Preprint of article published in:
International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, Vol. 9, No. 3 (June 2001)
Errata in the printed version have been corrected in this version. Last updated 15 October 2002.

A LOGIC FOR UNCERTAIN PROBABILITIES

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Received February 1998
Revised February 2000

We first describe a metric for uncertain probabilities called opinion, and subsequently a set of logical operators that can be used for logical reasoning with uncertain propositions. This framework which is called subjective logic uses elements from the Dempster-Shafer belief theory and we show that it is compatible with binary logic and probability calculus.

Keywords: Belief, evidence, reasoning, uncertainty, probability, logic.

1. Introduction

In standard logic, propositions are considered to be either true or false. However, a fundamental aspect of the human condition is that nobody can ever determine with absolute certainty whether a proposition about the world is true or false. In addition, whenever the truth of a proposition is assessed, it is always done by an individual, and it can never be considered to represent a general and objective belief. This indicates that important aspects are missing in the way standard logic captures our perception of reality, and that it is more designed for an idealised world than for the subjective world in which we are all living.

Several alternative calculi and logics which take uncertainty and ignorance into consideration have been proposed and quite successfully applied to practical problems where conclusions have to be made based on insufficient evidence (see for example Hunter 1996[1] or Motro & Smets 1997[2] for an analysis of some uncertainty logics and calculi). Although including uncertainty in the belief model is a significant step forward, it only goes half the way in realising the real nature of human beliefs. It is also necessary to take into account that beliefs always are held by individuals and that beliefs for this reason are fundamentally subjective.

In this paper we describe *subjective logic* (see Jøsang 1997[3] for an earlier version) as a logic which operates on subjective beliefs about the world, and use the term *opinion* to denote the representation of a subjective belief. Subjective logic operates on opinions and contains standard logical operators in addition to some non-standard operators which specifically depend on belief ownership. An opinion can be interpreted as a probability measure containing secondary uncertainty, and as such subjective logic can be seen as an extension of both probability calculus and binary logic.

Subjective logic must not be confused with fuzzy logic. The latter operates on crisp and certain measures about linguistically vague and fuzzy propositions, whereas subjective logic operates on uncertain measures about crisp propositions.

2. Representing Uncertain Probabilities

2.1. *The Belief Model*

The representation of uncertain probabilities will be based on a belief model similar to the one used in the Dempster-Shafer theory of evidence. The Dempster-Shafer theory was first set forth by Dempster in the 1960s as a framework for upper and lower probability bounds, and subsequently extended by Shafer who in 1976 published *A Mathematical Theory of Evidence*[4]. A more concise presentation can be found in Lucas & Van Der Gaag 1991[5] from which Defs.1 & 2 below are taken.

The first step in applying the Dempster-Shafer belief model is to define a set of possible situations which is called the *frame of discernment*. A frame of discernment delimits a set of possible states of a given system, exactly one of which is assumed to be true at any one time. Fig.1 illustrates a simple frame of discernment denoted by Θ with 4 elementary states $x_1, x_2, x_3, x_4 \in \Theta$.

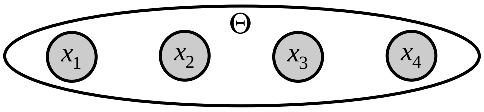

Fig. 1. Example of a frame of discernment

In the following, standard set theory will be used to describe frames of discernment, but the term *'state'* will be used instead of *'set'* because the former is more relevant to the field of application. It is assumed that the system can not be in more than one elementary state at the same time, or in other words, only one elementary state can be true at any one time. However, if an elementary state is assumed to be true, then all superstates can be considered true as well; e.g. if x_2 is assumed to be true then for example $x_2 \cup x_3$ and all other superstates of x_2 are also true. In fact Θ is by definition always true because it by definition contains a true state. This becomes more meaningful when assigning belief mass to states.

The elementary states in the frame of discernment Θ will be called atomic states because they do not contain substates. The powerset of Θ, denoted by 2^Θ, contains the atomic states and all possible unions of the atomic states, including Θ. A frame of discernment can be finite or infinite, in which cases the corresponding powerset is also finite or infinite respectively.

An observer who believes that one or several states in the powerset of Θ might be true can assign belief mass to these states. Belief mass on an atomic state $x \in 2^\Theta$ is interpreted as the belief that the state in question is true. Belief mass on a non-atomic state $x \in 2^\Theta$ is interpreted as the belief that one of the atomic states it contains is true, but that the observer is uncertain about which of them is true. The following definition is central in the Dempster-Shafer theory.

Definition 1 (Belief Mass Assignment) *Let Θ be a frame of discernment. If with each substate $x \in 2^{\Theta}$ a number $m_{\Theta}(x)$ is associated such that:*

$$\begin{array}{ll} \mathbf{1}. & m_{\Theta}(x) \geq 0 \\ \mathbf{2}. & m_{\Theta}(\emptyset) = 0 \\ \mathbf{3}. & \sum_{x \in 2^{\Theta}} m_{\Theta}(x) = 1 \end{array}$$

then m_{Θ} is called a belief mass assignment[1] *on Θ, or BMA for short. For each substate $x \in 2^{\Theta}$, the number $m_{\Theta}(x)$ is called the* belief mass[2] *of x.*

Fig.2 illustrates a part of the powerset of the frame of discernment of Fig.1 with the atomic states $x_1,\ x_2,\ x_3$ and x_4, and the non-atomic states x_5, x_6 and Θ. All the states in Fig.2 are in fact elements in 2^{Θ} and it can be imagined that belief mass is assigned to these states according to Def.1.

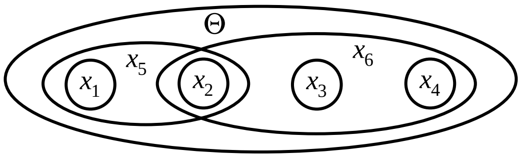

Fig. 2. Part of the powerset of Θ

A belief mass $m_{\Theta}(x)$ expresses the belief assigned to the state x and does not express any belief in substates of x in particular. If for example belief mass is assigned to x_5 in Fig.2 it must be interpreted as the belief that either x_1 or x_2 is true but that the observer is uncertain about which of them is true.

In contrast to belief mass, the *belief* in a state must be interpreted as an observer's total belief that a particular state is true. The next definition from the Dempster-Shafer theory will make it clear that belief in x not only depends on belief mass assigned to x but also on belief mass assigned to substates of x.

Definition 2 (Belief Function) *Let Θ be a frame of discernment, and let m_{Θ} be a BMA on Θ. Then the* belief function *corresponding with m_{Θ} is the function $b:\ 2^{\Theta} \longmapsto [0,1]$ defined by:*

$$b(x) = \sum_{y \subseteq x} m_{\Theta}(y), \quad x, y \in 2^{\Theta}\ .$$

Similarly to belief, an observer's *disbelief* must be interpreted as the total belief that a state is **not** true. The following definition is ours.

Definition 3 (Disbelief Function) *Let Θ be a frame of discernment, and let m_{Θ} be a BMA on Θ. Then the* disbelief function *corresponding with m_{Θ} is the function $d:\ 2^{\Theta} \longmapsto [0,1]$ defined by:*

$$d(x) = \sum_{y \cap x = \emptyset} m_{\Theta}(y), \quad x, y \in 2^{\Theta}\ .$$

[1] called *basic probability assignment* in Shafer 1976[4]
[2] called *basic probability number* in Shafer 1976[4]

The disbelief in for example state x_5 in Fig.2 is the sum of the belief masses on the states x_3 and x_4, i.e. all those that have and empty intersection with x_5. The disbelief of x corresponds to the *doubt* of x in Shafer's book. However, we choose to use the term 'disbelief' because we feel that for example the case when it is certain that a state is false can better be described by 'total disbelief' than by 'total doubt'.

Our next definition expresses uncertainty regarding a given state as the sum of belief masses on superstates or on partly overlapping states.

Definition 4 (Uncertainty Function) *Let Θ be a frame of discernment, and let m_Θ be a BMA on Θ. Then the* uncertainty function *corresponding with m_Θ is the function u : $2^\Theta \longmapsto [0,1]$ defined by:*

$$u(x) = \sum_{\substack{y \cap x \neq \emptyset \\ y \not\subseteq x}} m_\Theta(y), \quad x, y \in 2^\Theta \ .$$

The uncertainty regarding for example state x_5 in Fig.2 would be the sum of belief masses on the states x_6 and Θ.

Total uncertainty can be expressed by assigning all the belief mass to Θ. The belief function corresponding to this situation is called the vacuous belief function.

A BMA with zero belief mass assigned to Θ is called a dogmatic BMA. In later sections it is argued that dogmatic BMAs are unnatural in practical situations and strictly speaking can only be defended in idealised hypothetical situations.

With the concepts defined so far a simple theorem can be stated.

Theorem 1 (Belief Function Additivity)

$$b(x) + d(x) + u(x) = 1, \quad x \in 2^\Theta, \ \ x \neq \emptyset \ . \tag{1}$$

Proof 1
The sum of the belief, disbelief and uncertainty functions is equal to the sum of the belief masses in a BMA which according to Def.1 sums up to 1.

□

Eq.(1) is fundamental to our model of uncertain probabilities. The uncertainty function represents an observer's uncertainty regarding the truth of a given state, and can be interpreted as something that fills the void in the absence of both belief and disbelief.

For the purpose of expressing uncertain probabilities we will show that the relative number of atomic states is also needed in addition to belief functions. Assume for example that belief mass $m_\Theta(\Theta) = 1$ is assigned to Θ of Fig.2. Intuitively the probability of for example x_1 being true can then be estimated to 1/4 because any of the four atomic states can be true, and none is more probable than the others. Assume now that belief mass $m_\Theta(x_5) = 1$ is assigned to x_5. The probability of x_1 being true can now be estimated to 1/2 because only x_1 or x_2 can be true, and one is equally probable as the other.

For any particular state x the *atomicity* of x is the number of states it contains, denoted by $|x|$. In Fig.2 we have for example that $|x_5| = 2$. If Θ is a frame of discernment, the atomicity of Θ is equal to the total number of atomic states it contains. Similarly, if $x, y \in 2^\Theta$ then the overlap between x and y relative to y can be expressed in terms of number of atomic states. Our next definition captures this idea:

Definition 5 (Relative Atomicity) *Let Θ be a frame of discernment and let $x, y \in 2^\Theta$. Then for any given $y \neq \emptyset$ the* relative atomicity *of x to y is the function $a : 2^\Theta \longmapsto [0, 1]$ defined by:*

$$a(x/y) = \frac{|x \cap y|}{|y|}, \quad x, y \in 2^\Theta,\ y \neq \emptyset\ .$$

It can be observed that $x \cap y = \emptyset \ \Rightarrow\ a(x/y) = 0$, and that $y \subseteq x \ \Rightarrow\ a(x/y) = 1$. In all other cases the relative atomicity will be a value between 0 and 1. The relative atomicity of for example x_6 to x_5 in Fig.2 is given by:

$$\begin{aligned} a(x_6/x_5) &= |x_6 \cap x_5|/|x_5| \\ &= 1/2\ . \end{aligned}$$

The relative atomicity of an atomic state to its frame of discernment, denoted by $a(x/\Theta)$, can simply be written as $a(x)$. If nothing else is specified, the relative atomicity of a state then refers to the frame of discernment.

