Generalising Bayes' Theorem in Subjective Logic

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***Abstract*— Bayes' theorem provides a method of inverting conditional probabilities in probability calculus and statistics. Subjective logic generalises probability calculus whereby arguments are represented as opinions that can contain degrees of uncertainty. This paper presents Bayes' theorem in the formalism of subjective logic.**

I. INTRODUCTION

Subjective logic is a formalism for reasoning under uncertainty which generalises probability calculus and probabilistic logic. In subjective logic analysts can express their arguments as subjective opinions which can express probability distributions affected by degrees of uncertainty.

The concept of probabilistic logic was first proposed by Nilsson [11] with the aim of combining the capability of deductive logic to exploit the structure and relationship of arguments and events with the capacity of probability theory to express degrees of truth about those arguments and events.

A fundamental limitation of probabilistic logic (and of binary logic likewise) is the inability to take into account the analyst's levels of confidence in the probability arguments, and the inability to handle the situation when the analyst fails to produce probabilities for some of the input arguments.

An analyst might for example want to produce an input argument like *"I don't know"*, which expresses total ignorance and uncertainty about some statement. However, an argument like that can not be expressed if the formalism only allows input arguments in the form of Booleans or probabilities. An analyst who has little or no evidence for providing input probabilities could be tempted or even encouraged to set probabilities with little or no confidence. This practice would generally lead to unreliable conclusions, often described as the problem of 'garbage in, garbage out'. What is needed is a way to express lack of confidence in probabilities. In subjective logic, the lack of confidence in probabilities is expressed as *uncertainty mass*.

Another limitation of logic and probability calculus and probabilistic logic is that these formalisms are not designed to handle situations where multiple agents have different beliefs about the same statement. Subjective logic was introduced in [4] and described more thoroughly in [5] in order to overcome these limitations. In subjective logic, *subjective belief ownership* can be explicitly expressed, and different beliefs about the same statements can be combined through trust fusion and discounting whenever required.

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The general idea of subjective logic is to extend probabilistic logic by explicitly including: 1) uncertainty about probabilities and 2) subjective belief ownership in the formalism, as illustrated in Figure 1.

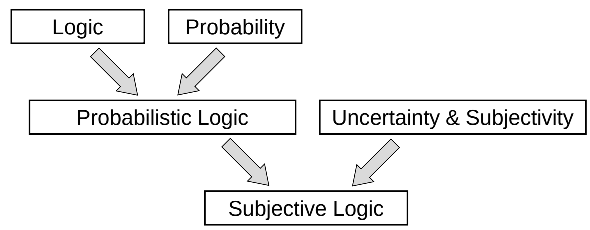

Fig. 1. General idea of subjective logic

Arguments in subjective logic are called *subjective opinions*, or *opinions* for short. An opinion can contain uncertainty mass in the sense of *uncertainty about probabilities*. This paper presents a method for inverting conditional opinions. Since conditional opinions generalise conditional probabilities. Bayes' theorem is a fundamental construct in probabilistic reasoning, and serves the purpose of inverting conditional probabilities. Since conditional opinions represent generalisations of probabilities, inverting conditional opinions represents a generalisationa of Bayes' theorem. An earlier publication presented a simple and slightly different method for inverting conditionals [9]. The present paper provides an improved method which is presented in the context of Bayes' theorem which highlights which makes it easier to relate the results to classical probability theory.

II. BAYES' THEOREM

Bayes' theorem is named after the English statistician and philosopher Thomas Bayes (1701–1761), who formally demonstrated how new evidence can be used to update beliefs. This formalism was further developed by the French mathematician Pierre-Simon Laplace (1749–1827), who first published the traditional formulation of Bayes' theorem in his 1812 *Théorie analytique des probabilités*. Bayes' theorem is traditionally expressed as in Eq.(1):

$$\text{Traditional statement of Bayes' theorem:} \quad p(x|y) = \frac{p(y|x)p(x)}{p(y)} \,. \qquad (1)$$

With Bayes' theorem, the inverted conditional $p(x|y)$ can be computed from the conditional $p(y|x)$. However, this traditional expression of Bayes' theorem hides some subtleties related to base rates, as explained below. People who have a basic knowledge of probability theory, but who are not familiar with Bayes' theorem, can easily get confused when confronted with it for the first time.

Assume for example the case of buying a lottery ticket with a relatively low probability of winning, expressed as the conditional probability $p(y|x) = 0.001$, where the statements are x: 'buying ticket' and y: 'winning prize'. Assume further that you actually bought a ticket, so that $p(x) = 1.0$, and actually won, so that $p(y) = 1.0$. An intuitive, but wrong, interpretation of Bayes' theorem would then be that $p(x|y) = (0.001 \times 1.0)/1.0 = 0.001$, i.e. that the probability of having bought a ticket given a win is only 0.001, which clearly is wrong. The obvious correct answer is that if you won a prize in the lottery then you certainly bought a ticket, expressed by $p(x|y) = 1.0$.

People who are familiar with Bayes' theorem know that $p(x)$ and $p(y)$ are base rates (prior probabilities), but this is typically not mentioned in explanations of Bayes' theorem in text books. It is often only when practical examples are presented that it becomes clear that Bayes' theorem requires base rates (priors) of x and y, and not situation-dependent probabilities of x and y.

In order to avoid confusion between the base rate of x, and the probability of x, we use the term $a(x)$ to denote the base rate of x. Similarly, the term $a(y)$ denotes the base rate of y. With this convention, Bayes' theorem can be formalised with base rates, as expressed in Theorem 1.

