedantic in our notation, an index k on the force could have been added to remind us that this is just the effect of a single phonon with wavenumber k .

$$F_{\perp}^P = -r \int_0^{2\pi} d\varphi [\sin(\varphi) P] \quad (3.14)$$

$$F_{\perp}^{\phi\phi} = -\frac{r}{2} \int_0^{2\pi} d\varphi [\partial_r \phi \partial_y \phi^*] \quad (3.15)$$

$$F_{\perp}^q = -\frac{q}{2} \int_0^{2\pi} d\varphi [\nu^* \partial_r \phi \cos(\varphi)] \quad (3.16)$$

These formulae are the common starting point for our calculations. To go further separate formulae for the case of Born approximation and partial wave analysis must be developed.

As mentioned earlier the force from a scattered wave is discussed in different articles ([Son96],[She98a] and [WT98]). All these references agree that the pressure part of the force, F_{\perp}^P is zero if the integration contour is far from the vortex. Much attention is due to a term proportional to the transverse scattering cross-section

$$\sigma_{\perp} = r \int_0^{2\pi} d\varphi \sin(\varphi) |a(\varphi)|^2 \quad (3.17)$$

which can be extracted from (3.15). There are two different expressions used for the scattering amplitude. One is the ordinary Born scattering amplitude $\sim \cot(\varphi/2)$, the other is a partial wave expression, which can be extracted by using the asymptotic form of the Bessel functions. Both these expressions create problems, since they make the transverse scattering cross-section divergent. The Born result gives a divergent integral near $\varphi = 0$ and the partial wave expression

gives a divergent series. The articles [Son96], [She98a] and [WT98] treat the divergences in different ways and end up with different conclusions. However in this thesis we will not go into this discussion. There are two reasons for that: First, we know that the scattering amplitude is not well defined for $\varphi \approx 0$. If we end up with a divergence, there is no need to regularize, we only need to *use the small angle result from section 2.3.3 instead*. This should remove all divergences in a painless way. Second, only the force to order α is of interest. The term σ_{\perp} is of order α^2 , so if we get a result proportional to α from this, it seems like the whole perturbative approach to the problem fails (!) and can no longer be used.

3.2.1 Circuit Far From the Vortex

In section 2.3 the scattering of a phonon on a vortex was studied by using the Born approximation, and now this solution is inserted into the derived formulae for the transverse force, (3.14), (3.15) and (3.16). The scattering formalism is just valid far from the vortex, and the integration curve will thus be put at a large distance. In a way this is also the most general limit, since it does not put any claims on exactly where the momentum transfer between the vortex and the wave takes place. It works fine both for point vortices and vortices where the core or the whole profile is kept fixed.

In the spirit of the Born approximation the calculation will be done just to first order in $\alpha = kq$. The corrections from the density profile of the vortex (section 2.5) will not be taken into account, since they are of higher order in k . The calculation is tedious and involves many steps and has a lot of pitfalls. There is no need to say that an accurate treatment of the scattering problem in section 2.3 is required to get the correct result. Since the calculation of the transverse force is the main goal for the thesis, the derivation is presented in great detail.

The solution found to the scattering problem was divided in one expression valid for finite angles, and one for small angles. These used together gave a solution which was continuous and single-valued. As discussed by Sonin [Son96], the small angle contribution is important for the force, but it is, as we shall see, not necessary to use the special solution derived for small angles to find this, since it actually is contained in the ordinary Aharonov-Bohm scattering amplitude as well.

The finite-angle solution can be written as the sum of a twisted plane wave, and a scattered wave disappearing at infinity. The twist of the incoming wave will prove to be essential to the result. No part of the waves ϕ or ν blows up far from the vortex¹. Since the integration contour is of length $\sim r$, all terms of order $\sim r^{-3/2}$ can be thrown away when r goes to infinity. The finite-angle Born

¹The scattering amplitude $a(\varphi)$ blows up near $\varphi \approx 0$, but this is not a physical property, since the scattering amplitude is just valid for finite angles.

solutions is

$$\phi = \phi_k [e^{ikx} (1 - i\alpha(\varphi - \pi)) + \delta\phi_a] \quad (3.18)$$

where $\delta\phi_a = \frac{a(\varphi)}{\sqrt{r}} \exp(ikx)$ is the scattered wave (2.35) expressed by the scattering amplitude. Because of the form of the scattered wave, only derivatives with respect to r count far from the vortex

$$\nabla\delta\phi_a = ik\delta\phi \mathbf{e}_r + \mathcal{O}(r^{-3/2}) \quad (3.19)$$

A reckless expansion in powers of $r^{-1/2}$ like this, is not to be recommended in all situations. It does indeed work rather good in the \mathbf{e}_r direction but in the \mathbf{e}_φ direction it fails badly, since it includes derivatives of $a(\varphi)$ with respect to φ , which gives very nasty divergences near $\varphi = 0$. For the calculation of the transverse force this does not matter, since only the \mathbf{e}_r direction counts, but when for example the circulation of the phonon gas is found in section 4.3, this is of importance. Problems like this is also a motivation to do numerical calculations as well.

The first part of the force to be discussed is the one from the pressure, (3.14). It is often said to vanish, and we shall see that a correct form of the incoming wave is required for this to happen. Since the density is $P = \frac{1}{4}(|\nu|^2 - |\nabla\phi|^2)$, the two things needed are

$$\nu = \phi_k \left[ik (e^{ikx} (1 - i\alpha(\varphi - \pi)) + \delta\phi_a) + i\alpha \frac{\sin(\varphi)}{r} e^{ikx} \right] + \mathcal{O}(\alpha^2)$$

and

$$\nabla\phi = \phi_k [ik e^{ikx} (1 - i\alpha(\varphi - \pi)) - i\alpha \nabla\varphi + ik\delta\phi_a \mathbf{e}_r] + \mathcal{O}(\alpha^2, r^{-3/2})$$

where $\mathbf{k} = k\mathbf{e}_x$. Next we take the absolute square of these two quantities. The gradient of the angle φ is $\nabla\varphi = \frac{1}{r}(-\sin(\varphi), \cos(\varphi))$, so that

$$|\nu|^2 = |\phi_k|^2 \left[k^2 (1 + 2\Re(e^{-ikx} \delta\phi_a)) + 2\alpha k \frac{\sin(\varphi)}{r} \right] + \mathcal{O}(\alpha^2, r^{-3/2})$$

and

$$|\nabla\phi|^2 = |\phi_k|^2 \left[k^2 (1 + 2\Re(e^{-ikx} \cos(\varphi) \delta\phi_a)) + 2\alpha k \frac{\sin(\varphi)}{r} \right] + \mathcal{O}(\alpha^2, r^{-3/2})$$

This is all which is needed to find the pressure, and we see that it can be expressed by the scattered wave $\delta\phi_a$ alone, since the constant terms and the terms proportional to $\sin(\varphi)$ cancel.

$$P = |\phi_k|^2 \frac{k^2}{2} \Re \{ (1 - \cos(\varphi)) e^{-ikx} \delta\phi_a \} \quad (3.20)$$

Now the importance of the extra factor $\exp(-i\alpha(\varphi - \pi))$ in the incoming wave is clearly seen. If it was not there, it would not cancel the $\sin(\varphi)$ factor from the density. In turn this would have given a non-zero contribution to force! If $a(\varphi) \sim \frac{\sin(\varphi)}{1-\cos(\varphi)}$ is inserted, the pressure part of the force (3.14) is proportional to

$$F_{\perp}^P \sim \sqrt{r} \int_{-\pi}^{\pi} d\varphi \sin^2(\varphi) e^{ikr(1-\cos(\varphi))} \xrightarrow{r \rightarrow \infty} 0 \quad (3.21)$$

When $r \rightarrow \infty$ the oscillations of the integrand become more and more rapid. Where the oscillations are fast it will average to zero, and only for φ near 0 the oscillations will be slow. On the other hand: near $\varphi = 0$ the $\sin^2(\varphi)$ term tends to zero. There is then no net contribution to the integral when r gets large, and hence no contribution to the force coming from the pressure term.

In addition to the pressure term, there are two more terms in the expression for the transverse force. First we take a look at (3.15). It contains a derivative with respect to y , which can be expressed by the derivatives of polar angles as $\partial_y = \sin(\varphi)\partial_r + \frac{1}{r}\cos(\varphi)\partial_{\varphi}$. The plane wave has no dependency of y , so to first order in α what we need is

$$\partial_r \phi = \phi_k ik \cos(\varphi) e^{ikr \cos(\varphi)} + \mathcal{O}(\alpha)$$

and

$$\partial_y \phi = \phi_k \left[-i\alpha \frac{\cos(\varphi)}{r} e^{ikr \cos(\varphi)} + ik \delta\phi_a \sin(\varphi) \right] + \mathcal{O}(r^{-3/2})$$

The product is

$$\partial_r \phi^* \partial_y \phi = |\phi_k|^2 k \left[-\alpha \frac{\cos^2(\varphi)}{r} + k \sin(\varphi) \cos(\varphi) e^{-ikr \cos(\varphi)} \delta\phi_a \right]$$

From first term in the product, the following contribution to the force is found

$$\frac{1}{2} k \alpha |\phi_k|^2 \int_0^{2\pi} d\varphi \cos^2(\varphi) = \frac{\pi}{2} |\phi_k|^2 k \alpha \quad (3.22)$$

This is again a term coming from the twist of the plane wave. The second term in $\partial_r \phi^* \partial_y \phi$ gives

$$\begin{aligned} & -\frac{r}{2} |\phi_k|^2 k^2 \int_{-\pi}^{\pi} d\varphi \sin(\varphi) \cos(\varphi) e^{-ikr \cos(\varphi)} \delta\phi_a \\ & = -\frac{1}{2} |\phi_k|^2 k^2 \sqrt{r} \frac{\pi \alpha}{\sqrt{2\pi i k}} \int_{-\pi}^{\pi} d\varphi \frac{\cos(\varphi) \sin^2(\varphi)}{1 - \cos(\varphi)} e^{ikr(1-\cos(\varphi))} \end{aligned}$$

where the Born result for the scattered wave $\delta\phi_a = \frac{\pi \alpha}{\sqrt{2\pi i k r}} \frac{\sin(\varphi)}{1-\cos(\varphi)} e^{ikr}$ is used. The integrand is, as in the pressure term, oscillating faster and faster as r goes

to infinity. Only around $\varphi \approx 0$ the oscillations are rather slow. The main contribution is expected to come from a small region, $(-\Lambda, \Lambda)$, near zero. The parameter Λ is thought to be a small number. The integration limit can nevertheless be dragged to infinity, since the main contribution anyway comes from the small values of φ . The integral is then Gaussian and can be solved with the standard formula (appendix A)

$$-|\phi_k|^2 k^2 \sqrt{r} \frac{\pi \alpha}{\sqrt{2\pi i k}} \int_{-\Lambda}^{\Lambda} d\varphi e^{-\frac{k r}{2i} \varphi^2} \xrightarrow{\Lambda \rightarrow \infty} -\pi |\phi_k|^2 k \alpha \quad (3.23)$$

The above contribution comes from the small scattering angles. It could as well be found from the small angle expression for ϕ , just as done by Sonin [Son96]. In a way it is surprising that it is contained in the ordinary scattering amplitude as well. The two contributions to $F_{\perp}^{\phi\phi}$ are added, so that

$$F_{\perp}^{\phi\phi} = -\frac{\pi}{2} |\phi_k|^2 k \alpha \quad (3.24)$$

There is then one contribution left to study, and that is what we labeled F_{\perp}^q in (3.16). Unlike the expression for $F_{\perp}^{\phi\phi}$, F_{\perp}^q is in itself proportional to α . The only possibility for a first order result in α is when plane waves are inserted for ν and ϕ , i.e.

$$\phi = \phi_k e^{ikr \cos(\varphi)} + \mathcal{O}(\alpha) \quad \nu = ik\phi_k e^{ikr \cos(\varphi)} + \mathcal{O}(\alpha)$$

The last contribution to the force then becomes

$$F_{\perp}^q = -\frac{1}{2} |\phi_k|^2 k \alpha \int_0^{2\pi} d\varphi \cos^2(\varphi) = -\frac{\pi}{2} |\phi_k|^2 k \alpha \quad (3.25)$$

Again Sonin's article is actually the only one of the discussed articles mentioning this contribution.

So far $\alpha = qk$ is used as a scattering parameter, since that is the natural choice when the analogy to Aharonov-Bohm scattering is considered. This is however not the most convenient choice when doing a thermal average. Then the best quantity to use is the average mass flux (1.2.2) of a plane wave, which is

$$\langle \mathbf{j}_k \rangle = \frac{1}{2} |\phi_k|^2 k \mathbf{k} \quad (3.26)$$

The circulation is, in dimensionless units, given as $\kappa_q = 2\pi q$. Expressed by these quantities the result from the Born approximation can be summarized as follows

$$F_{\perp}^P = 0 \quad F_{\perp}^{\phi\phi} = -\frac{1}{2} \kappa_q \langle \mathbf{j}_k \rangle \quad F_{\perp}^q = -\frac{1}{2} \kappa_q \langle \mathbf{j}_k \rangle \quad (3.27)$$

This is in agreement with Sonin [Son96], Stone [Sto99] and Shelankov [She98a], but differs from the result by Wexler and Thouless [WT98].

3.2.2 Circuit Near the Vortex

In section 3.2.1 the force was found by considering the asymptotic behavior of the Born solution at a large distance from the vortex. In this section we will turn the other way round, and try to extract information when the integration contour is placed *close to the vortex*. In a way this limit is more sinister than the large distance limit. When the integration contour is far from the vortex it does not matter where the force on the vortex is applied, since it anyway is inside the circuit. If the area enclosed by the integration loop is tiny it is however crucial that we know exactly where the force is applied. If the integration radius is taken to zero, the only possibility is to consider a point vortex, since a finite vortex core kept at rest must be affected by a force over a finite area. So far the point vortex approximation is presented as something which is just valid far from the vortex. This is just partly true, and actually it is just as much an approximation which holds for waves with long wavelengths. The distance r must be well away from the vortex core, but at a distance where kr can get arbitrarily small.

As a consequence of this, the solution of the scattering on a point vortex will now be used for small integration contours to find the force. Since the integration contour is of length $\sim r$, only terms of order $\sim r^{-1}$ can be contributing when the integration contour is small. In the Born approximation, the solution was only found asymptotically at large distances from the vortex. But the partial wave analysis result is valid near a vortex, and the partial wave expression for ϕ is nonsingular as kr approaches zero. This means that all terms containing just ϕ or derivatives of ϕ can be neglected in this limit, and $F^{\phi\phi} \rightarrow 0$. The density fluctuation is however singular in the the point vortex approximation

$$\nu = ik\phi - q\nabla\varphi \cdot \nabla\phi = ik\phi + i\alpha\phi_k \frac{\sin(\varphi)}{r} e^{ikx} + \mathcal{O}(\alpha^2) \quad (3.28)$$

The pressure is $P = \frac{1}{4}(|\nu|^2 - |\nabla\phi|^2)$. In the small r limit the $|\nabla\phi|^2$ term vanishes. The same happens to the $k^2|\phi|^2$ part of $|\nu|^2$. So when the limit is taken we get

$$P \xrightarrow{r \rightarrow 0} \frac{1}{2} |\phi_k|^2 k\alpha \frac{\sin(\varphi)}{r} \quad (3.29)$$

This gives

$$F_{\perp}^P = -r \int_0^{2\pi} d\varphi P \sin(\varphi) \xrightarrow{r \rightarrow 0} -\frac{1}{2} \kappa_q \langle j_k \rangle \quad (3.30)$$

where the circulation quantum is $\kappa_q = 2\pi q$ and the average mass current is $\langle j_k \rangle = \frac{1}{2} |\phi_k|^2 k^2$. In the last term, F_{\perp}^q , only the plane waves are inserted for ϕ and ν to lowest order in α . The result is hence independent of r , and the result from last section is still valid. The analytical calculation of the force in the limit near the vortex can hence be summarized as

$$F_{\perp}^P = -\frac{1}{2} \kappa_q \langle j_k \rangle \quad F_{\perp}^{\phi\phi} = 0 \quad F_{\perp}^q = -\frac{1}{2} \kappa_q \langle j_k \rangle \quad (3.31)$$

This must be compared with the large r limit (3.27). The terms are not the same, but their sum is equal. The conclusion is that in the point vortex approximation, the force on the system is just in one point, and hence it does not matter where the integration contour is placed.

