Principles of Subjective Networks

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Abstract—This paper focuses on representation and reasoning in conditional inference networks, combined with trust networks, thereby introducing subjective networks as graph-based structures of variables combined with conditional opinions. Subjective networks generalize Bayesian networks from being based on probability calculus, to being based on subjective logic. In addition, subjective networks generalise Bayesian networks from assuming a global view of input arguments by a single analyst, to taking subjective view of input arguments by multiple agents who might have conflicting opinions. The result is a highly flexible and expressive framework for modelling and analysing realistic situations, which is also backwards compatible with traditional Bayesian networks.

I. INTRODUCTION

A Bayesian network [1] is a compact representation of a joint probability distribution of random variables in the form of DAG (directed acyclic graph) and a set of conditional probability distributions associated with each node.

The goal of inference in Bayesian networks is to derive a conditional probability distribution of any set of (target) variables in the network, given that the values of any other set of (evidence) variables have been observed. Bayesian network reasoning algorithms provide a way to propagate the probabilistic information through the graph, from the evidence to the target. For a general introduction to Bayesian networks, see e.g. [2].

One serious limitation of traditional Bayesian network reasoning, is that all the input conditional probability distributions in the network must be assigned precise probabilities in order for the inference algorithms to work, and for the model to be analysed. This is problematic in situations where probabilities cannot be reliably elicited and one needs to do inference with uncertain or incomplete probabilistic information, inferring the most accurate conclusions possible.

Subjective opinions can express uncertain probabilistic information of any kind (high or low confidence, or even zero confidence) by varying the uncertainty mass between 0 and 1.

A straightforward generalization of Bayesian networks in subjective logic retains the network structure, but replaces conditional probability distributions with conditional subjective opinions at every node of the network. We call this structure a Subjective Bayesian Network, where its application for modelling and reasoning becomes a generalization of classical Bayesian reasoning. An example of the inference process for a subjective Bayesian network is provided in a companion paper [3].

Computational trust networks based on subjective logic are well studied, see e.g. [4]–[7], where the formalism is called subjective trust networks [8], [9]. Subjective trust networks make it possible to model and analyse trust relationships between active reasoning agents that can be humans or computerised entities. The fundamental idea is that agents have subjective opinions about the trustworthiness of other agents, and about the state of variables. This reflects the reality when analysing complex situations, where different pieces of evidence originate from different sources with varying degrees of reliability, often via multiple hops of indirection.

The capacity of subjective logic for reasoning in the presence of uncertainty, and for modelling trust networks, combined with the power of Bayesian networks for modelling conditional knowledge structures, creates a very potent combination that we call a Subjective Network. Early ideas about subjective networks were presented in [10]. The conceptual structure of elements that constitutes subjective networks is illustrated in Figure 1.

The next sections first give a brief overview of subjective Bayesian networks, as well as of trust network analysis with subjective logic. Finally, their combination in the form of subjective networks is presented, with an application example. We describes properties and aspects resulting from the generalisation to subjective networks, and how it can be applied.
II. SUBJECTIVE BAYESIAN NETWORKS

A. Bayesian Networks

Bayesian networks represent a powerful framework for modelling and analysing practical situations, where the analyst needs to make probabilistic inference about a set of variables with unknown values. Initially proposed by Pearl in 1988 [1], Bayesian network tools are currently being used in important applications in many areas like medical diagnostics, risk management, marketing, military planning, etc. For a general introduction into the field, see e.g. [2]. This section only provides a brief introduction.

When events and states are related in time and space, they are conditionally dependent. For example, the state of carrying an umbrella is typically influenced by the state of rain. These relationships can be expressed in the form of graphs, consisting of nodes connected with directed edges. To be practical, the graphs must be acyclic to prevent loops, so that the graph is a DAG (directed acyclic graph), to be precise. The nodes are variables that represent possible states or events. The directed edges represent the (causal) relationships between the nodes.

Associated with the Bayesian network graph, are various (conditional) probability distributions, that formally specify selected local (conditional) relationships between nodes. Missing probability distributions for specific target nodes can be derived through various algorithms that take as input arguments the existing known probability distributions and the structure of the Bayesian network graph.

A subjective Bayesian network models the random variables of nodes, and their conditional dependencies represented as probability distributions, where the conditional relationships between nodes are represented via a DAG. By specifying the values of a set of input variables, it is possible to compute probability distributions for the remaining variables/nodes.

