Conditional Reasoning with Subjective Logic*

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Conditional inference plays a central role in logical and Bayesian reasoning, and is used in a wide range of applications. It basically consists of expressing conditional relationship between parent and child propositions, and then to combine those conditionals with evidence about the parent propositions in order to infer conclusions about the child propositions. While conditional reasoning is a well established part of classical binary logic and probability calculus, its extension to belief theory has only recently been proposed. Subjective opinions represent a special type of general belief functions. This article focuses on conditional reasoning in subjective logic where beliefs are represented in the form of binomial or multinomial subjective opinions. Binomial conditional reasoning operators for subjective logic have been defined in previous contributions. We extend this approach to multinomial opinions, thereby making it possible to represent conditional and evidence opinions on frames of arbitrary size. This makes subjective logic a powerful tool for conditional reasoning in situations involving ignorance and partial information, and makes it possible to analyse Bayesian network models with uncertain probabilities.

Key words: Subjective logic, Conditional, Deduction, Abduction, Belief theory, Bayesian networks

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1 INTRODUCTION

Conditionals are propositions like *"If we don't hurry we'll be late for the show"* or *"If it rains, Michael will carry an umbrella"* which are of the form "IF x THEN y" where x denotes the antecedent and y the consequent proposition. The truth value of conditionals can be evaluated in different ways, e.g. as binary TRUE or FALSE, as a probability measure or as an opinion. Conditionals are complex propositions because they contain an antecedent and a consequent that are also propositions with truth values that can be evaluated in the same way. Conditionals can be linked e.g. by letting a conditional proposition be the antecedent of another conditional proposition.

The idea of having a conditional connection between an antecedent and a consequent proposition can be traced back to Ramsey [21] who articulated what has become known as Ramsey's Test: *To decide whether you believe a conditional, provisionally or hypothetically add the antecedent to your stock of beliefs, and consider whether to believe the consequent*. This idea was translated into a formal language by Stalnaker [27] in the form of the so-called Stalnaker's Hypothesis, formally expressed as: $p(\text{IF } x \text{ THEN } y) = p(y|x)$. The interpretation of Stalnaker's Hypothesis is that the probability of the conditional proposition "IF x THEN y" is equal to the probability of the proposition y given that the proposition x is TRUE. A more precise expression of Stalnaker's hypothesis is therefore $p(\text{IF } x \text{ THEN } y) = p(y|(p(x) = 1))$, but the bulkiness of this notation would make it impractical.

An alternative viewpoint to that of Stalnaker was put forward by Lewis [18] who argued that conditional propositions do not have truth-values and that they do not express propositions. This would mean that for any propositions x and y, there is no proposition z for which $p(z) = p(y|x)$, so the conditional probability can not be the same as the probability of conditionals.

In our opinion Stalnaker's Hypothesis is sound and applicable for conditional reasoning. We would argue against Lewis' view by simply saying that it is meaningful to assign a probability to a conditional proposition like "$y|x$", which is defined in case x is true, and undefined in case x is false.

Meaningful conditional deduction requires relevance between antecedent and consequent, i.e. that the consequent depends on the antecedent. Conditionals that are based on the dependence between consequent and antecedent are universally valid, and are called *logical conditionals* [3]. Deduction with logical conditionals reflect human intuitive conditional reasoning.

Both binary logic and probability calculus have mechanisms for conditional reasoning. In binary logic, Modus Ponens (MP) and Modus Tollens

(MT) are the classical operators which are used in any field of logic that requires conditional deduction. In probability calculus, binomial conditional deduction is expressed as:

$$p(y\|x) = p(x)p(y|x) + p(\overline{x})p(y|\overline{x}) \quad (1)$$

where the terms are interpreted as follows:

$p(y\|x)$	:	the conditional probability of y given x is TRUE
$p(y\|\overline{x})$	:	the conditional probability of y given x is FALSE
$p(x)$	:	the probability of the antecedent x
$p(\overline{x})$	:	the probability of the antecedent's complement ($= 1 - p(x)$)
$p(y\\|x)$	:	the deduced probability of the consequent y

The notation $y\|x$, introduced in [15], denotes that the truth or probability of proposition y is deduced as a function of the probability of the antecedent x together with the conditionals. The expression $p(y\|x)$ thus represents a derived value, whereas the expressions $p(y|x)$ and $p(y|\overline{x})$ represent input values together with $p(x)$. Below, this notational convention will also be used for opinions in subjective logic.

This article describes how the same principles for conditional inference outlined above can be formulated in subjective logic. The advantage of this approach is that conditional reasoning models can be analysed with subjective opinions as input and output values, i.e. in the presence of uncertainty and partial ignorance. This will also allow the analyst to appreciate the relative proportions of firm evidence and uncertainty as contributing factors to the derived probabilistic likelihoods.

This article is structured as follows. Section 2 reviews probabilistic conditional reasoning in order to provide a benchmark for subjective logic described later. Section 3 reviews the belief representation used in classical Dempster-Shafer belief theory as a background for subjective opinions. Section 4 provides a brief review of previous approaches to conditional belief reasoning. Section 5 describes subjective opinions which are used as arguments in subjective logic. Section 6 describes conditional deduction and abduction in subjective logic, and Section 7 describes how Bayesian networks can be based on subjective logic. Section 8 suggests application domains of conditional reasoning with subjective logic, and concludes the presentation.

2 PROBABILISTIC CONDITIONAL REASONING

Classical results from probabilistic conditional reasoning are briefly reviewed below in order to provide a benchmark for conditional reasoning with subjective logic, described in Sec.6.

2.1 Binomial Conditional Reasoning

Probabilistic conditional reasoning is used extensively in areas where conclusions need to be derived from probabilistic input evidence, such as for making diagnoses from medical tests. A pharmaceutical company that develops a test for a particular infection disease will typically determine the reliability of the test by letting a group of infected and a group of non-infected people undergo the test. The result of these trials will then determine the reliability of the test in terms of its *sensitivity* $p(x|y)$ and *false positive rate* $p(x|\overline{y})$, where x: *"Positive Test"*, y: *"Infected"* and $\overline{y}$: *"Not infected"*. The conditionals are interpreted as:

- $p(x|y)$: *"The probability of positive test given infection"*
- $p(x|\overline{y})$: *"The probability of positive test in the absence of infection"*.

The problem with applying these reliability measures in a practical setting is that the conditionals are expressed in the opposite direction to what the practitioner needs in order to apply the expression of Eq.(1). The conditionals needed for making the diagnosis are:

- $p(y|x)$: *"The probability of infection given positive test"*
- $p(y|\overline{x})$: *"The probability of infection given negative test"*

but these are usually not directly available to the medical practitioner. However, they can be obtained if the base rate of the infection is known.

The base rate fallacy [17] in medicine consists of making the erroneous assumption that $p(y|x) = p(x|y)$. While this reasoning error often can produce a relatively good approximation of the correct diagnostic probability value, it can lead to a completely wrong result and wrong diagnosis in case the base rate of the disease in the population is very low and the reliability of the test is not perfect. The required conditionals can be correctly derived by inverting the available conditionals using Bayes rule. The inverted conditionals are obtained as follows:

$$\left\{ \begin{array}{l} p(x|y) = \frac{p(x \wedge y)}{p(y)} \\ \\ p(y|x) = \frac{p(x \wedge y)}{p(x)} \end{array} \right. \quad \Rightarrow \quad p(y|x) = \frac{p(y)p(x|y)}{p(x)}\ . \tag{2}$$

On the right hand side of Eq.(2) the base rate of the disease in the population is expressed by $p(y)$. By applying Eq.(1) with x and y swapped in every term, the expected rate of positive tests $p(x)$ in Eq.(2) can be computed as a function of the base rate $p(y)$. In the following, $a(x)$ and $a(y)$ will denote the base rates of x and y respectively. The required conditional is:

$$p(y|x) = \frac{a(y)p(x|y)}{a(y)p(x|y) + a(\overline{y})p(x|\overline{y})} \ . \tag{3}$$

A medical test result is typically considered positive or negative, so when applying Eq.(1) it can be assumed that either $p(x)$ = 1 (positive) or $p(\overline{x})$ = 1 (negative). In case the patient tests positive, Eq.(1) can be simplified to $p(y\|x) = p(y|x)$ so that Eq.(3) will give the correct likelihood that the patient actually has contracted the disease.

2.2 Example 1: Probabilistic Medical Reasoning

Let the sensitivity of a medical test be expressed as $p(x|y) = 0.9999$ (i.e. an infected person will test positive in 99.99% of the cases) and the false positive rate be $p(x|\overline{y}) = 0.001$ (i.e. a non-infected person will test positive in 0.1% of the cases). Let the base rate of infection in population A be 1% (expressed as $a(y_A)$=0.01) and let the base rate of infection in population B be 0.01% (expressed as $a(y_B)$=0.0001). Assume that a person from population A tests positive, then Eq.(3) and Eq.(1) lead to the conclusion that $p(y_A\|x) = p(y_A|x) = 0.9099$ which indicates a 91% likelihood that the person is infected. Assume that a person from population B tests positive, then $p(y_B\|x) = p(y_B|x) = 0.0909$ which indicates only a 9% likelihood that the person is infected. By applying the correct method the base rate fallacy is avoided in this example.

2.3 Deductive and Abductive Reasoning

In the general case where the truth of the antecedent is expressed as a probability, and not just binary TRUE and FALSE, the opposite conditional is also needed as specified in Eq.(1). In case the negative conditional is not directly available, it can be derived according to Eq.(3) by swapping x and $\overline{x}$ in every term. This produces:

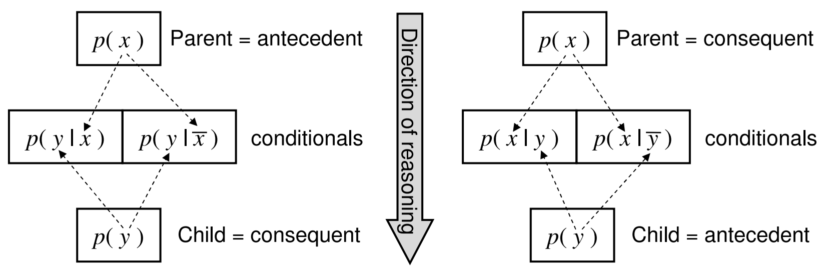
$$\begin{aligned} p(y|\overline{x}) &= \frac{a(y)p(\overline{x}|y)}{a(y)p(\overline{x}|y)+a(\overline{y})p(\overline{x}|\overline{y})} \\ &= \frac{a(y)(1-p(x|y))}{a(y)(1-p(x|y))+a(\overline{y})(1-p(x|\overline{y}))} \ . \end{aligned} \tag{4}$$

Eq.(3) and Eq.(4) enables conditional reasoning even when the required conditionals are expressed in the reverse direction to what is needed.