A frame of discernment with a corresponding BMA can be used to determine a probability expectation value for any given state. Uncertainty contributes to the probability expectation but will have different weight depending on the relative atomicities. When considering for example x_1 in Fig.2 the belief masses on x_5 and Θ both count as uncertainty but belief mass on Θ will have less weight than belief mass on x_5 because the atomicity of x_1 is smaller relative to Θ than it is to x_5.

Definition 6 (Probability Expectation) *Let Θ be a frame of discernment with BMA m_Θ, then the* probability expectation function *corresponding with m_Θ is the function $\mathrm{E} : 2^\Theta \longmapsto [0, 1]$ defined by:*

$$\mathrm{E}(x) = \sum_y m_\Theta(y)\ a(x/y), \quad y \in 2^\Theta\ . \tag{2}$$

This definition is equivalent with the pignistic probability described in e.g. Smets & Kennes 1994[6], and is based on the principle of insufficient reason; A belief mass assigned to the union of n atomic states is split equally among these n states.

The probability expectation of a given state is thus determined by the BMA and the atomicities. It should be noted that the probability expectation function removes information and that there can be infinitely many different BMAs that correspond to the same probability expectation value.

Shaferian belief functions and possibility measures have been interpreted as upper and lower probability bounds respectively (see e.g. de Cooman & Ayles 1998[7]). In our view belief functions can only be used to **estimate** probability values and not to set bounds, because the probability of a real event can never be determined with certainty, and neither can upper and lower bounds to it.

2.2. The Focused Frame of Discernment

This section describes how to derive from an arbitrary frame of discernment a binary frame of discernment and a corresponding BMA that for a given state will produce the same belief, disbelief and uncertainty functions as with the original frame of discernment and BMA.

Definition 7 (Focused Frame of Discernment) *Let Θ be a frame of discernment and let $x \in 2^{\Theta}$. The frame of discernment , denoted by $\widetilde{\Theta}^x$ and containing the two atomic states x and $\neg x$, where $\neg x$ is the complement of x in Θ, is then called the focused frame of discernment with focus on x.*

For example, the transition from the original frame of discernment of Fig.1 to a focused frame of discernment which focuses on the state $x_7 = (x_2 \cup x_3)$ is illustrated in Fig.3. It can be imagined that belief mass is assigned to all states drawn with solid lines in the left part of the figure, and that x_7 (drawn with dashed line) is defined as one of the two atomic states in the focused frame of discernment.

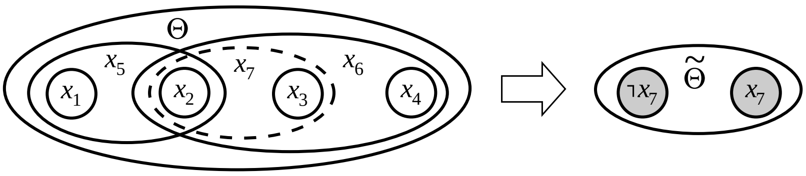

Fig. 3. Deriving the focused frame of discernment with focus on x_7

Definition 8 (Focused BMA and Relative Atomicity) *Let Θ be a frame of discernment with BMA m_Θ and let $b(x)$, $d(x)$ and $u(x)$ be the belief, disbelief and uncertainty functions of x in 2^{Θ}. Let $\widetilde{\Theta}^x$ be the the focused frame of discernment with focus on x. The focused BMA $m_{\widetilde{\Theta}^x}$ on $\widetilde{\Theta}^x$ is defined according to:*

$$\begin{aligned} m_{\widetilde{\Theta}^x}(x) &= b(x) \\ m_{\widetilde{\Theta}^x}(\neg x) &= d(x) \\ m_{\widetilde{\Theta}^x}(\widetilde{\Theta}^x) &= u(x)\ . \end{aligned} \tag{3}$$

The focused relative atomicity of x is defined by the following equation:

$$a_{\widetilde{\Theta}^x}(x) = [\mathrm{E}(x) - b(x)]/u(x)\ . \tag{4}$$

It can be seen that the belief, disbelief and uncertainty functions of x are identical in in 2^{Θ} and $2^{\widetilde{\Theta}^x}$. The focused relative atomicity is defined so that the probability expectation value of the state x is equal in Θ and $\widetilde{\Theta}^x$, and the expression for $a_{\widetilde{\Theta}^x}(x)$ in Def.8 can be determined by using Def.6.

The focused relative atomicity will in general be different from $\frac{1}{2}$ although $\widetilde{\Theta}^x$ contains exactly two states. It is in fact a constructed value which represents the weighted average of relative atomicities of x to all other states in 2^{Θ} as a function of their uncertainty mass.

A focused frame of discernment with corresponding focused BMA and relative atomicity makes it possible to work with binary frames of discernment instead of the full state space, and this is a great advantage when operators on belief functions are introduced in Sec.3.

2.3. *Example A: Assigning Belief Mass*

As example we will consider the frame of discernment to the left of Fig.3 with BMA m_Θ according to:

$$m_\Theta : \begin{cases} m_\Theta(x_1) &= 0.10 \\ m_\Theta(x_2) &= 0.20 \\ m_\Theta(x_3) &= 0.20 \\ m_\Theta(x_4) &= 0.00 \\ m_\Theta(x_5) &= 0.10 \\ m_\Theta(x_6) &= 0.30 \\ m_\Theta(\Theta) &= 0.10 \ . \end{cases} \quad \text{This produces the following belief, disbelief and uncertainty functions for } x_7\text{:} \quad \begin{aligned} b(x_7) &= 0.40 \\ d(x_7) &= 0.10 \\ u(x_7) &= 0.50 \ . \end{aligned} \tag{5}$$

Applying Def.6 produces the probability expectation value $\mathrm{E}(x_7) = 0.70$. The focused BMA on $\widetilde{\Theta}^{x_7}$ can be determined from Eqs.(3) and (5) resulting in:

$$m_{\widetilde{\Theta}^{x_7}} : \begin{cases} m_{\widetilde{\Theta}^{x_7}}(x_7) &= 0.40 \\ m_{\widetilde{\Theta}^{x_7}}(\neg x_7) &= 0.10 \\ m_{\widetilde{\Theta}^{x_7}}(\widetilde{\Theta}^{x_7}) &= 0.50 \ . \end{cases}$$

The focused relative atomicity of x_7 can be computed by using Eq.(4) to produce:

$$a_{\widetilde{\Theta}^{x_7}}(x_7) = 0.60 \ . \tag{6}$$

It can be seen that the focused BMA is more compact than the original BMA because it only represents the belief mass that is relevant for the state in focus.

2.4. *The Opinion Space*

After having presented some fundamental concepts in the previous sections the challenge is now to find a simple intuitive representation of uncertain probabilities. For this purpose we will define a 3-dimensional metric called *opinion* but which will contain a 4th redundant parameter in order to be simple to use in combination with logical operators.

Definition 9 (Opinion) *Let Θ be a binary frame of discernment with 2 atomic states x and $\neg x$, and let m_Θ be a BMA on Θ where $b(x)$, $d(x)$, $u(x)$, and $a(x)$ represent the belief, disbelief, uncertainty and relative atomicity functions on x in 2^Θ respectively. Then the* opinion *about x, denoted by ω_x, is the tuple defined by:*

$$\omega_x \equiv (b(x),\ d(x),\ u(x),\ a(x)) \ . \tag{7}$$

For compactness and simplicity of notation we will in the following denote the belief, disbelief, uncertainty and relative atomicity functions as b_x, d_x, u_x and a_x respectively. The notation $\mathrm{E}(\omega_x)$ will be equivalent to $\mathrm{E}(x)$ as a representation of probability expectation value of opinions.

The three coordinates (b, d, u) are dependent through Eq.(1) so that one is redundant. As such they represent nothing more than the traditional $($*Belief, Plausibility*$)$ pair of Shaferian belief theory. However, it is useful to keep all three coordinates in order to obtain simple expressions when introducing operators. Eq.(1) defines a triangle that can be used to graphically illustrate opinions as shown in Fig.4.

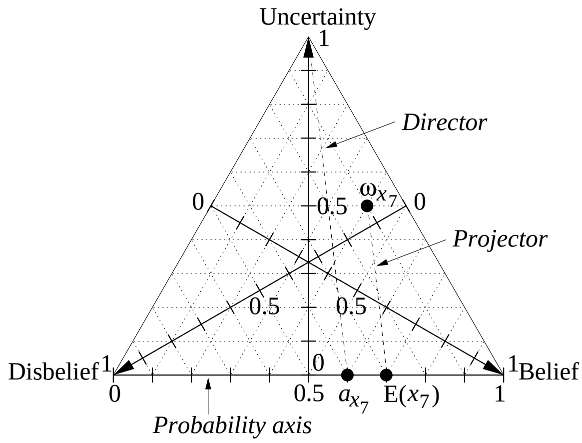

Fig. 4. Opinion triangle with ω_{x_7} as example

As an example the position of the opinion $\omega_{x_7} = (0.40,\ 0.10,\ 0.50,\ 0.60)$ from Example A in Sec.2.3 is indicated as a point in the triangle. Also shown are the probability expectation value and the relative atomicity.

The horizontal bottom line between the belief and disbelief corners in Fig.4 is called the *probability axis*. The relative atomicity can be graphically represented as a point on the probability axis. The line joining the top corner of the triangle and the relative atomicity point becomes the *director*. In Fig.4 $a(x_7) = 0.60$ is represented as a point, and the dashed line pointing at it represents the director.

The *projector* is parallel to the director and passes through the opinion point. Its intersection with the probability axis defines the probability expectation value which otherwise can be computed by the formula of Def.6. The position of the probability expectation $\mathrm{E}(x_7) = 0.70$ is shown.

Opinions situated on the probability axis are called *dogmatic opinions*. They represent situations without uncertainty and correspond to traditional frequentist probabilities. The distance between an opinion point and the probability axis can be interpreted as the degree of uncertainty.

Opinions situated in the left or right corner, i.e. with either $b = 1$ or $d = 1$ are called *absolute opinions*. They represent situations where it is absolutely certain that a state is either true or false, and correspond to 'TRUE' or 'FALSE' proposition in binary logic. With the definitions established so far we are able to derive the fundamental Kolmogorov axioms of traditional probability theory as a theorem.