Bayesian network computations frequently apply Bayes’ theorem of Eq.(1), that can be represented in terms of base rates: $p(x|y) = \frac{p(y|x)p(x)}{p(y)}$ (1)

Bayes’ theorem with base rates: $p(x|y) = \frac{a(x)p(y|x)}{a(y)}$ (2)

Bayes’ theorem with marginal base rates: $p(x|y) = \frac{a(x)p(y|x)}{a(x)p(y|x) + a(\overline{x})p(y|\overline{x})}$ (3)

Consider a Bayesian network containing $K$ nodes/variables, $X_1$ to $X_K$, and a joint probability distribution over all the variables. A variable $X_l$ can thus have index $I = 1, \ldots, K$. Now, a specific value of variable $X_l$ can have index $i = 1, \ldots, k$. The value with index $i$, of the variable with index $I$, is then denoted $x_{l,i}$. We write $X_l = x_{l,i}$ to denote that variable $X_l$ takes the particular value $x_{l,i}$.

As a shorthand notation, the term $x_I$: means a particular value of variable $X_I$, by omitting its explicit value index.

A particular probability in the joint distribution is represented by $p(X_1 = x_{11}, X_2 = x_{21}, \ldots, X_K = x_{K1})$, which in the shorthand notation can be expressed more concisely as $p(x_{11}, x_{21}, \ldots, x_{K1})$.

The chain rule of conditional probability reasoning expresses the joint probability in terms factorisation of conditional probabilities as:

$$p(x_1, x_2, \ldots, x_K) = p(x_1)p(x_2|x_1)p(x_3|x_1, x_2) \ldots, p(x_K|x_1, \ldots, x_{(K-1)})$$

$$= \prod_{l=1}^{K} p(x_l|x_{1:l-1})$$

$$= p(x_1)p(x_2|x_1)\prod_{l=3}^{K} p(x_l|pa(X_l))$$

(4)

where $pa(X_l)$ are the values for the direct parents of variable $X_l$, which is a subset of $\{x_1, \ldots, x_{l-1}\}$.

The application of Eq.(4) together with Bayes’ theorem of Eq.(1), a set of independence properties, as well as various computation algorithms, provide the basis for analysing complex Bayesian networks. Figure 2 illustrates typical reasoning models supported by Bayesian networks [2]. The ovals represent variables/nodes, and the double-lined arrows represent causal conditional relationships, where antecedent parent nodes are at the top, and consequent child nodes are at the bottom.

In predictive reasoning models, the analyst applies deduction with causal conditionals, with evidence on parent nodes to deduce conclusions about child nodes. For example assume that: $X_1$ is the variable for air pollution, $X_2$ is the variable for cigarette smoking, and $Y$ is the variable for lung cancer. Then, evidence about $X_1$ (cigarette smoking) and $X_2$ (exposure to air pollution) can be used to predict increased probability of $Y$ (having lung cancer).
In diagnostic reasoning models, the analyst applies abduction with causal conditionals, with evidence on child nodes to abduce conclusions about parent nodes. For example assume that: $Z_1$ is the variable for dispnoea (shortness of breath), $Z_2$ is the variable for characteristic signs of lung cancer on X-ray images, and $Y$ is the variable for lung cancer. Then, evidence about the symptoms $Z_1$ (dispnoea) and $Z_2$ (characteristic signs on X-ray) can be used to assess the probability of $Y$ (having lung cancer).

In intercausal reasoning models, the analyst applies abduction, followed by marginalisation or division, to infer conclusions about one of the parent nodes. For example, evidence about $Y$ (having lung cancer) and about the negation of $X_2$ (i.e. being non-smoker) can be used to assign the probability that $X_1$ (exposure to air pollution) is the cause of $Y$ (lung cancer).

In combined reasoning models, the analyst applies a combination of deduction and abduction to infer conclusions about a variable which is both child and parent at the same time. For example, evidence about $X_2$ (being smoker) and about $Z_2$ (characteristic signs on X-ray) can be used to estimate the likelihood of $Y$ (lung cancer). Note that $Y$ is child of $X_2$ and parent of $Z_2$.

These reasoning models, that are common in Bayesian reasoning, can also be generalised with subjective logic. In order to make this paper self-contained, the next sections briefly describe subjective opinions as well as the notation for deduction and abduction.

