Transient and steady drift currents in waves damped by surfactants

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(Received 24 March 2004; accepted 18 January 2005; published online 15 March 2005)

In this paper we study the Lagrangian mean drift induced by spatially damped capillary-gravity waves on a surface covered by an elastic film. The analysis is developed with regard to a typical laboratory setup, and explicit solutions for both transient and steady horizontal drift velocities are given. We consider a situation where the film covers the entire surface and is prevented from drifting away, e.g., by a film barrier. The drift below an inextensible film resembles the drift under an ice cover, with a jetlike current in the wave propagation direction just below the surface. If the film is elastic the solution changes drastically. For certain values of the film elasticity parameter the mean flow is in the direction opposite to that of wave propagation in the upper part of the water column.

I. INTRODUCTION

Surface-active materials can form large slicks on the ocean surface that influence the formation and evolution of capillary-gravity waves. Such surface films both inhibit wind wave growth and cause rapid damping of existing waves. The reason for this is the film’s ability to restrain tangential motion and the accompanying strong shear in the surface boundary layer. Nonlinearly, surface films cause enhanced transfer of momentum from the waves to a mean motion.

A marked maximum in the transverse wave damping rate for a finite value of film elasticity is connected to the excitation of dilational waves, which are characterized by alternating compression and dilation of the surface. During the course of the present study it became clear that we can regard the film and the surface boundary layer as a solid stretched elastic membrane. The capillary-gravity waves force the excitation of longitudinal (i.e., the dilational) waves in this membrane, and the frequency for which we find maximum damping of the transverse waves corresponds to the natural frequency of longitudinal waves in the membrane. As these findings provide valuable physical insight of the problem, we will discuss the linear system in some detail prior to the nonlinear analysis.

Vorticity will diffuse from the surface boundary layer into the lower parts of the fluid, and a mean Eulerian drift current develops. Previous studies have shown that both elastic and inextensible films significantly modify the drift currents in capillary-gravity waves. Unfortunately, there is a lack of experimental data, especially for elastic films. Furthermore, the above mentioned theoretical studies treat freely drifting films, hence the results apply to large slicks on the ocean surface rather than to a typical laboratory situation. It is the main aim of this paper to present theoretical results that can be validated in experiments. For this reason we focus on the part of the drift current that reaches below the thin surface boundary layer and thus easily can be measured directly. We use a Lagrangian description of motion and present both a transient and a time independent solution.

The outline of the paper is as follows: In Sec. II we formulate the problem mathematically. The first-order periodic solutions are presented in Sec. III, and in Sec. III A we discuss some properties of the waves. In Sec. IV we derive the equations for the wave drift. Analytical solutions are found for the horizontal drift velocity: a transient solution is presented in Sec. IV A, while a steady-state solution is presented in Sec. IV B. In Sec. IV C we discuss how the waves may affect the film cover in view of the assumptions the analysis rests upon. Finally, Sec. V contains a summary and some concluding remarks.

II. MATHEMATICAL FORMULATION

We consider monochromatic waves on the surface of a viscous, incompressible fluid of density \( \rho \). An insoluble monomolecular film covers the entire surface. The depth is such that the deep-water limit applies, i.e., at least larger than half a wavelength. The fluid motion is two dimensional, and we use a Cartesian coordinate system with the \( x \) axis aligned along the undisturbed surface and with the \( z \) axis positive upwards.

Changes in the area of a material surface (film) element lead to local variations in surface tension, and we must allow for this in our boundary conditions. Denoting the Eulerian fluid velocities by \( (U, W) \), pressure by \( P \), and the surface elevation by \( \zeta \), the horizontal and vertical dynamic boundary conditions at the surface, correct to the second order, become

\[
\rho \nu (U_x + W_z) + P \zeta_{xx} - 2 \rho \nu U_z \zeta_{xx} + \sigma_{xx} \zeta_{xx} - \sigma_{zz} - \sigma_{x} = \hat{\tau} - \sigma_{xx},
\]

(1)

\[
-P + 2 \rho \nu W_z - \rho \nu (U_z + W_x) \zeta_{xx} - \sigma_{xx} \zeta_{xx} - \sigma_{xx} = \hat{\sigma} + \gamma_{xx},
\]

(2)

respectively. Here \( \nu \) is the kinematic viscosity of the fluid, \( \sigma \) is the surface tension, and \( \hat{\tau} \) and \( \hat{\sigma} \) constitute external tan-
tential and normal stresses acting on the surface. Furthermore, we have let subscripts denote partial differentiation.

