A modified nonlinear Schrödinger equation for broader bandwidth gravity waves on deep water

Karsten Trulsen *, Kristian B. Dysthe
Department of Mathematics, University of Bergen, Johannes Brunsgate 12, N-5008 Bergen, Norway
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Abstract
The modified nonlinear Schrödinger equation of Dysthe [Proc. Roy. Soc. Lond. Ser. A, 369, 105–114 (1979)] is extended by relaxing the narrow bandwidth constraint to make it more suitable for application to a realistic ocean wave spectrum. The new equation limits the bandwidth of unstable wave number perturbations of a Stokes wave in good agreement with the exact results of McLean et al. [Phys. Rev. Lett. 46, 817–820 (1981)]. Results are presented for the parametric bifurcation boundary between collinear and oblique most unstable side band perturbations of a Stokes wave.

1. Introduction
The nonlinear spatio-temporal evolution of gravity water-surface waves can be predicted by the nonlinear Schrödinger equation provided the steepness is small $ka \ll 1$, and the bandwidth is narrow $|\Delta k|/k \ll 1$. Here $a$ and $k$ denote a characteristic amplitude and wave number, while $\Delta k$ is a modulation wave vector. Typically, one assumes that the steepness and the bandwidth are of the same order of magnitude $O(\epsilon)$, such that the leading nonlinear and dispersive effects balance at the third order $O(\epsilon^3)$. The resulting nonlinear Schrödinger equation was pioneered by Benney and Newell [1] for nonlinear dispersive waves in general, by Zakharov [16], Hasimoto and Ono [7] and Davey [5] for gravity waves on deep water, and by Benney and Roskes [2] for gravity waves on finite depth. A modification of the nonlinear Schrödinger equation to fourth-order accuracy $O(\epsilon^4)$, which we shall denote the MNLS equation, was derived by Dysthe [6] for gravity waves on infinite depth, with minor modifications for gravity waves on deep water by Lo and Mei [8], and by Brinch-Nielsen and Jonsson [3] for gravity waves on finite depth. We here define finite depth, deep water and infinite depth as $(kh)^{-1}$ being $O(1)$, $O(\epsilon)$ and 0, respectively, where $h$ is the depth.

The limitation in bandwidth seriously limits the applicability of the nonlinear Schrödinger equation and its fourth-order extension for three-dimensional ocean waves. Firstly, ocean wave spectra from the continental shelf are often band limited, but have bandwidths exceeding the above limitation. Secondly, these equations have regions of instability for a uniform Stokes wave extending outside the narrow bandwidth constraint.

* Corresponding author.
The nonlinear Schrödinger equation predicts that a uniform Stokes wave is unstable for perturbation wave numbers of any magnitude, and there is a continuum of most unstable side band perturbations distributed along a hyperbola in the plane $\Delta k = (\lambda, \mu)$. As a result, Martin and Yuen [10] showed that for some initial conditions, energy can be leaked to high spectral components, rendering the third-order theory potentially useless for three-dimensional computations.

The fourth-order equation of Dysthe [6] (MNLS) for infinite depth has two instability regions for a uniform Stokes wave; a primary instability region bounded by $|\Delta k| < 1.5k$, and a secondary instability region for $|\Delta k| > 1.5k$. We show that the primary instability region is actually more confined than previous studies have suggested, due to a high-order correction that has often been neglected, yet this instability region still extends outside the narrow bandwidth constraint. Three-dimensional numerical computations based on the MNLS equation were done by Lo and Mei [9]. They showed that the perturbations can be contained within the bandwidth constraint for a long time, thus demonstrating that such computations are in fact feasible with the fourth-order equation.

The Zakharov integral equation (Zakharov [16], Crawford et al. [4], Stiassnie and Shen [15]) has been developed to avoid the limitation in bandwidth. While being more general, the Zakharov equation is more expensive to solve numerically than the nonlinear Schrödinger equation or the MNLS equation. In order to maintain the relative simplicity of the MNLS equation, it is desirable to look for ways to relax the bandwidth constraint, while keeping the same accuracy in nonlinearity.

After reviewing the MNLS equation for waves on deep water in Section 2, we increase the bandwidth in Section 3 by requiring $|\Delta k|/k = O(\varepsilon^{1/2})$ while keeping the same accuracy in nonlinearity. We then compare the stability properties with respect to bandwidth of the new equation with the MNLS equation and the exact results of McLean [11]. In Section 4 we discuss the parametric boundary for the bifurcation happening for decreasing depth when the most unstable side band perturbation changes from being collinear to being oblique to the carrier wave vector.