Theorem 2 (Kolmogorov Axioms) *Given a frame of discernment* Θ *with a BMA* m_Θ*, the probability expectation function* E *with domain* 2^Θ *satisfies:*

1. $\mathrm{E}(x) \geq 0$ *for all* $x \in 2^\Theta$
2. $\mathrm{E}(\Theta) = 1$
3. *If* $x_1, x_2 \ldots \in 2^\Theta$ *are pairwise disjoint, then* $\mathrm{E}(\cup_{i=1}^{|2^\Theta|} x_i) = \sum_{i=1}^{|2^\Theta|} \mathrm{E}(x_i)$

Proof 2 *Each property can be proved separately.*

1. *Immediate results of Defs.1 & 2 are that* $b_x \geq 0$*, that* $u_x \geq 0$*, and that* $a_x \geq 0$ *for all* x*. As a consequence any probability expectation according to Def.6 will satisfy* $0 \leq \mathrm{E}(x) \leq 1$*.*

2. *Immediate results of Defs.1 are that* $b_\Theta = 1$ *and that* $u_\Theta = 0$*, resulting in* $\mathrm{E}(\Theta) = 1$*.*

3. *Let* $x_1, x_2 \ldots \in 2^\Theta$ *be a set of disjoint states, i.e. so that* $x_i \cap x_j = \emptyset$ *for* $i \neq j$*. According to Def.6 we can write:*

$$\mathrm{E}((x_i \cup x_j)) = \sum_y m_\Theta(y)\ a((x_i \cup x_j)/y), \quad y \in 2^\Theta\ . \tag{8}$$

Because x_i *and* x_j *are disjoint the following holds:*

$$a((x_i \cup x_j)/y) = a(x_i/y) + a(x_j/y)\ . \tag{9}$$

The sum in (8) can therefore be split in two so that $\mathrm{E}((x_i \cup x_j))$ *can be written:*

$$\begin{aligned}\mathrm{E}((x_i \cup x_j)) &= \textstyle\sum_y m_\Theta(y) a(x_i/y) + \sum_y m_\Theta(y) a(x_j/y),\ y \in 2^\Theta \\ &= \mathrm{E}(x_i) + \mathrm{E}(x_j)\ .\end{aligned} \tag{10}$$

This can be generalised to cover arbitrary sets of disjoint states. □

Opinions can be ordered according to probability expectation value, but additional criteria are needed in case of equal probability expectation values. The following definition determines the order of opinions:

Definition 10 (Ordering of Opinions) *Let* ω_x *and* ω_y *be two opinions. They can be ordered according to the following criteria by priority:*

1. *The opinion with the greatest probability expectation is the greatest opinion.*
2. *The opinion with the least uncertainty is the greatest opinion.*
3. *The opinion with the least relative atomicity is the greatest opinion.*

The first criterion is self evident. The second criterion is less so, but it is supported by experimental findings described by Ellsberg cited in Example B below. The third criterion is more an intuitive guess and so is the priority between the second and third criteria, and before these assumptions can be supported by evidence from practical experiments we invite the readers to judge whether they agree. An application of the third criterion will be illustrated by Example C in Sec.2.6.

2.5. *Example B: The Ellsberg Paradox*

The Ellsberg paradox[8] is a classical example of how traditional probability theory is unable to express uncertainty. Suppose you are shown an urn with 90 balls in it and you are told that 30 are red and that the remaining 60 balls are either black or yellow. One ball is going to be selected at random and you are given the following choice. Option I will give you $100 if a red ball is drawn and nothing of either a black or a yellow ball is drawn; option II will give you $100 if a black ball is drawn and nothing if a red or a yellow is drawn. Here is a summary of the options:

Table 1. First pair of betting options

	Red	Black	Yellow
Option I:	$100	0	0
Option II:	0	$100	0

Make a note of your choice and then consider another two options based on the same random draw from this urn:

Table 2. Second pair of betting options

	Red	Black	Yellow
Option III:	$100	0	$100
Option IV:	0	$100	$100

Which of option III and IV would you choose?

Ellsberg reports that, when presented with these pairs of choices, most people select options I and IV. Adopting the approach of expected utility theory this reveals a clear inconsistency in probability assessments. On this interpretation, when a person chooses option I over option II, he or she is revealing a higher subjective probability assessment of a *'Red'* than a *'Black'*. However, when the same person prefers option IV to option III, he or she reveals that his or her subjective probability assessment of *'Black or Yellow'* is higher than a *'Red or Yellow'*, implying that *'Black'* has a higher probability assessment than *'Red'*.

When representing the uncertain probabilities as opinions the choice of the majority becomes perfectly logic. Fig.5 shows a part of the powerset of the Ellsberg paradox example with corresponding BMA.

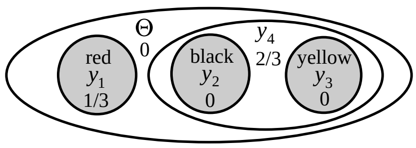

Fig. 5. Frame of discernment in the Ellsberg paradox

Utilities of option I and II depend on the opinions about y_1:*'Red'* and y_2:*'Black'*:

Option I: $\omega_{y_1} = (\frac{1}{3},\ \frac{2}{3},\ 0,\ \frac{1}{3})$ Option II: $\omega_{y_2} = (0,\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{2})$

By using Def.6 we find that: $\begin{cases} \mathrm{E}(\omega_{y_1}) & = 1/3 \\ \mathrm{E}(\omega_{y_2}) & = 1/3 \end{cases} \Rightarrow \quad \mathrm{E}(\omega_{y_1}) = \mathrm{E}(\omega_{y_2})$

The probability expectations are equal in both cases but ω_{y_2} contains uncertainty whereas ω_{y_1} does not. Fig.6 clearly shows the difference in uncertainty between the two opinions.

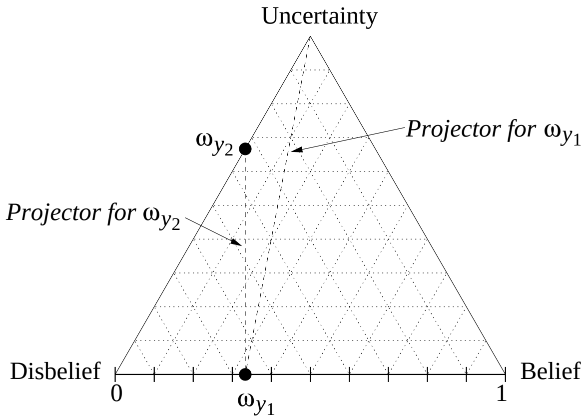

Fig. 6. Opinions about y_1 and y_2

It can be concluded that option I is the best choice because its corresponding probability of winning $100 is certain whereas option II represents an uncertain probability of winning.

Let us now turn to the next pair of options, namely option III and IV. This is equivalent to choosing between the states $(y_1 \cup y_3)$: *'Red or Yellow'* and y_4: *'Black or Yellow'* respectively. The corresponding opinions are:

Option III: $\omega_{y_1 \cup y_3} = (\frac{1}{3},\ 0,\ \frac{2}{3},\ \frac{1}{2})$ Option IV: $\omega_{y_4} = (\frac{2}{3},\ \frac{1}{3},\ 0,\ \frac{2}{3})$

From Def.6 we find that: $$\left\{ \begin{array}{ll} \mathrm{E}(\omega_{y_1 \cup y_3}) & = 2/3 \\ \mathrm{E}(\omega_{y_4}) & = 2/3 \end{array} \right. \Rightarrow \quad \mathrm{E}(\omega_{y_1 \cup y_3}) = \mathrm{E}(\omega_{y_4})$$

The probability expectations are again equal but $\omega_{y_1 \cup y_3}$ contains uncertainty whereas ω_{y_4} does not. Fig.7 clearly shows the difference in uncertainty between the two opinions.

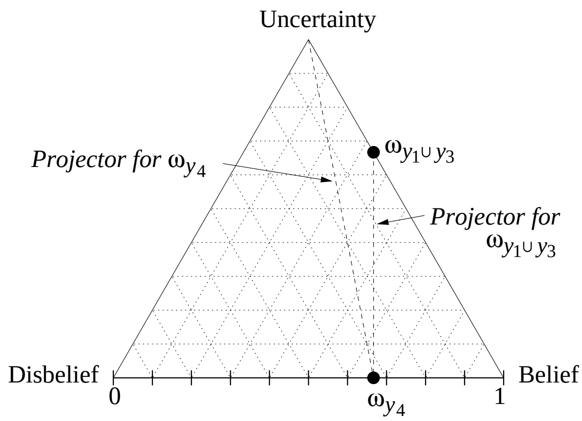

Fig. 7. Opinions about $(y_1 \cup y_3)$ and y_4

Based on the above it can be conclude that $\omega_{y_1 \cup y_3} < \omega_{y_4}$ and that opinion IV is the best choice because its probability of winning $100 is more certain than with option III.

We have shown that preferring option I to option II and preferring option IV to option III is perfectly rational and does not represent a paradox within the opinion model. Other models of uncertain probabilities are also able to explain the Ellsberg paradox, such as e.g. Choquet capacities(Choquet 1953[9], Chateauneuf 1991[10]). However, the next example presents a case which as far as we know can not be explained by any other model.

2.6. *Example C: Ordering Opinions*

In this example we describe a situation similar to the one in the Ellsberg paradox, namely an urn filled with balls, but this time having 9 different colours.

Suppose you are shown an urn with 80 balls in it and you are told that the urn was first filled with 20 *red* balls, then with 10 balls that were either *red*, *black* or *yellow*, then with 20 balls that were either *blue, white, green, pink, brown* or *orange*, and finally with 30 balls of any of the 9 mentioned colours. Fig.8 represents the frame of discernment of the situation where also the BMA is indicated.

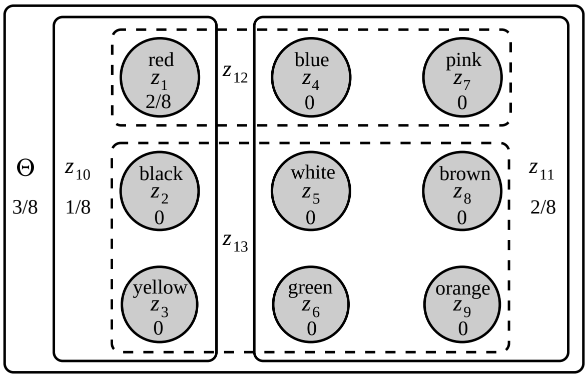

Fig. 8. Frame of discernment and BMA of urn with balls of 9 different colours

One ball is going to be drawn at random and we will compare the opinions about the states $z_{10}, z_{11}, z_{12}, z_{13} \in 2^{\Theta}$ defined by:

z_{10} : *'red, black or yellow'*, $\omega_{z_{10}} = (\frac{3}{8}, \frac{2}{8}, \frac{3}{8}, \frac{1}{3})$

z_{11} : *'blue, white, green, pink, brown or orange'*, $\omega_{z_{11}} = (\frac{2}{8}, \frac{3}{8}, \frac{3}{8}, \frac{2}{3})$

z_{12} : *'red, blue or pink'*, $\omega_{z_{12}} = (\frac{2}{8}, 0, \frac{6}{8}, \frac{1}{3})$

z_{13} : *'black, yellow, white, green, brown or orange'*, $\omega_{z_{13}} = (0, \frac{2}{8}, \frac{6}{8}, \frac{2}{3})$

By computing the respective probability expectation values it can be observed that:

$$\mathrm{E}(\omega_{z_{10}}) = \mathrm{E}(\omega_{z_{11}}) = \mathrm{E}(\omega_{z_{12}}) = \mathrm{E}(\omega_{z_{13}}) = 1/2 \ .$$

The problem is now to order the opinions about the 4 states that all have equal probability expectation. Fig.9 clearly shows that although the probability expectation values are equal the opinions have different levels of uncertainty and different relative atomicities.