The term *frame*⋆ will be used with the meaning of a traditional state space of mutually disjoint states. We will use the term *"parent frame"* and *"child frame"* to denote the reasoning direction, meaning that the parent frame is what the analyst has evidence about, and probabilities over the child frame is what the analyst needs. Defining parent and child frames is thus equivalent with defining the direction of the reasoning.

Forward conditional inference, called *deduction*, is when the parent frame is the antecedent and the child frame is the consequent of the available conditionals. Reverse conditional inference, called *abduction*, is when the parent frame is the consequent, and the child frame is the antecedent.

Deductive and abductive reasoning situations are illustrated in Fig.1 where x denotes a state in the parent frame and y denotes a state in the child frame. Conditionals are expressed as $p(\text{consequent}\,|\text{antecedent})$.



(a) Deduction (b) Abduction

FIGURE 1
Visualising deduction and abduction

The concepts of *"causal"* and *"derivative"* reasoning can be meaningful for clearly causal conditional relationships. By assuming that the antecedent causes the consequent, then causal reasoning is equivalent to deductive reasoning, and derivative reasoning is equivalent to abductive reasoning.

In medical reasoning for example, the infection causes the test to be positive, not the other way. The reliability of medical tests is expressed as causal conditionals, whereas the practitioner needs to apply the derivative inverted conditionals. Starting from a positive test to conclude that the patient is infected therefore represents derivative reasoning. Most people have a tendency to reason in a causal manner even in situations where derivative reasoning is

⋆ Usually called *frame of discernment* in traditional belief theory required. In other words, derivative situations are often confused with causal situations, which provides an explanation for the tendency of the base rate fallacy in medical diagnostics. In legal reasoning, the same type of reasoning error is called *the prosecutor's fallacy*.

2.4 Multinomial Conditional Reasoning

So far in this presentation the parent and child frames have consisted of binary sets $\{x, \overline{x}\}$ and $\{y, \overline{y}\}$. In general, both the parent and child frames in a conditional reasoning situation can consist of an arbitrary number of disjoint states. Let $X = \{x_i | i = 1 \ldots k\}$ be the parent frame with cardinality k, and let $Y = \{y_j | j = 1 \ldots l\}$ be the child frame with cardinality l. The deductive conditional relationship between X and Y is then expressed with k vector conditionals $p(Y|x_i)$, each being of l dimensions. This is illustrated in Fig.2.

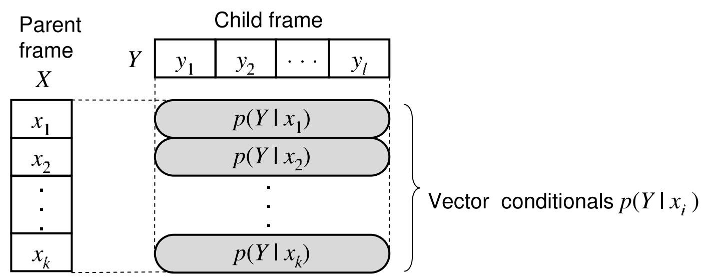


FIGURE 2
Multinomial deductive vector conditionals between parent X and child Y

The vector conditional $\vec{p}(Y|x_i)$ relates each state x_i to the frame Y. The elements of $\vec{p}(Y|x_i)$ are the scalar conditionals expressed as:

$$p(y_j|x_i), \qquad \text{where } \sum_{j=1}^{l} p(y_j|x_i) = 1 \ . \tag{5}$$

The probabilistic expression for multinomial conditional deduction from X to Y, generalising that of Eq.(1), is the vector $p(Y\|X)$ over Y where each scalar vector element $p(y_j\|X)$ is:

$$p(y_j\|X) = \sum_{i=1}^{k} p(x_i)p(y_j|x_i) \ . \tag{6}$$

The multinomial probabilistic expression for inverting conditionals, generalising that of Eq.(3), becomes:

$$p(y_j|x_i) = \frac{a(y_j)p(x_i|y_j)}{\sum_{t=1}^{l} a(y_t)p(x_i|y_t)} \tag{7}$$

where $a(y_j)$ represents the base rate of y_j.

By substituting the conditionals of Eq.(6) with inverted multinomial conditionals from Eq.(7), the general expression for probabilistic abduction emerges:

$$p(y_j \overline{\|} X) = \sum_{i=1}^{k} p(x_i) \left(\frac{a(y_j)p(x_i|y_j)}{\sum_{t=1}^{l} a(y_t)p(x_i|y_t)} \right) . \tag{8}$$

This will be illustrated by a numerical example below.

2.5 Example 2: Probabilistic Intelligence Analysis

Two countries A and B are in conflict, and intelligence analysts of country B want to find out whether country A intends to use military aggression. The analysts of country B consider the following possible alternatives regarding country A's plans:

$$\begin{aligned} y_1 &: \text{ No military aggression from country } A \\ y_2 &: \text{ Minor military operations by country } A \\ y_3 &: \text{ Full invasion of country } B \text{ by country } A \end{aligned} \tag{9}$$

The way the analysts will determine the most likely plan of country A is by trying to observe movement of troops in country A. For this, they have spies placed inside country A. The analysts of country B consider the following possible movements of troops.

$$\begin{aligned} x_1 &: \text{ No movement of country } A\text{'s troops} \\ x_2 &: \text{ Minor movements of country } A\text{'s troops} \\ x_3 &: \text{ Full mobilisation of all country } A\text{'s troops} \end{aligned} \tag{10}$$

The analysts have defined a set of conditional probabilities of troop movements as a function of military plans, as specified by Table 1.

The rationale behind the conditionals are as follows. In case country A has no plans of military aggression (y_1), then there is little logistic reason for troop movements. However, even without plans of military aggression against country B it is possible that country A expects military aggression from country B, forcing troop movements by country A. In case country A

Probability vectors	Troop movements x_1 No movemt.	x_2 Minor movemt.	x_3 Full mob.
$\vec{p}(X\|y_1)$:	$p(x_1\|y_1) = 0.50$	$p(x_2\|y_1) = 0.25$	$p(x_3\|y_1) = 0.25$
$\vec{p}(X\|y_2)$:	$p(x_1\|y_2) = 0.00$	$p(x_2\|y_2) = 0.50$	$p(x_3\|y_2) = 0.50$
$\vec{p}(X\|y_3)$:	$p(x_1\|y_3) = 0.00$	$p(x_2\|y_3) = 0.25$	$p(x_3\|y_3) = 0.75$

TABLE 1
Conditional probabilities $p(X|Y)$: troop movement x_i given military plan y_j

prepares for minor military operations against country B (y_2), then necessarily troop movements are required. In case country A prepares for full invasion of country B (y_3), then significant troop movements are required.

Based on observations by spies of country B, the analysts determine the likelihoods of actual troop movements to be:

$$p(x_1) = 0.00 \ , \qquad p(x_2) = 0.50 \ , \qquad p(x_3) = 0.50 \ . \tag{11}$$

The analysts are faced with an abductive reasoning situation and must first derive the conditionals $p(Y|X)$. The base rate of military plans is set to:

$$a(y_1) = 0.70 \ , \qquad a(y_2) = 0.20 \ , \qquad a(y_3) = 0.10 \ . \tag{12}$$

The expression of Eq.(7) can now be used to derive the required conditionals, which are given in Table 2 below.

Military plan	Probabilities of military plans given troop movement $\vec{p}(Y\|x_1)$ No movemt.	$\vec{p}(Y\|x_2)$ Minor movemt.	$\vec{p}(Y\|x_3)$ Full mob.
y_1: No aggr.	$p(y_1\|x_1) = 1.00$	$p(y_1\|x_2) = 0.58$	$p(y_1\|x_3) = 0.50$
y_2: Minor ops.	$p(y_2\|x_1) = 0.00$	$p(y_2\|x_2) = 0.34$	$p(y_2\|x_3) = 0.29$
y_3: Invasion	$p(y_3\|x_1) = 0.00$	$p(y_3\|x_2) = 0.08$	$p(y_3\|x_3) = 0.21$

TABLE 2
Conditional probabilities $p(Y|X)$: military plan y_j given troop movement x_i

The expression of Eq.(6) can then be used to derive the probabilities of military plans of country A, resulting in:

$$p(y_1\|X) = 0.54\ , \qquad p(y_2\|X) = 0.31\ , \qquad p(y_3\|X) = 0.15\ . \tag{13}$$

Based on the results of Eq.(13), it seems most likely that country A does not plan any military aggression against country B. Analysing the same example with subjective logic in Sec.6.4 will show that these results give a misleading estimate of country A's plans because they hide the underlying uncertainty.

3 BELIEF REPRESENTATIONS

Traditional probabilities are not suitable for expressing ignorance about the likelihoods of possible states or outcomes. If somebody wants to express ignorance as *"I don't know"* this would be impossible with a simple scalar probability value. A probability 0.5 would for example mean that the event will take place 50% of the time, which in fact is quite informative, and very different from ignorance. Alternatively, a uniform probability density function over all possible states would more closely express the situation of ignorance about the outcome of an event. Subjective opinions which can be interpreted as probability density functions, and which are related to belief functions, can be used to express this type of ignorance. As a background for subjective opinions, the theory of belief functions will be briefly described.

Belief theory represents an extension of classical probability by allowing explicit expression of ignorance. Belief theory has its origin in a model for upper and lower probabilities proposed by Dempster in 1960. Shafer later proposed a model for expressing beliefs [22]. The main idea behind belief theory is to abandon the additivity principle of probability theory, i.e. that the sum of probabilities on all pairwise disjoint states must add up to one. Instead belief theory gives observers the ability to assign so-called belief mass to any subset of the frame, i.e. to non-exclusive possibilities including the whole frame itself. The main advantage of this approach is that ignorance, i.e. the lack of information, can be explicitly expressed e.g. by assigning belief mass to the whole frame.

The term uncertainty can be used to express many different aspects of our perception of reality. In this article, it will be used in the sense of uncertainty about probability values. This is different from imprecise probabilities which are normally interpreted as a pair of upper and lower probability values. A philosophical problem with imprecise probabilities is described in Sec.4.3.