Our main aim is to investigate the mean drift velocities induced by the wave motion, and these are directly obtained by our choice of a Lagrangian description of motion. We label each fluid particle with coordinates \((a,c)\), chosen as the initial position when the fluid is at rest. Our dependent variables become the particle displacement \((x,z)\) and the pressure \(p\) in the vicinity of the particle, while our independent variables are \((a,c)\) and time, \(t\). Spatial derivatives in the Eulerian description may be transformed into derivatives in Lagrangian independent coordinates by

\[
 f_x = J(f,z)/J(x,z) = (f_x f_c - f_z f_a)(x_a z_c - x_c z_a),
\]

\[
 f_z = J(x,f)/J(x,z) = (x_a f_c - x_c f_a)(x_a z_c - x_c z_a),
\]

introducing the Jacobian \(J\). An advantage of using the Lagrangian description is that the free surface is given by \(c = 0\) at all times. For a similar analysis in Eulerian description one would have to use curvilinear coordinates.\(^7\) The velocity and acceleration of a fluid particle are simply \((x,t,z)\) and \((x,t,z)\). Because the time scale of the present problem is much less than an inertial period we disregard Coriolis forces. The equations for conservation of momentum and volume then read\(^3,10\)

\[
 \rho x_{at} = -J(p,z) + \rho v[J(J(x,z),z) + J(x,J(x,z))],
\]

\[
 \rho z_{at} = -J(x,p) - \rho g + \rho v[J(J(z,z),z) + J(x,J(x,z))],
\]

\[
 J(x,z) = 1.
\]

Here \(g\) is the acceleration due to gravity. We note that by (7) the denominators in (3) and (4) equal unity.

For insoluble films, as we consider here, there is no exchange of film material with the fluid, and we must require that the total amount of film material at the surface is conserved, hence\(^8\)

\[
 (x_a^2 + z_a^2)^{1/2} = \Gamma_0^2. \quad c = 0.
\]

Here \(\Gamma\) is the concentration of film material and \(\Gamma_0\) the concentration equilibrium value. Surface tension variations will be related to the fluid motion through the dilational modulus of the film, defined by

\[
 E = -\frac{d\sigma}{d(\ln \Gamma)}.
\]

The real part of \(E\) describes the elastic properties of the film, while the surface viscosity is incorporated in the imaginary part (e.g., Hansen and Ahmad\(^14\)). One distinguishes between the surface dilational and shear viscosity, which are associated to resistance against changes in area and shape of a surface element, respectively. For insoluble or sparingly soluble monolayers we have in general\(^12\) \(\text{Re}(E) \gg \text{Im}(E)\), and we will take \(E\) to be real. Experimental evidence shows that effects of the surface viscosity are negligible as far as the wave dynamics are concerned.\(^13\) As will become clear later, the nonlinear drift depends on the wave motion, hence the surface viscosity should not influence the induction of drift currents if the waves are unaffected. Furthermore, we take \(E\) to be constant, which means that \(\Gamma\) is close to \(\Gamma_0\). The validity of this assumption is discussed in Sec. IV C 1. The possible role of the surface shear viscosity in stabilizing the film cover is discussed in Sec. IV C 2.

To facilitate the calculations we make use of series expansions of the dependent variables after an ordering parameter \(\epsilon\) (e.g., Pierson\(^16\)):

\[
 x = a + \epsilon x^{(1)} + \epsilon^2 x^{(2)} + \cdots,
\]

\[
 z = c + \epsilon z^{(1)} + \epsilon^2 z^{(2)} + \cdots,
\]

\[
 p = -\rho gc + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \cdots,
\]

\[
 \sigma = T + \epsilon \sigma^{(1)} + \epsilon^2 \sigma^{(2)} + \cdots,
\]

\[
 \Gamma = \Gamma_0 + \epsilon \Gamma^{(1)} + \epsilon^2 \Gamma^{(2)} + \cdots.
\]

Using (10)–(14), we substitute the variables in the governing equations and collect terms of the same order in \(\epsilon\). The equations to \(O(\epsilon)\) yield the periodical motion or the primary wave, whereas the equations to \(O(\epsilon^2)\) yield the particle drift. It should be noted that \(\epsilon\) is dimensional, and for waves of amplitude \(\xi_0\), wave frequency \(\omega\), and wave number \(k\), we have \(\epsilon = \xi_0 \omega / k\). The equivalent nondimensional form is the wave steepness \(\xi_0 k\) which is assumed to be a small quantity.\(^9\)

Finally, in order to obtain a consistent set of equations, we must expand the external stresses in series:

\[
 \hat{\tau} = \epsilon \hat{\tau}^{(1)} + \epsilon^2 \hat{\tau}^{(2)} + \cdots,
\]

\[
 \hat{\sigma} = \epsilon \hat{\sigma}^{(1)} + \epsilon^2 \hat{\sigma}^{(2)} + \cdots.
\]

## III. LINEAR MOTION: TRANSVERSE AND DILATIONAL WAVES

The linear problem is extensively studied and we will not go into details on the derivation of the first order solutions (e.g., Lucassen and Lucassen-Reyninders\(^14\)). They are nonetheless needed to derive the equations to \(O(\epsilon^2)\) and will therefore be stated below. The solutions are obtained by separating the flow field into an irrotational and a rotational part, representing a capillary-gravity and a dilational wave, respectively, as further discussed in Sec. III A. For a progressive wave component the solutions are given by the real parts of\(^15\)

\[
 x^{(1)} = -\frac{i \kappa}{n} \left( e^{\kappa c} - i B e^{-\kappa c} \right) e^{i \kappa a + n t},
\]

\[
 z^{(1)} = -\frac{\kappa}{n} \left( e^{\kappa c} - i B e^{-\kappa c} \right) e^{i \kappa a + n t},
\]

\[
 p^{(1)} = \frac{\rho}{n} \left( n^2 + g \kappa \right) e^{\kappa c} - i g \kappa B e^{-\kappa c} \right) e^{i \kappa a + n t}.
\]