The asymptotic calculation presented here is remarkably simple. *It was not even necessary to solve the scattering problem to get the result.* The only conditions it relies on, is that ϕ is nonsingular when $r \rightarrow 0$, while ν is not.

3.2.3 Numerical Calculations

In addition to the Born solution of the Aharonov-Bohm problem, there is also an expression by partial wave analysis. In the previous sections 3.2.1 and 3.2.2 the asymptotes when the integration contour was placed at large distance and close to the vortex, were considered. The same could be done with the partial wave solution. There should however not be any need to do the same calculations again², so instead we will make formulae valid for integration contours at all distances, and then study them numerically. Our intention is to clutter as little as possible with the original result, and hence we will just plug in the full partial wave expression into the formula for the transverse force. All the formulae, as presented here, are complex, but from the derivation of the transverse force we remember that the force was given as the real part of the expressions. All numbers presented from now on are real parts of the expressions. It must be noted that the imaginary parts are usually very small. In the analytical calculations done earlier, they disappeared completely.

In the numerical handling of the problem, the substitution $z = kr$ is done. The wave $\phi = \phi(z, \varphi)$ is

$$\phi = \sum_l \phi_l e^{il\varphi} \quad (3.32)$$

where

$$\phi_l = (-1)^l e^{-i\frac{\pi}{2}\Omega_l} J_{\Omega_l}(z) \quad (3.33)$$

The subscript of the Bessel functions are in the acoustic Aharonov-Bohm scattering $\Omega_l = \sqrt{l^2 + 2\alpha l}$, with $\alpha = qk$. In the calculations the density ν is also needed. It is found from the phase in (2.4). In the point vortex approximation it is

$$\begin{aligned} \nu &= [ik - q\nabla\varphi \cdot \nabla] \phi \\ &= ik \sum_l \left[1 - \frac{\alpha l}{z^2} \right] \phi_l e^{il\varphi} \equiv ik \sum_l R_l e^{il\varphi} \end{aligned} \quad (3.34)$$

²The Born and partial wave analysis solutions have been carefully compared, and found to be the same.

which defines $R_l = R_l(z)$. The derivative of ϕ_l is

$$\frac{\partial \phi_l}{\partial r} = \frac{k}{2} (-1)^l e^{-i\frac{\pi}{2}\Omega_l} (J_{\Omega_l-1}(z) - J_{\Omega_l+1}(z)) \quad (3.35)$$

The integrals with respect to φ are easy to perform in the partial wave analysis. They will always be one of the two

$$\begin{aligned} \int_0^{2\pi} d\varphi \sin(\varphi) e^{i(l-m)\varphi} &= i\pi(\delta_{l-m,1} - \delta_{l-m,-1}) \\ \int_0^{2\pi} d\varphi \cos(\varphi) e^{i(l-m)\varphi} &= \pi(\delta_{l-m,1} + \delta_{l-m,-1}) \end{aligned} \quad (3.36)$$

Let us now find the force by handling the equations (3.14), (3.15) and (3.16) one by one. From the calculation of the momentum-flux tensor, the pressure was found to be $P = \frac{1}{4}(|\nu|^2 - |\nabla\phi|^2)$. In partial wave analysis this is

$$P = \frac{k^2}{4} \sum_{lm} \left[R_l R_m^* - \frac{\partial \phi_l}{\partial z} \frac{\partial \phi_m^*}{\partial z} - \frac{lm}{z^2} \phi_l \phi_m^* \right] e^{i(l-m)\varphi}$$

Integrating this according to (3.36) the pressure part of the force is

$$F_{\perp}^P = -r \int_0^{2\pi} d\varphi P \sin(\varphi) = -\frac{\pi}{4} k (S_1^P + S_2^P + S_3^P) \quad (3.37)$$

where the tree sums are

$$S_1^P = iz \sum_l R_l (R_{l-1}^* - R_{l+1}^*) \quad (3.38)$$

$$S_2^P = -iz \sum_l \frac{\partial \phi_l}{\partial z} \left(\frac{\partial \phi_{l-1}^*}{\partial z} - \frac{\partial \phi_{l+1}^*}{\partial z} \right) \quad (3.39)$$

$$S_3^P = -\frac{1}{z} \sum_l l \phi_l ((l-1)\phi_{l-1}^* - (l+1)\phi_{l+1}^*) \quad (3.40)$$

An important thing to notice is that the only dependency of k in these sums is through the flux $\alpha = kq$, so lowest order in q also means lowest order in k .

The partial derivative with respect to y is in polar coordinates $\partial_y = \sin(\varphi)\partial_r + \frac{1}{r}\cos(\varphi)\partial_{\varphi}$. The integrand in $F_{\perp}^{\phi\phi}$ is then

$$\frac{\partial \phi}{\partial r} \frac{\partial \phi^*}{\partial y} = k^2 \sum_{lm} \frac{\partial \phi_l}{\partial z} \left(\sin(\varphi) \frac{\partial \phi_m^*}{\partial z} - i \cos(\varphi) \frac{m}{z} \phi_m^* \right) e^{i(l-m)\varphi}$$

With the integration formulae (3.36) the double sums reduce to single sums

$$F_{\perp}^{\phi\phi} = -\frac{1}{2} \int_0^{2\pi} d\varphi \frac{\partial \phi}{\partial r} \frac{\partial \phi^*}{\partial y} = -\frac{\pi}{2} k (S_1^{\phi\phi} + S_2^{\phi\phi}) \quad (3.41)$$

where

$$S_1^{\phi\phi} = iz \sum_l \frac{\partial\phi_l}{\partial z} \left(\frac{\partial\phi_{l-1}^*}{\partial z} + \frac{\partial\phi_{l+1}^*}{\partial z} \right) \quad (3.42)$$

$$S_2^{\phi\phi} = -i \sum_l \frac{\partial\phi_l}{\partial z} \left((l-1)\phi_{l-1}^* - (l+1)\phi_{l+1}^* \right) \quad (3.43)$$

The only term left now is the one that was called F_\perp^q . It involved a product of the kind

$$\frac{\partial\phi}{\partial r} \nu^* = -ik^2 \sum_{lm} \frac{\partial\phi_l}{\partial z} R_m^* e^{i(l-m)\varphi}$$

What made this term different from the other was that it was in itself proportional to α . Since we are just keeping first order terms, plane waves can actually be inserted for ϕ and ν . It can however be nice to have a formula at the same form as the other terms, so that

$$F_\perp^q = -\frac{\pi}{2} k \alpha S^q \equiv -\frac{\pi}{2} k \alpha \left[-i \sum_l \frac{\partial\phi_l}{\partial z} (R_{l-1}^* + R_{l+1}^*) \right] \quad (3.44)$$

The transverse force can now be found by summing $S_1^P, S_2^P, S_3^P, S_1^{\phi\phi}, S_2^{\phi\phi}$ and S^q . The only k dependency of the sums are through powers of α . Since we are only interested in the leading order terms, let us do some final definitions

$$A^P = \frac{1}{4} \frac{\partial}{\partial \alpha} [S_1^P + S_2^P + S_3^P]_{\alpha=0} \quad (3.45)$$

$$A^{\phi\phi} = \frac{1}{2} \frac{\partial}{\partial \alpha} [S_1^{\phi\phi} + S_2^{\phi\phi}]_{\alpha=0} \quad (3.46)$$

The last sum S^q is in itself proportional to α so it just needs to be evaluated at $\alpha = 0$. To first order in α , the force can then be written as

$$F_\perp^P = -\kappa_q \langle j_k \rangle A^P \quad F_\perp^{\phi\phi} = -\kappa_q \langle j_k \rangle A^{\phi\phi} \quad F_\perp^q = -\frac{1}{2} \kappa_q \langle j_k \rangle S^q|_{\alpha=0} \quad (3.47)$$

This is a convenient way of writing it, when it shall be compared with the analytical limits (3.27) and (3.31). Both A^P and $A^{\phi\phi}$ are functions of $z = kr$ only. When $z \rightarrow 0$ it can mean either that r gets small or k gets small (or both).

The above result has been studied analytically in the two limits $z \rightarrow \infty$ and $z \rightarrow 0$. In (3.27) the large contour limit was found to be

$$A^P \rightarrow 0 \quad A^{\phi\phi} \rightarrow \frac{1}{2}$$

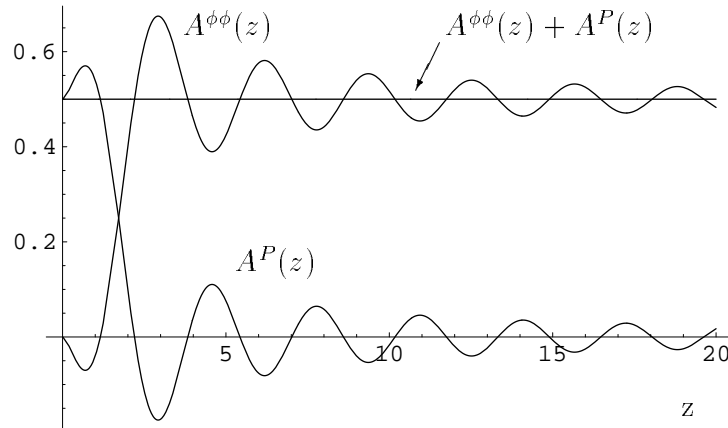


Figure 3.1: Shows $A^P(z)$, $A^{\phi\phi}(z)$ and $A^P(z) + A^{\phi\phi}(z)$ as function of z . The numerical expressions are done with $|l| < z + 10$. The two terms $A^P(z)$ and $A^{\phi\phi}(z)$ oscillate around their respective average values for large z , but switch places near $z = 0$. The sum $A^P(z) + A^{\phi\phi}(z)$ is constant.

The other limit, for small z , was found in (3.31) to be

$$A^P \rightarrow \frac{1}{2} \qquad A^{\phi\phi} \rightarrow 0$$

The last terms, F^q , was the same in both limits and since only plane waves enter into the calculation, $S^q|_{\alpha=0} = 1$ always. As said earlier it is expected that the sum $A^P + A^{\phi\phi}$ should remain constant in the point vortex approximation. This is because the momentum transfer between the vortex and the wave is just in one point, and hence it does not matter at what distance the integration contour is placed. This can be seen in plot 3.1 which shows numerical values of A^P , $A^{\phi\phi}$ and $A^P + A^{\phi\phi}$ as functions of z .

The numerical handling of the sums is easiest for small z . It is the Bessel functions $J_{\Omega_l}(z)$ that make our series convergent, since they converge rapidly to zero as $|l| \gg z$. This means the further out z is, the more terms must be evaluated. It is then a question of computer speed how far out we can go. There is also a problem with respect to numerical errors. The more terms that are added, the more numerical errors are accumulated.

The table 3.1 shows some values of A^P and $A^{\phi\phi}$ around $z = 100$, and where all terms with $|l| < 120$ are evaluated. The values are not far from the asymptotic values of $A^P = 0$ and $A^{\phi\phi}$. They seem to oscillate around these values, just as in the plot 3.1, but with lower amplitude since they are further out.

	$z = 100$	$z = 101$	$z = 102$	$z = 103$
$A^P =$	-0.0025	-0.0029	0.0048	-0.0011
$A^{\phi\phi} =$	0.5025	0.5029	0.4952	0.5011

Table 3.1: Some values A^P and $A^{\phi\phi}$ near $kr = 100$.

3.3 Conclusion

In this chapter three different calculations of the transverse force from a scattered wave has been done: one by putting the integration contour far from the vortex (section 3.2.1), one by putting the integration contour near the vortex (section 3.2.2) and one numerical at finite distances (section 3.2.3). The analytical limit with the integration contour at large distance gave the result

$$F_{\perp}^P = 0 \quad F_{\perp}^{\phi\phi} = -\frac{1}{2}\kappa_q \langle j_k \rangle \quad F_{\perp}^q = -\frac{1}{2}\kappa_q \langle j_k \rangle \quad (3.48)$$

This limit is the one usually considered when collecting the force. The calculation was tedious, and strongly dependent of the form of the wave. Both the special small angle contribution and the twist of the incident wave, proved to be crucial for the result.

Considering a point vortex, the force could also be found with an integration contour asymptotically near the vortex. This gave the result

$$F_{\perp}^P = -\frac{1}{2}\kappa_q \langle j_k \rangle \quad F_{\perp}^{\phi\phi} = 0 \quad F_{\perp}^q = -\frac{1}{2}\kappa_q \langle j_k \rangle \quad (3.49)$$

This calculation was very easy, but the limit is not so safe as the large contour limit, since it is just valid for a point vortex, and for small wavelengths.

The numerical result was obtained by summing up a finite number of terms from the partial wave analysis. The result is shown in figure 3.1, which also verifies the two analytical results.

The respective terms differ in different calculations, but the total force, when all terms are collected, is always the same. The conclusion from this chapter is that the transverse force from a single scattered wave is

$$F_{\perp} = -\kappa_q \langle j_k \rangle \quad (3.50)$$

The force is proportional to the circulation quantum and the mass flux of the incoming wave. When going the full two-fluid expression this gives a force proportional to the normal fluid velocity, which is in favor of the Iordanskii force. The full discussion of the consequences of this result, is held in chapter 5.

In the calculations of the force only the solution of the acoustic Aharonov-Bohm scattering problem was used. Some of the literature use the magnetic version instead, and to first order in the flux α , the magnetic analogy differs from

this by an additional s-wave, $\Delta = \Delta(r)$, found in (2.49). This is, however, not problematic to the transverse force, since any s-wave gives contributions anti-symmetric in φ , and thus vanishing in the integrals.

But even if the s-waves are not contributing to the transverse force, they are essential to the evaluation of the longitudinal force.

3.4 The Background Fluid Force

In this chapter the force on a vortex from a plane wave has been carefully discussed. The derivation involved many steps and terms, so to be able to devote all attention to the wave, only the simplest of all possible background fluids was considered, namely one vortex at rest. A more general background fluid could for example be the sum of a vortex at rest and a fluid moving with constant velocity \mathbf{v}_s^∞ far from the vortex. It is possible that the asymptotic superfluid velocity field, could as the plane waves, exert a force on the vortex. In section 1.2.4 a vortex moving with constant velocity, was considered. Because of galilean invariance this is equivalent with considering a stationary vortex and a constant velocity field far from the vortex. If we substitute $\dot{\mathbf{r}} \rightarrow -\mathbf{v}_s^\infty$ in the density correction from section 1.2.4, we get the solution

$$\mathbf{v}_s = q\nabla\varphi + \mathbf{v}_s^\infty \quad \rho = 1 - q\nabla\varphi \cdot \mathbf{v}_s \quad (3.51)$$

The force on the vortex can be found, in exactly the same manner as previously in this section, by an integral of the momentum-flux tensor (3.1). The only difference is that the above solutions are inserted for the plane wave solutions. The integration contour is a large circle with the vortex in the center. All terms of order r^{-2} can be neglected, and thus all derivatives of the density ρ can be forgotten too. The two velocity fields $\nabla\varphi$, and \mathbf{v}_s^∞ are of very different natures. Far from the vortex the constant field \mathbf{v}_s^∞ is dominating the local behavior since the vortex contribution goes as $\sim r^{-1}$. The vortex is on the contrary essential to the global properties of the fluid, such as the circulation, where the constant field \mathbf{v}_s^∞ does not contribute at all.