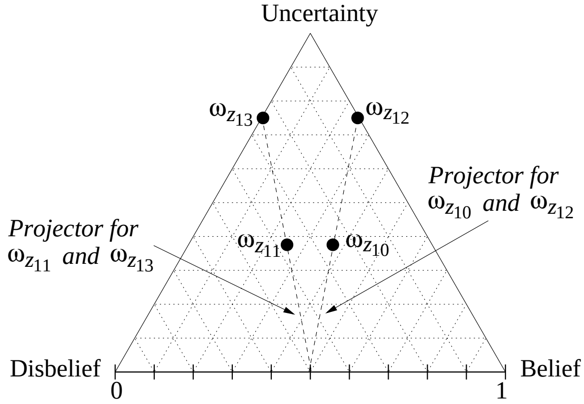

Fig. 9. Opinions about z_{10}, z_{11}, z_{12}, and z_{13}

According to the second criterion in Def.10 opinions positioned furthest down in the triangle are the greatest. According to the third criterion, those positioned furthest to the right are the greatest. We can therefore conclude that $\omega_{z_{10}} > \omega_{z_{11}} > \omega_{z_{12}} > \omega_{z_{13}}$. This result of course depends on the correctness of the priority between criteria 2 and 3 in Def.10, which needs to be verified by practical experiments on human judgement.

3. Logical Operators

So far we have described the elements of a frame of discernment as states. In practice states will verbally be described as propositions; if for example Θ consists of possible colours of a ball when drawn from an urn with red and black balls, and x designates the state when the colour drawn from the urn is red then it can be interpreted as the verbal proposition x: *'A ball drawn at random will be red'*.

Standard binary logic operates on binary propositions that can take the values 'TRUE' or 'FALSE'. *Subjective logic* operates on opinions about binary propositions, i.e. opinions about propositions that are assumed to be either true or false. In this section we describe the traditional logical operators 'AND', 'OR' and 'NOT' applied to opinions, and it will become evident that binary logic is a special case of subjective logic for these operators. An example of applying subjective logic to the problem of authentication and decision making for electronic transactions is described in Jøsang 1999[11].

Opinions are considered individual, and will therefore have an ownership assigned whenever relevant. In our notation, superscripts indicate ownership, and subscripts indicate the proposition to which the opinion applies. For example ω_x^A is an opinion held by agent A about the truth of proposition x.

3.1. *Propositional Conjunction and Disjunction*

Forming an opinion about the conjunction of two propositions from distinct frames of discernment consists of determining from the opinions about each proposition a new opinion reflecting the truth of both propositions simultaneously. This corresponds to 'AND' in binary logic.

Theorem 3 (Propositional Conjunction)
Let Θ_X and Θ_Y be two distinct binary frames of discernment and let x and y be propositions about states in Θ_X and Θ_Y respectively. Let $\omega_x = (b_x, d_x, u_x, a_x)$ and $\omega_y = (b_y, d_y, u_y, a_y)$ be an agent's opinions about x and y. Let $\omega_{x\wedge y} = (b_{x\wedge y}, d_{x\wedge y}, u_{x\wedge y}, a_{x\wedge y})$ be the opinion such that:

$$\begin{array}{ll}
\mathbf{1}. & b_{x\wedge y} = b_x b_y \\
\mathbf{2}. & d_{x\wedge y} = d_x + d_y - d_x d_y \\
\mathbf{3}. & u_{x\wedge y} = b_x u_y + u_x b_y + u_x u_y \\
\mathbf{4}. & a_{x\wedge y} = \frac{b_x u_y a_y + u_x a_x b_y + u_x a_x u_y a_y}{b_x u_y + u_x b_y + u_x u_y} \ .
\end{array}$$

Then $\omega_{x\wedge y}$ is called the propositional conjunction of ω_x and ω_y, representing the agents opinion about both x and y being true. By using the symbol '$\wedge$' to designate this operator, we define $\omega_{x\wedge y} \equiv \omega_x \wedge \omega_y$.

Forming an opinion about the disjunction of two propositions from distinct frames of discernment consists of determining from the opinions about each proposition a new opinion reflecting the truth of one or the other or both propositions. This corresponds to 'OR' in binary logic.

Theorem 4 (Propositional Disjunction)
Let Θ_X and Θ_Y be two distinct binary frames of discernment and let x and y be propositions about states in Θ_X and Θ_Y respectively. Let $\omega_x = (b_x, d_x, u_x, a_x)$ and $\omega_y = (b_y, d_y, u_y, a_y)$ be an agent's opinions about x and y. Let $\omega_{x\vee y} = (b_{x\vee y}, d_{x\vee y}, u_{x\vee y}, a_{x\vee y})$ be the opinion such that:

$$\begin{array}{ll}
\mathbf{1}. & b_{x\vee y} = b_x + b_y - b_x b_y \\
\mathbf{2}. & d_{x\vee y} = d_x d_y \\
\mathbf{3}. & u_{x\vee y} = d_x u_y + u_x d_y + u_x u_y \\
\mathbf{4}. & a_{x\vee y} = \frac{u_x a_x + u_y a_y - b_x u_y a_y - u_x a_x b_y - u_x a_x u_y a_y}{u_x + u_y - b_x u_y - u_x b_y - u_x u_y} \ .
\end{array}$$

Then $\omega_{x\vee y}$ is called the propositional disjunction of ω_x and ω_y, representing the agents opinion about x or y or both being true. By using the symbol '$\vee$' to designate this operator, we define $\omega_{x\vee y} \equiv \omega_x \vee \omega_y$.

Proof 3 and 4
Let Θ_X and Θ_Y be two binary frames of discernment, were $x, \neg x \in \Theta_X$ and $y, \neg y \in \Theta_Y$. The product frame of discernment of Θ_X and Θ_Y, denoted by $\Theta_{X\times Y}$ is obtained by conjugating each element of 2^{Θ_X} with each element of 2^{Θ_Y}. This produces:

$$\begin{array}{l}
\Theta_{X\times Y} \\
= \{x, \neg x, \Theta_X\} \times \{y, \neg y, \Theta_Y\} \\
= \{x\cap y, x\cap\neg y, x\cap\Theta_Y, \neg x\cap y, \neg x\cap\neg y, \neg x\cap\Theta_Y, \Theta_X\cap y, \Theta_X\cap\neg y, \Theta_X\cap\Theta_Y\} \ .
\end{array}$$

Let m_{Θ_X} and m_{Θ_Y} be BMAs on Θ_X and Θ_Y respectively. Because Θ_X and Θ_Y are binary, the belief masses can be expressed according to Eq.(3) as simple belief functions such that:

$$\begin{aligned} m_{\Theta_X}(x) &= b_x & \quad m_{\Theta_Y}(y) &= b_y \\ m_{\Theta_X}(\neg x) &= d_x & \quad m_{\Theta_Y}(\neg y) &= d_y \\ m_{\Theta_X}(\Theta_X) &= u_x & \quad m_{\Theta_Y}(\Theta_Y) &= u_y \ . \end{aligned}$$

The BMA on $\Theta_{X\times Y}$ is obtained by multiplying the respective belief masses on the elements of 2^{Θ_X} with the belief masses on the elements of 2^{Θ_Y}. This produces:

$$\begin{aligned} m_{\Theta_{X\times Y}}(x\cap y) &= b_x b_y & m_{\Theta_{X\times Y}}(\neg x\cap y) &= d_x b_y & m_{\Theta_{X\times Y}}(\Theta_X\cap y) &= u_x b_y \\ m_{\Theta_{X\times Y}}(x\cap \neg y) &= b_x d_y & m_{\Theta_{X\times Y}}(\neg x\cap \neg y) &= d_x d_y & m_{\Theta_{X\times Y}}(\Theta_X\cap \neg y) &= u_x d_y \\ m_{\Theta_{X\times Y}}(x\cap \Theta_Y) &= b_x u_y & m_{\Theta_{X\times Y}}(\neg x\cap \Theta_Y) &= d_x u_y & m_{\Theta_{X\times Y}}(\Theta_X\cap \Theta_Y) &= u_x u_y \ . \end{aligned}$$

- **Propositional Conjunction**

  The conjunction between $x \in \Theta_X$ and $y \in \Theta_Y$ is simply $x \cap y \in \Theta_{X\times Y}$. The derived frame of discernment with focus on $x \cap y$ then becomes $\widetilde{\Theta}^{x\cap y}_{X\times Y} = \{x\cap y, \neg\{x\cap y\}\}$, where $\neg\{x\cap y\} = \{x\cap \neg y, \neg x\cap y, \neg x\cap \neg y\}$. According to Def.8 the BMA $m_{\widetilde{\Theta}^{x\cap y}_{X\times Y}}$ is such that:

  $$\begin{aligned} \mathbf{1.} \quad & m_{\widetilde{\Theta}^{x\cap y}_{X\times Y}}(x\cap y) &&= b_{x\wedge y} \\ \mathbf{2.} \quad & m_{\widetilde{\Theta}^{x\cap y}_{X\times Y}}(\neg\{x\cap y\}) &&= d_{x\wedge y} \\ \mathbf{3.} \quad & m_{\widetilde{\Theta}^{x\cap y}_{X\times Y}}(\widetilde{\Theta}^{x\cap y}_{X\times Y}) &&= u_{x\wedge y} \ . \end{aligned}$$

  By using Eq.(4) it can also be observed that the derived relative atomicity of $x \cap y$ is such that:

  $$\mathbf{4.} \quad a_{\widetilde{\Theta}^{x\cap y}_{X\times Y}}(x\cap y) = a_{x\wedge y} \ .$$

  These four parameters define $\omega_{x\wedge y}$ as specified in Theorem 3.

- **Propositional Disjunction**

  Similarly to propositional conjunction, the propositional disjunction between $x \in \Theta_X$ and $y \in \Theta_Y$ is simply $x \cup y = \{x\cap y, x\cap\neg y, \neg x\cap y\}$, with $x \cup y \in \Theta_{X\times Y}$. The derived frame of discernment with focus on $x \cup y$ then becomes $\widetilde{\Theta}^{x\cup y}_{X\times Y} = \{x\cup y, \neg\{x\cup y\}\}$, where $\neg\{x\cup y\} = \{\neg x\cap\neg y\}$. According to Def.8 the belief mass assignment $m_{\Theta_{x\cup y}}$ is such that:

  $$\begin{aligned} \mathbf{1.} \quad & m_{\widetilde{\Theta}^{x\cup y}_{X\times Y}}(x\cup y) &&= b_{x\vee y} \\ \mathbf{2.} \quad & m_{\widetilde{\Theta}^{x\cup y}_{X\times Y}}(\neg\{x\cup y\}) &&= d_{x\vee y} \\ \mathbf{3.} \quad & m_{\widetilde{\Theta}^{x\cup y}_{X\times Y}}(\widetilde{\Theta}^{x\cup y}_{X\times Y}) &&= u_{x\vee y} \ . \end{aligned}$$

  By using Eq.(4) it can also be observed that the derived relative atomicity of $x \cup y$ is such that:

  $$\mathbf{4.} \quad a_{\widetilde{\Theta}^{x\cup y}_{X\times Y}}(x\cup y) = a_{x\vee y} \ .$$

  These four parameters define $\omega_{x\vee y}$ as specified in Theorem 4. □

As would be expected, propositional conjunction and disjunction of opinions are both commutative and associative. Idempotence is not defined because it would mean that the propositions x and y are identical and therefore belong to the same frame of discernment. It must always be assumed that the arguments are independent and refer to distinct frames of discernment.