General belief functions allow complex belief structures to be expressed on arbitrarily large frames. Shafer's book [22] describes many aspects of belief theory, but the two main elements are 1) a flexible way of expressing beliefs, and 2) a conjunctive method for fusing beliefs, commonly known as Dempster's Rule. We will not be concerned with Dempster's rule here.

In order for this presentation to be self contained, central concepts from Dempster-Shafer theory of evidence [22] are recalled. Let $X = \{x_i, i = 1, \cdots, k\}$ denote a frame (of discernment) consisting of a finite set of exhaustive and disjoint possible values for a state variable of interest. Let further 2^X denote its powerset, i.e. the set of all possible subsets of X. The frame can for example be the set of six possible outcomes of throwing a dice, and the (unknown) outcome of a particular instance of throwing the dice becomes the state variable. A bba (basic belief assignment†), denoted by m is defined as a belief mass distribution function from 2^X to $[0, 1]$ satisfying:

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{x \subseteq X} m(x) = 1 \; . \tag{14}$$

Values of a bba are called *belief masses*. Each subset $x \subseteq X$ such that $m(x) > 0$ is called a focal element.

The probability expectation projection [4], also known as the pignistic transformation [25, 26], produces a probability expectation value, denoted by $\mathrm{E}(x)$, defined as:

$$\mathrm{E}(x) = \sum_{y \in 2^X} m(y) \, \frac{|x \cap y|}{|y|}, \quad x \in 2^X \; . \tag{15}$$

A few special bba classes are worth mentioning. A vacuous bba has $m(X) = 1$, i.e. no belief mass committed to any proper subset of X. A *Bayesian* bba is when all the focal elements are singletons, i.e. one-element subsets of X. If all the focal elements are nestable (i.e. linearly ordered by inclusion) then we speak about *consonant* bba. A *dogmatic* bba is defined by Smets [24] as a bba for which $m(X) = 0$. Let us note, that trivially, every Bayesian bba is dogmatic.

4 REVIEW OF BELIEF-BASED CONDITIONAL REASONING

In this section, previous approaches to conditional reasoning with beliefs and related frameworks are briefly reviewed.

† Called *basic probability assignment* in [22], and *Belief Mass Assignment* (BMA) in [8].

4.1 Smets' Disjunctive Rule and Generalised Bayes Theorem

An early attempt at articulating belief-based conditional reasoning was provided by Smets (1993) [23] and by Xu & Smets [31, 30]. This approach is based on using the so-called Generalised Bayes Theorem as well as the Disjunctive Rule of Combination, both of which are defined within the Dempster-Shafer belief theory.

In the binary case, Smets' approach assumes a conditional connection between a binary parent frame Θ and a binary child frame X defined in terms of belief masses and conditional plausibilities. In Smets' approach, binomial deduction is defined as:

$$\begin{aligned}
pl(x) &= m(\theta)pl(x|\theta)+m(\overline{\theta})pl(x|\overline{\theta})+m(\Theta)(1-(1-pl(x|\theta))(1-pl(x|\overline{\theta})))\\
pl(\overline{x}) &= m(\theta)pl(\overline{x}|\theta)+m(\overline{\theta})pl(\overline{x}|\overline{\theta})+m(\Theta)(1-(1-pl(\overline{x}|\theta))(1-pl(\overline{x}|\overline{\theta})))\\
pl(X) &= m(\theta)pl(X|\theta)+m(\overline{\theta})pl(X|\overline{\theta})+m(\Theta)(1-(1-pl(X|\theta))(1-pl(X|\overline{\theta})))
\end{aligned} \tag{16}$$

The next example illustrate a case where Smets' deduction operator produces inconsistent results. Let the conditional plausibilities be expressed as:

$$\Theta \longmapsto X: \quad \left| \begin{matrix} pl(x|\theta) = 1/4 & pl(\overline{x}|\theta) = 3/4 & pl(X|\theta) = 1 \\ pl(x|\overline{\theta}) = 1/4 & pl(\overline{x}|\overline{\theta}) = 3/4 & pl(X|\overline{\theta}) = 1 \end{matrix} \right| \tag{17}$$

Eq.(17) expresses that the plausibilities of x are totally independent of θ because $pl(x|\theta) = pl(x|\overline{\theta})$ and $pl(\overline{x}|\theta) = pl(\overline{x}|\overline{\theta})$. Let now two bbas, m_Θ^A and m_Θ^B on Θ be expressed as:

$$m_\Theta^A: \begin{cases} m_\Theta^A(\theta) & = 1/2 \\ m_\Theta^A(\overline{\theta}) & = 1/2 \\ m_\Theta^A(\Theta) & = 0 \end{cases} \qquad m_\Theta^B: \begin{cases} m_\Theta^B(\theta) & = 0 \\ m_\Theta^B(\overline{\theta}) & = 0 \\ m_\Theta^B(\Theta) & = 1 \end{cases} \tag{18}$$

This results in the following plausibilities pl, belief masses m_X and pignistic probabilities E on X in Table 3:

Because X is totally independent of Θ according to Eq.(17), the bba on X should not be influenced by the bbas on Θ. It can be seen from Table 3 that the probability expectation values E are equal for both bbas, which seems to indicate consistency. However, the belief masses are different, which shows that Smets' method [23] can produce inconsistent results. It can be mentioned that the framework of subjective logic described in Sec.6 does not have this problem.

State	Result of m^A_Θ on Θ			Result of m^B_Θ on Θ		
	pl	m_Θ	E	pl	m_Θ	E
x	1/4	1/4	1/4	7/16	1/16	1/4
$\overline{x}$	3/4	3/4	3/4	1/16	9/16	3/4
X	1	0	n.a.	1	6/16	n.a.

TABLE 3
Inconsistent results of deductive reasoning with Smets' method

In Smets' approach, binomial abduction is defined as:

$$\begin{aligned}
pl(\theta) = \quad & m(x)pl(x|\theta) + m(\overline{x})pl(\overline{x}|\theta) + m(X)(pl(X|\theta)))\ , \\
pl(\overline{\theta}) = \quad & m(x)pl(x|\overline{\theta}) + m(\overline{x})pl(\overline{x}|\overline{\theta}) + m(X)pl(X|\overline{\theta})))\ , \\
pl(\Theta) = \quad & \ \ m(x)(1-(1-pl(x|\theta))(1-pl(x|\overline{\theta}))) \\
& +m(\overline{x})(1-(1-pl(\overline{x}|\theta))(1-pl(\overline{x}|\overline{\theta}))) \\
& +m(X)(1-(1-pl(X|\theta))(1-pl(X|\overline{\theta})))\ .
\end{aligned} \tag{19}$$

Eq.(19) fails to take the base rates on Θ into account and would therefore unavoidably be subject to the base rate fallacy, which would also be inconsistent with probabilistic reasoning as e.g. described in Example 1 (Sec.2.2). It can be mentioned that abduction with subjective logic described in Sec.6 is always consistent with probabilistic abduction.

4.2 Halpern's Approach to Conditional Plausibilities

Halpern (2001) [5] analyses conditional plausibilities from an algebraic point of view, and concludes that conditional probabilities, conditional plausibilities and conditional possibilities share the same algebraic properties. Halpern's analysis does not provide any mathematical methods for practical conditional deduction or abduction.

4.3 Conditional Reasoning with Imprecise Probabilities

Imprecise probabilities are generally interpreted as probability intervals that contain the assumed real probability values. Imprecision is then an increasing function of the interval size [28]. Various conditional reasoning frameworks based on notions of imprecise probabilities have been proposed.

Credal networks introduced by Cozman [1] are based on credal sets, also called convex probability sets, with which upper and lower probabilities can be expressed. In this theory, a credal set is a set of probabilities with a defined upper and lower bound. There are various methods for deriving credal sets, e.g. [28]. Credal networks allow credal sets to be used as input in Bayesian networks. The analysis of credal networks is in general more complex than the analysis of traditional probabilistic Bayesian networks because it requires multiple analyses according to the possible probabilities in each credal set. Various algorithms can be used to make the analysis more efficient.

Weak non-monotonic probabilistic reasoning with conditional constraints proposed by Lukasiewicz [19] is also based on probabilistic conditionals expressed with upper and lower probability values. Various properties for conditional deduction are defined for weak non-monotonic probabilistic reasoning, and algorithms are described for determining whether conditional deduction properties are satisfied for a set of conditional constraints.

The surveyed literature on credal networks and non-monotonic probabilistic reasoning only describe methods for deductive reasoning, although abductive reasoning under these formalisms would theoretically be possible.

A philosophical concern with imprecise probabilities in general, and with conditional reasoning with imprecise probabilities in particular, is that there can be no real upper and lower bound to probabilities unless these bounds are set to the trivial interval $[0,1]$. This is because probabilities about real world propositions can never be absolutely certain, thereby leaving the possibility that the actual observed probability is outside the specified interval. For example, Walley's Imprecise Dirichlet Model (IDM) [29] is based on varying the base rate over all possible outcomes in the frame of a Dirichlet distribution. The probability expectation value of an outcome resulting from assigning the total base rate (i.e. equal to one) to that outcome produces the upper probability, and the probability expectation value of an outcome resulting from assigning a zero base rate to that outcome produces the lower probability. The upper and lower probabilities are then interpreted as the upper and lower bounds for the relative frequency of the outcome. While this is an interesting interpretation of the Dirichlet distribution, it can not be taken literally. According to this model, the upper and lower probability values for an outcome x_i are defined as:

$$\text{IDM Upper Probability:} \quad \overline{P}(x_i) = \frac{r(x_i) + C}{C + \sum_{i=1}^{k} r(x_i)} \tag{20}$$

$$\text{IDM Lower Probability:} \quad \underline{P}(x_i) = \frac{r(x_i)}{C + \sum_{i=1}^{k} r(x_i)} \tag{21}$$

where $r(x_i)$ is the number of observations of outcome x_i, and C is the weight of the non-informative prior probability distribution.

It can easily be shown that these values can be misleading. For example, assume an urn containing nine red balls and one black ball, meaning that the relative frequencies of red and black balls are $p(\text{red}) = 0.9$ and $p(\text{black}) = 0.1$. The *a priori* weight is set to $C = 2$. Assume further that an observer picks one ball which turns out to be black. According to Eq.(21) the lower probability is then $\underline{P}(\text{black}) = \frac{1}{3}$. It would be incorrect to literally interpret this value as the lower bound for the relative frequency because it obviously is greater than the actual relative frequency of black balls. This example shows that there is no guarantee that the actual probability of an event is inside the interval defined by the upper and lower probabilities as described by the IDM. This result can be generalised to all models based on upper and lower probabilities, and the terms "upper" and "lower" must therefore be interpreted as rough terms for imprecision, and not as absolute bounds.