The background fluid force must be first order in \mathbf{v}_s^∞ and first order in the circulation quantum κ_q , if it exists (hopefully it does). Terms of the kind $(\mathbf{v}_s^\infty)^2$ and $(\nabla\varphi)^2$ are symmetric about the origin and do not contribute. The relevant terms in the energy momentum tensor (3.4) are

$$\Pi_{ij}^0 = -(q\nabla\varphi \cdot \mathbf{v}_s^\infty) \delta_{ij} + qv_{si}^\infty \partial_j\varphi + qv_{sj}^\infty \partial_i\varphi \quad (3.52)$$

The first term is diagonal so the coefficient is identified as the pressure. Let us then let the asymptotic superfluid velocity be along the x -axis, $\mathbf{v}_s^\infty = v_s^\infty \mathbf{e}_x$. The transverse force can then be written as

$$F_\perp^0 = -qv_s^\infty \oint [(-\partial_x\varphi) dS_y + \partial_y\varphi dS_x + \partial_x\varphi dS_x]$$

where $\mathbf{dS} = r(\cos(\varphi), \sin(\varphi))d\varphi$ and $\nabla\varphi = \frac{1}{r}(-\sin(\varphi), \cos(\varphi))$. Inserting this leads to

$$F_{\perp}^0 = -qv_s^{\infty} \int_0^{2\pi} d\varphi [\sin^2(\varphi) + \cos^2(\varphi)] = -2\pi qv_s^{\infty}$$

We notice that there are two equal contributions to the force, one from the pressure and one from the cross term $v_{si}^{\infty}\partial_j\varphi$. With vector notation this gives

$$\mathbf{F}_{\perp}^0 = -\boldsymbol{\kappa} \times \mathbf{v}_s^{\infty} \quad (3.53)$$

At zero temperature this is the only transverse force, and it is identical to the classical Magnus force (1.10). In full units, the force is proportional to the total density ρ .

This derivation was not so careful as the one with a plane wave passing the vortex. Physical arguments were used instead of mathematical calculations to get the result as fast as possible. The question now is: is this result still valid at nonzero temperature? In chapter 4 helium II at nonzero temperature is studied. The conclusion from that chapter is that with Ginzburg-Landau theory, helium II looks like a phonon gas on a background fluid, where the background fluid is a classical solution just like this. The result from this section is hence not altered when temperature is introduced. But we might get modifications to the scattering problem in chapter 2, and this could change the force from the scattered plane waves calculated earlier in this chapter.

Chapter 4

The Two-Fluid Description

4.1 Phenomenology

With normal pressure helium is the only existing fluid at temperatures near the absolute zero, and consequently helium is the only possibility of studying liquids at temperatures where quantum phenomenon becomes crucial. In nature there are two species of helium, ${}^3\text{He}$ obeying Fermi statistics and ${}^4\text{He}$ obeying Bose statistics, where in this thesis only the latter will be considered. Two phase transitions occur in ${}^4\text{He}$. The first is at the boiling point at $T = 5.2\text{K}$. Below this temperature ${}^4\text{He}$ behaves as a viscous liquid, called helium I, down to the lambda point at $T_\lambda = 2.172\text{K}$. Below this temperature we speak of helium II, in which a new phenomenon appears: flow through narrow capillaries without viscosity. This property is called *superfluidity*, and was first discovered by P.L.Kapitza in 1938. But regarding helium II as a pure superfluid is not correct either, since viscous phenomena are observed. For example will a rotating disk in helium II be brought to rest because of friction with the liquid. These two properties were first explained by Landau [Lan41] in 1941, by treating helium II as a mixture of two fluids: a superfluid with density ρ_s and a normal viscous fluid with density ρ_n , where the physical density of the liquid is the sum of these. There is no entropy associated with the superfluid, so all heat transfer is in the normal fluid. This explains the superfluid flow through capillaries since a temperature gradient is needed to set up a net normal fluid flow. The two densities are strongly dependent of temperature, so at the absolute zero helium II is purely superfluid while at the lambda temperature only the normal fluid is present. The total density is approximately constant.

The total mass flux of the fluid is $\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$ with superfluid velocity \mathbf{v}_s and normal fluid velocity \mathbf{v}_n . The superfluid velocity is without rotation so that $\nabla \times \mathbf{v}_s = 0$ everywhere (except vortex points).

The microscopic picture of helium II is that the superfluidity is due to the Bose-Einstein condensation below the lambda temperature. Landau [Lan41] did

not believe in a relation between superfluidity and Bose-Einstein condensation, and he commented that “nothing could prevent atoms in a normal state from colliding with excited atoms, i.e. when moving through the liquid they would experience a friction, and there would be no superfluidity at all”. The objection is later regarded as false and Bogoliubov [Bog47] showed that in the case of a weakly interacting Bose gas there was no friction between the excitations and the ground state atoms. This is also believed to be true for strongly interacting helium II ¹. In [LL87] it is also emphasized that we must be careful not to think of the two fluids as two physical fluids existing independently of each other. Instead it must be thought of as if the quantum fluids possess the property of having two flows simultaneously. They therefore recommends to use the terms superfluid and normal *flow* instead of *part*.

Many textbooks give a brief introduction to the two-fluid model. A nice general description can be found in [WB87], while [LL87] gives a macroscopic description with much attention to thermodynamics. For our situation with quantized vortices, [Don91] must be consulted, since he discusses forces in the two-fluid model, and the two-fluid model in presence of vortices.

4.1.1 Excitations

It is possible to begin a discussion of helium II by describing the individual atoms and their interactions. The interaction between the atoms is however strong, even at zero temperature, so such an approach is likely to miss the more collective nature of the excitations in the liquid. Two kinds of elementary collective excitations are usually spoken of: phonons and rotons. A discussion of these excitations was first held by Landau [Lan41]. Sometimes the phrase phonon means “sound wave”. Speaking of a gas of sound waves is however not too instructive, so phonon means in our context just a kind of collective excitation of the fluid. If the excitations have momentum \mathbf{p} , their energies are

$$E^{ph} = cp \qquad E^r = \frac{(p - p_0)^2}{\mu} + \Delta \qquad (4.1)$$

where c is the speed of sound, μ is the effective mass of the roton and Δ is the roton energy gap. After the first article by Landau it has been pointed out by Bogoliubov [Bog47] that the division in phonons and rotons is artificial. The phonons should be regarded as different region of the same excitation spectrum. The same conclusion was also reached by Landau in [Lan47], where he also sketches the curve 4.1. Near zero momentum, the curve is almost linear. This is the phonon region. The roton region is near the minimum of the curve at $Q = 1.9\text{\AA}^{-1}$. Even though the phonons and rotons can be regarded as different regimes of an excitation curve, it is often convenient to speak of phonon

¹There are, as we will see, interactions between the normal fluid and the superfluid, but these are only through vortex lines.

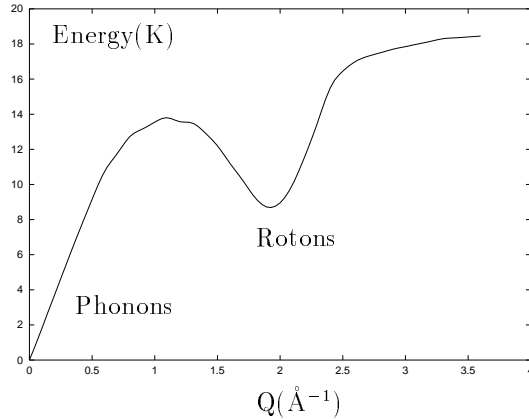


Figure 4.1: Experimental result for the excitation energy as a function of momentum. Found from inelastic neutron scattering data in [DDH81].

regions and roton regions. It is still unclear what causes the non-linear form of the excitation curve. For temperatures above $1K$ the thermodynamical behavior is almost completely due to the rotons. For low temperatures mainly excitations with small energies are created. Because of the energy gap, Δ , this means that phonons are dominating for very low temperatures, usually below $0.4K$. *In the rest of this thesis we will forget about rotons, and only the phonon region will be considered.*

The picture of a gas of excitation on a background also leads to the interesting feature of second sound. First sound is what previously has been called just “sound”: waves moving through the condensate. Second sound is sound waves propagating in the gas of excitations. There is no room (nor need) for a discussion of the properties of second sound in this thesis. More about excitations in helium II can be found in [Gly94] or other text-books.

4.1.2 Superfluidity

Since superfluidity is such an important phenomenon in helium II a short argument for superfluidity, as it was first done by Landau [Lan41], is presented here. The argument is very simple and it shows that superfluidity is just a consequence of the excitation spectrum and the galilean invariance of the system. Landau’s argument has been modified and generalized, but the basic idea is always the same. Let us consider a superfluid at the absolute zero with uniform velocity \mathbf{v} , flowing through a narrow capillary. Friction with the walls is, at a microscopic level, excitation of phonons and rotons in the fluid in contact with the boundaries. Thus let a phonon be excited with momentum \mathbf{p} in the rest system of the fluid, where the walls are moving with velocity $-\mathbf{v}$. The total momentum is \mathbf{P}_0

and E_0 is the total energy of the system (fluid+excitation).

$$\mathbf{P}_0 = \mathbf{p} \qquad E_0 = cp \qquad (4.2)$$

In the rest system of the capillary, the energy and momentum is

$$\mathbf{P} = \mathbf{P}_0 + M\mathbf{v} \qquad E = E_0 + \mathbf{P}_0 \cdot \mathbf{v} + \frac{1}{2}M\mathbf{v}^2 \qquad (4.3)$$

where M is the mass of the fluid. The last term in the energy is the kinetic energy of the fluid. The first two terms are the excitation energy $\Delta E = E_0 + \mathbf{P}_0 \cdot \mathbf{v}$ in the rest system of the capillary. For a phonon to be excited, this energy must be negative

$$\Delta E = cp + \mathbf{p} \cdot \mathbf{v} < 0 \qquad (4.4)$$

This means that there cannot be any excitation of phonons unless the fluid is moving faster than the speed of sound, i.e.

$$v > c \qquad (4.5)$$

if the fluid velocity is slow, there are no excitations and hence no friction with the walls. Exactly the same argument can be applied to the roton excitations. It leads to the condition that $v > \sqrt{\frac{2\Delta}{\mu}}$, where Δ is the energy gap of the roton and μ is the chemical potential. As long as the energy gap is non-zero, superfluidity is possible. In contrast to liquid helium, there is no energy gap for a free Bose gas. It is thus not a superfluid. A discussion of the energy spectrum of a weakly interacting Bose gas is held by Griffin [Gri96]

At finite temperatures helium II consists of a condensate and particles in excited states. The above argument which shows that for a slowly moving fluid, no new phonons and rotons are excited, should still be valid. There are, however, interactions between the walls and the thermal excitations already existing in the liquid. The superfluid is thus associated with the condensate, while the normal fluid is regarded as an ordinary viscous fluid.

4.1.3 The Equations

The intention with this introduction to the two-fluid model is not to give a complete discussion of all its properties and peculiarities, but to give enough information to understand the discussion of the forces on a vortex in chapter 5. So far, the important properties of the two-fluid model has been described mainly with words, but it can also be useful to see the equations of motion. As described in [Don91], they are

$$\rho_s \frac{d\mathbf{v}_s}{dt} = -\frac{\rho_s}{\rho} \nabla P + \rho_s S \nabla T + \frac{\rho_n \rho_s}{2\rho} \nabla (\mathbf{v}_n - \mathbf{v}_s)^2 - \mathbf{F}_{ns} \qquad (4.6)$$

$$\rho_n \frac{d\mathbf{v}_n}{dt} = -\frac{\rho_n}{\rho} \nabla P - \rho_s S \nabla T - \frac{\rho_n \rho_s}{2\rho} \nabla (\mathbf{v}_n - \mathbf{v}_s)^2 + \mathbf{F}_{ns} + \eta \nabla^2 \mathbf{v}_n^2 \qquad (4.7)$$

So the superfluid is described by a variant of Euler's equation (1.3) and the normal fluid is described by a kind of Navier-Stoke's equations (1.2), as expected. In addition each of the fluids respect a continuity equation. The most interesting feature, from our point of view, are the mutual friction terms between the normal fluid and superfluid, \mathbf{F}_{ns} . They are necessary to get a correct description of the two fluid model at a macroscopic level. The link to the rest of this thesis is that on a microscopic level, the mutual friction forces are through vortex lines. The superfluid Magnus force and the Iordanskii force discussed in chapter 5 are then both contributing to the mutual friction terms.

A detailed description of the thermodynamics of the two-fluid model is either not needed. One feature worth mentioning, though, is that there is no entropy associated with the superfluid flow. This must be kept in mind when discussing adiabatic motion in section 5.3.

4.2 Derivation from Ginzburg-Landau Theory

The two-fluid description, as sketched in this chapter, is pure phenomenology; this is how helium II looks in experiments. The point of the following is to link the two-fluid model to Ginzburg-Landau theory. In the chapters 1, 2 and 3 the Gross-Pitaevskii approximation was considered, which means that the quantum fields were approximated with classical functions. This approximation is valid near the absolute zero, when the liquid is in a condensed state. A better approximation for low temperatures is to consider small deviations from the classical fields, and expand in these quantities.

A second quantized description of liquid helium could start with the Ginzburg-Landau Hamiltonian density (4.8), which is a function of the atom creation and annihilation operators, $\hat{\Psi}^\dagger$ and $\hat{\Psi}$. For an interacting Bose gas, is it convenient to write $\hat{\Psi} = \Psi + \delta\hat{\Psi}$, where $\Psi = \langle \hat{\Psi} \rangle$ is the condensate wavefunction. In a Bose gas the condensed state is usually the non-interacting zero momentum state. The problem with liquid helium is that it is not easy to identify the condensate in such way, since even the ground state is strongly interacting. A way to avoid such problems is to forget about condensate and non-condensate, and describe the liquid as a phonon gas at the top of a background fluid instead. In such a description the opportunity to do direct identification of the condensate or normal fluid is lost, but the collective character of the background fluid and the excitations is emphasized.

In the chapters 1, 2 and 3 the polar form of the classical fields is used, $\Psi = \sqrt{\rho} \exp(i\theta)$, where ρ and θ are real. We will now quantize these fields. A quantization of the phonon gas makes it possible to find the normalization of the fields, and thus determine the mass flux and the energy of the liquid. The time averages of the classical quantities, become assembly averages in the second quantized formalism.

The simplest case is to consider a homogeneous background moving with velocity \mathbf{v}_s . As in most of the calculations in Ginzburg-Landau theory, dimensionless units are used.