B. Subjective Opinions

Subjective logic is a formalism that represents uncertain probabilistic information in the form of subjective opinions, and that defines a variety of operations for subjective opinions. In this section we present in detail the concept of subjective opinion which is the argument format used in subjective logic and which can represent an agent’s opinion about a variable or trust in an agent.

In subjective logic a domain is a state space consisting of two or more values. The values of the domain can e.g. be observable or hidden states, events, hypotheses or propositions, just like in traditional Bayesian modeling. Domains are typically specified to reflect realistic situations for the purpose of being practically analysed in some way.

The different values of the domain are assumed to be mutually exclusive and exhaustive, which means that the variable can take only one value at any time, and that all possible values of interest are included in the domain. For example, if the variable is the weather, we can assume its domain to be the set \{rainy, sunny, overcast\}. The available information about the particular value of the variable is very often of a probabilistic type, in the sense that we do not know the particular value, but we might know its probability. Probabilities express likelihoods with which the variable takes the specific values and their sum over the whole domain is 1. A variable together with a probability distribution defined on its domain is a random variable.

For a given variable of interest, the values of its domain are assumed to be the real possible states the variable can take in the situation to be analysed. In some cases, certain observations may indicate that the variable takes one of several possible states, but it is not clear which one in particular. For example, we might know for sure that the weather is either rainy or sunny, but do not know its exact state. For this reason it is very often more practical to consider subsets of the domain as possible values of the variable, i.e. instead of the original domain to consider a hyperdomain, which would contain all the singletons like \{rainy\}, but also composites like \{rainy, sunny\}; and assign beliefs to these values according to the available information, instead of providing a probability distributions on the original domain. In this case we are talking about a hypervariable in contrast to a random variable.

In the case of the WEATHER seen as a hypervariable, a possible value can be \{rainy, sunny\} which means that the actual weather is either rainy or sunny, but not both at the same time. Composites are only used as an artifact for assigning belief mass when the observer believes that one of several values is the case, but is confused about which one in particular is true. If the analyst wants to include the realistic possibility that there can be rain and sunshine simultaneously, then the domain would need to include a corresponding singleton value such as \{rainy&\&sunny\}. It is thus a question of interpretation how the analyst wants to separate between different types of weather, and thereby define the relevant domain.

A subjective opinion distributes a belief mass over the values of the hyperdomain. The sum of the belief masses is less than or equal to 1, and is complemented with an uncertainty mass which reflects the opinion’s confidence level. Subjective opinions also contain a base rate probability distribution expressing prior knowledge about the specific class of random variables, so that in case of significant uncertainty about a specific variable, the base rates provide a basis for default likelihoods. We give formal definitions of these concepts in what follows.

Let $X$ be a variable over a domain $\mathcal{X} = \{x_1, x_2, \ldots, x_k\}$ of cardinality $k$, where $x_i \ (1 \leq i \leq k)$ represents a specific value from the domain. Let $\mathcal{P}(\mathcal{X})$ be the powerset of $\mathcal{X}$. The hyperdomain is the reduced powerset of $\mathcal{X}$, denoted by $\mathcal{A}(\mathcal{X})$, and defined as:

$$\mathcal{A}(\mathcal{X}) = \mathcal{P}(\mathcal{X}) \setminus \{\emptyset\}.$$ \hspace{1cm} (5)

All proper subsets of $\mathcal{X}$ are elements of $\mathcal{A}(\mathcal{X})$, but $\emptyset$ and $\mathcal{X}$ are not, since they are not considered possible observations to which we can assign beliefs. The hyperdomain has cardinality $2^k - 2$. We use the same notation for the elements of the domain and the hyperdomain, and consider $X$ a hypervariable when it takes values from the hyperdomain.