We consider spatially damped waves, so we take \(n = -i \omega\), with the wave frequency \(\omega\) real and positive. Furthermore, we let \(\kappa = k + i \alpha\), where both the wave number \(k\) and the spatial damping coefficient \(\alpha\) are real and positive. \(B\) is a
complex coefficient to be determined from the boundary conditions. The solutions (17)–(19) are based on the assumption that the waves are sufficiently short, i.e., with vanishing motion near the bottom. The value of \( m \) reflects the vertical scale of the surface boundary layer and is given by

\[
m^2 = k^2 - i \omega / \nu. \tag{20}
\]

We have approximately

\[
m = (1 - i) \gamma, \tag{21}
\]

where \( \gamma = (\omega / 2 \nu)^{1/2} \) is the inverse boundary layer thickness. It follows from (20) that this approximation is valid for \( k / \gamma \ll 1 \). For wavelengths between 1 and 10 cm the value of \( k / \gamma \) is between 0.02 and 0.08.

From (8) and (9) we obtain to \( O(\epsilon) \)

\[
\sigma^{(1)} = E a_0 \tag{22}
\]

Using (22), the dynamic boundary conditions (1) and (2) become

\[
\rho \nu \xi(x^*_c + z^{(1)}_a) = E a_{ua}, \quad c = 0, \tag{23}
\]

\[
- \rho \nu x^{(1)}_c + 2 \rho \nu z^{(1)}_a = T_a, \quad c = 0. \tag{24}
\]

We have here set \( \xi^{(1)} = \xi^{(1)} = 0 \), assuming that there are no external stresses acting on the surface to this order, i.e., no periodic stresses due to the presence of air above the film. Inserting the solutions (17) and (18) into (23), we find that \( B \) is given by

\[
B = \frac{\kappa}{\gamma} \left( \frac{\kappa \gamma^{1/2} + i \mathcal{E}}{1 - (1 - i) \mathcal{E}} \right), \tag{25}
\]

where \( \mathcal{E} \) is a nondimensional elasticity parameter defined by

\[
\mathcal{E} = \frac{E k^2 \gamma}{\rho \omega^2}. \tag{26}
\]

The coefficient \( B \) is small, ranging from \( O(k^2 / \gamma^2) \) for a clean surface to \( O(k / \gamma) \) for a film covered surface. The real and imaginary parts of \( B \) will in the following be denoted \( B_r \) and \( B_i \), respectively. Because both \( k \) and \( \gamma \) are fixed for a given wave frequency \( \omega, \mathcal{E} \) is a linear measure of film elasticity. Using (18), (19), and (24) we obtain the dispersion relation and the damping coefficient

\[
\omega^2 = \omega_0^2 (1 + B_r), \quad \omega_0^2 = g k + (T / \rho) k^3, \tag{27}
\]

\[
\frac{\alpha}{k} = \frac{1}{2 C_p \gamma} \left( \frac{F^2}{\gamma} + \frac{k 2 (1 - \mathcal{E})}{F} \right). \tag{28}
\]

Here \( F = 1 - 2 \mathcal{E} + 2 \mathcal{E}^2 \), \( C_p = \omega / k \) is the phase velocity, and \( C_a = \partial \omega / \partial k \) is the group velocity of the waves. It can be shown that the spatial damping coefficient is related to the temporal damping coefficient \( \beta \) (e.g., Weber and Saetra\(^9\)) by \( \beta = C_a \alpha \), in accordance with the results of Gaster\(^{16}\). For not too low values of \( \mathcal{E} \) the damping coefficient is proportional to \( \mathcal{E}^2 / F \), hence \( \alpha \) has a maximum for \( \mathcal{E} = 1 \) at a frequency \( \Omega \) given by

\[
\Omega = (E / \rho d)^{1/2} k. \tag{29}
\]

Here \( d = \gamma^4 \) is the boundary layer thickness. By (29) we may write \( \mathcal{E} = \Omega^2 / \omega^2 \). The physical significance of \( \mathcal{E} \) will be discussed in the following section.

### A. Resonant behavior of the two wave modes

Ermakov\(^6\) has argued that we can regard the irrotational part of the wave field as a capillary-gravity wave and the rotational part as a dilational wave which is excited by the capillary-gravity wave. We denote the irrotational part by \( \tilde{\varepsilon} \) and the rotational part by \( \tilde{\varepsilon} \) so that \( \tilde{\varepsilon}^{(1)} = \tilde{\varepsilon} + \tilde{\varepsilon}^{(1)} \), \( \tilde{\varepsilon}^{(1)} = \tilde{\varepsilon} + \dot{\xi} \). In the following subscript 0 denotes surface values, and \( x_0 = \tilde{x}_0 + \tilde{\xi}_0 \) is the total horizontal surface displacement. Inserting for \( \epsilon \) we obtain to the leading order from (17) and (18)

