Diffusivity and viscosity dependence in the linear thermocline

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ABSTRACT

We re-visit the linear model of a thermally-driven ocean in a single hemisphere, to understand dependencies on viscosity and mixing. We focus in particular on the vertical and surface velocities. In all cases, the meridional and vertical flows are intensified near the boundaries; as a result, the overturning depends on viscosity. The two types of viscosity we examine (Rayleigh damping and diffusion with the no-slip condition) yield significantly different boundary transports. This in turn can cause large changes in the surface velocities. We observe a single-gyre surface circulation with a diffusive viscosity but two gyres with the Rayleigh.

We also examine how the solutions change when the vertical mixing itself is intensified near the boundaries. With (spatially) constant mixing, a significant fraction of the vertical transport occurs in the thermocline interior where viscosity is unimportant. But the localized mixing increases the boundary intensification of the upwelling, making viscosity even more important. There is evidence of similar boundary intensification and viscosity dependence in numerical models.

1. Introduction

The buoyancy-driven ocean circulation has time scales of 10s to 1000s of years and spatial scales comparable to the earth’s radius, making it difficult to observe and monitor the system. As a result, much of our intuition derives from numerical models. Models permit us to examine parameter dependencies (e.g. freshwater input) and to evaluate, for example, the likelihood of a reversal in the meridional overturning circulation (MOC). But the range of scales is demanding on numerical simulations; while the largest spatial scales are global, there is important dissipation on scales of meters. At present we model the large
scales and parameterize the small. But how to parameterize small-scale dissipation is not well understood and worse, models employing different parameterizations often produce different results.

There is, for example, diapycnal mixing, which increases the oceanic potential energy (Munk, 1966; Munk and Wunsch, 1998; Huang, 1999). In early numerical studies, mixing was parameterized as a small-scale diffusion with a constant diffusivity (e.g. Bryan, 1987; Colin de Verdiere, 1988). Observations suggest however that mixing is intensified near boundaries and over rough topography (Polzin et al., 1997; Ledwell et al., 2000; Wuest and Lorke, 2003); models which employ boundary-intensified mixing yield qualitatively different flows from those with constant mixing (Marotzke, 1997; Samelson, 1998). Mixing also depends on the ambient stratification and taking this into account can change even the sign of the response to buoyancy forcing (Nilsson and Walin, 2001).

Then there is viscosity. Small-scale eddies dissipate momentum in the ocean and this too is parameterized in models. In a perfect world, the overturning would not depend on viscosity or the choice of boundary conditions (e.g. no-slip or free-slip). But numerical evidence suggests it does (e.g. Huck et al., 1999).

How do we understand such dependencies? Ideally we would turn to theoretical models which are easier to understand than numerical models. But most of our theoretical models are insufficient in this regard. Scaling theories (e.g. Bryan and Cox, 1967; Marotzke, 1997; Park and Bryan, 2001) treat the dependence on vertical diffusivity but do not incorporate viscosity or boundary conditions. The abyssal circulation models (Stommel and Arons, 1960; Kawase, 1987) are dynamically more complete, but they also rely on assumptions about the vertical flow into the (unmodeled) upper layers.

However, one type of analytical model exists which simulates the full water column and can be solved in a closed basin. This is the linear, planetary geostrophic (PG) model, described by Pedlosky (1969; P69 hereafter) and Salmon (1986; S86 hereafter). The model represents an idealized ocean basin, forced by heating/cooling and a surface wind stress. The model’s main simplification is that its density is linearized about a base state which is invariant in \((x, y)\). This inhibits realistically strong or deep overturning; however it does permit full basin solutions, something not yet possible with nonlinear analytical models. In small basins the linear model mimics well nonlinear numerical models (Pedlosky, 2003; Spall, 2003).

Here we revisit the linear model in the context of the large-scale circulation, to examine dependencies on mixing and viscosity. As in most numerical models, the vertical transport is intensified near the lateral boundaries; so that transport occurs in boundary layers and is therefore affected by viscosity. We consider two types of viscosity to illustrate the dependence. The model also permits a spatially-variable diffusivity and we examine the case of a diffusivity which is intensified near the lateral boundaries, as is thought to be the case in the ocean.
2. The model

The model equations and boundary layer formulations follow closely those of P69 and of S86 who considered flow driven by an imposed surface temperature gradient and surface winds. The primary differences are that we will ignore wind forcing and will use a spatially variable vertical diffusivity. In addition, we will treat two different types of viscosity: diffusion (as in P69) and Rayleigh damping (as in S86). Note Pedlosky (2003) likewise employed the linear model with only buoyancy forcing, but on the $f$-plane (in a small basin).

The model involves the linearized planetary geostrophic equations. An excellent discussion of these equations (as well as their shortcomings) is given by Salmon (1998). We consider a square ocean basin under the assumptions of (1) steady flow, (2) a small Rossby number, (3) a small aspect ratio, (4) a linear variation of the Coriolis parameter (the $\beta$-plane) and (5) a flat bottom. The nondimensional equations are then:

$$
-fv = -\partial_x \phi + \mathcal{D}_x
$$

$$
fu = -\partial_y \phi + \mathcal{D}_y
$$

$$
0 = -\partial_z \phi + \theta + \delta^2 \partial_z^2 \theta
$$

$$
\partial_x u + \partial_y v + \partial_z w = 0
$$

$$
Sw = \kappa_H \nabla^2 \theta + \kappa_v \partial_z^2 \theta
$$

where $(u, v, w)$ are the velocities, $\phi$ is the pressure, $f$ the Coriolis parameter, $\delta = D/L$ the aspect ratio and $\theta$ the perturbation to a background temperature profile, i.e.:

$$
T = (\Delta T_v)(z/D) + (\Delta T_h)\theta(x, y, z).
$$

The density equation is linearized by demanding $\Delta T_v \gg \Delta T_h$. As noted this cannot be justified rigorously (e.g. Salmon, 1998) but facilitates analytical progress. In addition, $S$ is the Burger number.

$$
S = \frac{\alpha g \Delta T_v D}{f^2 \delta L^2}
$$

We employ two types of viscosity, $\mathcal{D}$: a Rayleigh drag and diffusion. Density is only damped diffusively. An alternative is to use Rayleigh mixing in the density equation (S86); such mixing is dynamically consistent but requires the introduction of a body force in the density equation for consistency with the surface forcing. We choose instead to retain diffusive mixing.$^2$

With both viscosities we require no-normal flow at the lateral walls. With diffusion, we

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2. The PG numerical model of Samelson and Vallis (1997) likewise employs diffusive mixing with a Rayleigh viscosity.
impose in addition the no-slip condition, the most commonly used condition in numerical simulations. The domain is such that \( x = [0, 1], \ y = [y_0, 1] \) and \( z = [-1, 0] \). The southern boundary latitude, \( y_0 \), is a small but positive number (since the PG equations do not permit the Coriolis parameter to vanish).