Let $A$ denote an agent which can be an individual, source, sensor, etc. A subjective opinion $\omega^A_X$ of the agent $A$ on the variable $X$ is a tuple

$$\omega^A_X = (b^A_X, \mu^A_X, a^A_X),$$ \hspace{1cm} (6)
where \( b^i_X: \mathcal{R}(\mathbb{X}) \rightarrow [0, 1] \) is a belief mass distribution, the parameter \( u^i_X \in [0, 1] \) is an uncertainty mass, and \( a^i_X: \mathbb{X} \rightarrow [0, 1] \) is a base rate probability distribution satisfying the following additivity constraints:

\[
\begin{align*}
    u^i_X + \sum_{x \in \mathcal{R}(\mathbb{X})} b^i_X(x) &= 1, \quad (7) \\
    \sum_{x \in \mathbb{X}} a^i_X(x) &= 1. \quad (8)
\end{align*}
\]

In the notation of the subjective opinion \( \omega_X \), the subscript is the target variable \( X \), the object of the opinion while the superscript is the opinion owner \( A \), the subject of the opinion. Explicitly expressing subjective ownership of opinions makes it possible to express that different agents have different opinions on the same variable. Indication of opinion ownership can be omitted when the subject is clear or irrelevant, for example, when there is only one agent in the modelled scenario.

The belief mass distribution \( b^i_X \) has \( 2^k - 2 \) parameters, whereas the base rate distribution \( a^i_X \) only has \( k \) parameters. The uncertainty parameter \( u^i_X \) is a simple scalar. A general opinion thus contains \( 2^k + k - 1 \) parameters. However, given that Eq.(7) and Eq.(8) remove one degree of freedom each, opinions over a domain of cardinality \( k \) only have \( 2^k + k - 3 \) degrees of freedom.

A subjective opinion in which \( u^i_X = 0 \), i.e. an opinion without uncertainty, is called a dogmatic opinion. A dogmatic opinion for which \( b^i_X(x) = 1 \), for some \( x \), is called an absolute opinion. In contrast, an opinion for which \( u^i_X = 1 \), and consequently, \( b^i_X(x) = 0 \), for every \( x \in \mathcal{R}(\mathbb{X}) \), i.e. an opinion with complete uncertainty, is called a vacuous opinion.

Every subjective opinion ‘projects’ to a probability distribution \( P_X \) over \( \mathbb{X} \) defined through the following function:

\[
P_X(x_i) = \sum_{x_j \in \mathcal{R}(\mathbb{X})} a_X(x_i | x_j) b_X(x_j) + a_X(x_i) u_X, \quad (9)
\]

where \( a_X(x_i | x_j) \) is the relative base rate of \( x_i \in \mathbb{X} \) with respect to \( x_j \in \mathcal{R}(\mathbb{X}) \) defined as follows:

\[
a_X(x_i | x_j) = \frac{a_X(x_i \land x_j)}{a_X(x_j)}, \quad (10)
\]

where \( a_X \) is extended on \( \mathcal{R}(\mathbb{X}) \) additively. For the relative base rate to be always defined, it is enough to assume \( a_X^i(x_i) > 0 \), for every \( x_i \in \mathbb{X} \). This means that everything we include in the domain has a non-zero probability of occurrence in general.

Binomial opinions apply to binary random variables where the belief mass is distributed over two elements. Multinomial opinions apply to random variables in \( n \)-ary domains, and where the belief mass is distributed over the elements of the domain. General opinions, also called hyper-opinions, apply to hypervariables where belief mass is distributed over elements in hyperdomains obtained from \( n \)-ary domains. A binomial opinion is equivalent to a Beta probability density function, a multinomial opinion is equivalent to a Dirichlet probability density function, and a hyper-opinion is equivalent to a hyper-Dirichlet probability density function [11]. Binomial opinions thus represent the simplest opinion type, which can be generalised to multinomial opinions, which in turn can be generalised to hyper-opinions. Simple visualisations for binomial and trinomial opinions are based on barycentric coordinate systems as illustrated in Figures 3 below.

In general, a multinomial opinion can be represented as a point inside a regular simplex. In particular, a trinomial opinion can be represented inside a tetrahedron (a 4-axis barycentric system), as shown in Figure 3.

Assume the random variable \( X \) on domain \( \mathbb{X} = \{ x_1, x_2, x_3 \} \). Figure 3 shows multinomial opinion \( \omega_X \) with belief mass distribution \( b_X = (0.20, 0.20, 0.20) \), uncertainty mass \( u_X = 0.40 \) and base rate distribution \( a_X = (0.750, 0.125, 0.125) \).

C. Notation for Subjective Conditional Inference

This section simply introduces the notation used for conditional deduction and abduction in subjective logic. The operator for deduction is described in [12], [13]; abduction is described in [12].