Theorem 1 (Bayes' Theorem With Base Rates): Bayes' theorem can be expressed as a function of base rates (prior probabilities) according to:

$$p(x|y) = \frac{p(y|x)a(x)}{a(y)} \ . \tag{2}$$

Proof:
Formally, a conditional probability is defined as

$$p(y|x) = \frac{p(x \wedge y)}{p(x)} \ . \tag{3}$$

Conditionals represent general dependence relationships between statements, so the terms $p(x \wedge y)$ and $p(x)$ on the right-hand side of Eq.(3) must necessarily represent general prior probabilities, and not for example probabilities of specific observations. A general prior probability is the same as a base rate, as explained in Section III-A. Hence, more explicit versions of Eq.(3) can be expressed as

$$\text{Conditional probabilities based on base rates:} \quad \begin{cases} p(y|x) = \frac{a(x \wedge y)}{a(x)} \ , \\ \\ p(x|y) = \frac{a(x \wedge y)}{a(y)} \ . \end{cases} \tag{4}$$

Bayes' theorem (of base rates) can easily be derived from the definition of conditional probability of Eq.(4) which expresses the conditional probability of $p(y|x)$ and $p(x|y)$ in terms of base rates:

$$\begin{cases} p(y|x) = \frac{a(x \wedge y)}{a(x)} \\ \\ p(x|y) = \frac{a(x \wedge y)}{a(y)} \end{cases} \Rightarrow \quad p(x|y) = \frac{p(y|x)a(x)}{a(y)} \ . \tag{5}$$

■

With Bayes' theorem of Eq.(2), the inverted conditional $p(x|y)$ can be computed from the conditional $p(y|x)$ and the pair of base rates $a(x)$ and $a(y)$. The ability to invert conditionals is an essential feature of Bayesian networks, where evidence can propagate through a network of conditionally connected variables, irrespective of the direction of the input conditionals.

However, Bayes' theorem in the form of Eq.(2) hides the fact that the base rate $a(y)$ must be expressed as a function of $a(x)$. The base rate $a(y)$ is then the *marginal base rate* of the value y. Theorem 2 below implements this requirement.

Theorem 2 (Bayes' Theorem with Marginal Base Rate): Bayes' theorem can be expressed in a form where the marginal base rate of y is a function of the base rate of x:

$$p(x|y) = \frac{p(y|x)a(x)}{p(y|x)a(x) + p(y|\overline{x})a(\overline{x})} \ . \tag{6}$$

Proof:
The marginal base rate $a(y)$ is derived by expressing the independent Bayes' theorem formulas for $p(y|x)$ and $p(y|\overline{x})$:

$$\begin{cases} p(y|x) = \frac{p(x|y)a(y)}{a(x)} \\ p(y|\overline{x}) = \frac{p(\overline{x}|y)a(y)}{a(\overline{x})} = \frac{(1-p(x|y))a(y)}{a(\overline{x})} \end{cases} \tag{7}$$

$$\Longleftrightarrow \quad \begin{cases} p(x|y) = \frac{p(y|x)a(x)}{a(y)} \\ p(x|y) = \frac{a(y)-p(y|\overline{x})a(\overline{x})}{a(y)} \end{cases} \tag{8}$$

$$\Longrightarrow \quad a(y) = p(y|x)a(x) + p(y|\overline{x})a(\overline{x}). \tag{9}$$

Eq.(6) emerges by inserting Eq.(9) in Eq.(2). ■

Note that Eq.(9) simply is an instance of the law of total probability.

The traditional formulation of Bayes' theorem of Eq.(1) is unnecessarily ambiguous and confusing because it does not distinguish between base rates (priors) and probabilities (posteriors), and because it does not show the dependency between the base rates of x and y. Bayes' theorem with marginal base rate expressed in Eq.(6) rectifies this problem by expressing the marginal base rate of y as a function of the base rate of x.

Let us revisit the example of the lottery, where the probability of winning given that you bought a ticket is $p(y|x) = 0.001$, and where intuition dictates that the probability of having bought a ticket given winning must be $p(x|y) = 1$. We assume that the probability of winning given no ticket is zero, expressed by $p(y|\overline{x}) = 0$. The correct answer then emerges directly from Eq.(6), expressed by

$$\text{Probability of ticket given win:} \quad p(x|y) = \frac{p(y|x)a(x)}{p(y|x)a(x) + p(y|\overline{x})a(\overline{x})} = 1. \tag{10}$$

In fact, neither the base rate of winning, nor the base rate of having bought a ticket, have any influence on the result, due to the fact that the probability of winning given no ticket is always zero.

III. SUBJECTIVE LOGIC

Subjective logic applies to uncertain probabilistic information in the form of *subjective opinions*, and defines a variety of operations for subjective opinions. In this section we present in detail the concept of subjective opinion, and, in particular, the *binomial opinion*, which is used in the subsequent sections for developing the generalised Bayes' theorem for binomial conditional opinions.

A. Opinion Representations

In subjective logic a *domain* is a state space consisting of two or more values. The values of the domain can e.g. be observable or hidden states, events, hypotheses or propositions, just like in traditional Bayesian modeling.

The available information about the particular value of the variable is typically of a probabilistic type, in the sense that we do not now the particular value, but we might know its probability. Probabilities express likelihoods with which the variable takes the specific values and their sum over the whole domain is 1. A variable together with a probability distribution defined on its domain is a *random variable*.

B. Binomial Opinions

Binomial opinions have a special notation which is used in the subsequent sections. The more general forms of multinomial or hypernomial opinions are not used here.

Let X be a random variable on domain $\mathbb{X} = \{x, \bar{x}\}$. The binomial opinion of agent A about variable X can be seen as an opinion about the truth of the statement "X is x" (denoted by $X = x$, or just x). This binomial opinion is expressed as the ordered quadruple

$$\omega_x^A = (b_x^A, d_x^A, u_x^A, a_x^A) \ , \quad \text{where} \tag{11}$$

b_x^A (*belief*)	belief mass in support of x,
d_x^A (*disbelief*)	belief mass in support of $\bar{x}$ (NOT x),
u_x^A (*uncertainty*)	uncertainty about probability of x,
a_x^A (*base rate*)	non-informative prior probability of x.