The forcing is imposed via an imposed surface temperature distribution. In most numerical models the surface heat flux is applied by relaxing the surface temperature to a specified distribution with a certain relaxation time constant (Haney, 1971). The present forcing thus pertains to the limit when the relaxation occurs instantaneously.

The constants in the temperature equation in (1) can be evaluated using typical oceanic values of \( \alpha \approx 10^{-4} \deg^{-1}, \Delta T_v = 30 \deg, \ D = 5000 \text{ m}, \ L = 5 \times 10^6 \text{ m} \) and \( f_0 \approx 10^{-4} \text{ sec}^{-1} \). These yield \( S \approx 5 \times 10^{-4} \). The vertical diffusivity can range from \( 10^{-5} \) to \( 10^{-2} \text{ m}^2/\text{sec} \); a value of \( 10^{-4} \) yields a scaled \( \kappa_v = \kappa_v^* f_0 D^2 \approx 5 \times 10^{-8} \). For the horizontal diffusivity, a typical value of \( 10^3 \text{ m}^2/\text{sec} \) yields a scaled \( \kappa_H = \kappa_H^* f_0 L^2 \approx 10^{-7} \). So the scaled diffusivities are similar in magnitude.

If Rayleigh damping is used in the momentum equations, then

\[
\mathcal{D} = -\epsilon \mathbf{u}.
\]

Using quoted values of the Rayleigh damping coefficient (e.g. Huck et al., 1999), the rescaled \( \epsilon \) is on the order of \( 10^{-2} \). If diffusion is used, then

\[
\mathcal{D} = E_H \nabla^2 \mathbf{u},
\]

(we neglect vertical momentum diffusion). Scaling a typical dimensional value by \( f_0 L^2 \) yields a horizontal Ekman number of \( E_H \approx 5 \times 10^{-5} \). As such, the horizontal Prandtl number, \( \sigma = E_H / \kappa_H \), is of order 100.

Taking the curl of the horizontal momentum equations and invoking continuity yields the vorticity relation:

\[
\beta \nu = f_0 w + \nabla \times \mathcal{D}.
\]

Dissipation permits no-normal flow at the western, northern and southern boundaries. With the Rayleigh viscosity, there is no boundary layer on the eastern wall, so the interior solution must itself satisfy no-normal flow there. With a diffusive viscosity, a boundary layer is required on the eastern wall to satisfy \( \nu = 0 \) (see below). In addition, there is a nonhydrostatic layer on all lateral walls which is required to bring the vertical velocity to rest. But because the aspect ratio, \( \delta \), is small, this layer is very thin and contributes only negligibly to the total overturning. As such, the no-slip condition on \( w \) will be neglected hereafter.4

3. No-slip also ensures no heat transport through the lateral walls.
4. Salmon (S86, 1998) finds that for sufficiently strong stratification, the vertical friction term is negligible in the linear solution. However, it can become important in nonlinear solutions where the stratification is weak, e.g. at high latitudes.
3. Solution-constant $\kappa_V$

a. Abyss

The ratios $\kappa_V/S$ and $\kappa_H/S$ in the temperature equation in (1) are small. So the temperature equation implies $w = 0$ away from the surface and the lateral boundaries. It follows that $v = 0$ in the abyss, from the vorticity equation (2). One can show that the deep zonal flow also must be zero, to satisfy no-normal flow at the lateral boundaries. So the thermally-forced circulation is trapped near the surface.

b. The thermocline interior

Near the surface, the imposed temperature distribution produces a thermocline boundary layer. Away from the lateral walls, we may neglect viscosity and horizontal mixing; only vertical mixing will enter. Combining the vorticity and temperature equations from (1) yields the thermocline equation:

$$\partial_t \phi_f - \frac{f^2 \kappa_V}{\beta S} \partial_z^4 \phi_f = 0,$$

written in terms of the pressure, $\phi$. Scaling this (Stommel and Veronis, 1957), we obtain the thermocline thickness:

$$l_T = \left( \frac{\kappa_V}{S} \right)^{1/4}.$$

If $\kappa_V = 10^{-4} \text{ m}^2/\text{sec}$, the nondimensional thermocline depth is about 0.1, or 500 m in a 5000 m deep basin. Note that the depth depends only weakly on $\kappa_V$; with $\kappa_V = 10^{-5} \text{ m}^2/\text{sec}$, the thermocline still has a thickness $l_T = 0.06$ or about 280 m.

Scaling Eqs. (1) with a vertical scale of $l_T$ yields:

$$\theta_f = O(1), \quad u_T, v_T, \phi_T \propto l_T, \quad w_T \propto l_T^2.$$

The thermocline temperature anomaly is order one, to match the surface forcing. Note the horizontal velocities are greater than the vertical velocity by $O(l_T)$. Note too that the area-integrated vertical velocity is $O(l_T^2)$.

Eq. (3) is separable and because the bottom is flat, the variables may be decomposed into vertical modes:

$$\phi_f(x, y, n) = 2 \int_{-1}^0 \phi_f(x, y, z) \cos (n \pi z) dz.$$

Then (3) becomes:

$$\partial_t \phi_f - \alpha_n \phi_f = -\frac{2\alpha_n}{n^2 \pi^2} T_0,$$

(5)
(after invoking the hydrostatic relation); here $T_0$ is the nondimensional surface temperature distribution and
\[
\alpha_n = \frac{f^3 \kappa_s n^4 \pi^4}{\beta S}.
\] (6)

For simplicity, we will assume $T_0 = T_0(y)$ (although this is not necessary). The horizontal velocities are likewise transformed via the cosine transform but the temperature and the vertical velocity by the sine transform.