Propositional conjunction and disjunction are equivalent to the 'AND' and 'OR' operators of Baldwin's support logic[12] except for the relative atomicity parameter which is absent in Baldwin's logic. When applied to absolute opinions, i.e with either $b = 1$ or $d = 1$, propositional conjunction and disjunction are equivalent to 'AND' and 'OR' of binary logic, that is; they produce the truth tables of logical 'AND' and 'OR' respectively. When applied to dogmatic opinions, i.e opinions with zero uncertainty, they produce the same results as the product and co-product of probabilities respectively. It can be observed that for dogmatic opinions the denominator becomes zero in the expressions for the relative atomicity in Theorems 3 and 4. However, the limits do exist and can be computed in such cases. See also comment about dogmatic opinions in Sec.5.2 below.

Propositional conjunction and disjunction must not be confused with the conjunctive and disjunctive rules of combination described by e.g. Smets 1993[13] and Smets & Kennes, 1994[6]. Propositional conjunction represents belief about the conjunction (i.e. logical 'AND') of distinct propositions whereas the conjunctive rule of combination is just another name for Dempster's rule for combining separate beliefs about the same proposition. The latter is described in Sec.5.4 below.

Propositional conjunction and disjunction of opinions are not distributive on each other. If for example ω_x, ω_y and ω_z are independent opinions we have:

$$\omega_x \wedge (\omega_y \vee \omega_z) \quad \neq \quad (\omega_x \wedge \omega_y) \vee (\omega_x \wedge \omega_z) \tag{11}$$

This result which may seem surprising is due to the fact that ω_x appears twice in the expression on the right side so that it in fact represents the propositional disjunction of partially dependent arguments. Only the expression on the left side is thus correct.

Propositional conjunction decreases the relative atomicity whereas propositional disjunction increases it. What really happens is that the product of the two frames of discernment produces a new frame of discernment with atomicity equal to the product of the respective atomicities. However, as opinions only apply to binary frames of discernment, a new frame of discernment with corresponding relative atomicity must be derived both for propositional conjunction and propositional disjunction. The expressions for relative atomicity in Theorems 3 and 4 are in fact obtained by forming the product of the two frames of discernment and applying Eq.(4) and Def.6 .

In order to show that subjective logic is compatible with probability calculus regarding product and co-product of probabilities we will prove the following theorem.

Theorem 5 (Product and Co-product)
Let Θ_X and Θ_Y be two distinct binary frames of discernment and let x and y be propositions about states in Θ_X and Θ_Y respectively. Let $\omega_x = (b_x, d_x, u_x, a_x)$ and $\omega_y = (b_y, d_y, u_y, a_y)$ be an agent's opinions about the propositions x and y respectively, and

let $\omega_{x\wedge y} = (b_{x\wedge y}, d_{x\wedge y}, u_{x\wedge y}, a_{x\wedge y})$ *and* $\omega_{x\vee y} = (b_{x\vee y}, d_{x\vee y}, u_{x\vee y}, a_{x\vee y})$ *be their respective propositional conjunction and disjunction. The probability expectation function* E *satisfies:*

$$\begin{array}{ll} \mathbf{1}. & \mathrm{E}(\omega_{x\wedge y}) = \mathrm{E}(\omega_x)\mathrm{E}(\omega_y) \\ \mathbf{2}. & \mathrm{E}(\omega_{x\vee y}) = \mathrm{E}(\omega_x) + \mathrm{E}(\omega_y) - \mathrm{E}(\omega_x)\mathrm{E}(\omega_y)\ . \end{array}$$

Proof 5 *Each property can be proved separately.*

1. *Equation 1 corresponds to the product of probabilities. By using Def.6 and Theorem 3 we get:*

$$\begin{aligned} \mathrm{E}(\omega_{x\wedge y}) &= b_{x\wedge y} + u_{x\wedge y}\ a_{x\wedge y} \\ &= b_x b_y + b_x u_y a_y + u_x a_x b_y + u_x a_x u_y a_y \\ &= (b_x + u_x a_x)(b_y + u_y a_y) \\ &= \mathrm{E}(\omega_x)\mathrm{E}(\omega_y)\ . \end{aligned} \tag{12}$$

2. *Equation 2 corresponds to the co-product of probabilities. By using Def.6, Theorem 4 and Eq.(1) we get:*

$$\begin{aligned} \mathrm{E}(\omega_{x\vee y}) &= b_{x\vee y} + u_{x\vee y}\ a_{x\vee y} \\ &= b_{x\vee y} + (d_x u_y + u_x d_y + u_x u_y)\ a_{x\vee y} \\ &= b_{x\vee y} + (u_x + u_y - b_x u_y - u_x b_y - u_x u_y)\ a_{x\vee y} \\ &= b_x + b_y - b_x b_y + u_x a_x + u_y a_y - b_x u_y a_y - u_x a_x b_y - u_x a_x u_y a_y \\ &= b_x + u_x a_x + b_y + u_y a_y - (b_x + u_x a_x)(b_y + u_y a_y) \\ &= \mathrm{E}(\omega_x) + \mathrm{E}(\omega_y) - \mathrm{E}(\omega_x)\mathrm{E}(\omega_y)\ . \end{aligned} \tag{13}$$

□

3.2. *Example D: Reliability Analysis*

A newly designed industrial process Z depends on two subprocesses X and Y to produce correct result. This conjunctive situation is illustrated in Fig.10, and the analysis of this system illustrates the use of the propositional conjunction operator.

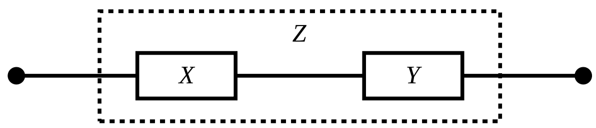

Fig. 10. Conjunctive system

The analysis is based on expressing opinions about proposition such as for example x: '*Process X will produce correct result*'. The propositions y and z are defined accordingly. A's perception of the reliability of Z can be expressed as $\omega^A_z = \omega^A_{x\wedge y}$. From earlier experience, agent *A* has the following opinions about the subprocesses X and Y:

$$\omega^A_x = (0.8,\ 0.1,\ 0.1,\ 0.5) \qquad\qquad \omega^A_y = (0.1,\ 0.8,\ 0.1,\ 0.5)\ .$$

By using the propositional conjunction operator, the opinion about the reliability of process Z can be computed as:

$$\omega_z^A \quad = (0.08,\ 0.82,\ 0.10,\ 0.475)\ .$$

The corresponding probability expectation value gives $\mathrm{E}(\omega_z^A) = 0.1275$. It can be verified that:

$$\mathrm{E}(\omega^A_{x\wedge y}) + \mathrm{E}(\omega^A_{x\wedge\neg y}) + \mathrm{E}(\omega^A_{\neg x\wedge y}) + \mathrm{E}(\omega^A_{\neg x\wedge\neg y}) = 1.$$

which shows that propositional conjunction preserves probability additivity.

3.3. *Negation*

The negation of an opinion about proposition x represents the opinion about x being false. This corresponds to 'NOT' in binary logic.

Theorem 6 (Negation)
Let $\omega_x = (b_x, d_x, u_x, a_x)$ be an opinion about the proposition x. Then $\omega_{\neg x} = (b_{\neg x}, d_{\neg x}, u_{\neg x}, a_{\neg x})$ is the negation of ω_x where:

$$\begin{array}{ll} \mathbf{1}. & b_{\neg x} = d_x \\ \mathbf{2}. & d_{\neg x} = b_x \\ \mathbf{3}. & u_{\neg x} = u_x \\ \mathbf{4}. & a_{\neg x} = 1 - a_x\ . \end{array}$$

By using the symbol '$\neg$' to designate this operator, we define $\neg\omega_x \equiv \omega_{\neg x}$.

Proof 6
The opinion about the negation of the proposition is the opinion about the complement state in the frame of discernment. An immediate result of Eq.(3) is then that $b_{\neg x} = d_x$, $d_{\neg x} = b_x$ and $u_{\neg x} = u_x$. The probability expectation values of x and $\neg x$ must satisfy $\mathrm{E}(\omega_x) + \mathrm{E}(\omega_{\neg x}) = 1$ which when used in Eq.4 results in $a_{\neg x} = 1 - a_x$.

□

Negation can be applied to expressions containing propositional conjunction and disjunction, and it can be shown that De Morgans's laws are valid.

4. The Evidence Space

This section describes an alternative representation of uncertain probabilities, namely by probability density functions over a probability variable. Similar ideas have been described by e.g. Gärdenfors & Sahlin 1982[14], Chávez, 1996[15] and Walley 1997[16]. In addition we define a mapping between the density function representation and the Shaferian belief model representation described in Sec.2 so that results from one space can be used in the other.

4.1. *Probability Density Functions*

The mathematical analysis leading to the expression for posteriori probability estimates of binary events can be found in many text books on probability theory, e.g. Casella & Berger 1990[17] p.298, and we will only present the results here.

It can be shown that posteriori probabilities of binary events can be represented by the beta distribution. The beta-family of density functions is a continuous family of functions indexed by the two parameters α and β. The beta(α, β) distribution can be expressed using the gamma function Γ as:

$$f(p \,|\, \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \ , \qquad \text{where } 0 \leq p \leq 1, \ \alpha > 0, \ \beta > 0 \ . \tag{14}$$

with the restriction that the probability variable $p \neq 0$ if $\alpha < 1$, and $p \neq 1$ if $\beta < 1$. The expectation value of the beta distribution is given by $\mathrm{E}(p) = \alpha/(\alpha+\beta)$.

The usual acronym for a probability density function is *'pdf'*. In our case the variable will always be a probability variable, so the pdf can be called a probability pdf, or *'ppdf'* for short.

The ppdf expression of Eq.(14) is valid for binary events, that is when the relative atomicity of the actual event is $\frac{1}{2}$. We generalise Eq.(14) to cover event spaces of arbitrary atomicity, and thus events of arbitrary relative atomicity through the following equations:

$$\begin{array}{ll} \alpha = r + 2a, & \text{where } \ 0 < a < 1, \ r \geq 0 \\ \beta = s + 2(1-a), & \text{where } \ 0 < a < 1, \ s \geq 0 \ . \end{array} \tag{15}$$

Here a is defined to be the relative atomicity of the actual event to which the ppdf applies, and shall be interpreted equivalently with the relative atomicity of opinions. The parameters r represents the amount of evidence supporting the actual event and the parameters s represents the amount of evidence supporting its negation. Our next definition captures this idea.