The Hamiltonian is

$$\hat{\mathcal{H}} = \frac{1}{4} \left(\hat{\rho}(\nabla\hat{\theta})^2 + (\nabla\hat{\theta})^2\hat{\rho} \right) + \frac{1}{8\hat{\rho}}(\nabla\hat{\rho})^2 + \frac{1}{2}(\hat{\rho} - 1)^2 \quad (4.8)$$

where $-\hat{\rho}$ is the conjugate momentum of $\hat{\chi}$. The Hamiltonian is Hermitian since the fields are self-conjugate. Working with the full fields is impossible because of the non-linear structure of the Hamiltonian density. As described above, small deviations from the classical solutions will now be considered. A very simple classical solution of the above equations is $\rho = 1 + \mathcal{O}(\mathbf{v}_s^2)$ and $\theta = \theta_0$, where the superfluid velocity $\mathbf{v}_s = \nabla\theta_0$ is small and constant. The operators are then

$$\hat{\rho} = 1 + \hat{\eta} \quad \hat{\theta} = \theta_0 + \hat{\chi} \quad (4.9)$$

The Hamiltonian (4.8) can now be expanded to second order in the fields $\hat{\eta}$ and $\hat{\chi}$. The zeroth and first order terms will not contribute to the dynamics of the system, so only the quadratic terms must be taken into account

$$\hat{\mathcal{H}}_2 = \frac{1}{2}(\nabla\hat{\chi})^2 + \hat{\mathbf{j}}^{ph} \cdot \mathbf{v}_s + \frac{1}{8}(\nabla\hat{\eta})^2 + \frac{1}{2}\hat{\eta}^2 \quad (4.10)$$

where the phonon flux operator is

$$\hat{\mathbf{j}}^{ph} = \frac{1}{2} (\hat{\eta}\nabla\hat{\chi} + \nabla\hat{\chi}\hat{\eta}) \quad (4.11)$$

The classical solutions of the above Hamiltonian are plane waves, as described in (1.25). The fields can hence be expanded by the classical waves as

$$\hat{\chi}(\mathbf{x}, t) = \sum_{\mathbf{k}} \chi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - \omega t} \hat{\alpha}_{\mathbf{k}} + \chi_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{x} + \omega t} \hat{\alpha}_{\mathbf{k}}^\dagger \quad (4.12)$$

$$\hat{\eta}(\mathbf{x}, t) = \sum_{\mathbf{k}} \eta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - \omega t} \hat{\alpha}_{\mathbf{k}} + \eta_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{x} + \omega t} \hat{\alpha}_{\mathbf{k}}^\dagger \quad (4.13)$$

where $\hat{\alpha}_{\mathbf{k}}$ and $\hat{\alpha}_{\mathbf{k}}^\dagger$ are the phonon annihilation and creation operators obeying the ordinary Bose commutator

$$[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}_{\mathbf{l}}^\dagger] = \delta_{\mathbf{k}\mathbf{l}} \quad (4.14)$$

According to the classical solution found in section 1.2.2, the frequency is $\omega = \omega_0 + \mathbf{k} \cdot \mathbf{v}_s$, where $\omega_0 = k\sqrt{1 + \frac{k^2}{4}}$. The energy could be expanded in small k giving $\omega_0 \approx k$. In a way this is an honest thing to do, since the expression is not dealing right with the high momenta. The cost of keeping the full expression is

small, thought, so it will be kept to remind us that the energy spectrum is not simply linear. An experimental result for the energy spectrum in helium II is shown in figure 4.1. The field $\hat{\chi}$ and its conjugate momentum $-\hat{\eta}$ must satisfy the fundamental commutator

$$[\hat{\chi}(\mathbf{x}, t), \hat{\eta}(\mathbf{y}, t)] = -i\delta(\mathbf{x} - \mathbf{y}) \quad (4.15)$$

In the thermodynamical limit, the volume is taken to infinity. The sum over \mathbf{k} then becomes an integral, $\sum_{\mathbf{k}} \rightarrow V \int \frac{d^D k}{(2\pi)^D}$, where the dimension D is either 2 or 3. Inserting the fields (4.12) and (4.13) into the fundamental operator leads to

$$V2i\Im \int \frac{d^D k}{(2\pi)^D} \chi_{\mathbf{k}}^* \eta_{\mathbf{k}} e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} = i\delta(\mathbf{x} - \mathbf{y}) \quad (4.16)$$

where \Im indicates the imaginary part of the expression. The relation

$$\Im(\chi_{\mathbf{k}}^* \eta_{\mathbf{k}}) = \frac{1}{2V} \quad (4.17)$$

is then obtained. Together with the classical relation $\eta_{\mathbf{k}} = i\frac{k^2}{\omega_0}\chi_{\mathbf{k}}$, this uniquely, up to a phase factor, determines the normalization of the fields

$$\chi_{\mathbf{k}} = \sqrt{\frac{\omega_0}{2Vk^2}} \quad \eta_{\mathbf{k}} = i\frac{k}{\sqrt{2V\omega_0}} \quad (4.18)$$

where the frequency is given by $\omega_0 = k\sqrt{1 + \frac{k^2}{4}}$. This is the result for the simple situation of a homogeneous condensate moving with constant velocity. There are many ways to generalize this, with for example a more general background fluid or we could have tried to draw more information from higher order terms, but the result presented here should be sufficient to recognize the ordinary two-fluid formulae. The most serious objection, due to the fact that the goal of this thesis is to determine the Lordanskii force, is that this description does not include vortices. A quantization of the phonon wave with a vortex in the background, is a lot more complicated than the one presented here. Probably it is more easily achieved in polar coordinates than in the ordinary Cartesian used here.

Regarding the k -dependency, a few things must be kept in mind. In the low energy limit the asymptotic k dependency is $\eta_k \sim \sqrt{k}$ and $\chi_k \sim \frac{1}{\sqrt{k}}$. The phase is dominating the density, which is the reason why derivatives of the density is often ignored. So far it is presented as we do expansion in η and χ . This is not entirely correct since the phase χ is not small for small k . But all dependency of χ in our theory is through the derivatives, $\nabla\chi$ and $\dot{\chi}$ which are small quantities. The correct presentation is hence that we do expansion in the density and *derivatives* of the phase.

4.2.1 The Mass Flux

Helium II, as presented here, consists of a background fluid, described by a classical wave-function, and a bath of thermal excitations around it. To identify the central formulae of the macroscopic two-fluid descriptions from this, translation and interpretation are needed. The identification is best done by studying the *mass flux* of the whole system. The mass flux operator is defined by

$$\hat{\mathbf{j}} = \frac{1}{2} \left(\hat{\rho} \nabla \hat{\theta} + \nabla \hat{\theta} \hat{\rho} \right) \quad (4.19)$$

The assembly average of the operator is

$$\langle \hat{\mathbf{j}} \rangle = \mathbf{v}_s + \langle \hat{\mathbf{j}}^{ph} \rangle \quad (4.20)$$

since the first order terms vanish. The phonon flux operator is defined in (4.11). In this simplified description the phonon gas is thought to be homogeneous, so that

$$\mathbf{j}^{ph} = \frac{1}{V} \int d^D x \langle \hat{\mathbf{j}}^{ph} \rangle \quad (4.21)$$

inserting the fields (4.13) and (4.12), only the diagonal terms where $\mathbf{k} = \mathbf{k}'$ survive the integration. Using the normalization from last section the expected result is obtained

$$\mathbf{j}^{ph} = \frac{1}{V} \sum_{\mathbf{k}} \mathbf{k} \left[N(\epsilon) + \frac{1}{2} \right] \quad (4.22)$$

where the Bose-Einstein distribution function is

$$N(\epsilon) = \langle \hat{\alpha}_{\mathbf{k}}^\dagger \hat{\alpha}_{\mathbf{k}} \rangle = \frac{1}{\exp(\epsilon/k_B T) - 1} \quad (4.23)$$

The energy spectrum of phonons on a moving background was found in section 1.2.2. Giving the phonon gas the drift velocity \mathbf{v}_n , the energy is

$$\epsilon = \omega - \mathbf{k} \cdot \mathbf{v}_n = \omega_0 - \mathbf{k} \cdot (\mathbf{v}_n - \mathbf{v}_s) \quad (4.24)$$

where the rest frame energy is $\omega_0 = k \sqrt{1 + k^2/4}$. The energy of the phonon gas is thus dependent on the relative velocity $\mathbf{v}_n - \mathbf{v}_s$. The velocities are thought to be small, so that the distribution functions can be expanded. In the thermodynamical limit $V \rightarrow \infty$, we obtain

$$\mathbf{j}^{ph} = - \int \frac{d^D k}{(2\pi)^D} \mathbf{k} [\mathbf{k} \cdot (\mathbf{v}_n - \mathbf{v}_s)] \left. \frac{\partial N}{\partial \epsilon} \right|_{\epsilon=\omega_0}$$

By putting the k_x -axis along the direction of $\mathbf{v}_n - \mathbf{v}_s$, the components along all other directions in \mathbf{k} space disappear and the important formula identifying the normal fluid density is obtained:

$$\mathbf{j}^{ph} = -(\mathbf{v}_n - \mathbf{v}_s) \int \frac{d^D k}{(2\pi)^D} k_x^2 \frac{\partial N}{\partial \epsilon} \Big|_{\epsilon=\omega_0} \equiv \rho_n (\mathbf{v}_n - \mathbf{v}_s) \quad (4.25)$$

The phonon part of the mass flux is proportional to the normal fluid density. But the direction is along the relative velocity $\mathbf{v}_n - \mathbf{v}_s$, not the normal fluid velocity \mathbf{v}_n . The *phonon gas is thus not equivalent to the normal fluid*. The total mass flux is

$$\mathbf{j} = \mathbf{v}_s + \rho_n (\mathbf{v}_n - \mathbf{v}_s) = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n \quad (4.26)$$

The *superfluid mass-density* is defined to be $\rho_s = 1 - \rho_n$. The relation between Ginzburg-Landau theory and the two-fluid description should be established. In Ginzburg-Landau theory we speak about a background fluid and a phonon gas, not superfluid and normal fluid. The density of the background fluid is identical to the total density and must not be confused with the superfluid density. The normal fluid is not visible in the total density, which is constant, but is identified from the mass flux, and it will be seen to be proportional to the normal fluid rest energy.

It can be useful to integrate out the angle dependency, to get a new formula for the normal density. In the low temperature limit only the small k are relevant, and we put $\omega_k = k$. There is no angle dependency in the distribution function, so we get π from the angle integration in two dimensions and $4\pi/3$ in three dimensions. By doing a partial integration with respect to k , and transform back the angle integral $\int d\Omega = 2(D-1)\pi$, a formula for the normal fluid density in the low temperature limit is obtained

$$\rho_n = \frac{D+1}{D} \int \frac{d^D k}{(2\pi)^D} k N_0 \quad (4.27)$$

The temperature dependency of ρ_n is easily found by substituting $u = k/k_B T$. The integral then becomes independent of T and it is seen that $\rho_n \sim T^{D+1}$. In the same way it is immediately seen that integrals of higher orders in k will also be higher order in T , and in the low temperature limit they can be neglected.

The exact evaluations of the integrals are not important, since the normal density will always be identified by one of the integrals (4.25) or (4.27).

4.2.2 The Energy

In the same manner as we found the two-fluid expressions for the mass flux the energy density can be found. The total Hamiltonian of the excitations is an

integral of the Hamiltonian density over the whole quantization volume

$$\widehat{H}_2 = \int d^D x \widehat{\mathcal{H}}_2 \quad (4.28)$$

where the Hamiltonian density after a few partial integrations is

$$\widehat{\mathcal{H}}_2 = -\frac{1}{2}\widehat{\chi}\nabla^2\widehat{\chi} + \widehat{\mathbf{j}}^{ph} \cdot \mathbf{v}_s - \frac{1}{8}\widehat{\eta}\nabla^2\widehat{\eta} + \frac{1}{2}\widehat{\eta}^2 \quad (4.29)$$

Inserting the Fourier components of η (4.13) and χ (4.12), the Hamiltonian density is given as a double sum. Using that the integration just gives the volume V for the diagonal elements, and zero for the off-diagonal elements, we can write the Hamiltonian as

$$\widehat{H}_2 = \sum_{\mathbf{k}} \omega_k \left[\widehat{N}_{\mathbf{k}} + \frac{1}{2} \right] \quad (4.30)$$

where the number operator is $\widehat{N}_{\mathbf{k}} = \widehat{\alpha}_{\mathbf{k}}^\dagger \widehat{\alpha}_{\mathbf{k}}$. The average energy density of the system can be found as the operator average of the Hamiltonian divided with the volume. It can, if the vacuum energy is dropped and the continuity limit is taken, be written as

$$E = \frac{1}{V} \langle \widehat{H}_2 \rangle = \int \frac{d^D k}{(2\pi)^D} \omega_{\mathbf{k}} N(\epsilon) \quad (4.31)$$

where $N(\epsilon) = \langle \widehat{N}_{\mathbf{k}} \rangle = (\exp(\epsilon/k_B T) - 1)^{-1}$ is the Bose-Einstein distribution function. The phonon frequency is, as in the last section, $\epsilon = \omega_0 - \mathbf{k} \cdot (\mathbf{v}_n - \mathbf{v}_s)$. To find the two-fluid expressions for the mass flux the distribution function was expanded to first order in the velocities \mathbf{v}_n and \mathbf{v}_s . In the case of the energy we need to go to second order, since the first order terms fall out in the angle integration. The zeroth order term is the rest energy density of the normal fluid

$$E_n = \int \frac{d^D k}{(2\pi)^D} \omega_0 N(\omega_0) \quad (4.32)$$

which is proportional to the normal fluid density (4.27). So far it has been difficult to find an interpretation of the normal fluid density, but it is now clear that it is associated with the thermal energy. The terms of second order in \mathbf{v}_n and \mathbf{v}_s are

$$\int \frac{d^D k}{(2\pi)^D} \left[-(\mathbf{v}_s \cdot \mathbf{k}) \mathbf{k} \cdot (\mathbf{v}_n - \mathbf{v}_s) \frac{\partial N(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=\omega_0} + \frac{1}{2} k ((\mathbf{v}_n - \mathbf{v}_s) \cdot \mathbf{k})^2 \frac{\partial^2 N(\epsilon)}{\partial \epsilon^2} \Big|_{\epsilon=\omega_0} \right]$$

where in the low energy limit $\omega_0 \approx k$. In the last term the the substitution

$$k \frac{\partial^2}{\partial k^2} N(k) = \frac{\partial}{\partial k} \left(k \frac{\partial N(k)}{\partial k} \right) + \frac{\partial N(k)}{\partial k} \quad (4.33)$$

can be done. The first term gives a contribution of higher order in T . The last can be identified as proportional to the normal fluid density (4.25). The cross-terms between the velocities cancel out, so that the excitation kinetic energy density is

$$\frac{1}{2}\rho_n(v_n^2 - v_s^2) \quad (4.34)$$

This is the result from the quadratic terms in the Hamiltonian. In addition there is a zeroth order contribution from the background fluid, $\frac{1}{2}\rho_s v_s^2$, so the total energy density is exactly what we hoped for, namely

$$E = E_n + \frac{1}{2}\rho_s v_s^2 + \frac{1}{2}\rho_n v_n^2 \quad (4.35)$$

where E_n is the rest energy density of the normal fluid and the two other terms are respectively the superfluid and normal fluid kinetic energy densities.

4.3 The Circulation

In the two-fluid description of helium II, the circulation of the superfluid velocity \mathbf{v}_s is quantized. Opposed to this, the normal fluid is regarded as an ordinary viscous fluid, and the circulation should be allowed to possess any value. With a relative motion to the walls, the normal fluid will experience friction, so at equilibrium the normal fluid velocity at the boundary must be identical with the motion of the container. In our thesis we will always use a container at rest, and hence a normal fluid velocity and circulation which is zero at the boundaries.