Additivity for binomial opinions is expressed as:

$$b_x^A + d_x^A + u_x^A = 1 \ . \tag{12}$$

The projected probability of a binomial opinion is expressed as:

$$\mathrm{P}_x^A = b_x^A + a_x^A u_x^A \ . \tag{13}$$

A binomial opinion can be represented as a point inside an equilateral triangle, which is a 3-axis barycentric coordinate system representing belief, disbelief, and uncertainty masses, with a point on the baseline representing base rate probability, as shown in Fig. 2. The axes run through the vertices along the altitudes of the triangle. The belief, disbelief, and uncertainty axes run through the vertices denoted by x, $\bar{x}$, and u, correspondingly, which have coordinates coordinates $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, correspondingly. In Fig. 2, $\omega_x = (0.20,\ 0.40,\ 0.40,\ 0.75)$, with projected probability $\mathrm{P}_x = 0.50$, is shown as an example. A strong positive opinion, for example, would be represented by a point towards the bottom right belief vertex.

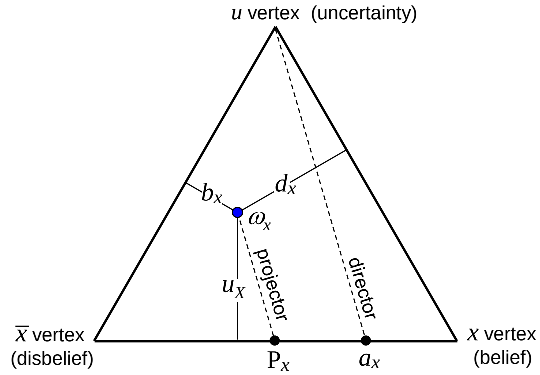

Fig. 2. Visualisation of a Binomial Opinion

In case the opinion point is located at the left or right vertex of the triangle, i.e. has $d_x = 1$ or $b_x = 1$ (and $u_x = 0$), the opinion is equivalent to boolean TRUE or FALSE, in which case subjective logic is reduced to binary logic. In case the opinion point is located on the base line of the triangle, i.e. has $u_x = 0$, then the opinion is equivalent to a probability distribution, in which case subjective logic is reduced to probability calculus. A vacuous binomial opinion denoted $\widehat{\omega}_x$ has $u_x = 1$ and is equivalent to a uniform PDF.

C. Marginal Base Rate

Conditional deduction with binomial opinions has previously been described in [8], and is not included here due to limited space. However, that description did not include the marginal base rate requirement of Eq.(17) below.

In general, the base rate of x and the conditionals on y put constraints on the base rate of y. The marginal base rate requirement is to derive a specific base rate, as described here. The expression for the base rate a_y in Eq.(19) and Eq.(20) is derived from the marginal base rate requirement of Eq.(14):

$$a_y = \mathrm{P}(y|x)\, a_x + \mathrm{P}(y|\bar{x})\, a_{\bar{x}} \ . \tag{14}$$

Assuming that $\omega_{y|x}$ and $\omega_{y|\bar{x}}$ are not both vacuous, i.e. that $u_{y|x} + u_{y|\bar{x}} < 2$ the simple expression for the marginal base rate a_y can be derived as follows:

$$a_y = \mathrm{P}(y|x)\, a_x + \mathrm{P}(y|\bar{x})\, a_{\bar{x}} \tag{15}$$

$$\Leftrightarrow \quad a_y = (b_{y|x} + a_y u_{y|x})\, a_x + (b_{y|\bar{x}} + a_y u_{y|\bar{x}})\, a_{\bar{x}} \tag{16}$$

$$\Leftrightarrow \quad a_y = \frac{b_{y|x}\, a_x + b_{y|\bar{x}}\, a_{\bar{x}}}{1 - u_{y|x}\, a_x - u_{y|\bar{x}}\, a_{\bar{x}}} \ . \tag{17}$$

With the marginal base rate of Eq.(17) it is guaranteed that the projected probabilities of binomial conditional opinions do not change after multiple inversions.

In case $\omega_{y|x}$ and $\omega_{y|\bar{x}}$ are both vacuous, i.e. when $u_{y|x} = u_{y|\bar{x}} = 1$, then there is no constraint on the base rate a_y.

Figure 3 is a screenshot of binomial deduction, involving a marginal base rate, which is equal to the deduced projected probability given a vacuous antecedent $\widehat{\omega}_x$, according to the requirement of Eq.(17).

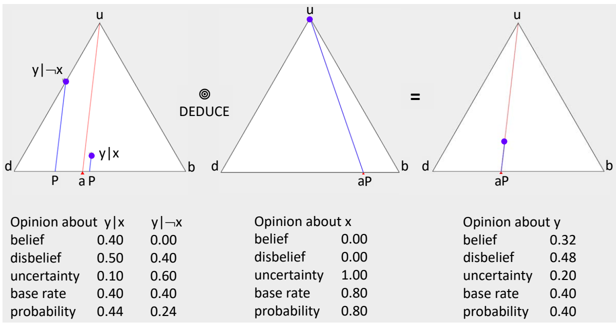

Fig. 3. Screenshot of deduction with vacuous antecedent $\widehat{\omega}_x$ and marginal base rate $a_y = 0.40$

To intuitively see why a marginal base rate is necessary, consider the case of a pair of dogmatic conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$, where both projected probabilities are $\mathrm{P}(y|x) = \mathrm{P}(y|\overline{x}) = 1$. In this trivial case we always have $\mathrm{P}(y) = 1$, independently of the projected probability $\mathrm{P}(x)$. It would then be totally inconsistent to e.g. have base rate $a_y = 0.5$ when we always have $\mathrm{P}(y) = 1$. The base rate must reflect reality, so the only consistent base rate in this case is $a_y = 1$, which emerges directly from Eq.(17).

D. Notation for Binomial Deduction and Abduction

This section simply introduces the notation used for conditional deduction and abduction in subjective logic. The notation is similar to the corresponding notation for probabilistic deduction and abduction. The detailed mathematical description of the deduction operators described in [8], [6]. The operator for abduction is based on that of deduction, as expressed in Eq.(47).