Definition 11 (Probability Density Function) *Let f be a probability density function over the probability variable p, then f is characterised by r, s and a according to:*

$$f(p \,|\, r, s, a) = \frac{\Gamma(r+s+2)}{\Gamma(r+2a)\Gamma(s+2(1-a))} p^{(r+2a-1)}(1-p)^{(s+2(1-a)-1)} \ , \qquad \textit{where } 0 \leq p \leq 1, \ 0 \leq r, \ 0 \leq s, \ 0 < a < 1 \ . \tag{16}$$

with the restriction that the probability variable $p \neq 0$ if $(r+2a) < 1$, and $p \neq 1$ if $(s + 2(1-a)) < 1$. Here r, s and a represent positive evidence, negative evidence and relative atomicity respectively. This function will be called a ppdf for short.

Justification

In order to justify Def.11 we will show that probability expectation values of ppdfs preserve additivity in a frame of discernment.

The probability expectation value of a ppdf can be directly derived from the expectation value of the beta distribution by using Eq.(15):

$$\mathrm{E}(p) = (r+2a)/(r+s+2) \ . \tag{17}$$

A process that can produce t different outcomes can be represented by the frame of discernment Θ with t atomic states. Over a period of time an observer has registered each *outcome n_i times where $0 \leq n_i$ and i represents the index of the outcome. The probability expectation value of each outcome indexed by i can be expressed as:*

$$\mathrm{E}(p_i) = \frac{n_i + 2/t}{\sum_{j=1}^{t} n_j + 2} \tag{18}$$

where $1/t$ is the relative atomicity of each event. The sum of the probability expectation values of all the events then becomes:

$$\sum_{i=1}^{t} \mathrm{E}(p_i) \quad = 1 \;. \tag{19}$$

A probability expectation value contain less information than the ppdf and the above result does not guarantee that a ppdf expresses the degree of uncertainty correctly. This will be discussed in Sec.4.4.

□

As an example, a process with two possible outcomes (i.e. binary event space, $a = 0.5$) that has produced $r = 7$ positive and $s = 1$ negative outcomes, will have a ppdf expressed as $f(p \,|\, 7.0, 1.0, 0.5)$ which is plotted in Fig.11.

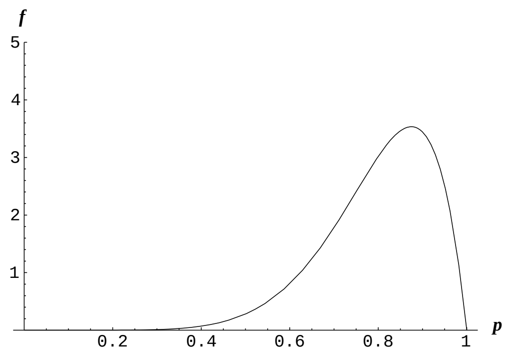

Fig. 11. Ppdf after 7 positive and 1 negative results

This curve expresses the uncertain probability that the process will give a positive outcome during future observations. The probability expectation value is given by $\mathrm{E}(p) = 0.8$. This can be interpreted as saying that the relative frequency of positive outcome is somewhat uncertain, and that the most likely value is 0.8.

4.2. *Mapping between the Evidence and Opinion Spaces*

The ppdf in Def.11 is a 3-dimensional representation of uncertain probabilities. This fits well with the 3-dimensional expression for opinions described in Sec.2.4, and we will in this section define a mapping between the two representations which leads to equivalent interpretations.

Definition 12 (Mapping) *Let $\omega = (b, d, u, a)$ be an agent's opinion about a proposition, and let $f(p \,|\, r, s, a)$ be the same agent's probability estimate regarding the same proposi-*

tion expressed as a ppdf, then ω *can be expressed as a function of* $f(p)$ *according to:*

$$\left\{ \begin{array}{l} b = \frac{r}{r+s+2} \\ d = \frac{s}{r+s+2} \\ u = \frac{2}{r+s+2} \end{array} \right. \quad \textit{where } u \neq 0 \ . \tag{20}$$

Justification

We start by requiring equality between the probability expectation values of ω *and* $f(p)$*, and by using Eq.(1).*

$$\left\{ \begin{array}{l} \mathrm{E}(f(p)) = \mathrm{E}(\omega) \\ b + d + u = 1 \end{array} \right. \tag{21}$$

$$(21) \quad \Rightarrow \quad \left\{ \begin{array}{l} b + ua = (r + 2a)/(r + s + 2) \\ b + d + u = 1 \end{array} \right. \tag{22}$$

$$(21) \quad \Rightarrow \quad \left\{ \begin{array}{l} b + ua = r/(r + s + 2) + 2a/(r + s + 2) \\ b + d + u = 1 \end{array} \right. \tag{23}$$

We require the solution to make b *an increasing function of* r*, and* d *an increasing function of* s*, so that there is an affinity between* b *and* r*, and between* d *and* s*. We also require* u *to be a decreasing function of* (r, s)*. By including this* 'affinity' *requirement we get:*

$$\left\{ \begin{array}{c} (21) \\ + \\ \textit{`affinity'} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} b = r/(r + s + 2) \\ d = s/(r + s + 2) \\ u = 2/(r + s + 2) \ . \end{array} \right.$$

□

Equivalently to Def.12 it is possible to express $f(p)$ as a function of ω:

$$\left\{ \begin{array}{c} (20) \\ (1) \end{array} \right. \Rightarrow \left\{ \begin{array}{l} r = 2b/u \\ s = 2d/u \end{array} \right. \quad \text{where } u \neq 0 \ .$$

We see for example that the uniform ppdf $f(p \,|\, 0.0,\ 0.0,\ 0.5)$ corresponds to the opinion $\omega = (0.0,\ 0.0,\ 1.0,\ 0.5)$ which expresses total uncertainty about a binary event, that $f(p \,|\, \infty, 0, a)$ or the absolute probability corresponds to $\omega = (1, 0, 0, a)$ which expresses absolute belief, and that $f(p \,|\, 0, \infty, a)$ or the zero probability corresponds to $\omega = (0, 1, 0, a)$ which expresses absolute disbelief. By defining ω as a function of f according to Eq.(20), the interpretation of ω corresponds exactly to the interpretation of $f(p)$.

Dogmatic opinions such as for example $\omega = (0.5, 0.5, 0.0, 0.5)$, do not have a clear equivalent representation as ppdf because the (r, s) parameters would explode and make it necessary to work with infinity ratios. In order to avoid this problem dogmatic and absolute opinions can be excluded, or in other words only allow opinions with $u \neq 0$. See also comments about dogmatic opinions in Sec.5.2.

Eq.(20) defines a bijective mapping between the evidence space and the opinion space so that any ppdf has an equivalent mathematical and interpretative representation as an opinion and vice versa, making it possible to produce opinions based on statistical evidence.

4.3. *Combination of Evidence*

Assume two observers A and B having observed a process over two different periods respectively. The event of producing a positive result is denoted by x and the event of producing a negative result is denoted by $\neg x$. The parameters r, and s represent the observed number of positive and negative results respectively. The parameter a represents the relative atomicity of the positive event. According to Def.11, the observers' respective ppdfs are then $f(p\,|\,r_x^A, s_x^A, a_x^A)$ and $f(p\,|\,r_x^B, s_x^B, a_x^B)$. Imagine now that they combine their observations to form a better estimate of the event's probability. This is equivalent to an imaginary observer $[A,B]$ having made all the observations and who therefore can form the ppdf defined by $f(p\,|\,r_x^A + r_x^B,\ s_x^A + s_x^B,\ a_x^{A,B})$. This is the basis for our definition of the consensus operator for combining evidence.

Definition 13 (Combining Evidence)
Let $f(p\,|\,r_x^A, s_x^A, a_x^A)$ and $f(p\,|\,r_x^B, s_x^B, a_x^B)$ be two ppdfs respectively held by the observers A and B regarding the truth of a proposition x. The ppdf $f(p\,|\,r_x^{A,B}, s_x^{A,B}, a_x^{A,B})$ defined by:

$$\begin{array}{ll} \mathbf{1.} & r_x^{A,B} = r_x^A + r_x^B \\ \mathbf{2.} & s_x^{A,B} = s_x^A + s_x^B \\ \mathbf{3.} & a_x^{A,B} = \frac{a_x^A(r_x^A + s_x^A) + a_x^B(r_x^B + s_x^B)}{r_x^A + s_x^A + r_x^B + s_x^B} \end{array}$$

is then called the consensus operator for combining A's and B's evidence, as if all the evidence was held by an imaginary observer $[A,B]$. By using the symbol '$\oplus$' to designate this operator, we get $f(p\,|\,r_x^{A,B}, s_x^{A,B}, a_x^{A,B}) = f(p\,|\,r_x^A, s_x^A, a_x^A) \oplus f(p\,|\,r_x^B, s_x^B, a_x^B)$.

The expression for the combined relative atomicity $a_x^{A,B}$ is not based on statistical analysis of evidence but is due to technical considerations. It could be imagined that the two observers have different views of the atomicity of the event space to which the observed event belongs, whereas a common view is required. A simple solution to this problem is to let the expression for $a_x^{A,B}$ be a weighted average of the respective relative atomicities, where the observer with the most observations has the greatest influence on $a_x^{A,B}$. The idea is that the observer with the most evidence about the event should also know the event space the best.

This operator for combining evidence will in Sec.5.2 form the basis for describing a consensus operator for opinions.

4.4. *Propositional Conjunction and Disjunction of Density Functions*

The mapping between the opinion space and the evidence space makes it possible to apply subjective logic to probability density functions over a probability variable, or ppdfs for short. This sections briefly describes some consequences of this.

A totally uncertain opinion $\omega_x = (0.0, 0.0, 1.0, 0.5)$ maps to $f_x(p\,|\,0.0, 0.0, 0.5)$ which is the uniform ppdf illustrated in Fig. 12.

Combining two totally uncertain opinions with the propositional conjunction operator 'AND' produces a new totally uncertain opinion with relative atomicity equal to the product of the operand relative atomicities. Let for example ω_x be defined as above and let $\omega_y = \omega_x$. Then $\omega_{x\wedge y} = (0.00, 0.00, 1.00, 0.25)$, and the corresponding ppdf

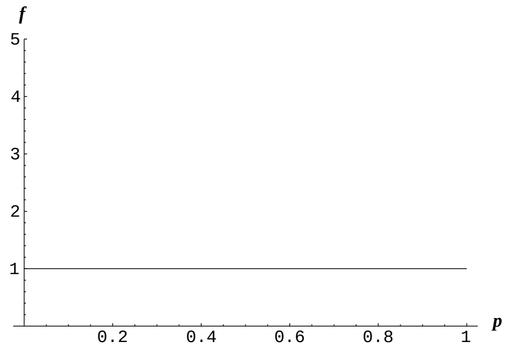

Fig. 12. Uniform ppdf

$f_{x\wedge y}(p\,|\,0.00,\,0.00,\,0.25)$. This result can be compared with the computation of simultaneous uniform density functions.

A method for computing simultaneous pdfs is described in e.g. Casella & Berger 1990[17] p.148. Let $g(p)$ represent simultaneous uniform pdfs over a probability variable p. This produces the simultaneous pdf described by:

$$\begin{aligned} g(p) &= f_x(p\,|\,0.0,\,0.0,\,0.5) \times f_y(p\,|\,0.0,\,0.0,\,0.5) \\ &= -\ln(p)\ . \end{aligned} \tag{24}$$

For comparison both $f_{x\wedge y}(p)$ and $g(p)$ are plotted in Fig. 13.