Let domain \( \mathbb{X} \) have cardinality \( k = |\mathbb{X}| \) and domain \( \mathbb{Y} \) have cardinality \( l = |\mathbb{Y}| \), where variable \( X \) plays the role of parent, and variable \( Y \) the role of child.

Assume the set of conditional opinions of the form \( \omega_{Y|x_i} \), where \( i = 1 \ldots k \). There is thus one conditional opinion for each element \( x_i \) of the parent variable. Each of these conditionals must be interpreted as the subjective opinion on \( Y \) given that \( x_i \) is TRUE. The subscript notation on each conditional opinion \( \omega_{Y|x_i} \) specifies not only the child variable \( Y \) it applies to, but also the element \( x_i \) of the parent variable it is conditioned on.

By extending the notation for binomial conditional deduction to the case of multinomial opinions, the general expression for multinomial conditional deduction is written as:

\[
\omega_{Y|x} = \omega_X \mathcal{O} \omega_{Y|x}
\]

where the symbol ‘\( \mathcal{O} \)’ denotes the conditional deduction operator for subjective opinions, and where \( \omega_{Y|x} \) is a set of \( k = |x| \) different opinions conditioned on each \( x_i \in \mathbb{X} \) respectively.
In case of abduction, the goal is to reason from the child variable $Y$ to the parent variable $X$. The multinomial expression for subjective logic conditional abduction is written as:

$$\omega_{X|Y} = \omega_Y \odot (\omega_{Y|X}, \omega_X)$$

$$= \omega_Y \odot \tilde{\phi}(\omega_{Y|X}, \omega_X)$$

where the symbol ‘$\tilde{\phi}$’ denotes the general conditional abduction operator, and the symbol ‘$\odot$’ denotes the inversion operator for conditional opinions. $\omega_{Y|X}$ is the set of $k = |X|$ different opinions conditioned on each $x_i \in X$ respectively. Similarly, $\omega_{X|Y}$ is the set of $l = |Y|$ different inverted opinions conditioned on each $y_j \in Y$ respectively.

In order to compute the abducted opinion according to Eq.(12) it is necessary to invert the set of conditional opinions $\omega_{Y|X}$ which produces the set of conditional opinions $\omega_{X|Y}$, so that the final part of the abduction computation can be based on multinomial deduction according to Eq.(11). Inversion of multinomial opinion conditionals is described in [14], and is in fact a generalisation of Bayes’ theorem of Eq.(3). The notation for multinomial opinion inversion is given below:

$$\omega_{X|Y} = \tilde{\phi}(\omega_{Y|X}, \omega_X)$$

Note that input conditionals do not necessarily have to be causal. However, for analysts it is typically easier to express conditionals in the causal direction, which is the reason why it is normally assumed that parent variables represent causes of child variables.

D. Subjective Bayesian Networks

A subjective Bayesian network models the random variables of nodes, and their conditional dependencies represented as conditional opinions, where the conditional relationships between nodes are represented via a directed acyclic graph (DAG). By assigning opinions to a set of input variables, it is possible to compute opinions for the remaining variables/nodes.

The goal of subjective Bayesian network modelling and analysis is to generalise traditional Bayesian network modelling and analysis, by including the uncertainty dimension, which involves some additional complexity. On the other hand, the advantage is that subjective Bayesian networks explicitly express the inherent uncertainty of realistic situation during the formal modelling, thereby allowing the analysis and the results to better reflect the situation as seen by the analysts. In other words, the inherent uncertainty of situations can no longer be ‘hidden under the carpet’, which is good news for policy and decision makers.

Consider a subjective Bayesian network containing a set $\tilde{X}$ of $K$ nodes/variables $X_1$ to $X_K$. The opinion is expressed by $\omega_X = \omega_{X_1} \ldots \omega_{X_K}$.

The chain rule of subjective conditionals describe how chained conditionals are combined in terms of iterative deduction of conditional opinions, expressed as

$$\omega_{X_k|X_{1:k-1}} = (\cdot \cdot \cdot (\omega_{X_k|X_{k-2}} \otimes \omega_{X_{k-1}|X_{k-2}}) \cdot \cdot \cdot \otimes \omega_{X_1|X_2})$$

where ‘$\otimes$’ denotes chained conditionals with the $\otimes$-operator for conditional deduction. Deduction with the serial conditional $\omega_{X_k|X_1}$ is expressed as

$$\omega_{X_k|X_1} = \omega_{X_1} \odot \omega_{X_k|X_1} .$$

Eq.(14) provides a basis for generalising chained deduction in subjective Bayesian networks, which of course assumes that the deduced opinion $\omega_{X_k|X_1}$ is a function of all the intermediate conditionals $\omega_{X_1|X_2}, \ldots, \omega_{X_K|X_{K-1}}$.