Opinions used in subjective logic do not define upper and lower probability bounds. As opinions are equivalent to general Dirichlet probability density functions, they always cover any probability value except in the case of dogmatic opinions which specify discrete probability values.

5 THE OPINION REPRESENTATION IN SUBJECTIVE LOGIC

Subjective logic[7, 8] is a type of probabilistic logic that explicitly takes uncertainty and belief ownership into account. Arguments in subjective logic are subjective opinions about states in a frame. A binomial opinion applies to a single proposition, and can be represented as a Beta distribution. A multinomial opinion applies to a collection of propositions, and can be represented as a Dirichlet distribution. Subjective logic also corresponds to a specific type of belief functions which are described next.

5.1 The Dirichlet bba

A special type of bba called *Dirichlet bba* corresponds to opinions used in subjective logic. Dirichlet bbas are characterised by allowing only mutually disjoint focal elements, in addition to the whole frame X itself. This is defined as follows.

Definition 1 (Dirichlet bba) *Let X be a frame and let (x_i, x_j) be arbitrary subsets of X. A bba m_X where the only focal elements are X and/or mutually exclusive subsets of X is a Dirichlet belief mass distribution function, called Dirichlet bba for short. This constraint can be expressed mathematically as:*

$$((x_i \neq x_j) \wedge (x_i \cap x_j \neq \emptyset)) \Rightarrow ((m_X(x_i) = 0) \vee (m_X(x_j) = 0)) \ . \quad (22)$$

The name "Dirichlet" bba is used because bbas of this type correspond to Dirichlet probability density functions under a specific mapping. A bijective mapping between Dirichlet bbas and Dirichlet probability density functions is described in [10, 11].

5.2 The Base Rate

Let X be a frame and let m_X be a Dirichlet bba on X. The relative share of the uncertainty mass $m_X(X)$ assigned to subsets of X when computing their probability expectation values can be defined by a function a. This function is the *base rate function*, as defined below.

Definition 2 (Base Rate Function) *Let $X = \{x_i | i = 1, \dots k\}$ be a frame and let m_X be a Dirichlet bba on X. The function $a : X \longmapsto [0, 1]$ satisfying:*

$$a(\emptyset) = 0 \quad \textit{and} \quad \sum_{x \in X} a(x) = 1 \tag{23}$$

that defines the relative contribution of the uncertainty mass $m_X(X)$ to the probability expectation values of x_i is called a base rate function on X.

The introduction of the base rate function allows the derivation of the probability expectation value to be independent from the internal structure of the frame. In the default case, the base rate function for each element is $1/k$ where k is the cardinality, but it is possible to define arbitrary base rates for all mutually exclusive elements of the frame, as long as the additivity constraint of Eq.(23) is satisfied.

The probability expectation value $\mathrm{E}(x_i)$ derived from a Dirichlet bba m is a function of the bba and the base rate function a, as expressed by:

$$\mathrm{E}(x_i) = m(x_i) + a(x_i) m(X) \ . \tag{24}$$

A central problem when applying conditional reasoning in real world situations is the determination of base rates. A distinction can be made between events that can be repeated many times and events that can only happen once.

Events that can be repeated many times are frequentist in nature and the base rates for these can be derived from knowledge of the observed situation, or reasonably approximated through empirical observation. For example, if an observer only knows the number of different colours that balls in an urn can have, then the inverse of that number will be the base rate of drawing a ball of a specific colour. For frequentist problems where base rates cannot be known with absolute certainty, then approximation through prior empirical observation is possible.

For events that can only happen once, the observer must often decide what the base rates should be based on subjective intuition, which therefore can become a source of error in conditional reasoning. When nothing else is know, the default base rate should be defined to be equally partitioned between all disjoint states in the frame, i.e. when there are k states, the default base rate should be set to $1/k$.

The difference between the concepts of subjective and frequentist probabilities is that the former can be defined as subjective betting odds – and the latter as the relative frequency of empirically observed data, where the two collapse in the case where empirical data is available [2]. The concepts of *subjective* and *empirical* base rates can be defined in a similar manner where they also converge and merge into a single base rate when empirical data is available.

5.3 Example 3: Base Rates of Diseases

The base rate of diseases within a community can be estimated. Typically, data is collected from hospitals, clinics and other sources where people diagnosed with the disease are treated. The amount of data that is required to calculate the base rate of the disease will be determined by some departmental guidelines, statistical analysis, and expert opinion about the data that it is truly reflective of the actual number of infections – which is itself a subjective assessment. After the guidelines, analysis and opinion are all satisfied, the base rate will be determined from the data, and can then be used with medical tests to provide a better indication of the likelihood of specific patients having contracted the disease [6].

5.4 Subjective Opinions

Subjective opinions, called *"opinions"* for short, represent a special type of belief functions used in subjective logic. Through the equivalence between subjective opinions and probability density functions in the form of Beta and Dirichlet distributions, subjective logic also provides a calculus for such probability density functions.

A subjective opinion consists of the combination of a Dirichlet bba and a base rate function contained in a single composite function. In order to have a simple and intuitive notation, the Dirichlet bba is split into a belief mass vector $\vec{b}$ and an uncertainty mass u. This is defined as follows.

Definition 3 (Belief Mass Vector and Uncertainty Mass)
Let m_X be a Dirichlet bba. The belief mass vector $\vec{b}_X$ and the uncertainty

mass u_X are defined as follows:

$$\begin{array}{llll} \textit{Belief masses:} & \vec{b}_X(x_i) & = m_X(x_i) \quad \textit{where} \quad x_i \neq X \ , \\ \textit{Uncertainty mass:} & u_X & = m_X(X) \ . \end{array} \tag{25}$$

It can be noted that Eq.(14) makes opinions satisfy the belief mass additivity criterion:

$$u_X + \sum_{x=1}^{k} \vec{b}_X(x_i) = 1 \ . \tag{26}$$

Belief mass additivity is different from probability additivity in that only elements of X can carry probability whereas the frame X as well as its elements can carry belief mass. The belief mass vector $\vec{b}_X$, the uncertainty mass u_X and the base rate vector $\vec{a}$ are used in the definition of subjective opinions.

Definition 4 (Subjective Opinions) *Let $X = \{x_i | i = 1 \ldots k\}$ be a frame and let m_X be a Dirichlet bba on X with belief mass vector $\vec{b}_X$ and uncertainty mass u_X. Let $\vec{a}_X$ be a base rate vector on X. The composite function $\omega_X = (\vec{b}_X, u_X, \vec{a}_X)$ is then a subjective opinion on X.*

We use the convention that the subscript on the opinion symbol indicates the frame to which the opinion applies, and that a superscript indicates the owner of the opinion. For example, the opinion ω^A_X represents subject entity A's opinion over the frame X. An alternative notation is $\omega(A : X)$. The owner can be omitted whenever irrelevant.

Opinions can be be geometrically represented as points in a pyramid with dimensions equal to the cardinality of the frame. For example Fig.3 illustrates an opinion pyramid on a ternary frame.

The uncertainty of the opinion is equal to the relative vertical distance from the base to the opinion point. Dogmatic opinions have zero uncertainty. The belief mass on a state x_i is equal to the relative distance from the triangular side plane to the opinion point when measured towards the vertex corresponding to the state. Specific belief and base rate parameters are referred to as:

$$\begin{cases} \text{Belief parameters:} & b_{x_i} = \vec{b}_X(x_i) \ , \\ \text{Base rate parameters:} & a_{x_i} = \vec{a}_X(x_i) \ . \end{cases} \tag{27}$$

The base rate vector $\vec{a}_X$ can be represented as a point on the pyramid base, and the line joining the pyramid apex with that point is called the director. The projection of the opinion onto the base parallel to the director determines the probability expectation value vector $\vec{\mathrm{E}}_X$.

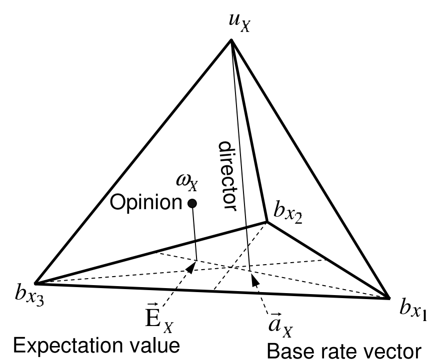


FIGURE 3
Visualisation of trinomial opinion

Assuming that the frame X has cardinality k, then the belief mass vector $\vec{b}_X$ and the base rate vector $\vec{a}_X$ will have k parameters each. The uncertainty mass u_X is a simple scalar. A subjective opinion over a frame of cardinality k will thus contain $(2k + 1)$ parameters. However, given the constraints of Eq.(14) and Eq.(23), the opinion will only have $(2k - 1)$ degrees of freedom. A binomial opinion will for example have three degrees of freedom.

Equivalently to the probability projection of Eq.(24), the probability transformation of subjective opinions can be expressed as a function of the belief mass vector, the uncertainty mass and the base rate vector.

Definition 5 (Probability Expectation) *Let $X = \{x_i | i = 1, \ldots k\}$ be a frame, and let ω_X be a subjective opinion on X consisting of belief mass vector $\vec{b}$, uncertainty mass u and base rate vector $\vec{a}$. The function E_X from ω_X to $[0, 1]$ defining the probability expectation values expressed as:*

$$\mathrm{E}_X(x_i) = \vec{b}_X(x_i) + \vec{a}_X(x_i)u_X \tag{28}$$

is then called the probability expectation function of opinions.

It can be shown that E_X satisfies the additivity principle:

$$\mathrm{E}_X(\emptyset) = 0 \quad \text{and} \quad \sum_{x \in X} \mathrm{E}_X(x) = 1 \ . \tag{29}$$

The base rate function of Def.2 expresses *a priori* probability, whereas the probability expectation function of Eq.(28) expresses *a posteriori* probability.

With a cardinality k, the default base rate for each element in the frame is $1/k$, but it is possible to define arbitrary base rates for all mutually exclusive elements as long as the additivity constraint of Eq.(23) is satisfied.

Two different subjective opinions on the same frame will normally share the same base rate functions. However, it is obvious that two different observers can assign different base rate functions to the same frame, and this could naturally reflect two different analyses of the same situation by two different persons.