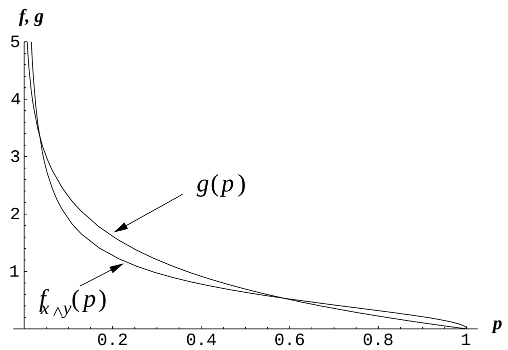

Fig. 13. Comparison between simultaneous uniform pdfs and conjunction of uniform ppdfs

From the expressions of $f_{x\wedge y}(p)$ and $g(p)$ it is easy to prove that $\mathrm{E}(f_{x\wedge y}(p)) = \mathrm{E}(g(p)) = 0.25$ although the curves are slightly different.

We have defined a ppdf to be a pdf over a probability variable. Intuitively one should think that simultaneous pdfs over a probability variable is equivalent with the propositional conjunction of ppdfs, and the difference seen in Fig.13 needs an explanation. Without going into detail possible reasons can for example be:

- The expression for ppdfs might only be an approximation in case the frame of discernment is larger than binary.

- The propositional conjunction operator could be imperfect.
- The computation of simultaneous pdfs could be imperfect.
- The mapping defined in Def.12 could be imperfect.

An investigation into these possibilities must be the subject future research. It should be noted that the greater the uncertainty the greater the difference between simultaneous pdfs and propositional conjunction of ppdfs becomes, so that the example in Fig.13 actually illustrates the biggest difference possible. As such the propositional conjunction operator provides at least a good approximation of simultaneous beta pdfs.

5. Evidential Operators

The propositional conjunction, disjunction and negation operators described in Sec.3 represent traditional logical operators. In this section two non-traditional operators are described, namely discounting and consensus of opinions.

5.1. *Discounting*

Assume two agents A and B where A has an opinion about B in the form of the proposition: '*B is knowledgeable and will tell the truth*'. In addition B has an opinion about a proposition x. Agent A can then form an opinion about x by discounting B's opinion about x with A's opinion about B. There is no such thing as physical belief discounting, and discounting of opinions therefore lends itself to different interpretations. The main difficulty lies with describing the effect of A disbelieving that B will give a good advice. This we will interpret as if A thinks that B is uncertain about the truth value of x so that A also is uncertain about the truth value of x no matter what B's actual advice is. Our next definition captures this idea.

Definition 14 (Discounting)
Let A and B be two agents where $\omega_B^A = (b_B^A, d_B^A, u_B^A, a_B^A)$ is A's opinion about B's advice, and let x be a proposition where $\omega_x^B = (b_x^B, d_x^B, u_x^B, a_x^B)$ is B's opinion about x expressed in an advice to A. Let $\omega_x^{AB} = (b_x^{AB}, d_x^{AB}, u_x^{AB}, a_x^{AB})$ be the opinion such that:

$$\begin{array}{ll} \mathbf{1}. & b_x^{AB} = b_B^A b_x^B, \\ \mathbf{2}. & d_x^{AB} = b_B^A d_x^B \\ \mathbf{3}. & u_x^{AB} = d_B^A + u_B^A + b_B^A u_x^B \\ \mathbf{4}. & a_x^{AB} = a_x^B \end{array}$$

then ω_x^{AB} is called the discounting of ω_x^B by ω_B^A expressing A's opinion about x as a result of B's advice to A. By using the symbol '$\otimes$' to designate this operator, we define $\omega_x^{AB} \equiv \omega_B^A \otimes \omega_x^B$.

The discounting function defined by Shafer[4] uses a *discounting rate* that can be denote by c, where the belief mass on each state in 2^Θ except the belief mass on Θ itself is multiplied by $(1 - c)$. By setting $(1 - c) = b_B^A$ our definition becomes equivalent to Shafer's definition. In our earlier publications(e.g. Jøsang 1997[3] and 1999[11]) the discounting operator was described as the recommendation operator, meaning exactly the same thing.

It is easy to prove that $\otimes$ is associative but not commutative. This means that in case of a chain of recommendations the discounting of opinions can start in either end of the chain, but that the order of opinions is significant. In a chain with more than one advisor, opinion independence must be assumed, which for example translates into not allowing the same entity to appear more than once.

5.2. *Consensus*

The consensus opinion of two opinions is an opinion that reflects both opinions in a fair and equal way. For example if two agents have observed a machine over two different time intervals they might have different opinions about it depending on the behaviour of the machine in the respective periods. The consensus opinion must then be the opinion that a single agent would have after having observed the machine during both periods.

Theorem 7 (Consensus)
Let $\omega_x^A = (b_x^A, d_x^A, u_x^A, a_x^A)$ and $\omega_x^B = (b_x^B, d_x^B, u_x^B, a_x^B)$ be opinions respectively held by agents A and B about the same proposition x. Let $\omega_x^{A,B} = (b_x^{A,B}, d_x^{A,B}, u_x^{A,B}, a_x^{A,B})$ be the opinion such that

$$\begin{array}{ll} \mathbf{1}. & b_x^{A,B} = (b_x^A u_x^B + b_x^B u_x^A)/\kappa \\ \mathbf{2}. & d_x^{A,B} = (d_x^A u_x^B + d_x^B u_x^A)/\kappa \\ \mathbf{3}. & u_x^{A,B} = (u_x^A u_x^B)/\kappa \\ \mathbf{4}. & a_x^{A,B} = \frac{a_x^B u_x^A + a_x^A u_x^B - (a_x^A + a_x^B) u_x^A u_x^B}{u_x^A + u_x^B - 2 u_x^A u_x^B} \end{array}$$

where $\kappa = u_x^A + u_x^B - u_x^A u_x^B$ such that $\kappa \neq 0$, and $a_x^{A,B} = (a_x^A + a_x^B)/2$ when $u_x^A, u_x^B = 1$. Then $\omega_x^{A,B}$ is called the consensus between ω_x^A and ω_x^B, representing an imaginary agent $[A, B]$'s opinion about x, as if she represented both A and B. By using the symbol '$\oplus$' to designate this operator, we define $\omega_x^{A,B} \equiv \omega_x^A \oplus \omega_x^B$.

Proof 7
The consensus operator for opinions is obtained by mapping the operator for combined evidence from Def.13 onto the opinion space using Def.12.

□

It is easy to prove that $\oplus$ is both commutative and associative which means that the order in which opinions are combined has no importance. Opinion independence must be assumed, which obviously translates into not allowing an agent's opinion to be counted more than once

The effect of the consensus operator is to reduce the uncertainty. For example the case where several witnesses give consistent testimony should amplify the judge's opinion, and that is exactly what the operator does. Consensus between an infinite number of independent non-dogmatic opinions would necessarily produce a consensus opinion with zero uncertainty.

Two dogmatic opinions can not be combined according to Theorem 7. This can be explained by interpreting uncertainty as *room for influence*, meaning that it is only possible to reach consensus with somebody who maintains some uncertainty. A situation with conflicting dogmatic opinions is philosophically counterintuitive, primarily because opinions about real situations can never be absolutely certain, and secondly, because if they were they would necessarily be equal. The consensus of two absolutely uncertain opinions results in a new absolutely uncertain opinion, although the relative atomicity is not well defined. The limit of the relative atomicity when both $u_x^A, u_x^B \to 1$ is $(a_x^A + a_x^B)/2$, i.e. the average of the two relative atomicities, which intuitively makes sense.

The consensus operator will normally be used in combination with the discounting operator, so that if dogmatic opinions are advised, the recipient should not have absolute trust in the advisor and thereby introduce uncertainty before combining the advice by the consensus operator, as illustrated in Example E below.

The consensus operator has the same purpose as Dempster's rule[4], but is quite different from it. Dempster's rule has been criticised for producing counterintuitive results (see e.g. Zadeh 1984[18] and Cohen 1986[19]), and in Sec.5.4 we will compare our consensus operator with Dempster's rule.

5.3. *Example E: Assessment of Testimony from Witnesses*

Imagine a court case where three witnesses W_1, W_2 and W_3 are giving testimony to express their opinions about a verbal proposition x which has been made about the accused. Assume that the verbal proposition is either true or false, and let each witness express his or her opinion about the truth of the proposition as an opinion ω_x^W, to the courtroom. The judge J then has to determine his or her own opinion about x as a function of her trust ω_W^J in the proposition: *'Witness W is reliable and will tell the truth'* in each individual witness. This situation is illustrated in Fig.14 where the arrows denote trust or opinions about truth.

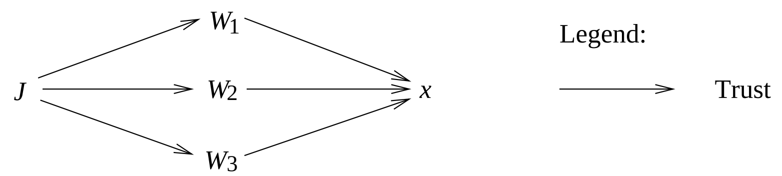

Fig. 14. Trust in testimony from witnesses

The effect of each individual testimony on the judge can be computed using the discounting operator, so that for example W_1's belief in x is discounted by the judge's trust in W_1. This causes the judge to have the opinion:

$$\omega_x^{JW_1} = \omega_{W_1}^J \otimes \omega_x^{W_1}$$

about the truth of x as a result of the testimony from W_1. Assuming that the opinions resulting from each witness are independent, they can finally be combined using the consensus operator to produce the judge's own opinion about x:

$$\omega_x^{J(W_1,W_2,W_3)} = (\omega_{W_1}^J \otimes \omega_x^{W_1}) \oplus (\omega_{W_2}^J \otimes \omega_x^{W_2}) \oplus (\omega_{W_3}^J \otimes \omega_x^{W_3}) \,. \tag{25}$$

As a numerical example, let J's opinion about the witnesses, and the witnesses' opinions about the truth of proposition x be given by:

$$\omega^J_{W_1} = (0.90, 0.00, 0.10, 0.50) \qquad \omega^{W_1}_x = (0.90, 0.00, 0.10, 0.50)$$

$$\omega^J_{W_2} = (0.00, 0.90, 0.10, 0.50) \qquad \omega^{W_2}_x = (0.90, 0.00, 0.10, 0.50)$$

$$\omega^J_{W_3} = (0.10, 0.00, 0.90, 0.50) \qquad \omega^{W_3}_x = (0.90, 0.00, 0.10, 0.50)$$

It can be seen that the judge has a high degree of trust in W_1, that she distrusts W_2, and that her opinion about W_3 is highly uncertain.