Eq. (5) implies that the depth-averaged flow ($n = 0$) must be zero. This follows from the vertical integral of (3), using $w = 0$ at the surface and bottom, and the no-flow condition at the eastern wall. So the flow is purely baroclinic (P69; S86; Colin de Verdiere, 1988). The general solution to (5) is:
\[
\phi_I(x, y, n) = \frac{2T_0}{n^2 \pi} \left(1 - e^{\alpha_n(x-1)}\right) + C_1(n)e^{\alpha_n(x-1)}.
\] (7)

The second term is called the “free mode” of the interior solution. The constant, $C_1$, determines the temperature on the eastern wall. It is independent of $y$, to preclude normal flow at $x = 1$, and cannot be determined at this point.

Solution (7) does not satisfy the no-slip condition on $v$ at the east wall. So, with a diffusive viscosity, we require a boundary layer there. One can show the layer width is $E_H^{1/3}$ and as such, horizontal mixing dominates vertical mixing. The following scalings apply in the layer:
\[
\theta_E \propto E_H^{1/3}, \quad u_E \propto l_T E_H^{1/3}, \quad v_E \propto l_T, \quad w_T \propto l_T^4 E_H^{-1/3},
\] (8)

assuming $\kappa_H$ is not greatly different from $\kappa_V$. Note $v_E$ is $O(l_T)$ to match $v_T$ in the interior. The integrated vertical velocity is very small, of $O(l_T^4)$. In fact, it is straightforward to show that the integrated vertical transport is actually zero (the same is true in the western layer under the no-slip condition; Sec. 3d). So the eastern layer cannot contribute to the overturning and we may neglect it (we relax the no-slip condition on $v$ at $x = 1$).

Notice that the meridional velocity implied by (7) is:
\[
v_I(x, y, n) = \frac{\alpha_n}{f} \left( C_1 - \frac{2T_0}{n^2 \pi} \right) e^{\alpha_n(x-1)}
\] (9)

and the vertical velocity is:
\[
w_I(x, y, n) = \frac{n^3 \pi^3 \kappa_v}{S} \phi_I - \frac{2n \pi \kappa_v}{S} T_0 = \frac{n^3 \pi^3 \kappa_v}{S} \left( C_1 - \frac{2T_0}{n^2 \pi} \right) e^{\alpha_n(x-1)}.
\] (10)

So the meridional and vertical flow in the thermocline is eastern-intensified. From (6), the decay from the eastern boundary is more pronounced in the north and with larger diffusivities.
The remainder of the solution involves the viscous boundary layers which close the circulation at the northern, western and southern walls.

c. Northern, southern boundaries

The boundary solutions at the north and south walls are essentially identical, so we need only treat the northern wall. We consider the diffusive viscosity first.

i. Diffusion. The boundary layer equation comes from the (viscous) vorticity and temperature equations:

\[ \frac{\partial \phi_N}{\partial t} - \frac{f_1^2 \kappa_H}{\beta S} \phi_N \partial_{\xi}^2 \phi_N - \frac{f_1^2 \kappa_V}{\beta S} \phi_N \partial_{\eta}^2 \phi_N - \frac{E_H}{\beta} \partial_{\eta}^4 \phi_N = 0, \]  

(11)

where \( f_1 \) is the Coriolis parameter at the northern wall and where only the highest order diffusion term has been retained. Because the flow impinging on the wall from the thermocline has a vertical scale of \( l_T \), the third term in (11) is of order unity. There are two layers. In the outer, thicker layer, the balance is among the first three terms and the layer has thickness \( l_{out} = l_T \) (assuming again \( \kappa_H \approx \kappa_V \)). In the inner layer, the second and fourth terms balance, yielding a layer of thickness \( l_{in} = E_H^{1/2} / l_T \).

In the outer layer, the primary balances can be inferred by scaling equations (1). Assuming that \( l_{out} \approx l_T \), we obtain:

\[ u_{out}, \theta_{out} = O(1), \quad v_{out}, \phi_{out} \propto l_T, \quad w_{out} \propto l_T^2. \]  

(12)

The layer velocity is thus dominated by the zonal component; the meridional and vertical velocities are comparable to those in the interior. Significantly, the horizontal velocities are geostrophic to order \( l_T \). Note the integrated vertical velocity in the layer is \( O(l_T^3) \).

For the inner layer, scaling suggests:

\[ u_{in} = O(1), \quad \phi_{in}, v_{in} \propto E_H^{1/2} l_T^{-1}, \quad \theta_{in} \propto E_H^{1/2} l_T^{-2}, \quad w_{in} \propto l_T^4 E_H^{-1/2}. \]

The zonal velocity is order one to bring that in the outer layer to rest. However, \( v_{in} \) is small and so can be ignored when demanding \( v = 0 \) at \( y = 1 \) (P69). The vertical velocity is larger than in the outer layer but the layer is narrow; nevertheless, the integrated vertical transport is of \( O(l_T^3) \), the same as in the outer layer. So we must also treat this layer.

The outer boundary layer equation, following the cosine transform in the vertical and changing variables to \( t = 1 - x \) and \( \zeta = 1 - y \), is:

\[ \partial_t \phi_{out} - \frac{1}{\mu_{n1}} \partial_{\zeta}^2 \phi_{out} + \alpha_n \phi_{out} = 0, \]  

(13)

where the variables with subscripts of one are evaluated at \( y = 1 \) and where:

\[ \mu_n = \left( \frac{\beta S}{n^2 \pi^2 f_1^2 \kappa_H} \right)^{1/2}. \]  

(14)
Eq. (13) is a diffusion equation and implies a westward thickening of the layer. We solve it by using the Laplace transform and by matching to the interior meridional velocity at \( y = 1 \). The result is:

\[
\phi_{\text{out}} = \alpha_n \left( C_1 - \frac{2T_1}{n^2 \pi^2} \right) \int_0^t e^{-\alpha_n \tau} \text{erfc} \left( \frac{\mu_n \zeta}{2 \sqrt{\tau}} \right) d\tau,
\]

(15)

where \( \text{erfc} \) is the complimentary error function. The vertical velocity turns out to be:

\[
w_{\text{out}} = \frac{n^3 \pi^3 \kappa_v}{S} \phi_{\text{out}} - \frac{n \pi \kappa_H}{S} \left( C_1 - \frac{2T_1}{n^2 \pi^2} \right) \frac{\alpha_n}{2 \sqrt{\tau}} \int_0^t \frac{e^{-\alpha_n \tau}}{\tau^{3/2}} \exp \left( -\frac{\mu_n \zeta^2}{4 \tau} \right) d\tau.
\]

(16)

The integral diverges for small \( \zeta \) \( (y \to 1) \), but this will not affect the overall solution because the more important areal integral of \( w \) is convergent. The vertical velocity integrals for the interior and boundary layers are given in the Appendix.