5.5 Binomial Subjective Opinions

A special notation is used to denote a binomial subjective opinion which consists of an ordered tuple containing the three specific belief masses *belief, disbelief, uncertainty* as well as the *base rate* of x_i.

Definition 6 (Binomial Subjective Opinion) *Let X be a frame where $x_i \in X$ is a state of interest. Assume m_X to be a Dirichlet bba on X, and a_X to be a base rate function on X. The ordered quadruple ω_{x_i} defined as:*

$$\omega_{x_i} = (b_{x_i}, d_{x_i}, u_{x_i}, a_{x_i}), \quad \textit{where} \begin{cases} \textit{Belief:} & b_{x_i} = m_X(x_i) \\ \textit{Disbelief:} & d_{x_i} = m_X(\overline{x}_i) \\ \textit{Uncertainty:} & u_{x_i} = m_X(X) \\ \textit{Base rate:} & a_{x_i} = a_X(x_i) \end{cases} \tag{30}$$

is then called a binomial opinion on x_i in the binary frame $X = \{x_i, \overline{x_i}\}$.

Binomial subjective opinions can be mapped to a point in an equal-sided triangle as illustrated in Fig.4.

The relative distance from the left side edge to the point represents belief, from the right side edge to the point represents disbelief, and from the base line to the point represents uncertainty. For an arbitrary binomial opinion $\omega_x = (b_x, d_x, u_x, a_x)$, the three parameters b_x, d_x and u_x thus determine the position of the opinion point in the triangle. The base line is the *probability axis*, and the base rate value can be indicated as a point on the probability axis. Fig.4 illustrates an example opinion about x with the value $\omega_x = (0.7,\ 0.1,\ 0.2,\ 0.5)$ indicated by a black dot in the triangle. The probability expectation value of a binomial opinion derived from Eq.(28), is:

$$\mathrm{E}(\omega_{x_i}) = b_{x_i} + a_{x_i} u_{x_i} \ . \tag{31}$$

The *projector* going through the opinion point, parallel to the line that joins the uncertainty corner and the base rate point, determines the probability expectation value of Eq.(31).

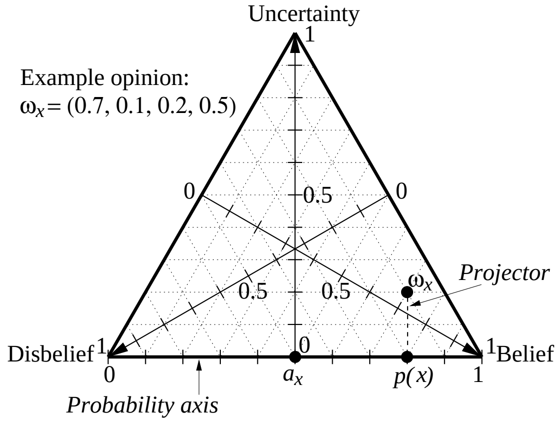


FIGURE 4
Opinion triangle with example binomial opinion

Although a binomial opinion consists of four parameters, it only has three degrees of freedom because the three components b_x, d_x and u_x are dependent through Eq.(14). As such they represent the traditional *Bel*(x) (Belief) and *Pl*(x) (Plausibility) pair of Shaferian belief theory through the correspondence *Bel*$(x) = b_x$ and *Pl*$(x) = b_x + u_x$.

The redundant parameter in the binomial opinion representation allows for more compact expressions of subjective logic operators than otherwise would have been possible. Various visualisations of binomial opinions are possible to facilitate human interpretation[‡] .

Binomial opinions are used in traditional subjective logic operators defined in [8, 9, 12, 14, 15, 20]. It can be shown that binomial opinions are equivalent to Beta distributions [8] and that multinomial opinions are equivalent to Dirichlet distributions [10].

6 CONDITIONAL REASONING IN SUBJECTIVE LOGIC

In sections 1 and 2 basic notions of classical probabilistic conditional reasoning were presented. This section extends the same type of conditional reasoning to subjective opinions. While conditional reasoning operators for

[‡] See for example the online demo of subjective logic at http://www.unik.no/people/josang/sl/

binomial opinions have already been described [15, 20], their generalisation to multinomial opinions will be described below.

6.1 Notation for Deduction and Abduction

Let $X = \{x_i | i = 1 \ldots k\}$ and $Y = \{y_j | j = 1 \ldots l\}$ be frames, where X will play the role of parent, and Y will play the role of child.

Assume the parent opinion ω_X where $|X| = k$. Assume also the conditional opinions of the form $\omega_{Y|x_i}$, where $i = 1 \ldots k$. There is thus one conditional for each element x_i in the parent frame. Each of these conditionals must be interpreted as the subjective opinion on Y, given that x_i is TRUE. The subscript notation on each conditional opinion indicates not only the frame Y it applies to, but also the element x_i on which it is conditioned. Similarly to Eq.(6), subjective logic conditional deduction is expressed as: .

$$\omega_{Y\|X} = \omega_X \circledcirc \omega_{Y|X} \tag{32}$$

where $\circledcirc$ denotes the general conditional deduction operator for subjective opinions, and $\omega_{Y|X} = \{\omega_{Y|x_i} | i = 1 \ldots k\}$ is a set of $k = |X|$ different opinions conditioned on each $x_i \in X$ respectively. Similarly, the expressions for subjective logic conditional abduction is expressed as:

$$\omega_{Y\overline{\|}X} = \omega_X \overline{\circledcirc}(\omega_{X|Y}, \vec{a}_Y) \tag{33}$$

where $\overline{\circledcirc}$ denotes the general conditional abduction operator for subjective opinions, and $\omega_{X|Y} = \{\omega_{X|y_j} | j = 1 \ldots l\}$ is a set of $l = |Y|$ different Dirichlet opinions conditioned on each $y_j \in Y$ respectively.

The mathematical methods for evaluating the general deduction and abduction operators of Eq.(32) and Eq.(33) are described next.

6.2 Subjective Logic Deduction

Assume that a conditional relationship exists between the two frames X and Y. Let $\omega_{Y|X}$ be the set of conditional opinions on the consequent frame Y as a function of the opinion on the antecedent frame X expressed as

$$\omega_{Y|X} : \left\{\omega_{Y|x_i},\ i = 1, \ldots k\right\} \ . \tag{34}$$

Each conditional opinion is a tuple composed of a belief vector $\vec{b}_{Y|x_i}$, an uncertainty mass $u_{Y|x_i}$ and a base rate vector $\vec{a}_Y$ expressed as:

$$\omega_{Y|x_i} = \left(\vec{b}_{Y|x_i}, u_{Y|x_i}, \vec{a}_Y\right) \ . \tag{35}$$

Note that the base rate vector $\vec{a}_Y$ is equal for all conditional opinions of Eq.(34). Let ω_X be the opinion on the antecedent frame X.

Traditional probabilistic conditional deduction can always be derived from these opinions by inserting their probability expectation values into Eq.(6), resulting in the expression:

$$\mathrm{E}(y_j \| X) = \sum_{i=1}^{k} \mathrm{E}(x_i)\mathrm{E}(y_j | x_i) \tag{36}$$

where Eq.(28) provides each factor.

The operator for subjective logic deduction takes the uncertainty of $\omega_{Y|X}$ and ω_X into account when computing the derived opinion $\omega_{Y\|X}$ as indicated by Eq.(32). The method for computing the derived opinion described below is based on a geometric analysis of the input opinions $\omega_{Y|X}$ and ω_X, and how they relate to each other.

The conditional opinions will in general define a sub-pyramid inside the opinion pyramid of the child frame Y. A visualisation of deduction with ternary parent and child pyramids and trinomial opinions is illustrated in Fig.5.

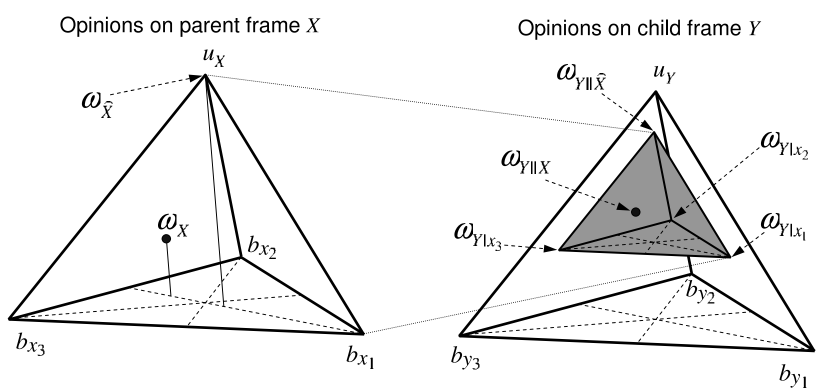


FIGURE 5
Sub-pyramid defined as the conditional projection of the parent pyramid.

The sub-pyramid formed by the conditional projection of the parent pyramid into the child pyramid is shown as the shaded pyramid on the right hand side in Fig.5. The position of the derived opinion $\omega_{Y\|X}$ is geometrically determined by the point inside the sub-pyramid that linearly corresponds to the opinion ω_X in the parent pyramid.

In general, the sub-pyramid will not appear as regular as in the example of Fig.5, and can be skewed in all possible ways. The dimensionality of the sub-pyramid is equal to the smallest cardinality of X and Y. For binary frames, the sub-pyramid is reduced to a triangle. Visualising pyramids larger than ternary is impractical on two-dimensional media such as paper and flat screens.

The mathematical procedure for determining the derived opinion $\omega_{Y\|X}$ is described in four steps below. The uncertainty of the sub-pyramid apex will emerge from the largest sub-triangle in any dimension of Y when projected against the triangular side planes, and is derived in steps 1 to 3 below. The following expressions are needed for the computations.

$$\begin{cases} \mathrm{E}(y_t|\widehat{X}) & = \sum_{i=1}^{k} a_{x_i}\mathrm{E}(y_t|x_i)\ , \\ \\ \mathrm{E}(y_t|(\widehat{x_r, x_s})) & = (1{-}a_{y_t})b_{y_t|x_s} + a_{y_t}(b_{y_t|x_r} + u_{Y|x_r})\ . \end{cases} \quad (37)$$

The expression $\mathrm{E}(y_t|\widehat{X})$ gives the expectation value of y_t given a vacuous opinion $\omega_{\widehat{X}}$ on X. The expression $\mathrm{E}(y_t|(\widehat{x_r, x_s}))$ gives the expectation value of y_t for the theoretical maximum uncertainty $u^{\mathrm{T}}_{y_t}$.