Let $\mathbb{X} = \{x, \overline{x}\}$ and $\mathbb{Y} = \{y, \overline{y}\}$ be two binary domains with respective variables X and Y, where there is a degree of relevance between X and Y. Let the analyst's respective opinions about x being true, about y being true given that x is true, and finally about y being true given that x is false be expressed as

$$\omega_x = (b_x, d_x, u_x, a_x), \tag{18}$$

$$\omega_{y|x} = (b_{y|x}, d_{y|x}, u_{y|x}, a_y) \tag{19}$$

$$\omega_{y|\overline{x}} = (b_{y|\overline{x}}, d_{y|\overline{x}}, u_{y|\overline{x}}, a_y) \tag{20}$$

Conditional deduction is computed with the deduction operator denoted '$\circledcirc$', so that binomial deduction is denoted

$$\text{Binomial deduction:} \quad \omega_{y\|x} = (\omega_{y|x}, \omega_{y|\overline{x}}) \circledcirc \omega_x \ . \tag{21}$$

Conditional abduction is computed with the abduction operator denoted '$\widetilde{\circledcirc}$', so that binomial abduction is denoted

$$\begin{aligned}\text{Binomial abduction: } \omega_{x\widetilde{\|}y} &= (\omega_{y|x}, \omega_{y|\overline{x}}) \mathbin{\widetilde{\circledcirc}} (a_x, \omega_y) \\ &= ((\omega_{y|x}, \omega_{y|\overline{x}}) \mathbin{\widetilde{\phi}} a_x) \circledcirc \omega_y \\ &= (\omega_{\widetilde{x|y}}, \omega_{\widetilde{x|\overline{y}}}) \circledcirc \omega_y .\end{aligned} \tag{22}$$

The conditionally abduced opinion $\omega_{x\widetilde{\|}y}$ expresses the belief about x as a function of the beliefs about y, the two conditionals $y|x$ and $y|\overline{x}$, as well as the base rate a_x.

In order to compute Eq.(22), it is necessary to invert the conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ using the generalised binomial Bayes' theorem. Abduction consists of applying the inverted conditional opinions $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ as for deduction according to Eq.(21).

IV. PREVIOUS ATTEMPTS OF REPRESENTING BAYES' THEOREM WITH BELIEFS

An early attempt at articulating belief-based conditional reasoning was provided by Xu & Smets [16], [15]. This approach is based on "Smets' generalised Bayes theorem" [14] as well as the Disjunctive Rule of Combination which are defined within the Dempster-Shafer belief theory [13].

In the binary case, Smets' approach assumes a conditional connection between a binary parent frame Θ and a binary child frame X defined in terms of belief masses and conditional plausibilities.

In Smets' approach, binomial abduction is defined as:

$$\begin{aligned} pl(\theta) = \;& m(x)pl(x|\theta) + m(\overline{x})pl(\overline{x}|\theta) + m(X)(pl(X|\theta))) \ , \\ pl(\overline{\theta}) = \;& m(x)pl(x|\overline{\theta}) + m(\overline{x})pl(\overline{x}|\overline{\theta}) + m(X)pl(X|\overline{\theta}))) \ , \\ pl(\Theta) = \;& m(x)(1-(1-pl(x|\theta))(1-pl(x|\overline{\theta}))) \\ & + m(\overline{x})(1-(1-pl(\overline{x}|\theta))(1-pl(\overline{x}|\overline{\theta}))) \\ & + m(X)(1-(1-pl(X|\theta))(1-pl(X|\overline{\theta}))) \ . \end{aligned} \tag{23}$$

Eq.(23) fails to take the base rates (prior probabilities) on Θ into account, which means that this method of abduction unavoidably falls victim to the base rate fallacy [2], [10], [12] and is therefore inconsistent with Bayes' theorem. "Smets' generalized Bayes' theorem" is therefore clearly not a generalisation of Bayes' theorem.

V. BINOMIAL SUBJECTIVE BAYES' THEOREM

This section describes the underlying mathematics of the binomial subjective Bayes' theorem.

A. Principles for Inverting Binomial Conditional Opinions

Assume that the available conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ are expressed in the opposite direction to that needed for applying the deduction operator of Eq.(24), here denoted

$$\omega_{x\|y} = (\omega_{x|y}, \omega_{x|\overline{y}}) \circledcirc \omega_y \ . \tag{24}$$

Binomial abduction simply consists of first inverting the pair of available conditionals $(\omega_{y|x}, \omega_{y|\overline{x}})$ to produce the pair of inverted conditionals $(\omega_{x|y}, \omega_{x|\overline{y}})$, and subsequently using these as input to binomial deduction.

Figure 4 illustrates the principle of conditional inversion, in the simple case of the conditionals $\omega_{y|x} = (0.80, 0.20, 0.00, 0.50)$ and $\omega_{y|\overline{x}} = (0.20, 0.80, 0.00, 0.50)$, and where $a_x = 0.50$. The inversion produces the pair of conditional opinions $\omega_{y|x} = (0.72, 0.12, 0.16, 0.50)$ and $\omega_{y|\overline{x}} = (0.16, 0.72, 0.12, 0.50)$, which are computed with the method of Definition 2 described below.

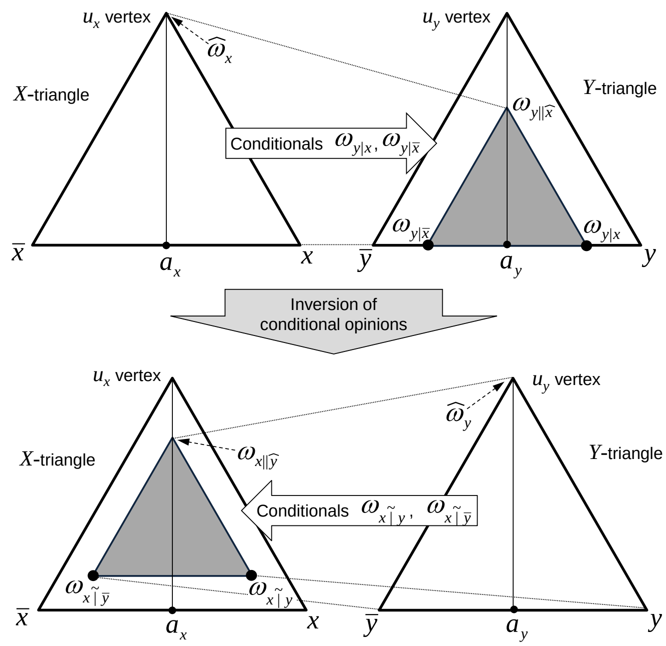

Fig. 4. Inversion of Binomial Conditionals

The shaded sub-triangle defined by the conditionals within the Y-triangle in the upper half of Figure 4 represents the image area for possible deduced opinions $\omega_{y\|x}$.