\[
\tilde{x} = \xi_0 e^{k x} e^i \theta, \tag{30}
\]

\[
\tilde{z} = -i \xi_0 e^{k x} e^i \theta, \tag{31}
\]

\[
\dot{\xi} = (\zeta / F) [\mathcal{E} (1 - 2 \mathcal{E}) - i \mathcal{E}] e^{(1 - i) \gamma} e^i \theta, \tag{32}
\]

\[
\dot{z} = -(k / \gamma)(\zeta / F) [\mathcal{E}^2 + i \mathcal{E} (1 - \mathcal{E})] e^{(1 - i) \gamma} e^i \theta, \tag{33}
\]

where \( \theta = \kappa a - \omega t \). We see that all the information on how the film influences the wave motion is incorporated in the parameter \( \mathcal{E} \). In the boundary layer there is a balance between inertia and viscous forces, so \( \dot{\xi} \) satisfies

\[
\dot{\xi}_0 = \nu \tilde{\varepsilon}_{xxx}. \quad \tag{34}
\]

Furthermore, we note that \( \xi \) is small, which means that the shape of the surface is virtually unaltered for any value of \( \mathcal{E} \). The horizontal surface displacement has a monotonically increasing phase lag for increasing \( \mathcal{E} \), and in the inextensible limit we have \( x_0 (\mathcal{E} \to \infty) / x_0 (\mathcal{E} = 0) = \exp[-(3 \pi / 4)] \).

A curious feature of dilational waves is that the phase velocity differs by a numerical factor for temporally and spatially damped waves, hence the group velocity is not well defined (Lucassen\(^5\), Weber and Christensen\(^7\)). Weber and Christensen considered forced dilational waves, i.e., when a fluctuating horizontal stress applied at the surface prevents wave amplitude decay. For such forced dilational waves the phase velocity differs from both temporally and spatially damped waves, in fact, using (29), the phase velocity can then be written as

\[
C_p = \Omega / k. \tag{35}
\]

As pointed out by Ermakov\(^6\), the transverse waves provide a forcing term in the horizontal boundary condition (23), which yields the dilational wave dispersion relation (e.g., Dysthe and Rabin\(^{18}\)). This term plays the same role as the stress term included by Weber and Christensen, allowing for no or weak damping of the dilational waves. Weber and Christensen also showed that the energy flux in dilational waves is negligible, there is rather a local balance between the work done by the elastic film and viscous dissipation. As a consequence the energy provided by the capillary-gravity waves to sustain the dilational waves is lost during the wave
cycle. Because the effect of film elasticity on transverse wave dispersion is insignificant, the dilational waves will have to adopt \( k \) and \( \omega \) from (27). Accordingly, the damping maximum occurs when the frequency of the transverse waves coincides with the natural frequency of forced dilational waves, as given by (29).

Now, define the mean boundary layer displacement

\[
\tilde{X} = \frac{1}{d} \int_{-\infty}^{0} \tilde{x} dc.
\]

(36)

From (34) and (36) we obtain \( d\tilde{X}_n = \nu(\tilde{X}_n - \tilde{x}_0) \). To the leading order the horizontal boundary condition (23) then reads

\[
\tilde{X}_n + \Omega^2 x_0 = 0.
\]

(37)

It follows that the mean boundary layer displacement is in phase with the total horizontal displacement of the film, a result which to the author’s knowledge has not been presented in the literature before. The film and the mean boundary layer displacement \( \tilde{X} \) act together as a solid elastic membrane so that the boundary layer responds instantaneously to changes in the position of the surface. This situation is unlike that of oscillatory motion near rigid boundaries (e.g., Schlichting and Gersten\(^{19}\)), and arises because the film is advected by the capillary-gravity waves. The phase velocity (35) is exactly that of longitudinal waves in an elastic membrane of width \( d \) and elastic modulus \( E \).

IV. THE NONLINEAR WAVE DRIFT

We derive the equations for the mean drift velocities by inserting the real parts of the first-order solutions (17)–(19) into (5)–(7), and collecting all terms of \( O(\varepsilon^2) \). Because we consider spatially damped waves we average the second-order equations over a wave period, denoting average values by an overbar. Defining nondimensional drift velocities \( u, w = (\varepsilon^2 f \nu)_u(\varepsilon^2 f \nu)_w \), where \( w = \frac{f \nu \omega}{\varepsilon^2} \) is the surface value of the classic inviscid Stokes drift, we obtain

\[
u u_t - \nu \tilde{V}^2 u = -\Pi_u - \nu \bar{K}^2 \left( 4 \left( 1 + \frac{x^2}{2k^2} \right) e^{2kx} + 6 \frac{\bar{K}}{k^2} \left( B_1^2 + B_2^2 \right) e^{2\gamma c} - 4 \frac{3}{k^2} \left( B_1 + B_2 \right) \cos \gamma c - \frac{2}{k^2} \left( B_1 - B_2 \right) \sin \gamma c \right) e^{-2\alpha a} ,
\]

(38)

\[
u w_t - \nu \tilde{V}^2 w = -\Pi_w + 2 \nu \bar{K}^2 \left( e^{2kx} - \frac{2}{k^2} \cos \gamma c + B_1 \sin \gamma c \right) e^{-2\alpha a} ,
\]

(39)

\[
u w_0 + w_c = 0 .
\]

(40)

Here \( \tilde{V}^2 = \partial^2 / \partial x^2 + \partial^2 / \partial c^2 \), and \( \Pi = \omega + \rho \left( \tilde{p}^{(2)} + \rho g \varepsilon^{(2)} \right) \) is the (nondimensional) dynamic pressure of \( O(\varepsilon^2) \).