The inverse rule to that of chained conditionals is the chain rule of the subjective Bayes’ theorem, which expresses chained inverted conditional opinions:

$$\omega_{X_k|X_{1:k-1}} = (\cdot \cdot \cdot (\omega_{X_k|X_{k-2}} \odot \omega_{X_{k-1}|X_{k-2}}) \cdot \cdot \cdot \odot \omega_{X_1|X_2})$$

Abduction with the serial inverted conditional $\omega_{X_i|X_k}$ is expressed as

$$\omega_{X_i|X_k} = \omega_{X_k} \odot (\omega_{X_i|X_k}) .$$

Eq.(16) and Eq.(17) provide a basis for generalising chained abduction in subjective Bayesian networks, which of course assumes that the abducted opinion $\omega_{X_i|X_k}$ is a function of all the intermediate conditionals $\omega_{X_1|X_2}, \ldots, \omega_{X_K|X_{K-1}}$.

In subjective Bayesian networks, (conditional) opinions and base rate distributions replace the (conditional) probability tables and priors used in traditional Bayesian networks. Based on the operators of subjective logic such as multiplication, division, deduction and abduction/inversion, models of subjective Bayesian networks can be nicely expressed and analysed.

The four reasoning categories of Figure 2 can be described within the framework of subjective logic.

III. SUBJECTIVE TRUST NETWORKS

A. Trust Discounting

The general idea behind trust discounting is to express degrees of trust in an information source and then to discount information provided by the source as a function of the trust in the source. We represent both the trust and the provided information in the form of subjective opinions, and then define an appropriate operation on these opinions to find the trust discounted opinion.

Let agent $A$ denote the relying party and agent $B$ denote an information source. Assume that agent $B$ provides information to agent $A$ about the state of a variable $X$ expressed as a subjective opinion on $X$. Assume further that agent $A$ has an opinion on the trustworthiness of $B$ with regard to providing
information about $X$. Based on the combination of $A$’s trust in $B$ and on $B$’s opinion about $X$ given as an advice to $A$, it is possible for $A$ to derive an opinion about $X$. This process is illustrated in Figure 4.

![Figure 4. Trust discounting of opinions](image)

Several trust discounting operators for subjective logic are described in the literature [4], [5]. The general representation of trust discounting is through conditionals [5], while special cases can be expressed with specific trust discounting operators. In this paper we use the specific case of uncertainty-favouring trust discounting which enables the uncertainty in $A$’s derived opinion about $X$ to increase as a function of the projected distrust in the advisor $B$.

Agent $A$’s trust in $B$ is formally expressed as a binomial opinion $\omega^A_B$ on domain $\mathcal{T} = \{i, \bar{i}\}$ where the values $i$ and $\bar{i}$ denote trusted and distrusted respectively.

Assume variable $X$ on domain $\mathcal{X}$, and let $\omega^B_X$ be $B$’s general opinion on $X$ as advisor to $A$. Trust discounting is expressed with the following notation:

$$\omega^{A:B}_X = \omega^A_X \otimes \omega^B_X .$$

(18)

Trust discounting of agent $B$ as a function of agent $A$’s trust in $B$, denoted by $A:B$, corresponds to transitive discounting of opinions with the operator denoted by $\otimes$. $\omega^{A:B}_X$ denotes $A$’s subjective opinion on $X$ derived as a function of $A$’s trust in $B$ and in $B$’s advice opinion about $X$.

**B. Trust Fusion**

Trust fusion means that the derived trust opinions resulting from multiple trust paths are fused. Let us consider an example where Alice needs to have her car serviced, and where she has received an advice from Bob to use the car mechanic Eric. We assume that Alice has doubts about Bob’s advice, so she would like to get a second opinion. She therefore asks her other colleague Claire for her opinion about Eric. The trust graph which includes both pieces of advice is illustrated in Figure 5.