The lower half of Figure 4 illustrates how the inverted conditionals define the shaded sub-triangle within the X-sub-triangle, which represents the image area for possible abduced opinions $\omega_{x\|y}$.

Note that in general, inversion produces increased uncertainty mass, as seen by the higher position of the shaded sub-triangle on the lower half of Figure 4.

The projected probabilities of the available conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ are

$$\begin{cases} \mathrm{P}(y|x) &= b_{y|x} + a_y u_{y|x} \ , \\ \mathrm{P}(y|\overline{x}) &= b_{y|\overline{x}} + a_y u_{y|\overline{x}} \ . \end{cases} \quad (25)$$

The projected probabilities of the inverted conditionals $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ are computed using the results of Eq.(25) and the base rate a_x:

$$\begin{cases} \mathrm{P}(x|y) &= \frac{\mathrm{P}(y|x)\,a_x}{\mathrm{P}(y|x)\,a_x + \mathrm{P}(y|\overline{x})\,a_{\overline{x}}}, \\ \mathrm{P}(x|\overline{y}) &= \frac{\mathrm{P}(\overline{y}|x)\,a_x}{\mathrm{P}(\overline{y}|x)\,a_x + \mathrm{P}(\overline{y}|\overline{x})\,a_{\overline{x}}}. \end{cases} \quad (26)$$

A pair of dogmatic conditional opinions can be synthesised from the projected probabilities of Eq.(26):

$$\begin{cases} \underline{\omega}_{x|y} = (\mathrm{P}(x|y), \mathrm{P}(\overline{x}|y), 0, a_x), \\ \underline{\omega}_{x|\overline{y}} = (\mathrm{P}(x|\overline{y}), \mathrm{P}(\overline{x}|\overline{y}), 0, a_x). \end{cases} \quad (27)$$

where $\mathrm{P}(\overline{x}|y) = (1 - \mathrm{P}(x|y))$ and $\mathrm{P}(\overline{x}|\overline{y}) = (1 - \mathrm{P}(x|\overline{y}))$.

The pair of dogmatic conditionals $\underline{\omega}_{x|y}$ and $\underline{\omega}_{x|\overline{y}}$ of Eq.(27) and the pair of inverted conditional opinions $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ have by definition equal projected probabilities. However, $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ do in general contain uncertainty, in contrast to $\underline{\omega}_{x|y}$ and $\underline{\omega}_{x|\overline{y}}$ which are void of uncertainty. The inverted conditional opinions $\omega_{x|y}$ and $\omega_{x|\overline{y}}$ are derived from the dogmatic opinions of Eq.(27) by determining their appropriate amounts of uncertainty mass.

B. Relevance and Irrelevance

This section formally defines the concept of relevance, both for probability distributions and for opinions. The definition of probabilistic relevance is given below.

In the case of binary probabilistic conditionals $p(y|x)$ and $p(y|\overline{x})$, the expression for relevance is expressed as

$$\Psi(y|X) = |p(y|x) - p(y|\overline{x})| \ . \quad (28)$$

The concept of relevance can be extended to conditional subjective opinions, simply by projecting conditional opinions to their corresponding projected probability functions, and applying Eq.(28).

Definition 1 (Subjective Relevance): Assume a pair of conditional opinions $\omega_{y|x}$ and $\omega_{y|\overline{x}}$, where each conditional opinion has a corresponding projected probability $\mathrm{P}_{y|x}$ and $\mathrm{P}_{y|\overline{x}}$. The relevance of X to each y is expressed as

$$\Psi(y|X) \ = \ |\mathrm{P}_{y|x} \ - \ \mathrm{P}_{y|\overline{x}}| \ . \quad (29)$$

Relevance is equivalent to *diagnosticity* which is a central concept in ACH (analysis om competing hypotheses) [3], [1]. Diagnosticity is the power of influence of a given evidence variable X over the truth of a specific hypothesis y. Through inversion of conditionals the same diagnosticity translates into diagnosticity of (symptom) variable Y over the truth of a specific hypothesis x.

It is useful to also define *irrelevance*, $\overline{\Psi}(y|X)$, as the complement of relevance:

$$\overline{\Psi}(y|X) = 1 - \Psi(y|X) \ . \tag{30}$$

The irrelevance $\overline{\Psi}(y|X)$ expresses the lack of diagnostic power of the conditionals $\omega_{y|x}$ and $\omega_{y|\bar{x}}$ over the value y, leading to uncertainty when applying the subjective Bayes' theorem.

Note that because X and Y are binary variables, where X takes its values from $\mathbb{X} = \{x, \bar{x}\}$ and Y takes its values from $\mathbb{Y} = \{y, \bar{y}\}$, we get the same relevance and irrelevance values for the two values of y and $\bar{y}$. We can therefore denote relevance of X to y and X to $\bar{y}$ by $\Psi(y|X)$ in both cases.

C. Uncertainty Mass of Inverted Binomial Conditionals

The amount of uncertainty mass in inverted binomial conditional opinions is a function of the following factors:

- The maximum uncertainty values $\ddot{u}_{X|y}$ and $\ddot{u}_{X|\bar{y}}$,
- The weighted proportional uncertainty $u^{\mathrm{w}}_{y|X}$,
- The irrelevance $\overline{\Psi}(y|X)$ and $\overline{\Psi}(\bar{y}|X)$.

These factors are applied in the in four-step procedure described below.