- **Step 1:** Determine the X-dimensions (x_r, x_s) that give the largest theoretical uncertainty $u^{\mathrm{T}}_{y_t}$ in each Y-dimension y_t, independently of the opinion on X. Each dimension's maximum uncertainty is:

$$u^{\mathrm{T}}_{y_t} = 1 - \mathrm{Min}\left[\left(1{-}b_{y_t|x_r}{-}u_{Y|x_r}{+}b_{y_t|x_s}\right), \forall(x_r, x_s)\in X\right]\ . \quad (38)$$

  The X-dimensions (x_r, x_s) are recorded for each y_t. Note that it is possible to have $x_r = x_s$.

- **Step 2:** First determine the triangle apex uncertainty $u_{y_t\|\widehat{X}}$ for each Y-dimension by assuming a vacuous opinion $\omega_{\widehat{X}}$ and the actual base rate vector $\vec{a}_X$. Assuming that $a_{y_t} \neq 0$ and $a_{y_t} \neq 1$ for all base rates on Y, each triangle apex uncertainty $u_{y_t\|\widehat{X}}$ can be computed as:

  Case A: $\mathrm{E}(y_t|\widehat{X}) \leq \mathrm{E}(y_t|(\widehat{x_r, x_s}))$:

$$u_{y_t\|\widehat{X}} = \left(\frac{\mathrm{E}(y_t|\widehat{X}) - b_{y_t|x_s}}{a_{y_t}}\right) \quad (39)$$

  Case B: $\mathrm{E}(y_t|\widehat{X}) > \mathrm{E}(y_t|(\widehat{x_r, x_s}))$:

$$u_{y_t\|\widehat{X}} = \left(\frac{b_{y_t|x_r} + u_{Y|x_r} - \mathrm{E}(y_t|\widehat{X})}{1 - a_{y_t}}\right) \quad (40)$$

Then determine the intermediate sub-pyramid apex uncertainty $u^{\text{Int}}_{Y\|\widehat{X}}$ which is equal to the largest of the triangle apex uncertainties computed above. This uncertainty is expressed as.

$$u^{\text{Int}}_{Y\|\widehat{X}} = \text{Max}\left[u_{y_t\|\widehat{X}}, \forall y_t \in Y\right] \ . \tag{41}$$

- **Step 3:** First determine the intermediate belief components $b^{\text{Int}}_{y_j\|\widehat{X}}$ in case of vacuous belief on X as a function of the intermediate apex uncertainty $u^{\text{Int}}_{Y\|\widehat{X}}$:

  $$b^{\text{Int}}_{y_j\|\widehat{X}} = \text{E}(y_j\|\widehat{X}) - a_{y_j} u^{\text{Int}}_{Y\|\widehat{X}} \ . \tag{42}$$

  For particular geometric combinations of the triangle apex uncertainties $u_{y_t\|\widehat{X}}$ it is possible that an intermediate belief component $b^{\text{Int}}_{y_j\|\widehat{X}}$ becomes negative. In such cases a new adjusted apex uncertainty $u^{\text{Adj}}_{y_t\|\widehat{X}}$ is computed. Otherwise the adjusted apex uncertainty is set equal to the intermediate apex uncertainty of Eq.(41). Thus:

  Case A: $b^{\text{Int}}_{y_j\|\widehat{X}} < 0: \quad u^{\text{Adj}}_{y_j\|\widehat{X}} = \text{E}(y_j\|\widehat{X})/a_{y_j}$ (43)

  Case B: $b^{\text{Int}}_{y_j\|\widehat{X}} \geq 0: \quad u^{\text{Adj}}_{y_j\|\widehat{X}} = u^{\text{Int}}_{Y\|\widehat{X}}$ (44)

  Then compute the sub-pyramid apex uncertainty $u_{Y\|\widehat{X}}$ as the minimum of the adjusted apex uncertainties according to:

  $$u_{Y\|\widehat{X}} = \text{Min}\left[u^{\text{Adj}}_{y_j\|\widehat{X}}, \forall y_j \in Y\right] \ . \tag{45}$$

  Note that the apex uncertainty is not necessarily the highest uncertainty of the sub-pyramid. It is possible that one of the conditionals $\omega_{Y|x_i}$ actually contains a higher uncertainty, which would simply mean that the sub-pyramid is skewed or tilted to the side.

- **Step 4:** Based on the sub-pyramid apex uncertainty $u_{Y\|\widehat{X}}$, the actual uncertainty $u_{Y\|X}$ as a function of the opinion on X is:

  $$u_{Y\|X} = u_{Y\|\widehat{X}} - \sum_{i=1}^{k}(u_{Y\|\widehat{X}} - u_{Y|x_i})b_{x_i} \ . \tag{46}$$

  Given the actual uncertainty $u_{Y\|X}$, the actual beliefs $b_{y_j\|X}$ are:

  $$b_{y_j\|X} = \text{E}(y_j\|X) - a_{y_j} u_{Y\|X} \ . \tag{47}$$

The belief vector $\vec{b}_{Y\|X}$ is expressed as:

$$\vec{b}_{Y\|X} = \left\{ b_{y_j\|X} \mid j = 1, \ldots l \right\} \ . \tag{48}$$

Finally, the derived opinion $\omega_{Y\|X}$ is the tuple composed of the belief vector of Eq.(48), the uncertainty belief mass of Eq.(46) and the base rate vector of Eq.(35) expressed as:

$$\omega_{Y\|X} = \left(\vec{b}_{Y\|X}, u_{Y\|X}, \vec{a}_Y \right) \ . \tag{49}$$

The method for multinomial deduction described above represents both a simplification and a generalisation of the method for binomial deduction described in [15]. In case of 2×2 deduction in particular, the methods are equivalent and produce exactly the same results.

6.3 Subjective Logic Abduction

Subjective logic abduction requires the inversion of conditional opinions of the form $\omega_{X|y_j}$ into conditional opinions of the form $\omega_{Y|x_i}$ similarly to Eq.(7). The inversion of probabilistic conditionals according to Eq.(7) uses the division operator for probabilities. While a division operator for binomial opinions is defined in [14], a division operator for multinomial opinions would be intractable because it involves matrix and vector expressions. Instead we define inverted conditional opinions as uncertainty-maximised opinions.

It is natural to define base rate opinions as vacuous opinions, so that the base rate vector $\vec{a}$ alone defines their probability expectation values. The rationale for defining inversion of conditional opinions as producing maximum uncertainty is that it involves multiplication with a vacuous base rate opinion which produces an uncertainty-maximised product. Let $|X| = k$ and $|Y| = l$, and assume the set of available conditionals to be:

$$\omega_{X|Y} : \ \left\{ \omega_{X|y_j}, \text{ where } j = 1 \ldots l \right\} \ . \tag{50}$$

Assume further that the analyst requires the set of conditionals expressed as:

$$\omega_{Y|X} : \ \left\{ \omega_{Y|x_i}, \text{ where } i = 1 \ldots k \right\} \ . \tag{51}$$

First compute the l different probability expectation values of each inverted conditional opinion $\omega_{Y|x_i}$, according to Eq.(7) as:

$$\mathrm{E}(y_j|x_i) = \frac{a(y_j)\mathrm{E}(\omega_{X|y_j}(x_i))}{\sum_{t=1}^{l} a(y_t)\mathrm{E}(\omega_{X|y_t}(x_i))} \tag{52}$$

where $a(y_j)$ denotes the base rate of y_j. Consistency requires that:

$$\mathrm{E}(\omega_{Y|x_i}(y_j)) = \mathrm{E}(y_j|x_i)\ . \tag{53}$$

The simplest opinions that satisfy Eq.(53) are the k dogmatic opinions:

$$\underline{\omega}_{Y|x_i} : \begin{cases} b_{Y|x_i}(y_j) & = \mathrm{E}(y_j|x_i), \quad \text{for } j = 1 \ldots k\ , \\ u_{Y|x_i} & = 0\ , \\ \vec{a}_{Y|x_i} & = \vec{a}_Y\ . \end{cases} \tag{54}$$

Uncertainty maximisation of $\underline{\omega}_{Y|x_i}$ consists of converting as much belief mass as possible into uncertainty mass while preserving consistent probability expectation values according to Eq.(53). The result is the uncertainty maximised opinion denoted as $\widehat{\omega}_{Y|x_i}$. This process is illustrated in Fig.6.

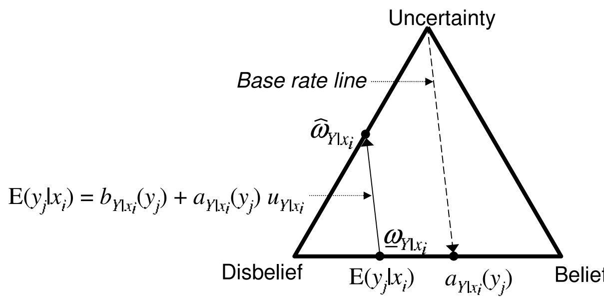


FIGURE 6
Uncertainty maximisation of dogmatic opinion

It must be noted that Fig.6 only represents two dimensions of the multinomial opinions on Y, namely y_j and its complement. The line defined by

$$\mathrm{E}(y_j|x_i) = b_{Y|x_i}(y_j) + a_{Y|x_i}(y_j) u_{Y|x_i} \tag{55}$$

that is parallel to the base rate line and that joins $\underline{\omega}_{Y|x_i}$ and $\widehat{\omega}_{Y|x_i}$ in Fig.6, defines the opinions $\omega_{Y|x_i}$ for which the probability expectation values are consistent with Eq.(53). An opinion $\widehat{\omega}_{Y|x_i}$ is uncertainty maximised when Eq.(55) is satisfied and at least one belief mass of $\widehat{\omega}_{Y|x_i}$ is zero. In general, not all belief masses can be zero simultaneously except for vacuous opinions.