The inner layer solution is simply a decaying exponential:

\[
\phi_{\text{in}} = -\mu_n^2 \left( \frac{\alpha_n E_H}{\beta} \right)^{1/2} \left( C_1 - \frac{2T_1}{n^2 \pi^2} \right) \text{erf} \left( \frac{\alpha_n t}{\sqrt{\tau}} \right) \exp \left( -\frac{\sqrt{\beta \zeta}}{\sqrt{E_H \mu_n}} \right).
\]

(17)

The amplitude is determined by matching to \( u_{\text{out}} \) \( (y = 1) \) and the vertical velocity is easily found from \( w = (\kappa_H/S) \phi_{z \zeta} \). Significantly, the area-integrated vertical velocity in the inner layer exactly cancels that due to horizontal mixing in the outer layer (the second term on the RHS of Eq. 16). This is because the vertical transport due to horizontal mixing is proportional to the tangential velocity at the wall and the latter vanishes because of the no-slip condition. The result is that the outer layer vertical transport comes solely from vertical mixing in the diffusive case.

**ii. Rayleigh damping.** Now consider Rayleigh damping. The boundary layer equation is:

\[
\partial_t \phi_{NR} - \frac{f_H^2 \kappa_H}{\beta S} \partial_z^2 \phi_{NR} - \frac{f_V^2 \kappa_V}{\beta S} \partial_x^2 \phi_{NR} - \frac{\epsilon}{\beta} \partial_x^2 \phi_{NR} = 0.
\]

(18)

As before, we must retain the third term. Assuming \( \kappa_H \approx \kappa_V \), the boundary layer can either have \( l = l_T \) or \( l = \sqrt{\epsilon} \), with the second or fourth term entering, respectively. But the order of the equation after the cosine transform is the same in either case, so we can treat all four terms simultaneously.

Scaling the equations of motion yields the *same* balances as in (12); so we have a single layer, identical to the outer layer in the diffusive case. The difference is that viscosity now enters in the dynamics. This occurs because Rayleigh damping is not scale selective. The solution to (18) is (15), but with \( \mu_n \) replaced by:
Rayleigh damping alters the relative contributions from the modes, but otherwise the layer structure is the same (and is of course identical in the limit of vanishing $\epsilon$). In addition, because there is no inner layer now, the integrated vertical velocity has contributions from both vertical and horizontal mixing.

The solutions at the southern boundary are exactly the same but with all variables evaluated at $y = y_0$. The same comments apply, although the flow amplitude and decay scales are different because of the change in the Coriolis parameter. Notice too that $T_0$ changes sign, being negative in the north and positive in the south ($C_1$, however is the same at both boundaries).

d. Western boundary

The western boundary layer is thinner than the northern one, with the result that lateral diffusion dominates the density mixing. The vorticity equation is:

$$\partial_x \phi_w - \frac{f^2 \kappa_H}{\beta S} \phi_w \partial_x^2 \phi_w - \frac{f}{\beta} \partial_y \phi_w = 0. \quad (20)$$

i. Diffusion. With $D_y = E_H \partial_x^2 v$, we have (demanding the solution be trapped at the wall):

$$\phi_w - \frac{f^2 \kappa_H}{\beta S} \phi_w \partial_x^2 \phi_w - \frac{E_H}{\beta} \partial_x^3 \phi_w = 0. \quad (21)$$

Assuming a vertical scale of $l_T$, the order of the equation is maintained for all parameters. Two situations are possible. In one, the first and third terms balance, yielding a single layer with scale $l = E_H^{1/3}$. In the second, there are two layers, with the first and second terms balancing in the outer layer (with $l \approx l_T^2$) and the second and third balancing in the inner (with $l \approx \sqrt{E_H}/l_T$). With typical parameters either case is plausible, but we will examine the first, for simplicity. With $l = E_H^{1/3}$, scaling yields:

$$\theta_w = O(1), \quad \phi_w, u_w \propto l_T, \quad v_w \propto \frac{l_T}{E_H^{1/3}}, \quad w_w \propto \frac{l_T^4}{E_H^{2/3}}. \quad (22)$$

The meridional velocity is large and the vertical velocity is larger than anywhere else in the basin. The meridional velocity is geostrophic but the zonal velocity is not.

The solution to (21), neglecting the second term, is like Munk’s (1950):

$$\phi_{wd} = A \exp\left(-\frac{\delta}{2} x\right) \cos\left(\frac{\sqrt{3} \delta}{2} x - \frac{\pi}{6}\right), \quad (23)$$

where $\delta = (\beta/E_H)^{1/3}$. The phase, $-\pi/6$, permits the no-slip condition on the meridional velocity at $x = 0$. The vertical velocity is:
\[ w_{wd} = -\frac{n\pi\kappa H \delta^2}{S} A \exp\left(-\frac{\delta}{2} x\right) \sin\left(\frac{\sqrt{3}\delta}{2} x - \frac{\pi}{3}\right). \]  

(24)

The amplitude, \( A(y, n) \), is determined by the no-normal flow condition, which yields a first order ODE. The solution is:

\[ A(y, n) = -f \frac{2}{\sqrt{3}} \int_y^1 u_j dy + \frac{f}{f_1} A(1, n) \]  

(25)

where \( u_j \) is the interior zonal velocity at \( x = 0 \). So \( A(y, n) \) is determined jointly by the inflow from the northern boundary and the outflow into the thermocline interior. We evaluate (25) by imposing \( A(1, n) = (2/\sqrt{3})\phi_{in}(0, 1) \) (the streamfunction at the north wall from the inner diffusive layer). The integral in (25) is calculated numerically.