Step 1: Maximum uncertainties $\ddot{u}_{x|y}$ and $\ddot{u}_{x|\bar{y}}$.

Figure 5 illustrates how the belief mass can be set to zero to determine the uncertainty-maximised conditional $\ddot{\omega}_{x|y}$.

The theoretical maximum uncertainties, $\ddot{u}_{X|y}$ for $\omega_{x|y}$, and $\ddot{u}_{X|\bar{y}}$ for $\omega_{x|\bar{y}}$, are determined by setting either the belief or the disbelief mass to zero, according to the simple IF-THEN-ELSE algorithm below.

Computation of $\ddot{u}_{X\|y}$	
IF	$\mathrm{P}(x\|y) < a_x$
THEN	$\ddot{u}_{x\|y} = \mathrm{P}(x\|y)/a_x$
ELSE	$\ddot{u}_{x\|y} = (1 - \mathrm{P}(x\|y))/(1 - a_x)$

(31)

Computation of $\ddot{u}_{X\|\bar{y}}$	
IF	$\mathrm{P}(x\|\bar{y}) < a_x$
THEN	$\ddot{u}_{X\|\bar{y}} = \mathrm{P}(x\|\bar{y})/a_x$
ELSE	$\ddot{u}_{X\|\bar{y}} = (1 - \mathrm{P}(x\|\bar{y}))/(1 - a_x)$

(32)

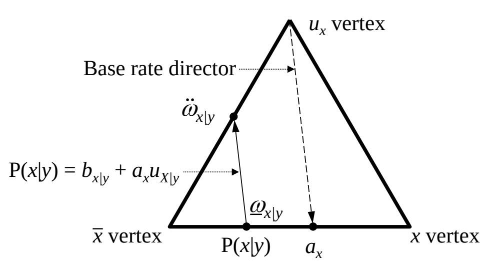

Fig. 5. Dogmatic Conditional $\underline{\omega}_{x|y}$, and Corresponding Uncertainty-Maximised Conditional $\ddot{\omega}_{x|y}$

Step 2: Weighted Proportional Uncertainty $u^{\mathrm{w}}_{y|X}$.

The sum of conditional uncertainty $u^{\Sigma}_{y|X}$ is computed as

$$u^{\Sigma}_{y|X} = u_{y|x} + u_{y|\bar{x}} \ . \tag{33}$$

The proportional uncertainty weights $\mathrm{w}^{u}_{y|x}$ and $\mathrm{w}^{u}_{y|\bar{x}}$ are computed as

$$\begin{cases} \mathrm{w}^{u}_{y|x} = \frac{u_{y|x}}{u^{\Sigma}_{y|X}} & \text{for } u^{\Sigma}_{y|X} > 0, \\ \mathrm{w}^{u}_{y|x} = 0 & \text{for } u^{\Sigma}_{y|X} = 0, \end{cases} \tag{34}$$

$$\begin{cases} \mathrm{w}^{u}_{y|\bar{x}} = \frac{u_{y|\bar{x}}}{u^{\Sigma}_{y|X}} & \text{for } u^{\Sigma}_{y|X} > 0, \\ \mathrm{w}^{u}_{y|\bar{x}} = 0. & \text{for } u^{\Sigma}_{y|X} = 0 \end{cases} \tag{35}$$

We also need the maximum theoretical uncertainty $\ddot{u}_{y|x}$ and $\ddot{u}_{y|\bar{x}}$. The theoretical maximum uncertainty masses $\ddot{u}_{y|x}$ and $\ddot{u}_{y|\bar{x}}$ are determined by setting either the belief or the disbelief mass to zero, according to the simple IF-THEN-ELSE algorithm below.

Computation of $\ddot{u}_{y\|x}$	
IF	$\mathrm{P}(y\|x) < a_y$
THEN	$\ddot{u}_{y\|x} = \mathrm{P}(y\|x)/a_y$
ELSE	$\ddot{u}_{y\|x} = (1 - \mathrm{P}(y\|x))/(1 - a_y)$

(36)

Computation of $\ddot{u}_{y\|\bar{x}}$	
IF	$\mathrm{P}(y\|\bar{x}) < a_y$
THEN	$\ddot{u}_{y\|\bar{x}} = \mathrm{P}(y\|\bar{x})/a_y$
ELSE	$\ddot{u}_{y\|\bar{x}} = (1 - \mathrm{P}(y\|\bar{x}))/(1 - a_y)$

(37)

The weighted proportional uncertainty components $u^{\mathrm{w}}_{y|x}$ and $u^{\mathrm{w}}_{y|\bar{x}}$ are computed as

$$\begin{cases} u^{\mathrm{w}}_{y|x} = \frac{\mathrm{w}^{u}_{y|x} u_{y|x}}{\ddot{u}_{y|x}} & \text{for } \ddot{u}_{y|x} > 0, \\ u^{\mathrm{w}}_{y|x} = 0 & \text{for } \ddot{u}_{y|x} = 0, \end{cases} \tag{38}$$

$$\begin{cases} u^{\mathrm{w}}_{y|\bar{x}} = \frac{\mathrm{w}^{u}_{y|\bar{x}} u_{y|\bar{x}}}{\ddot{u}_{y|\bar{x}}}, & \text{for } \ddot{u}_{y|\bar{x}} > 0, \\ u^{\mathrm{w}}_{y|\bar{x}} = 0 & \text{for } \ddot{u}_{y|\bar{x}} = 0. \end{cases} \tag{39}$$

The weighted proportional uncertainty $u^{\mathrm{w}}_{y|X}$ can then be computed as

$$u^{\mathrm{w}}_{y|X} = u^{\mathrm{w}}_{y|x} + u^{\mathrm{w}}_{y|\bar{x}} \ . \tag{40}$$

Step 3: Relative Uncertainties $\widetilde{u}_{x|y}$ and $\widetilde{u}_{x|\bar{y}}$.