In order to find the dimension(s) that can have zero belief mass, the belief mass will be set to zero in Eq.(55) successively for each dimension $y_j \in Y$, resulting in l different uncertainty values defined as:

$$u^j_{Y|x_i} = \frac{\mathrm{E}(y_j|x_i)}{a_{Y|x_i}(y_j)},\ \text{where } j = 1 \ldots l\ . \tag{56}$$

The minimum uncertainty expressed as $\mathrm{Min}[u^j_{Y|x_i}$, for $j = 1 \ldots l]$ determines the dimension which will have zero belief mass. Setting the belief mass to zero for any other dimension would result in negative belief mass for other dimensions. Assume that y_t is the dimension for which the uncertainty is minimum. The uncertainty-maximised opinion can then be determined as:

$$\widehat{\omega}_{Y|x_i} : \begin{cases} b_{Y|x_i}(y_j) & = \mathrm{E}(y_j|x_i) - a_Y(y_j) u^t_{Y|x_i}, \text{ for } j = 1 \ldots l \\ u_{Y|x_i} & = u^t_{Y|x_i} \\ \vec{a}_{Y|x_i} & = \vec{a}_Y \end{cases} \tag{57}$$

By defining $\omega_{Y|x_i} = \widehat{\omega}_{Y|x_i}$, the expressions for the set of inverted conditional opinions $\omega_{Y|x_i}$ (with $i = 1 \ldots k$) emerges. Conditional abduction according to Eq.(33) with the original set of conditionals $\omega_{X|Y}$ is now equivalent to conditional deduction according to Eq.(32) where the set of inverted conditionals $\omega_{Y|X}$ is used deductively. The difference between deductive and abductive reasoning with opinions is illustrated in Fig.7 below.

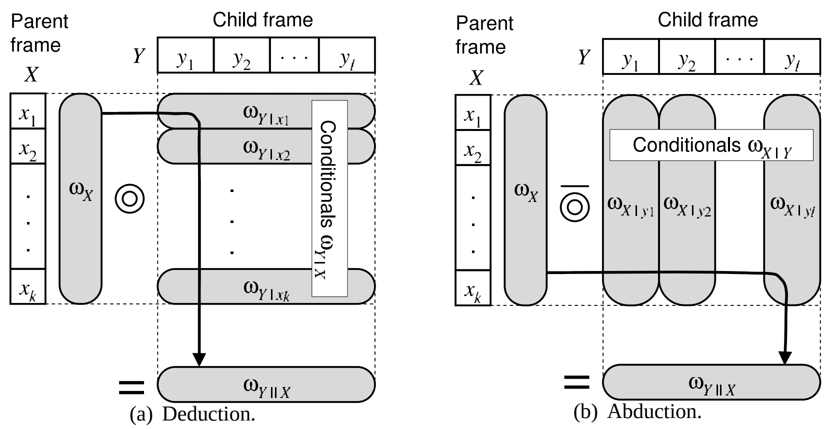


FIGURE 7
Visualising deduction and abduction with opinions

Fig.7 shows that deduction uses conditionals defined over the child frame, and that abduction uses conditionals defined over the parent frame.

6.4 Example 4: Military Intelligence Analysis with Subjective Logic

Example 2 is revisited, but now with conditionals and evidence represented as subjective opinions according to Table 4 and Eq.(58).

Opinions $\omega_{X\|Y}$	Troop movements x_1: No movemt.	x_2: Minor movemt.	x_3: Full mob.	X Any
$\omega_{X\|y_1}$:	$b(x_1) = 0.50$	$b(x_2) = 0.25$	$b(x_3) = 0.25$	$u = 0.00$
$\omega_{X\|y_2}$:	$b(x_1) = 0.00$	$b(x_2) = 0.50$	$b(x_3) = 0.50$	$u = 0.00$
$\omega_{X\|y_3}$:	$b(x_1) = 0.00$	$b(x_2) = 0.25$	$b(x_3) = 0.75$	$u = 0.00$

TABLE 4
Conditional opinion $\omega_{X|Y}$: troop movement x_i given military plan y_j

The opinion about troop movements is expressed as the opinion:

$$\omega_X = \begin{cases} b(x_1) & = 0.00, \quad a(x_1) = 0.70 \\ b(x_2) & = 0.50, \quad a(x_2) = 0.20 \\ b(x_3) & = 0.50, \quad a(x_3) = 0.10 \\ u & = 0.00 \end{cases} \tag{58}$$

First the conditional opinions must be inverted as expressed in Table 5.

Military plan	Opinions of military plans given troop movement $\omega_{Y\|x_1}$ No movemt.	$\omega_{Y\|x_2}$ Minor movemt.	$\omega_{Y\|x_3}$ Full mob.
y_1: No aggression	$b(y_1) = 1.00$	$b(y_1) = 0.00$	$b(y_1) = 0.00$
y_2: Minor ops.	$b(y_2) = 0.00$	$b(y_2) = 0.17$	$b(y_2) = 0.14$
y_3: Invasion	$b(y_3) = 0.00$	$b(y_3) = 0.00$	$b(y_3) = 0.14$
Y: Any	$u = 0.00$	$u = 0.83$	$u = 0.72$

TABLE 5
Conditional opinions $\omega_{Y|X}$: military plan y_j given troop movement x_i

Then the likelihoods of country A's plans can be computed as the opinion:

$$\omega_{Y\|X} = \begin{cases} b(y_1) & = 0.00, \quad a(y_1) = 0.70, \quad \mathrm{E}(y_1) = 0.54 \\ b(y_2) & = 0.16, \quad a(y_2) = 0.20, \quad \mathrm{E}(y_2) = 0.31 \\ b(y_3) & = 0.07, \quad a(y_3) = 0.10, \quad \mathrm{E}(y_3) = 0.15 \\ u & = 0.77 \end{cases} \tag{59}$$

These results can be compared with those of Eq.(13) which were derived with probabilities only, and which are equal to the probability expectation values given in the rightmost column of Eq.(59). The important observation to make is that although y_1 (no aggression) seems to be country A's most likely plan in probabilistic terms, this likelihood is based on uncertainty only. The only firm evidence actually supports y_2 (minor aggression) or y_3 (full invasion), where y_2 has the strongest support ($b(y_2) = 0.16$). A likelihood expressed as a scalar probability can thus hide important aspects of the analysis, which will only come to light when uncertainty is explicitly expressed, as done in the example above.

7 BAYESIAN NETWORKS WITH SUBJECTIVE LOGIC

A Bayesian network is a graphical model for conditional relationships. Specifically, a Bayesian network is normally defined as a directed acyclic graph of nodes representing variables and arcs representing conditional dependence relations among the variables.

Equipped with the operators for conditional deduction and abduction, it is possible to analyse Bayesian networks with subjective logic. For example, the simple Bayesian network:

$$X \longrightarrow Y \longrightarrow Z \tag{60}$$

can be modelled by defining conditional opinions between the three frames. In case conditionals can be obtained with X as antecedent and Y as consequent, then deductive reasoning can be applied to the edge $[X : Y]$. In case there are available conditionals with Y as antecedent and X as consequent, then abductive reasoning must be applied.

In the example illustrated in Fig.8 it is assumed that deductive reasoning can be applied to both $[X : Y]$ and $[Y : Z]$.

The frames X and Y thus represent parent and child of the first conditional edge, and the frames Y and Z represent parent and child of the second conditional edge respectively.

This chaining of conditional reasoning is possible because of the symmetry between the parent and child frames. They both consist of sets of mutually exclusive elements, and subjective opinions can be applied to both. In general it is arbitrary which frame is the antecedent and which frame is the consequent in a given conditional edge. Conditional reasoning is possible in either case, by applying the deductive or the abductive operator.

Frame pairs to consider as parent-child relationships must have a degree of relevance to each other. The relevance between two nodes can be formally

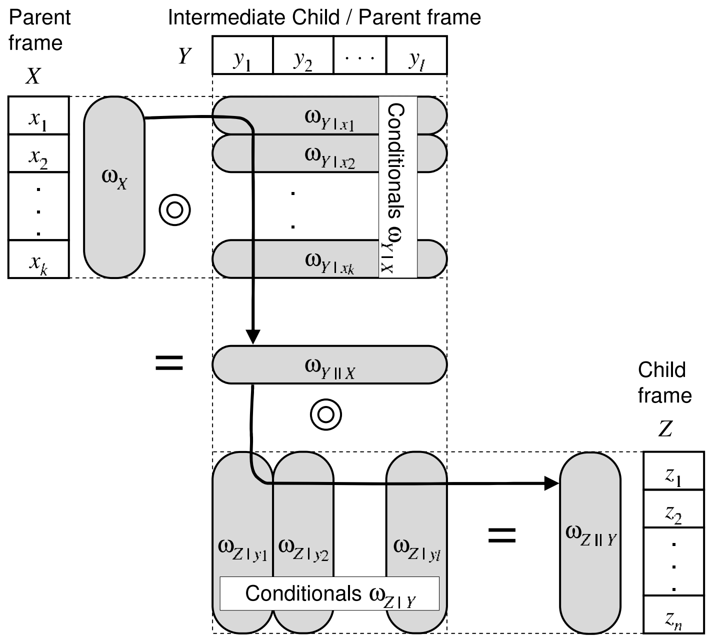


FIGURE 8
Deductive opinion structure for the Bayesian network of Eq.(60)

expressed as a relevance measure, and is a direct function of the conditionals. For probabilistic conditional deduction, the relevance denoted as $\mathrm{R}(y,x)$ between two states y and x can be defined as:

$$\mathrm{R}(y,x) = |p(y|x) - p(y|\overline{x})| \ . \tag{61}$$

It can be seen that $\mathrm{R}(y,x) \in [0,1]$, where $\mathrm{R}(y,x) = 0$ expresses total irrelevance, and $\mathrm{R}(y,x) = 1$ expresses total relevance between y and x.

For conditionals expressed as opinions, the same type of relevance between a given state $y_j \in Y$ and a given state $x_i \in X$ can be defined as:

$$\mathrm{R}(y_j,x_i) = |\mathrm{E}(\omega_{Y|x_i}(y_j)) - \mathrm{E}(\omega_{Y|\overline{x_j}}(y_j))| \ . \tag{62}$$

The relevance between a child frame Y and a given state $x_i \in X$ of a parent frame can be defined as:

$$\mathrm{R}(Y,x_i) = \mathrm{Max}[\mathrm{R}(y_j,x_i), \quad j = 1, \ldots l] \ . \tag{63}$$

Finally, the relevance between a child frame Y and a parent frame X can be defined as:

$$\mathrm{R}(Y, X) = \text{Max}[\mathrm{R}(Y, x_i), \quad i = 1, \ldots k] \ . \tag{64}$$

In our model, the relevance measure of Eq.(64) is the most general.

In many situations there can be multiple parents for the same child, which requires fusion of the separate child opinions into a single opinion. The question then arises which type of fusion is most appropriate. The two most typical situations to consider are the cumulative case and the averaging case.