From (38) and (40) we obtain \( w_c = 2\alpha a \). At the surface the mean vertical velocity must vanish, hence the boundary layer terms in \( w \) should be negligible. The main contribution in \( w \) should therefore come from the potential part of the flow. Then \( w_c \sim k w \), and using (40) we find \( w \sim (\alpha/k) u \ll u \).

We therefore neglect \( w \) in (39), and calculate \( \Pi \) from the reduced equation. In doing this we restrict ourselves to consider the drift some distance from the end walls where the motion is mainly horizontal. To the leading order we obtain

\[
\Pi_u = -2 \nu \bar{K}^2 \frac{\bar{K}^2}{k^2} e^{2\bar{K}^2 - 2(a\alpha)} + Q .
\]

(41)

In (41) a new variable \( Q \), independent of \( c \), is added to simulate the effect of a sloping surface. Using (25) and (41), the equation for the mean horizontal drift velocity becomes

\[
u u - nu_c = -Q - \nu \bar{K}^2 \left( 4 e^{2kx} + 6 \frac{\bar{K}}{k^2} \frac{\varepsilon^2}{F} e^{2\gamma c} - 4 \frac{2}{k^2} \sin \gamma c \right) e^{-2\alpha a} .
\]

(42)

In (42) terms of \( O(k/\gamma) \) compared to other terms are neglected, and the equation is therefore not valid for a film-free surface.

The first term inside the curly brackets on the right-hand side of (42) yields the Stokes drift. The other two terms are mainly second-order corrections to the frictional force, i.e., \( \nu u U_{\infty} \) in Eulerian description. Keeping in mind the subdivision of the wave field into a transverse and a dilational wave [Eqs. (30)–(33)], we realize that the first of these terms represents nonlinear effects of the dilational wave, while the second represents interaction between the transverse and the dilational wave. A careful retracing of the originating terms in (5) is not necessary, it suffices to note the difference in the factor of the exponents (we have used \( e^{kx} \approx e^{\gamma c} \)). If we neglect the contributions from the transverse wave on the right-hand side of (42) we are left with the \( e^{2\gamma c} \) term, which originates from a forced dilational wave. The frequency of such a wave is given by (29), thus \( \varepsilon = 1 \). Letting \( Q = 0 \) and \( \alpha \rightarrow 0 \), Eq. (42) becomes

\[
u u - nu_c = -6 \nu \bar{K}^2 e^{2\gamma c} ,
\]

(43)

which is precisely the equation for the drift induced by forced dilational waves.\(^{17}\) Letting \( \varepsilon \rightarrow \infty \), as for inextensible films, Eq. (42) becomes equal to the equation for the mean drift under an ice cover.\(^{20,21}\)

So far we have not made any assumptions about the conditions at the surface, and (42) is valid for both stagnant and freely drifting films. We will assume that the film is fixed, i.e., prevented from drifting away by a film barrier as in some laboratory situations.\(^{22,32}\)

A. Transient solution

For the time development of the drift current we will assume that the period of time in consideration is so short that the drift current never reaches the bottom. Accordingly, we must require
\[ u = 0, \quad c \to -\infty. \]  

(44)

In a tank of finite length, a return current driven by a pressure force \( \langle Q \rangle \) will develop after some time. Ünlüata and Mei\(^{23}\) provide an estimate for the time scale \( T_r \) required for the development of the return current. For a tank of length \( L \) we have \( T_r \sim L/\xi \delta \), or equivalently, \( \omega T_r \sim (L/\xi) / (\xi k) \). With \( L=100 \) cm and a wave steepness of \( \zeta_k = 0.1 \), we find that \( \omega T_r \sim O(10^4) \). For the present we consider the onset of the motion limited to a time scale \( \omega t = O(10^2) \), say, so we will neglect \( \langle Q \rangle \) in (42). A time independent solution with \( \langle Q \rangle \) non-zero is presented in Sec. IV B.

It proves convenient to separate \( u \) into three parts,

\[ u = u^{(S)} + u^{(p)} + u^{(h)}. \]  

(45)

The first two terms on the right-hand side of (45) are given by the particular solution of (42), with \( u^{(S)} \) representing the inviscid Stokes drift, and \( u^{(p)} \) a vorticity solution confined to the surface boundary layer. The homogeneous solution \( u^{(h)} \) represents a transient quasi-Eulerian mean current resulting from diffusion of vorticity from the surface (e.g., Craik\(^{24}\)).