The relative uncertainties $\widetilde{u}_{x|y}$ and $\widetilde{u}_{x|\bar{y}}$ are computed as

$$\begin{aligned} \widetilde{u}_{x|y} = \widetilde{u}_{x|\bar{y}} &= u^{\mathrm{w}}_{y|X} \sqcup \overline{\Psi}(y|X) \\ &= u^{\mathrm{w}}_{y|X} + \overline{\Psi}(y|X) - u^{\mathrm{w}}_{y|X}\, \overline{\Psi}(y|X). \end{aligned} \tag{41}$$

The relative uncertainty $\widetilde{u}_{X|y}$ is an increasing function of the weighted proportional uncertainty $u^{\mathrm{w}}_{y|X}$, because uncertainty in the initial conditionals is reflected by uncertainty in the inverted conditionals. A practical example is when Alice is ignorant about whether Bob carries an umbrella in sunny or rainy weather. Then observing Bob carrying an umbrella provides no information about the weather.

$\widetilde{u}_{x|y}$ is also an increasing function of the irrelevance $\overline{\Psi}(y|X)$, because if $\omega_{y|x}$ and $\omega_{y|\bar{x}}$ reflect irrelevance, then there is no basis for deriving belief about the inverted conditionals $\omega_{x|y}$ and $\omega_{x|\bar{y}}$, so they must be uncertainty-maximised. A practical example is when Alice knows that Bob always carries an umbrella both in rain and sun. Then observing Bob carrying an umbrella tells her nothing about the weather.

The relative uncertainty $\widetilde{u}_{x|y}$ is thus high in case the weighted proportional uncertainty $u^{\mathrm{w}}_{y|X}$ is high, or the irrelevance $\overline{\Psi}(y|X)$ is high, or both are high at the same time. This principle is modelled by computing the relative uncertainty $\widetilde{u}_{x|y}$ as the disjunctive combination of $u^{\mathrm{w}}_{y|X}$ and $\overline{\Psi}(y|X)$, denoted by the coproduct operator $\sqcup$ in Eq.(41). Note that in the binomial case we have $\widetilde{u}_{x|\bar{y}} = \widetilde{u}_{x|y}$.

Step 4: Uncertainty of inverted opinions.

Having computed $\ddot{u}_{X|y}$ and the relative uncertainty $\widetilde{u}_{X|y}$, the uncertainty levels $u_{x|y}$ and $u_{x|\bar{y}}$ can finally be computed:

$$\text{Inverted binomial uncertainty:} \begin{cases} u_{x\widetilde{|}y} = \ddot{u}_{X|y}\widetilde{u}_{X|y}, \\ \\ u_{x\widetilde{|}\bar{y}} = \ddot{u}_{X|\bar{y}}\widetilde{u}_{X|\bar{y}}. \end{cases} \quad (42)$$

This marks the end of the four-step procedure for the computation of uncertainty.

D. Deriving Binomial Inverted Conditionals

Having determined the uncertainty levels in the four-step procedure of Section V-C the computation of the inverted opinions is straightforward:

$$\omega_{x\widetilde{|}y} = \begin{cases} b_{x\widetilde{|}y} = \mathrm{P}(x|y) - a_x u_{x|y}\ , \\ d_{x\widetilde{|}y} = 1 - b_{x|y} - u_{x|y}\ , \\ u_{x\widetilde{|}y}\ , \\ a_x\ , \end{cases} \quad (43)$$

$$\omega_{x|\bar{y}} = \begin{cases} b_{x|\bar{y}} = \mathrm{P}(x|\bar{y}) - a_x u_{x|\bar{y}}\ , \\ d_{x|\bar{y}} = 1 - b_{x|\bar{y}} - u_{x|\bar{y}}\ , \\ u_{x|\bar{y}}\ , \\ a_x\ . \end{cases} \quad (44)$$

The process of inverting conditional opinions as described above is in fact Bayes' theorem generalised for binomial conditional opinions, as articulated in Definition 2:

Definition 2 (Binomial Subjective Bayes' Theorem): Let $(\omega_{y|x},\ \omega_{y|\bar{x}})$ be a pair of binomial conditional opinions, and let a_x be the base rate of x. The pair of conditional opinions, denoted $(\omega_{x\widetilde{|}y},\ \omega_{x\widetilde{|}\bar{y}})$, derived through the procedure described above, are the inverted binomial conditional opinions of the former pair. The symbol '$\widetilde{\phi}$' denotes the operator for conditional inversion and thereby for the generalised Bayes' theorem, so inversion of a pair of binomial conditional opinions can be expressed as

$$\text{Binomial Subjective Bayes' theorem:} \quad (\omega_{x\widetilde{|}y},\ \omega_{x\widetilde{|}\bar{y}}) = (\omega_{y|x},\ \omega_{y|\bar{x}})\ \widetilde{\phi}\ a_x\ . \quad (45)$$

□

The reason why both $\omega_{y|x}$ and $\omega_{y|\bar{x}}$ are needed in the binomial subjective Bayes' theorem of Definition 2 is that the two conditionals together are required to determine the uncertainty masses of the inverted conditionals $\omega_{x|y}$ and $\omega_{x|\bar{y}}$.

It can be shown that Bayes' theorem of Eq.(6) applied to the projected conditional probabilities $\mathrm{P}_{y|x}$ and $\mathrm{P}_{y|\bar{x}}$ produces the projected probabilities $\mathrm{P}_{x|y}$ and $\mathrm{P}_{x|\bar{y}}$, which means that Bayes' theorem is a special case of the subjective Bayes' theorem described above.