The integrated vertical velocity, from (22), is \( O(l_T^2 S H^{1/3}) \), which, with the previously noted values, is \( O(l_T^2) \). So the transport in the western boundary could be as large as in the interior. However, here too the integrated vertical velocity vanishes:

\[ \int w_w(x, y, n) dx = -\frac{n\pi\kappa H}{S} \int_0^\infty \delta^2 \phi_w dx = \frac{n\pi\kappa H}{S} \delta \phi_w(x = 0) = 0, \]  

(26)

because of the no-slip condition. So the western boundary layer does not contribute to the net transport in the diffusive case.\(^5\)

ii. Rayleigh damping. Now \( D_y = -\epsilon v \), the transformed western boundary equation is:

\[ \partial_x \phi_w + \mu_n^2 \phi_w = 0, \]  

(27)

where \( \mu_n \) is as in (19) (but with a variable \( f \)). As in the northern layer, the horizontal mixing and Rayleigh terms are of the same order and collapse into one term. The scaling depends on the relative sizes of \( \epsilon/\beta \) and \( \kappa H/S \), but given the aforementioned values, \( \epsilon/\beta \approx \sqrt{\kappa H/S} \) and the layer thickness is of order \( l_T^2 \). In this case, we have:

\[ \theta_w, w_w = O(1), \quad \phi_w, u_w \propto l_T, \quad v_w \propto l_T^{-1}. \]  

(28)

So the vertical and meridional velocities are larger than with the diffusive viscosity. The same scalings apply if \( \epsilon/\beta \) is smaller than \( l_T^2 \); if larger, the vertical and meridional velocities will be smaller than shown above.

The solution which decays away from the boundary is simply:

\[ \phi_{w}(x, y, n) = A^* e^{-\kappa x^2}. \]  

(29)

As with diffusive damping, the streamfunction amplitude follows from invoking no-normal flow at the western wall:

\(^5\) Note that the vertical transport is zero regardless of the boundary layer structure in (21).
We will also evaluate $A^*$ numerically. The vertical transport is again of $O[l_T^2]$, and thus potentially as large as in the interior. And, unlike with diffusion, the zonally-integrated $w$ is not necessarily zero:

$$
\int w_n^*(x, y, n) dx = \sum \frac{n \pi \kappa_B \mu_n^2}{S} A(y, n) \sin(n \pi z) \neq 0.
$$

So in the west too we anticipate significant differences between the Rayleigh and diffusive viscosities.

e. Determining $C_I$

Lastly, we calculate the free mode portion of the interior solution, by demanding that the basin-integrated vertical velocity be zero. So the vertical transports, mentioned repeatedly, are of central importance. We derive $C_I$ by dividing the (negative of the) contribution to the basin integrated vertical velocity due to the surface forcing, $T_0$, by that due to $C_1$. The transports are given in the Appendix.

4. Boundary-intensified $\kappa_V$

We may also consider a spatially varying vertical diffusivity. There are numerous cases one could consider but a relevant one is when mixing is greatest near the lateral boundaries (e.g. Marotzke, 1997; Samelson, 1998; Scott and Marotzke, 2000). The previous derivation is largely unchanged if we assume $\kappa_V$ is piecewise constant, e.g.:

$$
\kappa_V = \kappa_B \quad d \leq x \leq 1
$$

$$
\kappa_V = \kappa_I \quad x < d,
$$

where $\kappa_B \gg \kappa_I$. The solution, $\phi_{IE}$, in the region $d \leq x \leq 1$ is the same as in (7), with $\kappa_V$ replaced by $\kappa_B$. The solution for $x < d$, matching the zonal velocity at $x = d$, is:

$$
\phi_{IW}(x, y, n) = \frac{2T_0}{n^2 \pi^2} (1 - e^{\alpha_I(x-d)+\alpha_B(d-1)}) + C_i(n) e^{\alpha_I(x-d)+\alpha_B(d-1)},
$$

where $\alpha_I$ and $\alpha_B$ are defined using the corresponding values of $\kappa_V$. We also take $\kappa_{V} = \kappa_B$ along the north, south and west walls. And because $\kappa_H$ enters only in the boundary layers, we take $\kappa_H = \kappa_B$.

The northern and southern boundary solutions in the region $d \leq x \leq 1$ are as in Section 3c, but using $\alpha_B$ instead of $\alpha_n$. Where $x < d$, the inflow from the interior is weakened, so the flow is dominated by the throughflow from the east. The western boundary solutions
5. Results

a. Diffusion

We begin with the diffusive viscosity and spatially constant vertical mixing. We take \( T_0(y) = \mathcal{T}(y_m - y) \), where \( y_m \) is the basin mid-latitude and \( \mathcal{T} = 1 \). This yields cooling in north and heating in south. A representative case is shown in Figure 1. The contours indicate the vertically-integrated vertical velocity:

\[
\int_{-1}^{0} w(x, y, z)dz = w(x, y, n) \int_{-1}^{0} \sin(n\pi z)dz = -\frac{1}{n\pi} w(x, y, n)(1 - (-1)^n).
\]

Figure 1. The vertically-averaged vertical velocities for a case with a diffusive viscosity. The diffusivities and the horizontal Ekman number are as shown. The shading scheme has been chosen to highlight the interior fields; the western boundary transports are larger. Superimposed are the surface velocities, with an arbitrary scale.

are as before, but the flow amplitude changes due to the alteration of the interior zonal velocity and of the inflow from the northern/southern walls.
Table 1. The contributions to the area-integrated vertical velocity in the directly-forced solution by region. The columns show the vertical diffusivity, the horizontal diffusivity, the Rayleigh coefficient and the western boundary longitude of the eastern intensified-mixing region. Note the horizontal Ekman number, $E_H$, does not affect the transports. The last columns are the contributions from the interior and the north, south and west walls. The sums are as computed from the Appendix with 101 sine modes.