Cumulative fusion is applicable when independent evidence is accumulated over time such as by continuing observation of outcomes of a process. Averaging fusion is applicable when two sources provide different but independent opinions so that each opinion is weighed as a function of its certainty.

Both cumulative and averaging situations are encountered in practical situations, and their operators are provided below. The cumulative operator of fusing opinions [10] represents a generalisation of the consensus operator[9].

Definition 7 (Cumulative Fusion Operator)
Let ω^A and ω^B be opinions respectively held by agents A and B over the same frame $X = \{x_j | j = 1, \cdots l\}$. Let $\omega^{A\diamond B}$ be the opinion such that:

Case I: For $u^A \neq 0 \ \vee \ u^B \neq 0$:

$$\begin{cases} b^{A\diamond B}(x_j) & = \frac{b^A(x_j)u^B + b^B(x_j)u^A}{u^A + u^B - u^A u^B} \\ \\ u^{A\diamond B} & = \frac{u^A u^B}{u^A + u^B - u^A u^B} \end{cases} \tag{65}$$

Case II: For $u^A = 0 \ \wedge \ u^B = 0$:

$$\begin{cases} b^{A\diamond B}(x_j) & = \gamma^A \, b^A(x_j) + \gamma^B b^B(x_j) \\ \\ u^{A\diamond B} & = 0 \end{cases} \tag{66}$$

$$\textit{where} \quad \gamma^A = \lim_{\substack{u^A \to 0 \\ u^B \to 0}} \frac{u^B}{u^A + u^B} \quad \textit{and} \quad \gamma^B = \lim_{\substack{u^A \to 0 \\ u^B \to 0}} \frac{u^A}{u^A + u^B}$$

Then $\omega^{A\diamond B}$ is called the cumulatively fused bba of ω^A and ω^B, representing the combination of independent opinions of A and B. By using the symbol '$\oplus$' to designate this belief operator, we define $\omega^{A\diamond B} \equiv \omega^A \oplus \omega^B$.

The averaging operator for opinions [10] represents a generalisation of the consensus operator for dependent opinions [13, 16].

Theorem 1 (Averaging Fusion Rule)
Let ω^A and ω^B be opinions respectively held by agents A and B over the same frame $X = \{x_j \mid j = 1, \cdots, l\}$. Let $\omega^{A\underline{\diamond}B}$ be the opinion such that:

Case I: For $u^A \neq 0 \;\vee\; u^B \neq 0$:

$$\begin{cases} b^{A\underline{\diamond}B}(x_j) & = \frac{b^A(x_j)u^B + b^B(x_j)u^A}{u^A+u^B} \\ & \\ u^{A\underline{\diamond}B} & = \frac{2u^Au^B}{u^A+u^B} \end{cases} \tag{67}$$

Case II: For $u^A = 0 \;\wedge\; u^B = 0$:

$$\begin{cases} b^{A\diamond B}(x_j) & = \gamma^A b^A(x_j) + \gamma^B b^B(x_j) \\ & \\ u^{A\diamond B} & = 0 \end{cases} \tag{68}$$

$$\textit{where} \quad \gamma^A = \lim_{\substack{u^A \to 0 \\ u^B \to 0}} \frac{u^B}{u^A + u^B} \quad \textit{and} \quad \gamma^B = \lim_{\substack{u^A \to 0 \\ u^B \to 0}} \frac{u^A}{u^A + u^B}$$

Then $\omega^{A\underline{\diamond}B}$ is called the averaged opinion of ω^A and ω^B, representing the combination of the dependent opinions of A and B. By using the symbol '$\underline{\oplus}$' to designate this belief operator, we define $\omega^{A\underline{\diamond}B} \equiv \omega^A \underline{\oplus} \omega^B$.

In case of dogmatic opinions, the cumulative and the averaging operators are equivalent. This is so because dogmatic opinions must be interpreted as opinions based on infinite evidence, so that two different opinions necessarily must be dependent, in which case the averaging operator is applicable.

By fusing child opinions resulting from multiple parents, arbitrarily large Bayesian networks can be constructed. Depending on the situation it must be decided whether the cumulative or the averaging operator is applicable. An example with three grandparent frames X_1, X_2, X_3, two parent parent frames Y_1, Y_2 and one child frame Z is illustrated in Fig.9 below.

The nodes X_1, X_2, X_3 and Y_2 represent initial parent frames because they do not themselves have parents in the model. Opinions about the initial parent nodes represent the input evidence to the model.

When representing Bayesian networks as graphs, the structure of conditionals is hidden in the edges, and only the nodes consisting of parent and children frames are shown.

When multiple parents can be identified for the same child, there are two important considerations. Firstly, the relative relevance between the child and each parent, and secondly the relevance or dependence between the parents.

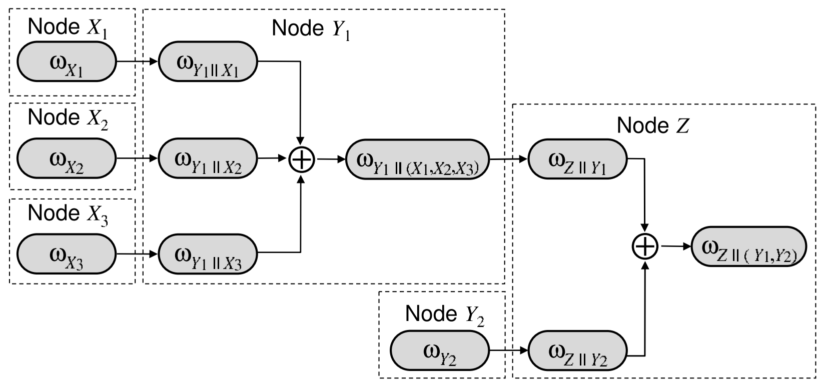


FIGURE 9
Bayesian network with multiple parent evidence nodes

Strong relevance between child and parents is desirable, and models should include the strongest child-parent relationships that can be identified, and for which there is evidence directly or potentially available.

Dependence between parents should be avoided as far as possible. A more subtle and hard to detect dependence can originate from hidden parent nodes outside the Bayesian network model itself. In this case the parent nodes have a hidden common grand parent node which makes them dependent. Philosophically speaking everything depends on everything in some way, so absolute independence is never achievable. There will often be some degree of dependence between evidence sources, but which from a practical perspective can be ignored. When building Bayesian network models it is important to be aware of possible dependencies, and try to select parent evidence nodes that have the lowest possible degree of dependence.

As an alternative method for managing dependence it could be possible to allow different children to share the same parent by fissioning the parent opinion, or alternatively taking dependence into account during the fusion operation. The latter option can be implemented by applying the averaging fusion operator.

It is also possible that evidence opinions provided by experts need to be discounted due to the analysts doubt in their reliability. This can be done with the trust transitivity operator[¶] of subjective logic.

[¶] Also called discounting operator

Definition 8 (Trust Transitivity) *Let A, B and be two agents where A's opinion about B's recommendations is expressed as a binomial opinion $\omega^A_B = \{b^A_B, d^A_B, u^A_B, a^A_B\}$, and let X be a frame where B's opinion about X is recommended to A with the opinion $\omega^B_X = \{\vec{b}^B_X, u^B_X, \vec{a}^B_X\}$. Let $\omega^{A:B}_X = \{\vec{b}^{A:B}_X, u^{A:B}_X, \vec{a}^{A:B}_X\}$ be the opinion such that:*

$$\begin{cases} b^{A:B}_X(x_i) = b^A_B b^B_X(x_i), \quad \text{for } i = 1 \dots k \ , \\ u^{A:B}_X = d^A_B + u^A_B + b^A_B u^B_X \ , \\ a^{A:B}_X(x_i) = a^B_X(x_i) \ . \end{cases}$$

then $\omega^{A:B}_X$ is called A's discounted opinion about X. By using the symbol $\otimes$ to denote this operator, trust transitivity can be expressed as $\omega^{A:B}_X = \omega^A_B \otimes \omega^B_X$. □

The transitivity operator is associative but not commutative. Discounting of opinions through transitivity generally increases the uncertainty mass, and reduces belief masses.

8 DISCUSSION AND CONCLUSION

When faced with complex situations combined with partial ignorance, pure human cognition and reasoning will often lead to inconsistent and unreliable conclusions. Practical situations where this can happen include medical diagnostic reasoning, the analysis of financial markets, criminal investigations, and military intelligence analysis, just to name a few examples. In such cases, reasoning based on subjective logic can complement human reasoning to derive more consistent and reliable conclusions. The challenge for applying subjective logic to the analysis of such situations, is to

- adequately model the situation, and
- determining the evidence needed as input to the model.

The modelling of a given situation includes defining the relevant parent and child frames, and defining the conditional opinions that relate parent and child frames to each other. Determining the evidence consists of determining the opinions on parent frames from adequate and reliable sources of information.

The results of the analysis are in the form of opinions on child frames of interest. These derived opinions can then for example assist a medical practitioner to make a more accurate diagnosis, can assist a financial market analyst to more realistically predict trends and consequences of actions, can assist police in uncovering crime scenarios, and can assist intelligence analysts in predicting military scenarios.

Multinomial subjective opinions consist of a Dirichlet bba and a base rate function. We have described methods for conditional deduction and conditional abduction with subjective opinions. These methods are based on the geometric interpretation of opinions as points in pyramids where the dimensionality of a pyramid is equal to the cardinality of the frame. This interpretation provides an intuitive basis for defining conditional reasoning operators for multinomial opinions. The ability to perform conditional reasoning with multinomial opinions gives many advantages, such as

- the parent and child frames can be of arbitrary size,
- the reasoning can go in any direction, meaning that for two frames where there are conditionally dependent subjective opinions, the choice of parent and child is arbitrary,
- conditional reasoning can be chained as in Bayesian networks,
- conditional reasoning can be done with arbitrary degrees of ignorance in the opinions,
- the computations are always compatible with classical probabilistic computations, and in fact
- the computations are reduced to classical probabilistic computations in case of only using dogmatic opinions.

The cumulative and averaging fusion operators for multinomial opinions makes it possible to have multiple parents for each child in Bayesian networks. In summary, the described methods provide a powerful tool set for analysing complex situations involving multiple sources of evidence and possibly long chains of reasoning. This allows uncertainty and incomplete knowledge to be explicitly expressed in the input opinions, and to be carried through the analysis to the conclusion opinions. In this way the analyst can better appreciate the level of uncertainty associated with the derived conclusions.

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