The application of inverted binomial conditionals for binomial conditional deduction is in fact binomial abduction. The difference between inversion and abduction is thus that abduction takes the evidence argument ω_y, whereas inversion does not. Eq.(46) and Eq.(47) compare the two operations.

$$\text{Inversion (Bayes' theorem):} \quad (\omega_{x\widetilde{|}y},\ \omega_{x\widetilde{|}\bar{y}}) = (\omega_{y|x},\ \omega_{y|\bar{x}})\ \widetilde{\phi}\ a_x\ , \quad (46)$$

$$\begin{aligned} \text{Abduction:} \quad \omega_{x\widetilde{\|}y} &= (\omega_{y|x},\ \omega_{y|\bar{x}})\ \widetilde{\circledcirc}\ (a_x,\ \omega_y) \\ \text{(using} \quad &= ((\omega_{y|x},\ \omega_{y|\bar{x}})\ \widetilde{\phi}\ a_x)\ \circledcirc\ \omega_y \\ \text{Bayes' theorem)} \quad &= (\omega_{x\widetilde{|}y},\ \omega_{x\widetilde{|}\bar{y}})\ \circledcirc\ \omega_y\ . \end{aligned} \quad (47)$$

E. Convergence of Repeated Inversions

An interesting question is, what happens when conditionals are repeatedly inverted? In the case of probabilistic logic, which is uncertainty agnostic, the inverted conditionals always remain the same after repeated inversion. This can be formally expressed as

$$\begin{aligned} (p(y|x),\ p(y|\bar{x})) &= (p(x|y),\ p(x|\bar{y}))\ \widetilde{\phi}\ a(y) \\ &= ((p(y|x),\ p(y|\bar{x}))\ \widetilde{\phi}\ a(x))\ \widetilde{\phi}\ a(y)\ . \end{aligned} \quad (48)$$

In the case of opinions with marginal base rates, the projected probabilities of conditional opinions also remain the same. However, repeated inversion of conditional opinions increases uncertainty in general.

The increasing uncertainty is of course limited by the theoretical maximum uncertainty mass for each conditional. In general, the uncertainty mass of conditional opinions converges towards their theoretical maximum, as inversions are repeated infinitely many times.

Figure 6 illustrates the process of repeated inversion of conditionals, based on the same example as in Figure 4, where the initial conditionals are $\omega_{y|x} = (0.80, 0.20, 0.00, 0.50)$ and $\omega_{y|\bar{x}} = (0.20, 0.80, 0.00, 0.50)$, and where the equal base rates are $a_x = a_y = 0.50$.

Table I lists a selection of the computed conditional opinions $\omega_{y|x}$ and $\omega_{x|y}$, consisting of: i) the convergence conditional opinion, ii) the first eight inverted conditional opinions, and iii) the initial conditional opinion, in that order.

The uncertainty increases relatively fast in the first few inversions, and rapidly slows down. Two convergence conditional opinions are $\omega_{y|x} = \omega_{x\widetilde{|}y} = (0.60, 0.00, 0.40, 0.50)$. The two others are $\omega_{y|\bar{x}} = \omega_{x\widetilde{|}\bar{y}} = (0.00, 0.60, 0.40, 0.50)$. The inverted opinions were computed with an office spreadsheet,

TABLE I
SERIES OF INVERTED CONDITIONAL OPINIONS

	Index	Opinion		Belief	Disbelief	Uncertainty	Base rate
Convergence	∞	$\ddot{\omega}_{y\tilde{\vert}x} = \ddot{\omega}_{x\tilde{\vert}y}$	=	(0.6,	0.0,	0.4,	0.5)
	.			…	…	…	…
	8	$\omega_{y\tilde{\vert}^8 x}$	=	(0.603358,	0.003359,	0.393282,	0.5)
	7	$\omega_{x\tilde{\vert}^7 y}$	=	(0.605599,	0.005599,	0.388803,	0.5)
	6	$\omega_{y\tilde{\vert}^6 x}$	=	(0.609331,	0.009331,	0.381338,	0.5)
	5	$\omega_{x\tilde{\vert}^5 y}$	=	(0.615552,	0.015552,	0.368896,	0.5)
	4	$\omega_{y\tilde{\vert}^4 x}$	=	(0.62592,	0.02592,	0.34816,	0.5)
	3	$\omega_{x\tilde{\vert}^3 y}$	=	(0.6432,	0.0432,	0.3136,	0.5)
	2	$\omega_{y\tilde{\vert}^2 x}$	=	(0.672,	0.072,	0.256,	0.5)
	1	$\omega_{x\tilde{\vert}y}$	=	(0.72,	0.12,	0.16,	0.5)
Initial	0	$\omega_{y\vert x}$	=	(0.8,	0.2,	0.0 ,	0.5)

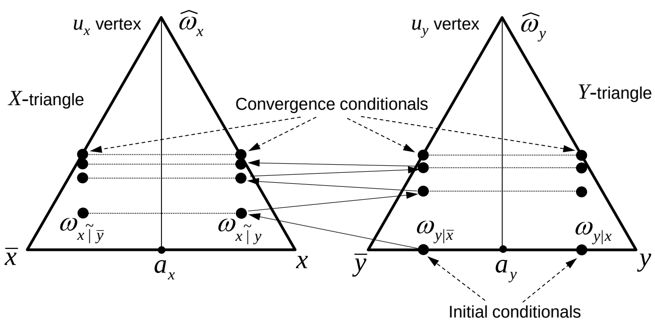

Fig. 6. Convergence of repeated inversion of binomial conditionals

which started rounding off results from index 6. The final convergence conditional was not computed with the spreadsheet, but was simply determined as the opinion with the theoretical maximum uncertainty.

The above example is rather simple, with its perfectly symmetrical conditionals and base rates of 1/2. However, the same pattern of convergence of increasing uncertainty occurs for arbitrary conditionals and base rates. In general, the two pairs of convergence conditionals are not equal. The equality in our example above is coincidentally due to the symmetric conditionals and base rates.

VI. DISCUSSION AND CONCLUSION

The generalisation of Bayes' theorem to binomial conditional opinions is useful because it enables the use of Bayesian classifiers and the modelling of Bayesian networks with subjective logic. The advantage of this approach is that classifiers and Baysian networks can be expressed with arbitrary degrees of uncertainty.

It is relatively straightforward to generalise Bayes' theorem to also apply to multinomial conditional opinions [7]. The approach is based on inverting multinomial opinions in a similar way to how it is described for binomial opinions above. The case of generalising Bayes' theorem to hypernomial opinions is more challenging, because the procedure for determining the uncertainty of inverted hypernomial opinions would be hard to formalise. This is then an interesting topic for future research.

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