$\kappa_V \quad \kappa_H \quad \epsilon \quad d \quad W_I \quad W_N \quad W_S \quad W_W \quad \Sigma W_T$

1e-04 1e-04 0 0 .0021 -.0042 .0034 0 .0013
1e-04 2e-04 0 0 .0021 -.006 - .0048 0 .0009
2e-04 1e-04 0 0 .0029 -.0061 .0051 0 .0019
1e-04 1e-04 0.01 0 .0021 -.0001 .0036 -.0003 .0053
1e-04 2e-04 0.01 0 .0021 0 .0029 -.0004 .0046
2e-04 1e-04 0.01 0 .0029 -.0002 .0059 -.0003 .0075
1e-04 1e-04 0.02 0 .0021 -.0003 .0061 -.0001 .0078
1e-03 1e-03 0 0.9 .0021 -.0879 .0813 0 -.0045
1e-03 1e-03 0.01 0.9 .0021 .0007 .005 -.001 .0068

Note only the odd modes contribute. Superimposed in the figure are vectors indicating the surface velocities (to which all modes contribute).

Despite the constant mixing, the vertical velocities are nonuniform and intensified near the boundaries. There is downwelling at the north wall and upwelling at the south. At the west wall there is upwelling near the wall and downwelling offshore (reflecting the decaying sinusoidal character of the Munk solution).

The surface velocities are greatest near the west wall. These are northward, and greater in the south than in the north. There is westward flow along the southern wall and eastward flow along the northern wall, the latter being the weaker flow. The flow in the interior is weaker still and zonal, toward the east. The flow is diverted near the eastern wall toward the north and south.

The solution in Figure 1 comprises both the directly-forced and the free mode solutions. As noted, the latter depends on the areal integral of $w$, and this is tabulated by region in Table 1. There is, for example, a net upwelling in the interior; this occurs because of the variation in the Coriolis parameter, i.e. there is more upwelling in the south, where $f$ is smaller, than downwelling in the north. There is downwelling at the northern boundary and upwelling at the southern boundary, and each transport is as large as the net interior transport. However it is more appropriate to compare with the sum of the north and south integrals; then the interior dominates. Taken together, the interior accounts for about 70% of the areal average while the north/south walls contribute about 30%. Recall there is no net transport in the western layer with diffusion.

The free mode is shown in Figure 2. The pattern of up- and downwelling here is largely the same as in the directly-forced solution but the mode nevertheless contributes a net downwelling. The surface velocities are also northward along the west wall, but eastward in the south and northward along the eastern wall. However, because the free mode is
weaker than the directly-forced solution (see below), these velocities do not alter the total velocities greatly.

b. Rayleigh damping

With a Rayleigh viscosity (Fig. 3), the depth-integrated vertical velocity is also boundary-intensified and comparable in magnitude to that with the diffusive viscosity. But the spatial distribution of $w$ has changed. The downwelling in the north now merges with that in the eastern interior, and the upwelling in the southern interior is more evident. There is upwelling near the northern part of the western wall and downwelling in the south, and no changes in sign of $w$ moving offshore.

As before, the surface velocities in the interior are largely zonal and eastward. But there is now a strong westward flow along both north and south walls and the west wall flow is...
toward the mid-basin. Notice the northern and southern boundary currents are fed by the eastern-intensified currents; these were present previously, but are more prominent here.

As noted, the Rayleigh damping changes the three non-eastern boundary layers. One consequence is that the transport in the northern layer entering the western layer is greatly altered (because now there is no inner, no-slip layer). This accounts for the difference in surface velocities at the west wall.

Secondly, the vertical transports in the layers have changed significantly. The net downwelling in the northern layer is less than with diffusion (Table 1), but the upwelling in the south has increased slightly; so the combined north/south wall contribution is a net upwelling, somewhat larger even than that from the interior. The western layer also contributes; it is however a relatively small contribution.

Because the total upwelling is greater, the free mode (Fig. 4) is stronger. Plotting the $C_1(n)$ for the diffusive and Rayleigh cases (Fig. 5), we see clearly the free mode coefficients are larger in the latter case. However, in both cases $C_1(n) \ll 2T/(n\pi)^2$, so the directly-forced solution dominates the full solution.
We now consider the case where mixing is intensified in the boundary layers. We may anticipate the effect of such intensification from the scalings. The vertical velocity is of $O(l_T^2)$ in the interior, so with a constant diffusivity the interior transport is of the same order. But if the mixing occurs primarily in a strip of width $d$ in the east, the transport will be of $O(dl_T^2)$. If $d \approx l_T$, then the interior transport will be comparable to that in the northern and southern layers. So the latter will become more important to total transport.

Consider a case in which mixing is intensified in the eastern region $0.9 < x \leq 1$ (so that the width is comparable to the thermocline depth in the previous examples). We take $\kappa_B = 10^{-3}$ and $\kappa_f = 10^{-5}$, with a diffusive viscosity. Thus the boundary diffusivities are 10 times larger than they were previously.\footnote{Recall that decreasing the diffusivity by a factor of 10 only decreases the interior thermocline depth by about 40%.

\[ \kappa_v = \kappa_H = 10^{-4}, \varepsilon = 10^{-2}; \text{Free mode} \]
The vertical and surface velocities are shown in Figure 6. The fields are very similar to those obtained with a constant diffusivity (Fig. 1). The largest vertical velocities occur near the non-eastern boundaries, and the surface flow is greatest near the south and west walls. However, the relative contributions to the area-averaged vertical velocity have changed (Table 1). The interior has net upwelling, with the same magnitude as before (because $\kappa_V$ is 10 times greater and the eastern strip covers 1/10th the interior); but the transports in the northern and southern layers, with the larger diffusivities, are increased by an order of magnitude; they now dominate the total integral. In particular the downwelling which occurs at the north wall overwhelms all other contributions (Table 1).

We require a net upwelling from the free mode (Fig. 7), and this also occurs primarily near the north wall. The free mode is stronger than with a constant diffusivity, but nevertheless is still weaker than the directly-forced solution (see below).
The solution with a Rayleigh viscosity (Fig. 8), like its constant mixing counterpart, has boundary-intensified upwelling and a two-gyre surface circulation. The interior surface flow is more zonal than before, reflecting a weaker $w$ in the interior (see Eq. 2). There are also differences in the distribution of $w$: the west is now primarily an upwelling region and the east a downwelling region. The directly-forced solution has a net upwelling, most of which occurs at the southern boundary (Table 1). The interior is the next largest contributor and then the western wall, with the northern wall being the weakest.