When integrating the mass flux along the boundaries, only the superfluid density is visible

$$\oint d\mathbf{l} \cdot \mathbf{j} = \oint d\mathbf{l} \cdot (\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n) = \rho_s \kappa \quad (4.36)$$

The densities ρ_s and ρ_n are thought to be constant by the boundaries. This formula appears explicitly in the calculation of the effective Magnus force in section 5.3.

In the phonon scattering in chapter 2 the constraint was that the scattered phonons should deviate as little as possible from plane waves. Plane waves give a non-rotating phonon gas. What is the consequence of this constraint for the normal fluid circulation in the presence of a vortex? Is it consistent with a non-rotating normal fluid by the boundaries?

Earlier in this chapter, the two-fluid model has been developed in the low temperature limit from the Ginzburg-Landau theory, without vortices. If, on the contrary, one vortex is present, it is known from chapter 2 that plane waves are not just scattered by the vortex, but the incoming plane waves themselves are modified as well. When using the same thermal distribution as in the case of

no vortices, is it possible that the phonon gas is rotating. Our situation is in a sense paradoxical: *We will find the circulation of the excitations in the presence of the vortex, which correspond to no circulation when there are no vortices.* In other words it is the equilibrium state of the thermal system with a vortex in the middle.

As before, the Ginzburg-Landau theory is presented in dimensionless units. Considering the situation of one vortex at rest at the origin, the integral of the mass current of the background fluid can easily be found,

$$\oint \mathbf{dl} \cdot \mathbf{j}_0 = q \oint \mathbf{dl} \cdot \nabla \varphi = 2\pi q = \kappa_q \quad (4.37)$$

which is not surprising. To find the contribution from the phonons we first calculate the contribution from one single wave scattered on the vortex, and then generalize the result to be valid for a thermal average of excitations. The mass flux of a wave with wave-number \mathbf{k} is

$$\langle \mathbf{j}_k \rangle = \frac{1}{2} \nu^* \nabla \phi \quad (4.38)$$

where ϕ and ν are plane waves scattered on a stationary vortex. Because of time averaging there are no contributions from the mixed terms between the background and the excitations, so that the leading order term contains the fields to second order. The phase ϕ was found in (2.35). We take the gradient

$$\nabla \phi = \phi_k \left[i\mathbf{k}e^{ikx}(1 - i\alpha(\varphi - \pi)) - i\alpha e^{ikx} \nabla \varphi + \nabla \delta \phi_a \right] \quad (4.39)$$

where $\alpha = kq$ as usual. The other quantity needed is the density which by (2.4) is

$$\nu = \phi_k \left[ik e^{ikx}(1 - i\alpha(\varphi - \pi)) + ik \delta \phi_a + i\alpha \frac{\sin(\varphi)}{r} e^{ikx} \right] + \mathcal{O}(\alpha^2) \quad (4.40)$$

If the integration contour is a circle, the line segment is $\mathbf{dl} = r \mathbf{e}_\varphi d\varphi$ and only the gradient in the \mathbf{e}_φ direction is needed in the contour integral. For large angles, the scattered wave is expressed by the scattering amplitude (2.35) so that the gradient is

$$\nabla \delta \phi_a = ik \delta \phi \mathbf{e}_r + \mathcal{O}(r^{-3/2}) \quad (4.41)$$

This formula is lying a bit. It is true that the derivative in the \mathbf{e}_φ direction is of order $r^{-3/2}$, but a pure expansion in powers of $r^{-1/2}$ misses the rather ugly singularities in φ coming from the derivatives of $a(\varphi)$ near $\varphi = 0$. This singularity will not, and must not, enter into the integrals, but its existence is a motivation to look for explicit small angle contributions later.

The incoming wave is along the x -axes, which in polar coordinates is $\mathbf{e}_x = \cos(\varphi)\mathbf{e}_r - \sin(\varphi)\mathbf{e}_\varphi$. This gives the mass current in the \mathbf{e}_φ direction to be

$$\langle \mathbf{j}_k \rangle_\varphi = \frac{1}{2} |\phi_k|^2 k \left[-k \sin(\varphi) - \frac{\alpha}{r} (1 + \sin^2(\varphi)) - k \sin(\varphi) e^{-ikx} \delta\phi_a \right] \quad (4.42)$$

Integrated over a whole period the plane wave part disappears as it should. When taking out the first term proportional to α and integrating it, we get

$$\frac{1}{2} |\phi_k|^2 r \int_0^{2\pi} d\varphi \left[-\frac{\alpha}{r} (1 + \sin^2(\varphi)) \right] = -\frac{3}{2} |\phi_k|^2 k \alpha \quad (4.43)$$

This is the contribution from the term proportional to α in the density and from the twist of the incoming wave in the phase.

We will now find the contributions from the scattered wave in (4.42). The scattered wave is

$$\delta\phi_a = \frac{\pi\alpha}{\sqrt{2\pi ikr}} \frac{\sin(\varphi)}{1 - \cos(\varphi)} e^{ikr} \quad (4.44)$$

The term $\exp(ikr(1 - \cos(\varphi)))$ is more and more rapidly oscillating as $r \rightarrow \infty$. Where the oscillations are fast, the integral will average to zero, and the major contribution will come for the small angles, where the oscillations are calm. We expand the integrand in this region

$$\begin{aligned} & \frac{1}{2} |\phi_k|^2 r \int_0^{2\pi} d\varphi \left[-k \frac{\pi\alpha}{\sqrt{2\pi ikr}} \frac{\sin^2(\varphi)}{1 - \cos(\varphi)} e^{ikr(1 - \cos(\varphi))} \right] \\ & = -k |\phi_k|^2 \sqrt{r} \frac{\pi\alpha}{\sqrt{2\pi k}} \int_{-\Lambda}^{\Lambda} d\varphi e^{-\frac{kr}{2r}\varphi^2} \end{aligned}$$

This situation is similar to that in the calculation of the transverse force. The parameter Λ is in this approximation small. But since the main contribution to the integral is for small φ , when $r \rightarrow \infty$, the integration limit can still be taken to infinity. The integral is then of a Gaussian kind, and with the standard formula (appendix A), we end up with the contribution

$$-\pi |\phi_k|^2 k \alpha \quad (4.45)$$

which must be added to the other contribution (4.43). But this is not the complete result since so far no contributions from derivatives of the scattered wave has been considered. The derivative with respect to φ of the finite angle contribution $\delta\phi_a$, although higher order in $r^{-3/2}$, had wild divergences when $\varphi \rightarrow 0$ and can not be used.

We will instead consider the special small angle expression for the scattered wave (2.34), which was

$$\delta\phi_s = \alpha \sqrt{\frac{2\pi ik}{x}} e^{ikx} \int_0^y du e^{\frac{ik}{2x} u^2} \quad (4.46)$$

The derivative of this with respect to y is

$$\frac{\partial \delta \phi_s}{\partial y} = \alpha \sqrt{\frac{2\pi i k}{x}} e^{ikx} e^{\frac{ik}{2x} y^2}$$

which is highly finite and well behaved near $y = 0$. For small angles $y \approx r\varphi$. The contribution to the circulation from this term can then be written as an integral over a small region near the origin. The integral obtained is again of the Gaussian kind, where the main contribution comes from small arguments. The integration limits can then, in the same manner as above, be taken to infinity. Since the scattered wave is proportional to α , a plane wave can be inserted for ν

$$\frac{1}{2} |\phi_k|^2 \int_{-\infty}^{\infty} dy (-ik) \alpha \sqrt{\frac{2\pi i k}{x}} e^{\frac{ik}{2x} y^2} = \pi |\phi_k|^2 k \alpha \quad (4.47)$$

This is the last contribution to the integral. The terms (4.43), (4.45) and (4.47) must be added. The conclusion is that the contour integral far from the vortex of one phonon scattered on a vortex is

$$\oint \mathbf{dl} \cdot \langle \mathbf{j}_k \rangle = -\frac{3}{2} \pi |\phi_k|^2 k \alpha = -\frac{3}{2} \kappa_q \langle j_k \rangle \quad (4.48)$$

where $\kappa_q = 2\pi q$ is the circulation quantum and $\langle j_k \rangle$ is the average mass flux of the plane wave. There is a bit abuse of notation in this formula, since mass current on the left side is the full scattered wave, while the one on the right side only is the plane wave part. This is not a vector result, and it is the magnitude of the mass flux which is entering into the right hand side. The whole contribution from all phonons is found by letting $\langle j_k \rangle \rightarrow k N(k)$, where $N(k)$ is the Bose-Einstein distribution function, and integrating over all k . This gives the phonon contribution

$$\oint \mathbf{dl} \cdot \mathbf{j}^{\text{ph}} = -\frac{3}{2} \kappa_q \int \frac{d^2 k}{(2\pi)^2} k N_0 = -\kappa \rho_n \quad (4.49)$$

where the last identification follows from the Landau expression for the normal density (4.27). Together with the contribution from the background fluid (4.37), this is the result from the Ginzburg-Landau theory. In our units the total (or background) density is $\rho = 1$. The superfluid density is $\rho_s = 1 - \rho_n$, so that

$$\oint \mathbf{dl} \cdot \mathbf{j} = \oint \mathbf{dl} \cdot (\mathbf{j}_0 + \mathbf{j}^{\text{ph}}) = \kappa_q (1 - \rho_n) = \kappa_q \rho_s \quad (4.50)$$

which proves that it is possible to derive this formula from Ginzburg-Landau theory. The calculation shows that the phonon gas is indeed rotating, and the direction is opposite of the background fluid. In a way this is analogous to Lenz'

law in electro-magnetism, which says the current induced by a change is in the direction opposing the change.

The calculation here is only to lowest order in k , which corresponds to order T^{D+1} . There is no reason to expect that this result should be valid at arbitrary power of T . At higher order in k , it is known from section 2.5 that the core structure of the vortex becomes important. The circulation by the boundaries will then also depend on the vortex core.

A calculation of the circulation, directly in the two-fluid model, is done in [TGV⁺01].

4.3.1 Numerical Calculation

In the last section the rotation of the phonon gas was found from the asymptotic behavior of the scattered phonons. The problem with that calculation was that it involved combinations of many terms. A numerical calculation can help us to show that no terms have been lost and that no contributions has been over-counted. The partial wave expression for the time average of the mass flux is

$$\langle \mathbf{j}_k \rangle = \frac{1}{2} \sum_{ml} \nu_l^* \left[\frac{\partial \phi_l}{\partial r} \mathbf{e}_r + \frac{il}{r} \phi_l \mathbf{e}_\varphi \right] e^{i(l-m)\varphi} \quad (4.51)$$

where the point vortex expressions are

$$\phi_l = (-1)^l e^{-\frac{\pi}{2}\Omega_l} J_{\Omega_l}(kr) \quad (4.52)$$

$$\nu_l = ik \left(1 - \frac{\alpha l}{(kr)^2} \right) \phi_l \quad (4.53)$$

The subscripts of the Bessel functions are in the acoustic Aharonov-Bohm scattering $\Omega_l = \sqrt{l^2 + 2\alpha l}$, $\alpha = kq$. An integral over a circle with radius r , gives 2π when $l = m$, else zero.

$$\oint \mathbf{dl} \cdot \langle \mathbf{j}_k \rangle = \pi \sum_l l \left(1 - \frac{\alpha l}{(kr)^2} \right) J_{\Omega_l}^2(kr)$$

This is a rather simple result which we unfortunately have not found any way to sum up exactly. The s-wave is not contributing to the integral, which is quite natural since it is symmetric about the origin. To first order in α there is hence no difference between acoustic and magnetic Aharonov-Bohm scattering since they only differ in the s-wave. To first order in α we write the result as

$$\oint \mathbf{dl} \cdot \langle \mathbf{j}_k \rangle = -\frac{3}{2} \kappa_q \langle j_k \rangle A(kr) \quad (4.54)$$

where $\kappa_q = 2\pi q$ and

$$A(kr) = -\frac{2}{3} \frac{\partial}{\partial \alpha} \left[\sum_l l \left(1 - \frac{\alpha l}{(kr)^2} \right) J_{\Omega_l}^2(kr) \right]_{\alpha=0} \quad (4.55)$$

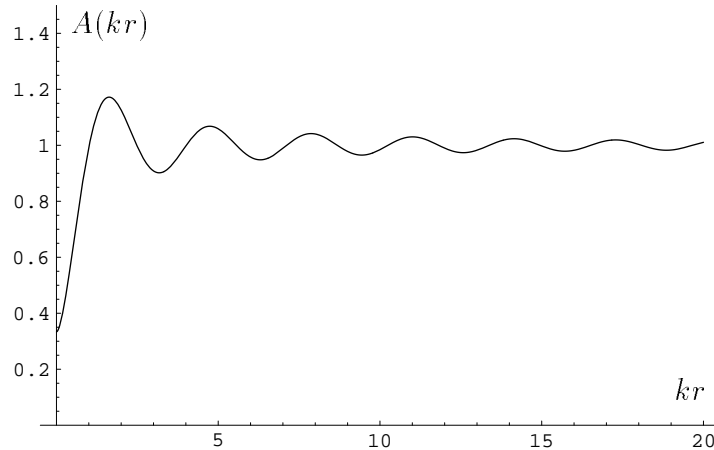


Figure 4.2: Numerical calculation of $A(kr)$ for $kr = 0..20$, with $|l| < kr + 10$.

The analytical result from the last section can be written $\lim_{kr \rightarrow \infty} A(kr) = 1$. A numerical plot of $A(kr)$ is in figure 4.2. If $r = R$ is the boundary of the system, no waves can have wavelengths longer than the size of the system and $kR > 1$ and to avoid complication with boundary effects we must assume $kR \gg 1$. The plot clearly approaches a constant value in this limit, and verifies the analytical result; the normal fluid circulation at the boundary is zero.

Except from the verification of the analytical value of the normal fluid circulation, the numerical result offers some interesting prospects for the interior too. The result is only valid for a point vortex, which means it must be used well outside the vortex core. But it is possible to have kr small and r outside the core as long as the phonon wave-numbers are small enough. The plot has the limit $\lim_{kr \rightarrow 0} A(kr) = \frac{1}{3}$. In this limit the contour integral of the phonon flux is $\oint \mathbf{dl} \cdot \mathbf{j}^{ph} = -\frac{1}{3}\rho_n$ and the total circulation is

$$\oint \mathbf{dl} \cdot (\mathbf{j}_0 + \mathbf{j}^{ph}) = \kappa_q - \frac{1}{3}\kappa_q\rho_n = \kappa_q\rho_s + \frac{2}{3}\kappa_q\rho_n \quad (4.56)$$

where the superfluid density is $\rho_s = 1 - \rho_n$. The surprise is that the *normal fluid seems to have a maximum circulation* $\kappa_n = \frac{2}{3}\kappa_q$ *in the interior*. The normal fluid circulation must change continuously from $\frac{2}{3}\kappa_q$ somewhere in the interior, to zero at the boundaries. The phenomenological two-fluid model offers a value for the circulation by the boundaries, but does not say anything about the interior. Therefore this result is interesting for the understanding of helium II.

Chapter 5

Forces on a Vortex

In the past years there has been a lot of controversy about the force on a quantized vortex in helium II, or more precisely: discussion about the existence of the Lordanskii force. Before entering into this discussion, let us take a brief look at the more general properties of forces in helium II (see [Don91]). The two-fluid description, as presented in chapter 4, did not include any forces, and there were no interactions between the superfluid and the normal fluid. Nor did it include vortices, and actually the vortices are crucial for the microscopic picture of forces acting in helium II. At a macroscopic level, the interaction between the superfluid and normal fluid appears as mutual friction. When going to a smaller length scale, the forces between the fluid and a single quantized vortex can be considered. This is an intermediate region, where classical quantities can mostly be used to describe the fluid, but where on the other hand quantized vortices, as pure quantum effects, must be included as well. The simplest situation to consider at this level is the presence of one, straight vortex line. Since the vortex is straight uniformity along the z -axis can be assumed, and the system is effectively two-dimensional. The three-dimensional result can be recovered by letting the two-dimensional results be regarded as per vortex string length instead.