With \( Q=0 \), we find from (42)

\[ u^{(S)} = e^{2kc - 2am}, \]  

(46)

\[ u^{(p)} = \frac{3\varepsilon^2}{2F} e^{2\gamma - 2am} + \frac{2\varepsilon}{F} [(1 - 2\varepsilon) \cos \gamma c - \sin \gamma c] e^{\gamma - 2am}. \]  

(47)

The complete particular solution of (42) is then

\[ u^{(p)} = u^{(S)} + u^{(p)}. \]

The wave motion does not start from rest, and it is therefore not obvious what initial condition we should apply for \( u \). One may assume that the first-order motion and the associated inviscid Stokes drift are established at any point after a few waves have passed by. Initially the surfactant is evenly distributed and there are no stresses at the surface opposing the drift. Thus the film is drifting freely and we take \( u \) equal to the inviscid Stokes drift (e.g., Weber and Saetra\(^{a}\)):

\[ u = u^{(S)}, \quad t = 0. \]  

(48)

The wave-induced mean stress on the film will compress it toward the film barrier, resulting in a higher concentration of film material away from the wave maker. In this way a surface tension gradient develops. After some time the surface tension gradient will be large enough to balance the viscous stress on the film, which becomes stagnant. The time required to reach this equilibrium depends on the physical properties of the particular surfactant in question. We will model this process by letting the surface drift velocity decay in time so that

\[ u = e^{-itR} e^{-2am}, \quad c = 0, \]  

(49)

and where the relaxation time \( R \) is specific to the film. Equation (49) implies that the film drifts a distance proportional to \( u_0 R \) before a steady state is reached.

The solution for \( u^{(h)} \) can be found by applying Laplace transforms. From (42), (44), (48), and (49), we obtain for the total drift velocity

\[ u = u^{(p)} - e^{-2am} \left\{ u^{(p)}_0 \exp(-c/2 \sqrt{\nu t}) \right. \]

\[ + \frac{c e^{-itR}}{2 \sqrt{\nu \pi}} \int_0^\infty \left[ \exp\left[-(c^2/4 \nu \xi) + (\xi/R)\right] \right. \]

\[ - \frac{2\varepsilon}{\pi F} \int_0^\infty \left[ \frac{3\xi}{4(\xi + 2\omega)} + \frac{1 - 2\varepsilon}{\xi^2 + \omega^2} \right] \]

\[ \times e^{-\xi} \sin(c/\sqrt{\nu}) d\xi = \left. \right\}, \]  

(50)

where \( u^{(p)}_0 \) is the surface value of the particular solution.

In Figs. 1 and 2 vertical profiles of \( u \) from (50) are shown using \( k=2\pi \) and \( 1 \) cm\(^{-1} \), \( \omega t = 100 \), and \( \omega R = 10\pi \). We have set \( a=0 \) in these and all subsequent figures. Figures 3–5 contain contour plots of \( u \) for \( k=2\pi \) cm\(^{-1} \), showing the time development for \( \omega t = 1 – 100 \), using the same value of \( R \) as in Figs. 1 and 2. The relaxation time \( R \) is chosen such that the surface drift is nonzero throughout the time interval.

The most striking feature is that the drift beneath the boundary layer is in the direction opposite to that of wave propagation for \( \varepsilon = 0.5 \). In order to explain this result we look
at the solution and boundary condition after the film has become stagnant. At large times \( t > R \), the boundary condition (49) becomes

\[ u = 0, \quad c = 0, \quad (51) \]

and for the homogeneous solution \( u^{(h)} \) we find from (50),

\[ u^{(h)} = -u_0^{(p)} \text{erfc}(c/2\sqrt{rt}). \quad (52) \]

Equation (52) describes the evolution of a drift current which may penetrate deep below the surface layer. This downward flux of momentum stems from the need to balance the contribution from the particular solution in (51), i.e., \( u^{(h)} = -u^{(p)} \) at the surface. Physically, the force that keeps the film from drifting (from the film barrier) is transferred to the bulk fluid and momentum diffuses into deeper layers by friction. In contrast, the particular solution is insignificant below the wave penetration depth at all times. The transient part of the drift thus develops in response to the no-slip condition (51), and the drift below the surface layer is against the wave propagation direction only when \( u_0^{(p)} \) is positive. When \( u_0^{(p)} \) is negative the transient response is a drift in the wave propagation direction. Even with a backward drift below the surface layer, the film is still compressed against the far end of the tank. The Lagrangian shear stress acting on the film is \( -p c u (c=0) \), which is a positive quantity for all \( E \) and \( t \).

The particular solution \( u^{(p)} \) represents the direct effect of the waves, and because \( u^{(p)} \) is completely determined by the \( O(\varepsilon) \) solutions, it has a strong dependence on \( E \). The major difference in Eq. (42) between elastic and inextensible films lies in the term representing interaction between the transverse and the dilational waves (i.e., the term proportional to \( c^2 \)). The phase lag between the horizontal and vertical surface displacements, representing the dilational and transverse waves, is determined by \( E \), as described in Sec. III A. Nonlinearly, the value of \( E \) then determines the transfer of momentum from periodic motion to a mean flow by altering the interaction between the two wave modes.