Interestingly, the free mode in this case (Fig. 9) is much stronger and closely resembles the full solution. Examining the amplitudes, we find a large contribution from the $n = 3$ free mode (Fig. 10), and this mode in turn dominates the whole solution. The result is that the velocities have a much stronger “nonlocal” character than in the previous examples.

d. Parameter dependences

The qualitative features of the solutions are relatively robust to parameter changes, the primary alterations occurring in the boundary transports and in the strength of the free
mode. One can infer parameter dependencies by focusing on the vertical transports in the directly-forced solution (Table 1).

Despite this being a linear solution, the parameter dependencies are nonlinear and are sometimes nonintuitive. For example, the interior vertical transport depends only weakly on $\kappa_V$ (because $w$ is proportional to $\kappa_V$ but the decay from the eastern boundary is inversely proportional to $\kappa_V$). Doubling $\kappa_V$ increases the net interior upwelling, but only by about 40% (Table 1).

The dependencies in the northern and southern boundaries are even subtler. The diffusive viscosity case is simpler because the contribution from horizontal mixing is canceled by the inner layer transport. Increasing $\kappa_V$ increases the vertical transport, in part because $w$ is greater and because the flow impinging on the northern wall from the interior is greater. But increasing $\kappa_H$ also increases the transport, because this widens the boundary layer.

With the Rayleigh viscosity, lateral mixing, vertical mixing and viscosity all enter. Worse, the contributions from vertical and horizontal mixing are comparable in magnitude.
Figure 8. The averaged vertical and surface horizontal velocities with the Rayleigh viscosity and the boundary-intensified $\kappa_V$.

and of opposite sign. The dependencies thus can be surprising. To illustrate this, we plot the integrated transport for two values of $\epsilon$ as a function of $\kappa_V$ and $\kappa_H$ (Fig. 11). With the smaller viscosity (lower panel), the northern layer is upwelling and the transport increases when either $\kappa_V$ or $\kappa_H$ increased. In contrast, with $\epsilon = 0.01$ (upper panel), the northern layer is downwelling. The transport is increased with larger $\kappa_V$, but is decreased with larger $\kappa_H$. These changes stem in part from the effect of viscosity on the boundary layer width. If the viscosity is small, the boundary layer width depends on $\kappa_H/S$; if large, the width is maintained by the viscosity.

The tendencies are similar at the southern boundary but are often more pronounced because $f$ is smaller. The southern layer transport also depends on the latitude of the southern boundary; shifting $y_0$ southward tends to increase the transport, by decreasing $f$ at the wall.

The western boundary layer does not contribute to the basin-integrated transport with a diffusive viscosity and no-slip boundaries (it would contribute with free-slip boundaries).
Changing either $\kappa_H$ or $E_H$ simply changes the width and structure of the layer. Vertical mixing does not enter directly in the layer dynamics, but it does affect the flow amplitude (by changing the interior transports). With the Rayleigh viscosity, the vertical velocity is proportional to the lateral mixing but the layer thickness, proportional to $\mu_n^{-2}$, increases with $\kappa_H$. The result is that the area-integrated transport is independent of $\kappa_H$ in the limit $\kappa_H/S \gg \epsilon$. In the other extreme, where $\kappa_H/S \ll \epsilon$, the transport is linearly dependent on $\kappa_H$ and inversely proportional to $\epsilon$ (which thickens the layer).

Essentially the same parameter dependencies occur with boundary-intensified mixing. The primary difference is the smaller relative contribution from the interior.

6. Summary

We have re-examined the linear model of an idealized ocean driven by thermal forcing at the surface, excluding winds. Similar models were studied by Pedlosky (1969) and Salmon (1986). The flow is confined to the surface thermocline and is composed of two
parts: a directly-forced solution and a free mode. In the former, velocities in the thermocline interior (away from boundaries) are determined by the local forcing. The free mode satisfies the requirement that the area-averaged vertical velocity be zero; its interior velocities are thus forced nonlocally.

The vertical velocity has central importance in these solutions. The vertical transport is intensified near the lateral boundaries and as such is affected by viscosity. We obtained qualitatively different solutions with a diffusive viscosity (with the no-slip condition) and a Rayleigh viscosity. The former yielded a single-gyre surface circulation while the latter produced a two-gyre circulation. The directly-forced solution in the interior was the same in both cases, so the differences come solely from the boundary currents. With diffusion, only vertical mixing contributes to the transport in the northern and southern layers and there is no net transport in the western layer. With Rayleigh damping however vertical and

Figure 10. The amplitude of the free mode for the diffusive and Rayleigh viscosities in the boundary-intensified mixing cases. Again the amplitude of the directly-forced solution, \(2T/(n\pi)^2\) is also shown.
horizontal mixing and viscosity all enter in the north and south, and the western layer does contribute.

We have also examined how the solutions change when the vertical diffusivity is intensified near the boundaries. When $\kappa_V$ is constant, the interior accounts for a large fraction of the area-integrated vertical transport. In addition, the directly-forced solution usually dominates over the free mode. But when mixing is intensified near the boundaries the boundary transport is the most important and the results depend more sensitively on viscosity. In some examples, the free mode dominates the directly-forced solution, yielding strongly nonlocal velocities.

7. Discussion

Among the advantages of the linear model are that the boundary layers are fully “resolved” and parameter dependencies can be evaluated without re-running lengthy numerical calculations. The linear model is also more complete dynamically than either the
scaling or abyssal circulation theories. But the model employs a base stratification which
does not vary with latitude and yields solutions trapped at the surface. Can such a model
really help us to understand ocean circulation and/or climate models?

As noted above, the linear model can produce single or double-gyre surface circulations,
depending on the choice of viscosity and boundary condition. Numerical models (without
wind forcing and with a flat bottom) generally produce only single-gyre circulations (e.g.
Colin de Verdiere, 1988; Marotzke, 1997; Park and Bryan, 2001), regardless of the
viscosity (Huck et al., 1999). The models usually exhibit upwelling in the western
boundary current and downwelling at the north and/or east walls. The location of the latter
appears to determine the character of the flow; northern boundary sinking requires an
eastward jet at the surface near the north wall (e.g. Huck et al., 1999) whereas eastern
sinking causes broader eastward flow in the interior (e.g. Marotzke, 1997). The down-
welling in these models goes to great depths, due at least in part to convection.