One of the fundamental assumptions in the two-fluid description of helium II, is that the superfluid and normal fluid velocities are independent of each other. As classical fluids, the two-fluids also obey the galilean invariance principle. This puts strong limitations on the form of force, and the only allowed expressions are dependent of the velocity differences $\mathbf{v}_n - \mathbf{v}_s$ and $\mathbf{v}_v - \mathbf{v}_n$, where \mathbf{v}_v is the vortex velocity, \mathbf{v}_s the superfluid velocity and \mathbf{v}_n the normal fluid velocity. With slow velocities, only contributions linear in the velocities need to be considered. The force can be divided in one longitudinal and one transverse component to \mathbf{v}_v , where in this thesis only the transverse part will be considered. The longitudinal force is interesting as well, but there is unfortunately no room to make a discussion of it here. The transverse force can be written as

$$\mathbf{F}_\perp = \boldsymbol{\kappa} \times [A(\mathbf{v}_v - \mathbf{v}_s) + B(\mathbf{v}_v - \mathbf{v}_n)] \quad (5.1)$$

where A and B are, at this point, unknown coefficients and $\boldsymbol{\kappa} = \pm \frac{h}{m} \mathbf{e}_z$ is the quantized circulation. At the absolute zero, there is no normal fluid, and there is general agreement that the coefficient A is the total density of the fluid. The force is thus fully analogous to the classical Magnus Force, section 1.1.2. At finite temperature the generally accepted continuation of this is that $A = \rho_s$, and this part of the force is called *the superfluid Magnus force*.

$$\mathbf{F}_M = \rho_s \boldsymbol{\kappa} \times (\mathbf{v}_v - \mathbf{v}_s) \quad (5.2)$$

The controversy is about the second parameter, B . The normal fluid can be constructed of thermal excitations: phonons and rotons. The transverse force for rotons was found by Lifshitz and Pitaevskii [LP58]. In this thesis we will however concentrate on the region dominated by the phonons, which means temperatures safely below 1K. The low temperature is also a reason for the disagreements, since experimental data are hard to obtain. In 1965 S.V. Iordanskii [Ior65] made a calculation of the transverse normal fluid force in this region, and found it to be $B = \rho_n$. The phonon contribution to the force transverse to \mathbf{v}_n is named after him: the *Iordanskii force*.

$$\mathbf{F}_I = \rho_n \boldsymbol{\kappa} \times (\mathbf{v}_v - \mathbf{v}_n) \quad (5.3)$$

The normal fluid density goes as T^{D+1} . The Iordanskii force is hence small compared to the superfluid Magnus force for low temperatures¹.

Instead of dividing the transverse force in one term proportional to $\mathbf{v}_s - \mathbf{v}_v$ and one proportional to $\mathbf{v}_n - \mathbf{v}_v$, the terms proportional to the vortex velocity can be isolated. This is the *effective Magnus force*

$$\mathbf{F}_v = (A + B) \boldsymbol{\kappa} \times \mathbf{v}_v \quad (5.4)$$

Iordanskii's calculations was regarded as true for many years, until a paper by P. Ao and D. Thouless in 1993 [AT93]. They present a derivation where the effective Magnus force is associated with the geometrical phase of the wave function, the *Berry phase*, which will be discussed in section 5.3.2. Applied on the two-fluid model, their result is that the effective Magnus force is proportional to ρ_s and hence *there is no Iordanskii force*. Later there have been a lot of articles polishing these arguments, where among these the articles [TAN96] and [GTRV00] should be considered. The conclusion that there is no Iordanskii force is often referred to as the *TAN result*. As a reaction to these topological arguments, a lot of articles have been written where the Iordanskii force is calculated directly from the scattering of phonons on a vortex, and usually these attempts end with Iordanskii's conclusion. Among these E.B. Sonin's article [Son96] and the article by M. Stone [Sto99] are authoritative.

¹At higher temperatures it is small compared to the transverse force from rotons.

5.1 The Iordanskii Force

As mentioned earlier, the Iordanskii force is the most controversial part of the transverse force. The general expression for the transverse force (5.1) has two parameters where the first seems to be established as $A = \rho_s$. The other parameter, B , determining the Iordanskii force, can either be found by calculating the effective Magnus force giving $A + B$, or it can be calculated directly. Here only the direct approaches of determining the Iordanskii force are discussed. This means that the force on a vortex from a single phonon wave, found in chapter 3, will be generalized to a thermal gas of phonons. There are several articles doing direct calculations. Of those concluding in favor of a Iordanskii force are in addition to Iordanskii [Ior65] himself: Sonin [Son96][Son01], Stone [Sto99] and Shelankov [She98a][She98b] the most important. The analogy between phonon scattering and the Aharonov-Bohm effect is central in their discussions, as well as in ours (chapter 2). The Iordanskii force is interpreted as a consequence of the left-right asymmetry of the phonon wave in the presence of a vortex. An imaginative analogy between the velocity field of the vortex and a spinning cosmic string is offered by Volovik [Vol98] (and also discussed by Stone). A relativistic generalization is discussed by Carter, Langlois and Prix [CLP01].

Those doing direct calculations and concluding against the existence of the Iordanskii force are mainly involved in the calculations of the effective Magnus force as well. The articles by Wexler and Thouless [WT98] and Demircan, Ao and Niu [DAN95] are not too convincing, since they do not apply the analogy to Aharonov-Bohm scattering, and they also miss the important small angle contribution to the force. A better attempt to calculate the force is done by Thouless et al. in [TGV⁺01]. There the calculation is done directly at the two-fluid model, without considering the Ginzburg-Landau theory.

Our conclusion from chapter 3 was that the force from one phonon wave could be written

$$-\boldsymbol{\kappa} \times \langle \mathbf{j}_k \rangle \quad (5.5)$$

where the quantized circulation vector is $\boldsymbol{\kappa} = \pm \frac{\hbar}{m} \mathbf{e}_z$, and $\langle \mathbf{j}_k \rangle$ is the time average of the mass flux of one single plane wave. From the discussion of the two-fluid description in chapter 4, we remember that the normal fluid could be constructed as an integral over plane waves, so that

$$\int \frac{d^D r}{(2\pi)^D} \langle \mathbf{j}_k \rangle = \mathbf{j}^{ph} = \rho_n (\mathbf{v}_n - \mathbf{v}_s) \quad (5.6)$$

The fundamental assumption is that the presence of the vortex does not change the occupation numbers of the waves, so that this formula is still valid. The total phonon force is thus the Iordanskii force

$$\mathbf{F}_I = -\rho_n \boldsymbol{\kappa} \times \mathbf{v}_n \quad (5.7)$$

The derivation of the normal fluid without vortices is not ideal for the direct calculation of the force. The best approach would of course be to do the quantization of the phonons directly with the vortex in the background. This is a difficult procedure, and our approach, where the presence of the vortex is thought of only as a slightly perturbation of the system, should also be reliable. The scattering problem, and the calculations of the force from one phonon were all done with a stationary vortex, $\mathbf{v}_v = 0$, and no asymptotic superfluid velocity², $\mathbf{v}_s = 0$. The direct calculations is then in favor of the Iordanskii force.

The most serious challenge to this result is not the direct calculations in [WT98], [DAN95] and [TGV⁺01], but the calculations of the effective Magnus force. A discussion of this will be held in section 5.3.

5.2 The Superfluid Magnus Force

The superfluid Magnus force is the least controversial force component, and most authors agree that the coefficient before $\mathbf{v}_v - \mathbf{v}_n$ in (5.1) is $A = \rho_s$, the superfluid density.

So far in this thesis no calculations of the superfluid Magnus force has been done at finite temperatures, but in section 3.4 the background fluid force was found. It corresponds to the superfluid Magnus force at the absolute zero. The superfluid Magnus force at finite temperatures can be found by combining the expression for the background fluid force with the force from the phonon gas. The force from the phonon gas has so far just been found with a non-moving background fluid which means that both the scattering problem in chapter 2 and the calculation of the phonon force in chapter 3 must be reconsidered to get the correct expression with a background fluid in motion. But a qualitative guess is that the changes of the phonon force is small, since the background fluid velocity field could, as seen in section 1.2.2, be taken as a harmless modification of the incident waves, which will not influence the scattering equations, and the modifications of the momentum-flux tensor in chapter 3 will be cross terms between \mathbf{k} and \mathbf{v}_s . If we suppose that all terms proportional to \mathbf{k} give terms proportional to \mathbf{j}^{ph} when the full phonon contribution is collected, these cross terms are second order in the velocities and can be neglected. These qualitative arguments save us from a lot of mathematics, but they are not sufficient if the superfluid Magnus force were to be severely derived.

Let us assume that the result obtained by phonon scattering on a static background can be straightforward generalized to a background fluid in motion by substituting $\mathbf{v}_n \rightarrow \mathbf{v}_n - \mathbf{v}_s$. We get

$$\begin{aligned} \mathbf{F}_\perp &= \mathbf{F}_{0\perp} + \mathbf{F}_\perp^{ph} = -\rho\boldsymbol{\kappa} \times \mathbf{v}_s - \rho_n\boldsymbol{\kappa} \times (\mathbf{v}_n - \mathbf{v}_s) \\ &= -\boldsymbol{\kappa} \times (\rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n) \end{aligned} \quad (5.8)$$

²The velocity coming from the vortex itself, $\mathbf{v}_s = \nabla\varphi$, is ignored

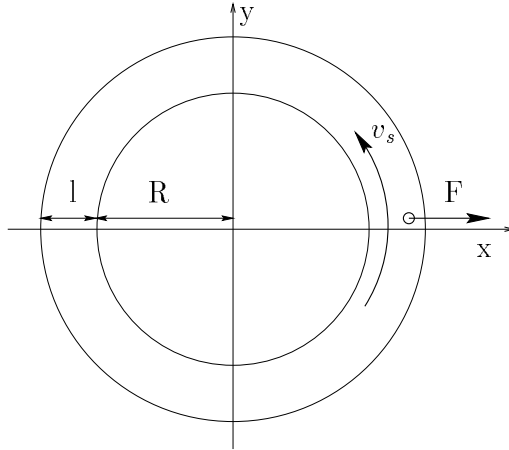


Figure 5.1: A superfluid streaming between two cylinders.

where the superfluid density is $\rho_s = \rho - \rho_n$. This is the total force on a static vortex, where both the superfluid Magnus force and the Lordanskii force are included.

Another way of determining the superfluid Magnus force is through an imaginary experiment, as presented by Wexler [Wex97]. It relies on thermodynamical properties of helium II. The system which is considered is a double cylinder where the fluid is trapped between the outer and inner wall (see figure 5.1). The radius of the inner circle R is large compared with the relative distances l between the walls. N quanta of circulation, where N is large, are trapped at the inner wall in such a way that they give rise to a *uniform superfluid velocity field*

$$v_s \approx N \frac{h}{2\pi Rm} \quad (5.9)$$

where m is the helium atom mass. The normal fluid is thought to be at rest, $\mathbf{v}_n = 0$. Suppose that a vortex is created near the outer wall at the x -axis, and is slowly dragged along the x -axis until it reaches the inner wall, where it is annihilated and gives rise to a modification of the superfluid velocity (5.9). If A is positive, the force on the vortex is towards the outer wall, and work must be done to move the vortex. If the motion is so slow that vortex velocity can be neglected, (5.1) gives the work as

$$W = -F_M l = A\kappa v_s l \quad (5.10)$$

On the other hand let us consider the Helmholtz free energy density $f = \varepsilon - Ts$, where s is the entropy density and ε is the energy density. Since the normal fluid is at rest, the entropy can be regarded as a constant, and any change in free energy is identical to the change in energy. The free energy density can also be written as

$$f = \varepsilon_0 + f_{ex} \quad (5.11)$$

where $\varepsilon_0 = \frac{1}{2}\rho v_s^2$ is the energy density of the background fluid, and f_{ex} is the free energy of the phonon excitations. The ordinary formula for the excitation free energy density is

$$f_{ex} = \frac{k_B T}{\mathcal{A}} \sum_{\mathbf{k}} \ln(1 - e^{-\epsilon/k_B T}) \quad (5.12)$$

where $\mathcal{A} \approx 2\pi Rl$ is the area in which the fluid lives. The excitation energy ϵ must be evaluated in the frame where the fluid is at rest. In the laboratory system the energy is

$$\epsilon = \hbar\omega - \hbar\mathbf{v}_s \cdot \mathbf{k} \quad (5.13)$$

According to the figure 5.1 we let \mathbf{v}_s be along the y -axis. To second order in v_s the excitation free energy is

$$f_{ex}(v_s) = f_{ex}(v_s = 0) + \frac{1}{2}v_s^2 \left. \frac{\partial^2 f_{ex}}{\partial v_s^2} \right|_{v_s=0} \quad (5.14)$$

since the first order term disappears. When differentiating the phonon free energy (5.12) with respect to \mathbf{v}_s and taking the continuity limit, we get

$$\frac{\partial f_{ex}}{\partial v_s} = \hbar \int \frac{d^2 k}{(2\pi)^2} \frac{k_y}{e^{-\epsilon/k_B T} - 1} = \hbar \int \frac{d^2 k}{(2\pi)^2} k_y N(\epsilon) \quad (5.15)$$

where we have identified $N(\epsilon)$ as the Bose-Einstein distribution function. By using the identity $\frac{\partial N}{\partial v_s} = -\hbar k_y \frac{\partial N}{\partial \epsilon}$, this gives for the second derivative evaluated in $v_s = 0$:

$$\frac{\partial^2 f_{ex}}{\partial v_s^2} = -\hbar^2 \int \frac{d^2 k}{(2\pi)^2} k_y^2 \frac{\partial N(\hbar\omega)}{\partial \omega} = -\rho_n \quad (5.16)$$

where the normal fluid density has been identified from the usual Landau formula (4.25). Including the background fluid kinetic energy, the free energy density is

$$f = \frac{1}{2}(\rho - \rho_n) v_s^2 = \frac{1}{2}\rho_s v_s^2 \quad (5.17)$$

When the vortex is pinned to the inner wall it gives rise to a change of the superfluid velocity, and hence to the free energy. Using the expression for the superfluid velocity field as a function of N , (5.9), the change in free energy density is

$$\begin{aligned} \Delta f &= f(N+1) - f(N) \\ &\approx \frac{1}{2}\rho_s \left(\frac{h}{2\pi Rm} \right)^2 ((N+1)^2 - N^2) \approx \frac{h}{2\pi Rm} \rho_s v_s \end{aligned} \quad (5.18)$$

The change in free energy is the same as the work performed on the system, (5.10), since the entropy is constant. This determines the superfluid Magnus force

$$(2\pi Rl) \Delta f = W \implies A = \rho_s \quad (5.19)$$

We write

$$\mathbf{F}_M = -\rho_s \boldsymbol{\kappa} \times \mathbf{v}_s \quad (5.20)$$

This derivation relies on simple thermodynamical properties of helium II. Again it is seen that the background fluid is not identical to the superfluid. The background fluid has density ρ while the superfluid has density ρ_s . The derivation is difficult to generalize to a moving normal fluid. What made this derivation easy, is the constant entropy making the change in energy and free energy identical. With a moving normal fluid, an expression for the entropy and entropy change must have been needed as well.