The surface value of \( u^{(p)} \) is of special significance, as it determines the magnitude and direction of the transient solution \( u^{(h)} \). From (46) and (47) we find (setting \( a=0 \))

\[ u_0^{(p)} = \frac{2 - c^2}{2F}. \quad (53) \]

It follows that \( u_0^{(p)} \) changes sign for \( E = \sqrt{2} \) (\( F > 0 \) for all \( E \)). The right-hand side of (53) has a sharp peak with a maximum value of 1.78 for \( E = 0.44 \), and a limiting value of -0.25 for \( E \to \infty \). If the elastic modulus of the film is \( E = 20 \text{ mN m}^{-1} \), which is a typical value for many films, we find that \( E \) increases from 0.5 to \( \sqrt{2} \) as the wavelength is shortened from 8.2 to 3.5 cm.

**B. Steady solution**

In practice the wave tank is closed in both ends, and there is a rigid bottom at \( c = -h \). The boundary condition at the bottom is

\[ u = 0, \quad c = -h. \quad (54) \]

The value of \( Q \) in (42) is determined by demanding that the net volume flux is zero, i.e.,
The solution for an inextensible film is quite different from the solution for elastic films. In comparison we have $u_0^{(p)}(c=-h/3).$ The maximum value of $u$ for $\mathcal{E}=0.5$ is higher for $k=2\pi$ cm$^{-1}$ than for $k=1$ cm$^{-1}$. This is explained by noting that $u^{(p)}$ depends on $k$ and $g$, whereas the other terms in (56) depend on the channel depth $h$, and that the vertical axis in Figs. 6 and 7 is dimensional. Below the wave penetration depth the profile is basically given by the third term on the right-hand side of (56), which is proportional to $u_0^{(p)},$ and the absolute value of this term increases monotonically from $c=-h/3$ to 0. Closer to the surface the particular solution $u^{(p)}$ becomes more important. As we approach the surface where $|c/h|<1$, we have $u=u^{(p)}-u_0^{(p)} \to 0$.

The solution for an inextensible film is quite different from the solution for elastic films. In comparison we have $u_0^{(p)}(\mathcal{E} \to \infty)/u_0^{(p)}(\mathcal{E}=0.5)=-1/7$. The magnitude of the drift in the interior is therefore much smaller in the limit $\mathcal{E} \to \infty$ than for $\mathcal{E}=0.5$, but now the boundary layer terms in the particular solution $u^{(p)}$ give rise to a jetlike current in the direction of wave propagation just below the surface. This is similar to the findings of Weber$^{20}$ and Melsom$^{21}$ who considered spatially damped gravity waves in the marginal ice zone.

Figure 6. Steady-state solution $u$ from (56), $k=2\pi$ cm$^{-1}$.

Figure 7. Steady-state solution $u$ from (56), $k=1$ cm$^{-1}$.

**C. Effect of the waves on the film cover**

In addition to the local compression and dilation of the film, the waves may influence the film in other ways. Two aspects of importance for this study, spatially varying surface tension and self-organized motion in the film, are discussed below.

1. **Spatially varying surface tension**

The waves will compress the film toward one end of the tank due to the wave-induced stress on the film. Because the waves are rapidly damped, this stress will vary along the tank and cause the surfactant concentration, and hence surface tension, to vary accordingly$^{22}$ We have assumed that the elastic modulus $E$ is approximately constant, but this may not be the case if the total change in surface tension along the tank is too large. As we already have expressions for both the oscillatory and secondary mean motion, we can calculate the change in surface tension from the appropriate form of the horizontal boundary condition (1). To $O(\mathcal{E}^2)$ we obtain

$$\int_0^L \rho u^2 \Delta \sigma - u_3 \Delta w. \tag{58}$$

Here $\Delta \sigma = \sigma(a=0) - \bar{\sigma}(a=L)$ denotes the total drop in surface tension along the tank. The wave-induced stress (acting on the fluid), is given by the integrand on the left-hand side of (58), and becomes to the leading order,$^{25}$
\[ \rho u_s u_e (c = 0) = - \rho u_s (E^2/F) \gamma e^{-2\alpha z}. \] (59)

The above result is experimentally verified for inextensible films by Kang and Lee,\textsuperscript{26} and for elastic films by Gushchin and Ermakov.\textsuperscript{23}

Let us suppose the tank is so long that the waves are virtually extinct before they reach the far end of the tank. With \( E = 30 \text{ mN m}^{-1} \) and wavelengths from 5 to 7 cm, it would suffice with a tank length of less than 1 m in order to reduce the amplitude by 95%. Using (59), and neglecting the vertical velocities, the total drop in surface tension becomes

\[ \Delta \sigma = \frac{\rho u_s C_s}{2k}. \] (60)