But the model circulations change when topography, in particular a continental slope, is
included. This suppresses the deep sinking, making the overturning shallower. The result is
a two gyre surface circulation (Winton, 1997; Park and Bryan, 2001; Spall and Pickart,
2001). So, as in the linear model, changing the vertical transport changes the surface
circulation. Note in particular that the linear model exhibited less vertical transport in the
north with Rayleigh damping than with diffusivity (Table 1), and also had two gyres instead
of one.

There are other similarities between the linear and nonlinear models. For one, the
vertical and meridional flows in simulations are usually boundary-intensified. As noted, the
sinking usually occurs in the north and/or east and upwelling in the west. The vertical
velocities are also sensitive to the choice of viscosity; using a Rayleigh drag rather than a
diffusive viscosity with the no-slip condition can change the overturning by 50% (Huck et
al., 1999). And the circulation is qualitatively similar with constant and boundary-
ingenerated mixing (Marotzke, 1997). While there are undeniable differences, the linear
model does exhibit similar tendencies.

What then of the observations? The North Atlantic Current, which extends from the Gulf
Stream, flows eastward in the interior and then northward, crossing the Iceland-Faroe
Ridge to feed the Norwegian Atlantic Current (NwAC). The NwAC is an eastern-
intensified boundary current which flows toward the Arctic (Mauritzen, 1996). The
primary core of the current lies near the shelfbreak and is strikingly narrow—only about
20 km wide (Skagseth and Orvik, 2002). The boundary current continues, with part
branching off to circulate in the Arctic and part returning southward toward the Denmark
Straits (Mauritzen, 1996). The current is cooled substantially in this cyclonic gyre,
becoming denser. It eventually exits the Nordic Seas through the Denmark Straits as the
North Atlantic Deep Water (NADW).

A robust feature of the linear model are eastern-intensified currents in the interior
(Section 3b). These are nonviscous currents which derive from vortex stretching due to
vertical mixing. The current in the north flows northward and feeds the northern boundary
current. The latter depends on the viscosity, but at least with Rayleigh damping it is an intense, westward current. If, moreover, the vertical mixing is intensified near the boundary (i.e. over the continental slope), the eastern current is as narrow as the mixing region. So the linear model may explain both the location and width of the NwAC. One difference of course is that the model’s current cannot plunge to great depths when it reaches the western wall, like the NADW. But the sense of circulation, at least with Rayleigh damping, is correct.

In the context of the linear model, the NwAC would most likely be associated with the free mode. The linear model often exhibits strong upwelling at the southern boundary and of course the North Atlantic has no such boundary (although similar upwelling might be happening at the equator). But with upwelling occurring in other regions, in the Pacific and Indian oceans, a free mode would be required to close the vertical velocities. In the interior, the free mode is an eastern-intensified current.

With a constant vertical diffusivity, a large portion of the upwelling occurs in the interior. This is permitted by the β-effect, which balances vortex stretching. On the f-plane, this cannot happen; Pedlosky (2003) finds in this case that upwelling is again boundary-trapped. However this does not mean that the present model is like the f-plane case when the mixing is boundary-intensified, because β enters the dynamics also in the boundary layers.

The dependence on viscosity and boundary conditions raises the question of which one should be used in climate simulations. A definitive answer probably lies in precise observations and comparisons with models. It is possible the linear model could provide insight in such an assessment.

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APPENDIX

Integrated vertical velocities

To determine $C_1(n)$, we require the areal integrals of $w$ for the different regions. Note that the vertical velocity is proportional to $\sin \left(n \pi \frac{z}{H}\right)$, so the integrated velocities are too. Values of these integrals for specific parameters are given in Table 1.

Interior:

$$
\int_0^1 \int_{y_0}^1 w dy dx = \frac{n^3 \pi^2 \kappa_v}{S} \left[ C_1 \int_{y_0}^1 \frac{1}{\alpha_n} \left(1 - \exp^{-\alpha_n} \right) dy - \frac{2T}{n^2 \pi} \int_{y_0}^1 (y_m - y) \frac{1}{\alpha_n} \left(1 - e^{-\alpha_n} \right) dy \right]
$$

(34)

where $y_m$ is the mid-latitude value.
North (outer):

\[
\int_0^1 \int_{y_0}^1 w \, dy \, dx = \left( C_1 - \frac{2 T_N}{n^2 \pi^2} \right) \left[ \frac{n^3 \pi^3 k_v}{S} \cdot \frac{2}{\mu_n \sqrt{\pi \alpha_n}} \int_0^1 \gamma \left( \frac{3}{2} \cdot \alpha_n \right) \, dt \right. \\
+ \left. \frac{-n \pi k_H}{S} \cdot \sqrt{\alpha_n} \mu_n \int_0^1 \text{erf} \left( \frac{y}{\sqrt{\alpha_n}} \right) \, dt \right]
\] 

(35)

where \( \gamma \) is the incomplete gamma function and \( T_N = \overline{f} (y_m - 1) \); note \( \alpha_n \) and \( \mu_n \) are also evaluated at \( y = 1 \). A similar expression obtains at the south wall. Including the inner, no-slip layer (in the diffusive case) cancels the second term on the RHS (Sec. 3c(i)).

West (Rayleigh):

\[
\int_0^1 \int_{y_0}^1 w \, dy \, dx = \frac{-n \pi k_H}{S} \mu_n^2 A,
\]

(36)

where \( A \) is given in either (25) or (30).

To calculate the free mode amplitude, \( C_1(n) \), we separate the contributions proportional to \( T \) and to \( C_1(n) \), sum and divide:

\[
C_1(n) = - \frac{W_{NT} + W_{ST} + W_{IT} + W_{WT}}{W_{NC} + W_{SC} + W_{WC} + W_{IC}},
\]

(37)

where the \( W \)’s are the transports and the \( T/C \) subscripts denote the contribution from the \( T_0 \) and \( C_1 \) portions. Note that \( W_{WT} \) in the numerator only contributes with the Rayleigh viscosity. Table 1 shows the sums for the directly-forced transports, i.e. the \( W_T \)’s.

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