The judge's separate opinions about the proposition x as a function of the advice from each witness then become:

$$\omega^{JW_1}_x = (0.81,\ 0.00,\ 0.19,\ 0.50)$$

$$\omega^{JW_2}_x = (0.00,\ 0.00,\ 1.00,\ 0.50)$$

$$\omega^{JW_3}_x = (0.09,\ 0.00,\ 0.91,\ 0.50)$$

It can be seen that $\omega^{JW_2}_x$ is totally uncertain due to the fact that the judge distrust W_2, and that $\omega^{JW_3}_x$ is highly uncertain because the judge is very uncertain about testimonies from W_3. Only $\omega^{JW_1}_x$ represents an opinion that can be useful for making a decision. By combining all three independent opinions into one the judge gets:

$$\omega^{J(W_1, W_2, W_3)}_x \quad = (0.8135, 0.0000, 0.1865, 0.5000)$$

It can be seen that the combined opinion is mainly based on the advice from W_1.

5.4. *Comparing the consensus operator with Dempster's rule*

In this section we will compare our consensus operator with the original Dempster's rule and with Smets' non-normalised version of Dempster's rule.

We start with the well known example that Zadeh 1984[18] used for the purpose of criticising Dempster's rule. Smets 1988[20] used the same example in defence of the non-normalised version of Dempster's rule.

Suppose that we have a murder case with three suspects; Peter, Paul and Mary and two witnesses W_1 and W_2 who give highly conflicting testimonies. The beliefs of the two witnesses can be combined using Dempster's rule and the non-normalised Dempster's rule defined below.

Definition 15 *Let Θ be a frame of discernment, and let m^A_Θ and m^B_Θ be BMAs on Θ. Then $m^A_\Theta \odot m^B_\Theta$ is a function $m^A_\Theta \odot m^B_\Theta : \ 2^\Theta \longmapsto [0,1]$ such that:*

1. $m^A_\Theta \odot m^B_\Theta(\emptyset) = 0,$ *and*
2. $m^A_\Theta \odot m^B_\Theta(x) = \frac{\sum_{y\cap z=x} m^A_\Theta(y)\cdot m^B_\Theta(z)}{1-\kappa}$ *for all* $x \neq \emptyset$

where $\kappa = \sum_{y\cap z=\emptyset} m^A_\Theta(y)\cdot m^B_\Theta(z)$ in Dempster's rule, and where $\kappa = 0$ in the non-normalised version.

Table 3 gives the belief masses of Zadeh's example and the resulting belief masses after applying Dempster's rule and its non-normalised version.

Table 3. Comparison of operators in Zadeh's example

	Witness 1	Witness 2	Dempster's rule	Non-normalised rule
Peter	0.99	0.00	0.00	0.00
Paul	0.01	0.01	1.00	0.0001
Mary	0.00	0.99	0.00	0.00
Θ	0.00	0.00	0.00	0.00

Dempster's rule selects the least suspected by both witnesses as the guilty. The non-normalised version acquits all the suspects and indicates that the guilty has to be someone else. This is explained by Smets 1988[20] with the so-called open world interpretation of the frame of discernment which says that there can be unknown possible states outside the known frame of discernment.

Although both Dempster's rule and the non-normalised version seem to give very counterintuitive results the main problem in this example is the witnesses' dogmatic BMAs which is philosophically meaningless, and no operator can be expected to give a meaningful answer in such cases.

Because the BMAs of W_1 and W_2 are dogmatic our consensus operator can not be applied to this example. The consensus operator requires operands with a non-zero uncertainty component. We will therefore introduce uncertainty by allocating some belief to the state $\Theta = \{\text{Peter, Paul, Mary}\}$. Table 4 gives the modified BMAs and the results of applying the rules.

Table 4. Comparison of operators after introducing uncertainty in Zadeh's example

	Witness 1	Witness 2	Dempster's rule	Non-normalised rule	Consensus operator
Peter	0.98	0.00	0.490	0.0098	0.492
Paul	0.01	0.01	0.015	0.0003	0.010
Mary	0.00	0.98	0.490	0.0098	0.492
Θ	0.01	0.01	0.005	0.0001	0.005

The column for the consensus operator is obtained by displaying the 'belief' coordinate from the consensus opinions. The consensus opinion values and their corresponding probability expectation values are:

$$\omega_{x_1}^{W_1,W_2} = (0.492,\ 0.503,\ 0.005,\ 1/3)\ , \qquad \mathrm{E}(\omega_{x_1}^{W_1,W_2}) = 0.494$$
$$\omega_{x_2}^{W_1,W_2} = (0.010,\ 0.985,\ 0.005,\ 1/3)\ , \qquad \mathrm{E}(\omega_{x_2}^{W_1,W_2}) = 0.012$$
$$\omega_{x_3}^{W_1,W_2} = (0.492,\ 0.503,\ 0.005,\ 1/3)\ , \qquad \mathrm{E}(\omega_{x_3}^{W_1,W_2}) = 0.494\ .$$

When uncertainty is introduced Dempster's rule corresponds well with intuitive human judgement. The non-normalised Dempster's rule however still indicates that none of the suspects are guilty and that new suspects must be found. Our consensus operator corresponds well with human judgement and gives almost the same result as Dempster's rule.

The belief masses resulting from Dempster's rule in Table 4 add up to 1. The 'belief' parameters of the consensus opinions do not add up to 1 because they are actually taken from 3 different focused frames of discernment, but the following holds:

$$\mathrm{E}(\omega_{x_1}^{W_1,W_2}) + \mathrm{E}(\omega_{x_2}^{W_1,W_2}) + \mathrm{E}(\omega_{x_3}^{W_1,W_2}) = 1 \ .$$

The above example indicates that Dempster's rule and the consensus operator give similar results. However, this is not always the case as illustrated by the following example. Let the two agents A and B have equal beliefs about the truth of a binary proposition x. The agents' BMAs and the results of applying the rules are give in Table 5.

Table 5. Comparison of operators i.c.o. equal beliefs

	Witness 1	Witness 2	Dempster's rule	Non-normalised rule	Consensus operator
x	0.90	0.90	0.99	0.99	0.947
$\neg x$	0.00	0.00	0.00	0.00	0.000
Θ	0.10	0.10	0.01	0.01	0.053

The consensus opinion about x and the corresponding probability expectation value are:

$$\omega_x^{A,B} = \ (0.947,\, 0.000,\, 0.053,\ 0.500)\ , \qquad \mathrm{E}(\omega_x^{A,B}) = 0.974\ .$$

Dempster's rule and the non-normalised version give the same result because the witnesses' BMAs are non-conflicting. It is interesting to notice that Dempster's rule amplifies the combined belief twice as much as the consensus operator. If belief is to be interpreted as resulting from evidence according to the mapping between the evidence space and the opinion space described in Sec.4.2, then the consensus operator seems to give the most correct result.

6. Uncertainty Maximisation

Although an opinion represents an uncertain probability its formal representation requires crisp values in the form of the four parameters b, d, u and a. The main problem when applying subjective logic and other uncertainty calculi is to determine input operand values from real world observations.

A distinction can be made between events that can be repeated many times and events that can only happen once. Frequentist probabilities with high certainty are meaningful in the first case but less so in the latter. For example assigning 0.5 belief mass to x: *'Oswald killed Kennedy'* and 0.5 belief mass to $\neg x$: *'Oswald did not kill Kennedy'* really shows that the observer is totally uncertain and therefore should have assigned 1.0 belief mass to Θ instead.

In Zadeh's example described in Sec.5.4 the actual event of having committed a particular murder must be considered as one that can only happen once, and the witnesses' dogmatic opinions are therefore partly meaningless. In such cases belief masses assigned to an event and its negation outweigh each other and should be transformed into uncertainty while keeping the probability expectation value unchanged. This idea is captured by the following definition.

Definition 16 (Uncertainty Maximisation) *Let $\omega_x = (b_x, d_x, u_x, a_x)$ be an opinion about a binary event that can only happen once. If belief mass is assigned to b_x and d_x simultaneously then uncertainty maximisation of ω_x consists of transforming it into the opinion $\widehat{\omega}_x = (\hat{b}_x, \hat{d}_x, \hat{u}_x, \hat{a}_x)$ defined by:*

$$\begin{array}{ll} \mathbf{1}. & \hat{b}_x = 0 \\ \mathbf{2}. & \hat{d}_x = 1 - u_x - b_x/a_x \\ \mathbf{3}. & \hat{u}_x = u_x + b_x/a_x \\ \mathbf{4}. & \hat{a}_x = a_x \end{array} \quad \textit{and} \quad \begin{array}{ll} \mathbf{1}. & \hat{b}_x = 1 - u_x - d_x/(1 - a_x) \\ \mathbf{2}. & \hat{d}_x = 0 \\ \mathbf{3}. & \hat{u}_x = u_x + d_x/(1 - a_x) \\ \mathbf{4}. & \hat{a}_x = a_x \end{array} \tag{26}$$

when $\mathrm{E}(\omega_x) \leq a_x$ *when* $\mathrm{E}(\omega_x) > a_x$.

Assigning belief mass simultaneously to an event and its negation in case of events that can only happen once is intuitively meaningless and can be avoided by uncertainty maximisation. This translates into not allowing opinions where $b \neq 0$ and $d \neq 0$ simultaneously. As a consequence only opinions situated on the outer left or right edge of the opinion triangle in Fig.4 are allowed. The only acceptable dogmatic opinions would then be the absolute opinions $(1, 0, 0, a)$ and $(0, 1, 0, a)$ which correspond to 'TRUE' and 'FALSE' propositions in binary logic. The purpose of uncertainty maximisation is to help observers realise their ignorance regarding the probability of events that can only happen once. Maximising the uncertainty in W_1's opinions from Zadeh's example in Table 3 would produce:

Table 6. Original and uncertainty maximised opinions of W_1 in Zadeh's example

	Original dogmatic opinions	Uncertainty maximised opinions
Peter	$(0.99,\ 0.01,\ 0.00,\ 1/3)$	$(0.985,\ 0.000,\ 0.015,\ 1/3)$
Paul	$(0.01,\ 0.99,\ 0.00,\ 1/3)$	$(0.000,\ 0.970,\ 0.030,\ 1/3)$
Mary	$(0.00,\ 1.00,\ 0.00,\ 1/3)$	$(0.000,\ 1.000,\ 0.000,\ 1/3)$

The uncertainty maximised opinions regarding Peter and Paul are now more meaningful. The opinion regarding Mary however remains unchanged because it is not only dogmatic but also absolute. Witnesses are likely to give absolute opinions when they feel very sure about something, and this indicates that the judge's or the jury's opinions about witnesses should always be included by using the discounting operator, and thereby introduce uncertainty, when combining testimonies of this type.

7. Conclusion

Uncertainty comes in many flavours. The opinion metric described here provides a new interpretation of the Shaferian belief model and allows secondary uncertainty about traditional frequentist probabilities to be expressed. Instead of interpreting the Shaferian belief functions as probability bounds we use them to represent uncertainty about probabilities and to estimate probability expectation values.

By applying standard and non-standard logical operators on the opinion metric a simple and powerful framework for artificial reasoning emerges which we have called subjective logic. We have for example shown that the propositional conjunction and disjunction operators for opinions are compatible with product and co-product of probabilities as well as with 'AND' and 'OR' of binary logic. This makes subjective logic very general and it is our belief that it can be successfully applied in a multitude of applications.

Acknowledgements

The work reported in this paper has been funded in part by the Co-operative Research Centre for Enterprise Distributed Systems Technology (DSTC) through the Australian Federal Government's CRC Programme (Department of Industry, Science & Resources).

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