A more complete discussion of the relation between the background fluid and the phonon gas on one side, and the superfluid and normal fluid on the other, was held in section 4.2.

5.3 The Effective Magnus Force

Much of the controversy about the forces acting on a vortex string in helium II originates in the articles by Ao and Thouless [AT93] and Thouless, Ao and Niu [TAN93] in 1993, where they derive an expression for Magnus force in a superconductor. In their derivation they interpret the force as a geometrical phase of the wave function, *the Berry-phase* ([Ber84], section 5.3.2), associated with an adiabatic motion of a vortex around a circuit. The calculation was modified in 1996 [TAN96] and applied on superfluids only.

The following derivation relies on the papers [TAN96] and [Ao96]. The goal is to calculate the *effective Magnus force*, and hence to determine the sum $A + B$ in (5.1). The superfluid and normal fluid velocities are put to zero, $\mathbf{v}_s = \mathbf{v}_n = 0$.

Consider a vortex moving in system of N particles in two dimensions. The number of particles is thought to be conserved so that ordinary wave mechanics can be used. We want to find an expression for the *force on the vortex* from the system, as we push it slowly around. If the vortex is at rest, the whole fluid is supposed to be at rest, and the fundamental assumption needed to perform the calculation, is that the total time evaluation of the system, is due to the motion of the vortex. This condition holds if the vortex motion is so slow that the system is in equilibrium at any time, and the motion is hence *adiabatic*. The force on the vortex is found as

$$\mathbf{F}_v = -\dot{\mathbf{p}} = -\text{Tr} \left[\dot{\hat{\rho}} \hat{\mathbf{p}} \right] \quad (5.21)$$

where the density operator is $\hat{\rho} = \sum_n f_n |\Psi_n\rangle\langle\Psi_n|$ and the sum is over all energy eigenstates of the system. The formalism with the density matrix is a generalization of the Schrödinger theory, and is used on large systems where it is impossible to find a wavefunction. The system is instead described by a statistic distribution over states. At equilibrium with temperature T the factors f_n are the Boltzmann factors $f_n = \exp(-E_n/k_b T)$. Considering adiabatic motion, these equilibrium factors can be used. The total momentum operator is simply the sum of the momentum operators for each particle

$$\hat{\mathbf{p}} = -i\hbar \sum_i \nabla_i \quad (5.22)$$

where ∇_i means derivatives with respect to the position of particle i . When the density matrix is inserted, the force becomes

$$\mathbf{F}_v = -\text{Tr} \left[\sum_n f_n \left\{ |\dot{\Psi}_n\rangle\langle\Psi_n| + |\Psi_n\rangle\langle\dot{\Psi}_n| \right\} \hat{\mathbf{p}} \right] \quad (5.23)$$

Now the full meaning of the adiabatic motion becomes visible. Since the Boltzmann factors f_n are independent of time, the whole time evaluation is through the fields. The trace of an operator is $\text{Tr} A = \sum_k \langle\Psi_k|A|\Psi_k\rangle$. The full expression for the force is then

$$\mathbf{F}_v = -\sum_n f_n \sum_k \left\{ \langle\Psi_k|\dot{\Psi}_n\rangle\langle\Psi_n|\hat{\mathbf{p}}|\Psi_k\rangle + \langle\Psi_k|\Psi_n\rangle\langle\dot{\Psi}_n|\hat{\mathbf{p}}|\Psi_k\rangle \right\}$$

The operator $\hat{\mathbf{p}}$ is hermitian, so that it can be taken inside to work on either the “bra” or the “ket” according to our choice. Rearranging the terms and using that the sum over all k just gives the identity operator $\sum_k |\Psi_k\rangle\langle\Psi_k| = \hat{\mathbf{1}}$, we get

$$\mathbf{F}_v = -\sum_n f_n \left\{ \langle\hat{\mathbf{p}}\Psi_n|\dot{\Psi}_n\rangle + \langle\dot{\Psi}_n|\hat{\mathbf{p}}\Psi_n\rangle \right\}$$

Now the expression for the momentum operator and for the time derivative can be inserted. The wave functions $\Psi_n = \Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N; \mathbf{r}_v(t))$ are dependent of the position of the particles and the vortex. A crucial condition to get any further, is that the only dependency of the vortex position is through the distance from particles to the vortex, so that $\Psi_n = \Psi_n(\mathbf{r}_1 - \mathbf{r}_v(t), \dots, \mathbf{r}_N - \mathbf{r}_v(t))$. This condition holds if the container is large, and the vortex is far from the boundaries. The time derivative can hence be transformed to derivatives of the particle positions

$$\dot{\Psi}_n = \nabla_{\mathbf{r}_v} \Psi_n \cdot \mathbf{v}_v = -\sum_j \nabla_j \Psi_n \cdot \mathbf{v}_v \quad (5.24)$$

with vortex velocity $\mathbf{v}_v = \dot{\mathbf{r}}_v$. With the definition of the momentum operator (5.22), the force can be written as

$$F_v = i\hbar \sum_n f_n \sum_{ij} \left[\langle\nabla_i \Psi_n | \nabla_j \Psi_n \cdot \mathbf{v}_v \rangle - \langle\nabla_j \Psi_n \cdot \mathbf{v}_v | \nabla_i \Psi_n \rangle \right]$$

The second term in the above expression is the conjugate of the first, which guarantees that the force is real. The result is now proportional to the vortex velocity. Only the force to first order in \mathbf{v}_v is of interest, so the wave-functions can from now on be evaluated as if the vortex was at rest, $\Psi_n = \Psi_n(\mathbf{r}_1 \dots \mathbf{r}_N)$. To express this in particle position space, $\hat{\mathbf{1}} = \prod_{k=1}^N (\int d^2 r_k) |\mathbf{r}_1 \dots \mathbf{r}_N\rangle \langle \mathbf{r}_1 \dots \mathbf{r}_N|$ is inserted. Each of the integrations is over the physical area. The many particle wave function is $\Psi_n(\mathbf{r}_1 \dots \mathbf{r}_N) = \langle \mathbf{r}_1 \dots \mathbf{r}_N | \Psi_n \rangle$. The dot products can by the use of a vector calculus formula be transformed to

$$\sum_{ij} [\nabla_i \Psi_n^* (\nabla_j \Psi_n \cdot \mathbf{v}_v) - \nabla_i \Psi_n (\nabla_j \Psi_n^* \cdot \mathbf{v}_v)] = \mathbf{v}_v \times \sum_{ij} (\nabla_i \Psi_n^* \times \nabla_j \Psi_n)$$

Here the sum over i and j are interchanged in the second term on the left hand side. The anti-symmetry of the cross product guarantees that the result is still purely imaginary. The force is here seen to be *perpendicular to the vortex velocity*. This is a consequence of the adiabatic motion. If it were a force component along the vortex path, it would do work on the system. Since the work cannot give a motion of the fluid (the fluid is at rest), the energy must be lost as heat in the system, changing the entropy. A final vector formula gives

$$\sum_{ij} [\nabla_i \Psi_n \times \nabla_j \Psi_n] = \sum_{ij} \frac{1}{2} \nabla_i \times (\Psi_n^* \nabla_j \Psi_n - \Psi_n \nabla_j \Psi_n^*)$$

So far no assumptions about the statistical properties of the particles has been made. We will now suppose that the system consists of N identical Bosons. The particles can then be interchanged without changing the sign of the wavefunction. The sum over i and j has N diagonal terms where $i = j$, and $N(N-1)$ off-diagonal terms with $i \neq j$. Because of the integration over all \mathbf{r}_k , the diagonal terms are identical to the $i = j = 1$ case, while all off-diagonal terms are identical to $i = 1, j = 2$. The final expression for the force can hence be split in a one-particle contribution and a two-particle contribution,

$$\mathbf{F}_v = \mathbf{F}_v^1 + \mathbf{F}_v^2 \quad (5.25)$$

where

$$\mathbf{F}_v^1 = \mathbf{v}_v \times \int d^2 r \nabla \times \left[\frac{i}{2} (\nabla - \nabla') \rho(\mathbf{r}' ; \mathbf{r}) \right]_{\mathbf{r}' = \mathbf{r}} \quad (5.26)$$

$$\mathbf{F}_v^2 = \mathbf{v}_v \times \int d^2 r_1 \int d^2 r_2 \nabla_1 \times \left[\frac{i}{2} (\nabla_2 - \nabla'_2) \Gamma(\mathbf{r}_1 \mathbf{r}'_2 ; \mathbf{r}_1 \mathbf{r}_2) \right]_{\mathbf{r}'_2 = \mathbf{r}_2} \quad (5.27)$$

The one-particle contribution is expressed by the one-particle density matrix,

$$\rho(\mathbf{r}' ; \mathbf{r}) = \hbar N \sum_n f_n \prod_{k=2}^N \left(\int d^2 r_k \right) \Psi_n^*(\mathbf{r}' \mathbf{r}_2 \dots \mathbf{r}_N) \Psi_n(\mathbf{r} \mathbf{r}_2 \dots \mathbf{r}_N) \quad (5.28)$$

while the two-particle contribution is expressed by the two-particle density matrix

$$\begin{aligned} \Gamma(\mathbf{r}'_1 \mathbf{r}'_2; \mathbf{r}_1 \mathbf{r}_2) &= \hbar N(N-1) \sum_n f_n \prod_{k=3}^N \left(\int d^2 r_k \right) \\ &\times \Psi^*(\mathbf{r}'_1 \mathbf{r}'_2 \mathbf{r}_3 \dots \mathbf{r}_N) \Psi(\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \dots \mathbf{r}_N) \end{aligned} \quad (5.29)$$

This is the TAN result. None of the special properties of the two-fluid model has been used. The only assumptions are that the motion is slow and adiabatic, and the system is large, so that boundary effects can be ignored.

5.3.1 Application on Helium II

The TAN result (5.25) is valid for a general N -particle system, but usually is applied on either superconductors or superfluids. We will use the formula on superfluid helium II only. In the previous section the one-particle density matrix was identified from a N -particle quantum mechanics expression. The same matrix can also be written as

$$\rho(\mathbf{r}'; \mathbf{r}) = \langle \hat{\Psi}(\mathbf{r}')^\dagger \hat{\Psi}(\mathbf{r}) \rangle \quad (5.30)$$

where $\hat{\Psi}^\dagger$ and $\hat{\Psi}$ are the helium atom creation and annihilation operators. The conserved current can be expressed by the one-particle density matrix

$$\begin{aligned} &\frac{-i\hbar}{2} (\nabla - \nabla') \rho(\mathbf{r}'; \mathbf{r})|_{\mathbf{r}'=\mathbf{r}} \\ &= \frac{-i\hbar}{2} \langle \hat{\Psi}^\dagger(\mathbf{r}) \nabla \hat{\Psi}(\mathbf{r}) - \nabla \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \rangle = \langle \hat{\mathbf{j}}(\mathbf{r}) \rangle \end{aligned} \quad (5.31)$$

This makes it possible to identify the mass current in the one particle contribution to the force (5.26). The expression for the force is proportional to the vortex velocity, so to first order in the velocity it is enough to consider the mass current of a vortex at rest. With Stoke's theorem, the integration can be transformed to a contour integral, where the integration contour is the physical boundary of the system. The one-particle contribution to the force is then

$$\begin{aligned} \mathbf{F}_v^1 &= -\mathbf{v}_v \times \oint \mathbf{dr} \cdot \langle \hat{\mathbf{j}}(\mathbf{r}) \rangle \\ &= -\mathbf{v}_v \times \oint \mathbf{dr} (\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n) = [\rho_s \boldsymbol{\kappa} + \rho_n \boldsymbol{\Gamma}_n] \times \mathbf{v}_v \end{aligned} \quad (5.32)$$

where $\boldsymbol{\kappa} = \frac{\hbar}{m} \mathbf{e}_z$ is the quantized superfluid circulation and $\boldsymbol{\Gamma}_n$ is the normal fluid circulation. The normal fluid circulation is not quantized, and by the boundary it must at equilibrium be equal to the circulation of the container. This is a consequence of the interpretation of the normal fluid as an ordinary viscous liquid.

A relative motion between the normal fluid and the wall would lead to a friction force counteracting the normal fluid motion. We will in the following use the boundary condition with non-rotating normal fluid, $\Gamma_n = 0$.

For the comparison with the direct calculation, it is crucial that the same boundary conditions are used. In the Ginzburg-Landau theory, the constraint was that the normal fluid in the presence of the vortex should be as close as possible to the non-vortex situation. In section 4.3 we derived that this led to no normal fluid circulation. *The Boundary conditions in the calculation of the phonon force are hence identical to the boundary conditions in the calculation of the effective Magnus force.*

The two-particle contribution (5.27) is difficult to calculate in general. Using the quantized fields, as in the one-particle contribution, leads to a product of four field operators. This product is hard to handle. We can thus not make a calculation of it, as in the one-particle contribution. But there are reasons to think that it will vanish. The two-particle density matrix describes the interactions and correlations between particles and the internal interaction between the particles should not contribute to the force between the vortex and the liquid.

The TAN result gives then an effective Magnus force proportional to ρ_s , implying no Iordanskii force. The conclusion is dependent of the two-particle part of the force to vanish, which we are unable to verify by calculation.

5.3.2 Berry's Phase

Central in the interpretation of the TAN result, is a geometrical phase of the wavefunction; the Berry phase. The idea of geometrical phases will not be sketched, as presented by Berry [Ber80]. Consider a Hamiltonian, $\hat{H}(\mathbf{r}_v(t))$, depending on a time-varying parameter, $\mathbf{r}_v(t)$. The state $|\Psi(t)\rangle$ satisfies the ordinary Schrödinger equation

$$\hat{H}(\mathbf{r}_v(t))|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle \quad (5.33)$$

The Hamiltonian has a set of time-dependent eigenfunctions, with eigenstates E_n which are (for simplicity) independent of time

$$H(\mathbf{r}_v(t))|\Psi_n(\mathbf{r}_v(t))\rangle = E_n|\Psi_n(\mathbf{r}_v(t))\rangle \quad (5.34)$$

Let us put the system in the state $|\Psi_n(\mathbf{r}_v(0))\rangle$ at $t = 0$. The system will continue to be in the state $|\Psi_n(\mathbf{r}_v(t))\rangle$, if the time-evolution is *adiabatic*

$$|\Psi(t)\rangle = e^{\frac{1}{i\hbar}E_n t + i\gamma_n(t)}|\Psi_n(\mathbf{r}_v(t))\rangle \quad (5.35)$$

where the $\exp(\frac{1}{i\hbar}E_n t)$ is the dynamical Dirac phase factor. The attention now is to the other phase factor, $\gamma_n(t)$. Inserting the above expression into the time-dependent Schrödinger equation (5.33), gives the equation

$$E_n|\Psi_n(\mathbf{r}_v(t))\rangle = i\hbar\left[\frac{1}{i\hbar}E_n + i\dot{\gamma}_n\right]|\Psi_n(\mathbf{r}_v(t))\rangle + \frac{\partial}{\partial t}|\Psi_n(\mathbf{r}_v(t))\rangle$$

The time dependency of the eigenstates are only through $\mathbf{r}_v(t)$ so the time derivative can be found as $\frac{\partial}{\partial t} = \dot{\mathbf{r}}_v \cdot \nabla_{\mathbf{r}_v}$. If the eigenstates are normalized to unity, this gives an expression for $\dot{\gamma}_n$

$$\dot{\gamma}_n = i \langle \Psi_n(\mathbf{r}_v(t)) | \nabla_{\mathbf{r}_v} \Psi(\mathbf{r}_v) \rangle \cdot \dot{\mathbf{r}}_v$$

The interesting point is the $\dot{\mathbf{r}}_v$ on the right hand side. When integrating over t the right hand side transforms to a contour integral over the path $\mathbf{r}_v(t)$. Of special interest is when the path is closed and $\mathbf{r}_v(t) = \mathbf{r}_v(0)$. The phase factor γ_n is then not zero, but has a value depending on the closed path C . The Berry phase is

$$\gamma_n(C) = \oint_C d\mathbf{r}_v \cdot \langle \Psi_n(\mathbf{r}_v) | \nabla_{\mathbf{r}_v} \Psi_n(\mathbf{r}_v) \rangle \quad (5.36)$$

The value of $\gamma_n(C)$ is independent of how the circuit C is traversed, as long as the motion is slow enough to be adiabatic.