From plots of \( E \) versus \( \sigma \) we can find acceptable limits for the total drop in surface tension so that the value of \( E \) will be approximately constant. As an example we consider oleyl alcohol which is commonly used in laboratory studies. For a mean surface tension \( \sigma_0 = 69 \text{ mN m}^{-1} \), the value of \( E \) is 30 mN m\(^{-1}\). Maximum damping (\( E = 1 \)) occurs for a wavelength of 6.55 cm. If the initial damping of the waves is \( \xi_0 = 1 \text{ mm} \) which is quite small, the drop in surface tension becomes \( \Delta \sigma = 2.93 \text{ mN m}^{-1} \). Judging from Fig. 10 of Mass and Milgram,\textsuperscript{13} the change in film elasticity would be \( \Delta E = E_{\text{max}} - E_{\text{min}} \approx 9 \text{ mN m}^{-1} \). It follows that the elastic modulus would vary as much as 30% along the tank, which is definitely unacceptable. By reducing the amplitude to \( \xi_0 = 0.5 \text{ mm} \), we find that \( \Delta E < 2 \text{ mN m}^{-1} \). It must be emphasized that the above example applies to oleyl alcohol only, whose value of \( E \) has a strong dependence on film pressure/surface tension. For most surfactants, the value of the elastic modulus becomes gradually more insensitive to changes in surface tension for increasingly dense films, i.e., higher concentration of film material.

\section*{2. Self-organized motion}

When an obstacle is placed on the surface of steady flowing water, surface-active substances will be trapped on the upstream side and form a film. The upstream front of the film is generally known as the Reynolds ridge.\textsuperscript{27} In laboratory experiments it is often observed that film material tend to recirculate in regularly spaced channels (usually two channels) oriented along the bulk flow (e.g., Mockros and Krone,\textsuperscript{28} Scott,\textsuperscript{29} and Vogel and Hirs\textsuperscript{30}). The recirculatory surface flow is usually an order of magnitude less than the bulk flow.\textsuperscript{30} Recently, Gushchin and Ermakov\textsuperscript{23} investigated the effect of waves in compressing a surface film toward a film barrier in a set of experiments much similar to the ones mentioned above. Gushchin and Ermakov observed recirculation patterns within the film in the part nearest to the Reynolds ridge, where the wave amplitudes were highest and the film least dense. Such recirculation of surface-active material must be avoided for the boundary condition (51) to hold. Common for all the above mentioned studies is that only a part of the surface is covered by the film, in contrast to, e.g., Mass and Milgram’s experiments. The role of the sidewalls and an unsteady Reynolds ridge in generating such motion is unclear. No recirculation of film material is reported by Mass and Milgram.

In the portion of the film near the ridge the surfactant concentration will be low, and resistance against surface shear is probably small. Surface-active substances that are solid in bulk phase show a higher resistance to shear than substances that are liquid,\textsuperscript{30} and the latter is used in all but one of the above studies. Vogel and Hirs\textsuperscript{a} examined monolayers prepared from both liquid and solid bulk. Furthermore, the fluid velocities were quite high in Mockros and Krone’s, Scott’s, and Vogel and Hirs’ experiments, of the order of 10 cm s\(^{-1}\). In Gushchin and Ermakov’s experiments the amplitudes of the incoming waves at the ridge predict a Stokes drift between 1 and 2 cm s\(^{-1}\) (\( \xi_0 = 2.5-3.5 \text{ mm} \), and a wavelength of 10 cm). It appears from their results that the film was stagnant in the parts of the film where the amplitudes were reduced to about 1 mm. Gushchin and Ermakov, studying a nonlinear effect, did not use small amplitude waves, and the recirculation patterns appeared in all the experimental runs. It is therefore not possible to say if the recirculation patterns would have disappeared for some lower wave amplitude, \( \xi_0 < 1 \text{ mm} \), say (Ermakov, personal communication, 2004).

A fundamental assumption in the present study is that the film covers the entire surface, hence there is no ridge. It appears that with a sufficiently dense film layer and low wave amplitudes, self-organized motion in the film can be avoided. Self-organized motion may also be prevented by using surface-active substances that are solid in bulk phase, and therefore have a higher surface shear viscosity.

\section*{V. SUMMARY AND CONCLUDING REMARKS}

The main aim of this paper has been to investigate the effect of surface films on the mean drift in spatially damped capillary-gravity waves. It has been shown that the existing linear theory agrees well with experiment for a variety of surfactants, but there is a lack of experimental data on the nonlinear wave-induced drift. The presented theory has been developed with a typical laboratory situation in mind, in contrast to previous studies. The horizontal drift velocity is shown to be strongly dependent on the value of the elasticity parameter \( E \). An intriguing result is that the horizontal drift in the upper layer of the water column is directed opposite to the direction of wave propagation for values of \( E < \sqrt{2} \). For inextensible films, the drift is similar to the wave-induced drift under an ice cover, with a jetlike current in the wave propagation direction just below the surface. For all values of \( E \) the viscous stresses on the film act in the wave propagation direction, compressing the film against the film barrier. It is concluded that the wave amplitudes must be kept small, and the film layer dense, to ensure that the elastic modulus of the film remains approximately constant. Using small amplitude waves and a dense film is also desirable in order to prevent self-organized motion in the film.
ACKNOWLEDGMENTS

This study was supported by The Research Council of Norway through Grant No. 151774/432. The author wish to thank Professor Jan Erik Weber for continuous support and guidance throughout the study. The author also wish to thank Professor Kristian Dysthe for pointing him to the experiments on the Reynolds ridge, and Dr. Stanislav Ermakov for providing detailed information on the experiments in Ref. 22.