Formal full notation and compact notation of the graph of Figure 5 is shown in Eq.(19). This trust network also involves trust transitivity which is computed with trust discounting.

**Full notation:** $[A, E] = ([A; B]; [B, E]) \circ ([A; C]; [C, E])$

**Compact notation:** $[A, E] = [A; B, E] \circ [A; C, E]$

(19)

The computation of trust fusion involves both trust discounting and belief fusion, because what is actually fused is a pair of opinions computed with trust discounting. The trust target $E$ in Figure 5 can be expressed as a binary variable $X = \{\text{“E is reliable”, “E is unreliable”}\}$ so that that $A$ in fact derives a (trust) opinion about the variable $X$.

In general, it is assumed that agent $A$ receives opinions about target $E$ from two sources $B$ and $C$, and that $A$ has referral trust opinions in both $B$ and $C$, as illustrated in Figure 5.

In case the target node is a variable $X$, then the edges from $B$ and $C$ to $X$ can be assumed to represent opinions, but the principle of discounting and fusion is the same as before.

The symbol $\odot$ denotes fusion between the two trust paths $[A; B, X]$ and $[A; C, X]$ (in compact notation). The expression for $A$’s derived opinion about $X$ as a function of trust fusion, is given by Eq.(20).

$$\omega^A_{[A; B, C]} = (\omega^A_B \otimes \omega^C_B) \odot (\omega^A_C \otimes \omega^C_X)$$

(20)

The input arguments to the model of Figure 6 are represented as the 4 opinion triangles at the top of the figure, and
the derived trust opinion is represented as the opinion triangle at the bottom of the figure.

This trust fusion example uses a combination of trust discounting and fusion. By combining fusion and trust discounting, complex trust networks can be modeled and analyzed, as described in [9].

IV. SUBJECTIVE NETWORKS

This section addressed the features and challenges within subjective networks. First, the concept of conditional independence in light of uncertain (non-dogmatic) opinions is addressed. Finally, the current challenges in propagating and fusion of opinions from multiple sources over multiple variables is addressed.

A. Conditional Independence

The concept of conditional independence in traditional Bayesian networks changes in the context of subjective networks. The criterion for conditional independence in Bayesian networks can be concisely expressed as:

**Definition 1 (Bayesian Conditional Independence.):**

Variables $X$ and $Z$ are conditionally independent given a value of $Y$ if and only if, given the value of $Y$, knowledge of the value of $X$ provides no information on the likelihood of values of $Z$, and knowledge of the value of $Z$ provides no information on the likelihood of values of $X$.

In case the criterion of Definition 1 is satisfied, then the graph of Eq.(21) satisfies the Markov property which means that it is an I-map (independence map):

$$ X \implies Y \implies Z. $$

In Bayesian theory, a probability distribution (not the base rate distribution) of variable $Y$ is typically known conditionally on another variable, e.g. $X$, whereas variable $X$ typically is known unconditionally. Knowing the value of a variable, say $X = x$, is equivalent to assigning a probability $p(X = x) = 1$, which is just a limit case of knowing its general probability distribution, where $p(X = x)$ can be an arbitrary probability. It is of course a philosophical question whether something can be known unconditionally, but if it can, then there is in principle no difference between knowing a probability distribution over a variable, and knowing the value of a variable, because knowing the value of a variable is equivalent to knowing that its probability is 1.

The independence criterion of Definition 1 can be expressed in terms of subjective opinions where ‘knowing the value’ e.g. of intermediate variable $Y$, is equivalent to having an absolute opinion about that variable.

**Definition 2 (Subjective Conditional Independence.):**

Variable $X$ and $Z$ are conditionally independent given an opinion on $Y$ if and only if, given an absolute opinion on $Y$, any opinion on $X$ provides no information on the likelihood of values of $Z$, and any opinion on $Z$ provides no information on the likelihood of values of $X$.

Variable $Z$ in Eq.(21) would be dependent on variable $X$ given a relatively uncertain opinion about variable $Y$ in Eq.(21), because an opinion on $X$ could then influence the opinion on $Y$, which in turn could influence the opinion on $Z$.

Assume that the Bayesian graph of Eq.(21) represents an epistemic situation, i.e. that the variables represent specific instances. Then the input argument opinions should also be epistemic, and thereby uncertainty-maximised. An absolute opinion on $Y$ produces independence between $X$ and $Z$, i.e. $Z$ is independent of $X$ given an absolute opinion on $Y$. However, an epistemic non-absolute opinion on $Y$ does not produce independence, i.e. $Z$ is dependent to some degree on $X$, given the relatively uncertain epistemic opinion on $Y$.