Now we want to see the connection between the Berry phase and the expression for the effective Magnus force in section 5.3. In that calculation the derivatives with respect to the vortex position was changed to derivatives with respect to the particle positions. Now we go the other way around, and express the force by derivatives of the vortex position only, which means that $\sum_i \nabla_i = -\nabla_{\mathbf{r}_v}$. Substituting this in the expression for the force, we get

$$F_v = i\hbar v_v \sum_n f_n \nabla_{\mathbf{r}_v} \times \langle \Psi_n | \nabla_{\mathbf{r}_v} \Psi_n \rangle \quad (5.37)$$

This looks like the integrand in the Berry phase. The force F_v is thought to be independent of the vortex position \mathbf{r}_v . An integral of \mathbf{r}_v over the left hand side only gives the volume. The right hand side gives, with the use of Stoke's theorem, a contour integral. So when integrating over a volume Ω , we get

$$F_v = \frac{i\hbar v_v}{\Omega} \oint_{\partial\Omega} d\mathbf{r}_v \cdot \langle \Psi_n | \nabla_{\mathbf{r}_v} \Psi_n \rangle = \hbar v_v \frac{\gamma(\partial\Omega)}{\Omega} \quad (5.38)$$

where the Berry phase as function of the boundary $\partial\Omega$ is

$$\gamma(\partial\Omega) = \sum_n f_n \gamma_n(\partial\Omega) \quad (5.39)$$

where $\gamma_n(\partial\Omega)$ is the Berry phase of the wavefunction which was in state Ψ_n at $t = 0$. We have thus two expressions for the effective Magnus force. In section 5.3.1 the force was found to be proportional to the superfluid density and now we have seen that it can also be expressed by the Berry phase. This relation was also found by Haldane and Wu [HW85]. The force is independent of which circuit $\partial\Omega$ that is chosen and how it is traversed, giving the result topological authority.

Impurities and other perturbations can thus not change the TAN result. *The superfluid Magnus force is hence a topological quantity.* This interpretation of the TAN result is the same both for superfluids and superconductors. Of special importance is this for dirty superconductors, where this result says that the superfluid Magnus force is not influenced by impurities, nor details in the vortex core.

5.4 Discussion of the Controversy

Opposed to the direct calculations, the TAN result implies no Iordanskii force. We will now make a summary of the assumptions done in both the direct calculations and the TAN calculation and point to the possible reasons for the conflicting results.

The direct calculations of the Iordanskii force are carried out in Ginzburg-Landau theory. There is of course a fundamental assumption that Ginzburg-Landau theory gives an adequate description of helium II. We saw in chapter 4 that the phenomenological two-fluid description of helium II could be derived from Ginzburg-Landau theory, in the absence of vortices. In the derivation, it was necessary to make some interpretations and identifications, but most of the two-fluid results came out rather easily. Thus it seems like the Ginzburg-Landau theory works, at least at a macroscopic length scale.

In the scattering of a phonon on a vortex, the presence of the vortex was treated as a perturbation of the non-vortex situation. In an ordinary scattering problem, this means that the solution far from the vortex can be divided in an incoming plane wave and a scattered wave. But in the scattering of a phonon on a vortex this was not possible, since the long range of the vortex forced a twist on the incoming wave. This twist, although long range, was proportional to the wave-number k , so in the low energy limit there should still be expected that the perturbative approach works. There are hence two condition on k : First, the wavelengths of the thermal phonons must be small compared with the size of the system, for boundary effects to be negligible. But the phonon wavelengths must also be large compared with the vortex core, for core effect to be ignored. Mathematically this means that $k \ll 1$ and $kR \gg 1$ when R is the radius of the container.

The TAN derivation of the effective Magnus force starts with a very general N -particle system. The force is found from the response in the liquid, as the vortex is pushed adiabatically around. The container is, as in the direct calculations, thought to be large, so that boundary effects can be ignored. The dependency of the vortex position is thus only through the distance from the particles to the vortex. Because of the adiabatic motion, all time-dependency is through the vortex position. Using this, we are able to express the force by the one- and two-particle density matrices. The one-particle contribution is proportional to ρ_s ,

provided that the normal fluid circulation is zero. The normal fluid circulation was also found to be zero in Ginzburg-Landau theory, proving that the Ginzburg-Landau theory and the TAN derivation have the same boundary conditions. The contribution from the two-particle density matrix is difficult to calculate exactly, but general arguments indicates that it is zero. The TAN expression for the effective Magnus force is proportional to ρ_s , implying no Iordanskii force.

There are two possible ways out of the disagreement. Of course there could be some technical error in one of the arguments, leading to a wrong answer. In this thesis we have carefully gone through the calculations, and not found any obvious mistakes. We will therefore concentrate about the other possibility, which is that both calculations are indeed correct, but calculating different quantities. The direct calculations are of course determining the Iordanskii force. The TAN calculation finds the adiabatic force as the vortex is pushed around. Is it possible that the Iordanskii force is not adiabatic? Let us consider the TAN situation with a vortex pushed around in a stationary liquid. The liquid is in contact with the surroundings at the vortex and by the boundaries. Since the Iordanskii force is perpendicular to the vortex motion, there is no work on the liquid at the vortex point. But we know that the thermal phonons are scattered on the vortex and these scattered phonons will in turn interact with the boundaries. There might be a net heat transfer between the boundaries and the liquid. This can be the reason why the Iordanskii force is not visible in the Berry phase and the adiabatic calculation of the force. If the Iordanskii force is not adiabatic, it is still possible to associate the superfluid Magnus force with the Berry phase. But it is not possible to find the effective Magnus force, only the superfluid Magnus force, from the calculation based on adiabatic motion.

Conclusion

The purpose of this thesis was to calculate the Lordanskii force in liquid helium II. The direct calculations of the force were carried out in Ginzburg-Landau theory. In order to find the force, the scattering of a phonon on a quantized vortex was examined, and the analogy to Aharonov-Bohm scattering was discussed. In the Born approximation, the conventional Aharonov-Bohm scattering amplitude was obtained. The scattering amplitude was not valid for small angles, though, and right behind the vortex a special small angle solution was developed. Hidden in the Born integral was also an important modification of the incoming wave, which was interpreted as a consequence of the long range of the vortex. The Born solution was thus consisting of three different contributions; a sum of two valid at finite angles, and one valid for small angles. A comparison in the overlapping region showed that the Born solution was continuous and single valued. The scattering problem was in addition solved as a sum of partial waves. By expressing Bessel functions as contour integrals, we were able to sum up. The solution, which was valid at all angles, was identical to the Born solution when compared in the different limits.

The transverse force on the vortex from one phonon was found, by examining the momentum change through a contour enclosing the vortex. Analytical answers were found in two limits: with the integration contour far from the vortex, and close to the vortex (but outside the core). The Born solution was used far from the vortex, where the calculation was laborious and involved many terms and limits. With the integration contour put near the vortex, the force was found almost with no calculations at all. In addition numerical expressions were obtained at finite distances from the vortex. All these calculations agreed on a transverse force proportional to the phonon mass flux, giving the Lordanskii force, when the full phonon contribution was collected.

The two-fluid description of helium II was derived from Ginzburg-Landau theory. In Ginzburg-Landau theory, liquid helium was modeled as a phonon gas on the top of a background fluid. The superfluid and normal fluid were identified from the expression for the mass flux. From the phonon scattering we were able to show that the normal fluid circulation was zero at the boundaries, proving that the boundary conditions used in Ginzburg-Landau theory were consistent with the two fluid model.

We have gone through the calculation of the effective Magnus force, found by pushing the vortex adiabatically around in a general N -particle system. Applied on the two-fluid model it gave a force proportional to ρ_s , implying no Iordanskii force. The superfluid Magnus force was identified with the geometrical Berry phase, giving the force a topological character.

Several ways out of the conflict were pointed out. One worth considering is the possibility of Iordanskii force being non-adiabatic, and hence lost in the Berry phase calculation.

Appendix A

Some Useful Integrals

Here is a list of some integrals which are used in the thesis. These integrals can also be found in many integral tables, for example *Table of integrals, series, and products* [GR94] or *Handbook of Mathematical Functions* [AS68].

First some integral representations of the Bessel functions

$$\int_{-\pi}^{\pi} d\varphi e^{-in\varphi + iz \sin(\varphi)} = 2\pi J_n(z) \quad (\text{A.1})$$

$$\int_{-\pi}^{\pi} d\varphi \sin(n\varphi) \sin(z \sin(\varphi)) = [1 - (-1)^n] \pi J_n(z) \quad (\text{A.2})$$

$$\int_{-\pi}^{\pi} d\varphi \cos(n\varphi) \cos(z \sin(\varphi)) = [1 + (-1)^n] \pi J_n(z) \quad (\text{A.3})$$

This leads to two integrals we need

$$\int_{-\pi}^{\pi} d\varphi \sin(\varphi) e^{iz \sin(\varphi)} = 2\pi i J_1(z) \quad (\text{A.4})$$

$$\int_{-\pi}^{\pi} d\varphi \cos(\varphi) e^{iz \sin(\varphi)} = 0 \quad (\text{A.5})$$

Another integral we need is

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + \gamma^2} = \frac{\pi}{|\gamma|} \quad (\text{A.6})$$

and the Gaussian integral

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \quad (\text{A.7})$$

This formula works for $\Re(\alpha) \geq 0$

We also need integrals of the kind

$$\int_0^\infty dx \frac{x^{\nu+1} J_\nu(ax)}{x^2 + \gamma^2} = \gamma^\nu K_\nu(a\gamma) \quad (\text{A.8})$$

For the integrals of the Hankel functions, there are not always available formulas, but the Hankel functions of the first kind can always be expressed by Bessel functions as

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-i\pi\nu} J_\nu(z)}{i \sin(\pi\nu)} \quad (\text{A.9})$$

For $b \leq 1$ we have the integral

$$\int_0^\infty J_\nu(t) e^{ibt} dt = \frac{1}{\sqrt{1-b^2}} e^{i\nu \arcsin(b)} \quad (\text{A.10})$$

Combining these formulas we get

$$e^{i\frac{\pi}{2}\nu} \int_0^\infty H_\nu^{(1)}(t) e^{-it \cos(t)} dt = -\frac{2 \sin[\nu(\varphi - \pi + 2\pi n)]}{\sin(\pi\nu) \sin(\varphi)} \quad (\text{A.11})$$

On the right hand side we must choose an integer n according to the value of φ .

Appendix B

Vector Calculus in 2-D

In two dimensions the usual vector calculus formulae can take some peculiar shapes. Let the the vectors $\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y$ and $\mathbf{B} = B_x\mathbf{e}_x + B_y\mathbf{e}_y$ live in the xy-plane, and let \mathbf{e}_z be the unit vector in the z-direction. Then we have

$$\mathbf{A} \times \mathbf{e}_z = (A_y, -A_x) \quad (\text{B.1})$$

$$\mathbf{A} \times \mathbf{B} = A_x B_y - A_y B_x \quad (\text{B.2})$$

$$\nabla \times \mathbf{A} = \nabla \cdot (\mathbf{A} \times \mathbf{e}_z) \quad (\text{B.3})$$

The two dimensional delta functions can also be identified as

$$\nabla \times \nabla \varphi = 2\pi \delta(\mathbf{r}) \quad (\text{B.4})$$

where $\varphi = \arctan(y/x)$. This can be seen by remembering that the curl of a gradient is zero everywhere except in singularities. An integral on a circle with radius r gives, when Stoke's theorem is applied

$$\int_{\omega} d^2r \nabla \times \nabla \varphi = \oint_{\partial\omega} d\mathbf{l} \cdot \nabla \varphi = r \int_0^{2\pi} d\varphi \frac{1}{r} = 2\pi$$

which proves that this is a delta function. The commutator $[\nabla, \frac{\partial}{\partial t}] \varphi_0$ will also be needed when $\varphi_0 = \arctan((y - y_0)/(x - x_0))$ and $\mathbf{r}_0 = \mathbf{r}_0(t)$. First we notice that all time dependency of φ_0 is through \mathbf{r}_0 , so that $\frac{\partial}{\partial t} \nabla \varphi_0 = -(\dot{\mathbf{r}}_0 \cdot \nabla) \nabla \varphi_0$. Using the ordinary vector formulae found in text-books we find

$$\begin{aligned} \nabla \left(\frac{\partial \varphi_0}{\partial t} \right) &= -\nabla [\dot{\mathbf{r}}_0 \cdot \nabla \varphi_0] \\ &= -(\dot{\mathbf{r}}_0 \cdot \nabla) \nabla \varphi_0 - \dot{\mathbf{r}}_0 \times (\nabla \times \nabla \varphi_0) \\ &= \frac{\partial}{\partial t} \nabla \varphi_0 - (\dot{\mathbf{r}}_0 \times \mathbf{e}_z) 2\pi \delta(\mathbf{r} - \mathbf{r}_0) \end{aligned}$$

By taking the cross product on both sides, we get

$$\left[\nabla, \frac{\partial}{\partial t} \right] \varphi_0 \times \mathbf{e}_z = 2\pi \dot{\mathbf{r}}_0 \delta(\mathbf{r} - \mathbf{r}_0) \quad (\text{B.5})$$

Appendix C

The Two-Dimensional Green's Function

The time-independent two-dimensional Green's function $D(\mathbf{r})$ is a solution of the equation

$$(k^2 + \nabla^2)D(\mathbf{r}) = -\delta^2(\mathbf{r}) \quad (\text{C.1})$$

The Fourier transform of the equation is

$$(k^2 - p^2)\tilde{D}(\mathbf{k}) = -1$$

The Green's function is then found as the double integral

$$D(\mathbf{r}) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 - k^2} e^{i\mathbf{p}\cdot\mathbf{r}}$$

If we introduce polar coordinates (p, θ) in phase space, the angle part of the integral can be easily carried out

$$\int_0^{2\pi} d\theta e^{ipr \cos(\varphi)} = \int_0^{2\pi} d\theta e^{ipr \sin(\varphi)} = 2\pi J_0(pr)$$

from equation (A.1). The restoring integral can be found by regularizing $-k^2 = (\epsilon \pm ik)^2$ for some small positive ϵ , and use formula E.H.II 96(58) in [GR94].

$$\int_0^\infty dx \frac{x}{x^2 + b^2} J_0(ax) = K_0(ab)$$

where K_0 is a Bessel function of second kind. The Green's function is then

$$D(\mathbf{r}) = \frac{1}{2\pi} K_0(r(\epsilon \pm ik)) \xrightarrow{\epsilon \rightarrow 0} \frac{i}{4} H_0^{(1)}(kr) \quad (\text{C.2})$$

The final result is independent of the choice of sign. $H_0^{(1)}$ is the zeroth order Hankel function of the first kind.

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