As an example, consider the practical situation of variable $X = \{x, \tau\}$, where $x$ represents weather forecast for rain, of variable $Y = \{y, \bar{y}\}$ where $y$ represents that Bob carries an umbrella when leaving from home in the morning, and of variable $Z = \{z, \bar{z}\}$ where $z$ represents that Bob forgets the umbrella on the train. Assume that Bob usually carries an umbrella when the weather forecast for the day says rain, expressed by the pair of conditional opinions $\omega_{YX}$. Assume further that Bob is rather forgetful, so he often forgets his umbrella on the train, expressed by the pair of conditional opinions $\omega_{ZY}$. Then, if the analyst wants to infer whether Bob will forget the umbrella on the train from the absolute opinion that he carries an umbrella, it can be assumed that the opinion on $Z$ independent of any opinion on $X$, because knowing the weather forecast does not change his likelihood of forgetting the umbrella. However, if the observer is uncertain about whether Bob carries an umbrella, expressed by a relatively uncertain epistemic opinion $\omega_Y(y)$, then the opinion on $Z$ is not independent of the opinion on $X$. If the analyst wants to infer whether Bob will forget an umbrella on the train from an uncertain opinion about whether he carries an umbrella, then knowing the weather forecast will make the opinion about actually carrying an umbrella more certain, which influences the opinion about whether he will forget an umbrella on the train.

The extreme case would be that the analyst has a vacuous opinion on node $Y$, which would have no effect of reducing the dependence of $Z$ on $X$. In general, the degree of dependence between $X$ and $Y$ is a function of the uncertainty in the opinion on $Y$. The dependence is also a function of the (ir)relevance in the pairs of conditionals $\omega_{YX}$ and $\omega_{ZY}$.

The second way of satisfying the subjective conditional independence requirement of Definition 2 is to have variables that are irrelevant to each other. The Bayesian network of Eq.(21) has total irrelevance between $X$ and $Z$ when either the set of conditional opinions $\omega_{YX}$ makes $X$ totally irrelevant to $Y$, or when the set of conditional opinions $\omega_{ZY}$ makes $Y$ totally irrelevant to $Z$. If $X$ is irrelevant to $Y$, then no opinion on $X$ can influence the opinion on $Y$. If $Y$ is irrelevant to $Z$, then no opinion on $Y$ can influence the opinion on $Z$.

B. Subjective Network Modelling

A fused subjective network models the random variables of nodes, and their conditional dependencies represented as conditional opinions, where the conditional relationships between
nodes are represented via a directed acyclic graph (DAG). Opinions can be directly assigned to any variable, which could be none, some or all variables. Whenever multiple (conflicting) opinions exist for a specific variable, fusion can be applied. It is then possible to compute opinions for all variables/nodes, based on an arbitrary set of input opinions. In case there are no input opinions, the derived opinions are of course vacuous.

It is possible that different analysts have different opinions about the same variables, or about different variables of the same Bayesian network. Traditional Bayesian networks are not designed to handle such situations, but subjective networks are well equipped for that purpose.

Subjective networks allow the analyst to model the frame of multiple sources that provide evidence about the frame of conditionally dependent variables. Figure 7 illustrates the integration of a subjective trust network and a subjective Bayesian network to form a subjective network.

![Subjective network diagram](image_url)

**Figure 7.** Subjective network, with trust network and Bayesian network

The investigation of theoretical models and practical methods for Bayesian network modelling based on subjective logic, combined with trust networks, opens up a highly fertile field of research in AI (Artificial Intelligence) and machine learning.

The subjective logic operators such as deduction, abduction, and fusion will play important roles to propagate opinions across variables and to resolve conflicting opinions from multiple sources.

V. CONCLUSION

Subjective Bayesian networks generalise traditional Bayesian networks, whereby subjective opinions are used as arguments instead of probabilities. Subjective trust networks represent models for computational trust reasoning in a network of agents.

Subjective networks combine subjective Bayesian networks and subjective trust networks. This combination provides a powerful framework for modelling and analysing realistic situations in affected by uncertainty and partially missing information.

**REFERENCES**