The instability of a uniform Stokes wave considered here belongs to the first member of class I, according to the terminology of McLean et al. [13]. The full instability region is bounded, but is too large to be described by a theory with a narrow-bandwidth constraint. If this entire instability region must be accounted for, the Zakharov integral equation is more appropriate. However, ocean waves are not uniform. It is therefore questionable if the Stokes wave is a good indicator for the ability of a model to predict a realistic ocean wave spectrum. Indeed it was found by Crawford et al. [4] that the effect of randomness was to reduce the extent of the instability region in the $\Delta k$ plane. Therefore we hope that the new equation may have sufficient bandwidth for application to realistic three-dimensional ocean wave problems.

2. The modified nonlinear Schrödinger equation (MNLS) for deep water

We start this review with the equations for the velocity potential $\phi(t, x, z)$ and surface displacement $\zeta(t, x)$ of an incompressible fluid with uniform depth $h$,

\begin{align}
\nabla^2 \phi &= 0 \quad \text{for } -h < z < \zeta, \\
\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} + \frac{\partial}{\partial t} (\nabla \phi)^2 + \frac{1}{2} \nabla \phi \cdot \nabla (\nabla \phi)^2 &= 0 \quad \text{at } z = \zeta, \\
\frac{\partial \zeta}{\partial t} + \nabla \phi \cdot \nabla \zeta &= \frac{\partial \phi}{\partial z} \quad \text{at } z = \zeta, \\
\frac{\partial \phi}{\partial z} &= 0 \quad \text{at } z = -h.
\end{align}

Here $g$ is the acceleration of gravity, $z$ is a vertical coordinate, the horizontal position vector is $x = (x, y)$ and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. 


Subject to the assumptions

\[ ka = O(\varepsilon), \quad |\Delta k|/k = O(\varepsilon), \quad (kh)^{-1} = O(\varepsilon). \]  

we employ the following harmonic expansions for the velocity potential and surface displacement:

\[ \phi = \tilde{\phi} + \frac{1}{2} (Ae^{i\theta+kz} + A_2e^{2i\theta+kz}) + \cdots + c.c., \]  

\[ \xi = \tilde{\xi} + \frac{1}{2} (Be^{i\theta} + B_2e^{2i\theta}) + \cdots + c.c., \]  

where c.c. denotes the complex conjugate, and the phase is \( \theta = kx - \omega t \) for a wave with carrier wave vector in the \( x \)-direction. The slow drift \( \tilde{\phi} \) and set-down \( \tilde{\xi} \) as well as the harmonic amplitudes \( A, A_2, \ldots, B, B_2, \ldots \) are functions of the slow modulation variables \( \varepsilon x \) and \( \varepsilon t \). Additionally, \( \tilde{\phi} \) depends on the slow vertical variable \( \varepsilon z \), while \( A, A_2, \ldots \) depend on the basic vertical coordinate \( z \).

From the leading-order perturbation problem for the first harmonic, we get the dispersion relation for deep-water gravity waves

\[ \omega^2 = gk. \]  

In the following, results have been made dimensionless by the substitutions

\[ \omega t \rightarrow t, \quad \frac{k(x, z)}{k} \rightarrow (x, z), \quad k(B, B_n, \tilde{\xi}) \rightarrow (B, B_n, \tilde{\xi}), \]  

\[ k^2 \omega^{-1}(A, A_n, \tilde{\phi}) \rightarrow (A, A_n, \tilde{\phi}). \]  

Hence the ordering parameter \( \varepsilon \) is implied in the variables, but does not appear explicitly.

At the fourth order \( O(\varepsilon^4) \), we get the coupled evolution equations governing the slow variation of \( A \) and \( \tilde{\phi} \),

\[ \frac{\partial A}{\partial t} + \frac{1}{2} \frac{\partial A}{\partial x} + i \frac{\partial^2 A}{\partial x^2} - \frac{i}{4} \frac{\partial^2 A}{\partial y^2} + \frac{1}{2} |A|^2 A - \frac{1}{16} \frac{\partial^3 A}{\partial x^3} + \frac{3}{8} \frac{\partial^3 A}{\partial x \partial y^2} + \frac{3}{2} |A|^2 \frac{\partial A}{\partial x} - \frac{1}{4} A^2 \frac{\partial A^*}{\partial x} + iA \frac{\partial \tilde{\phi}}{\partial x} = 0 \quad \text{at } z = 0, \]  

\[ \nabla^2 \tilde{\phi} = 0 \quad \text{for } -h < z < 0, \]  

\[ \frac{\partial \tilde{\phi}}{\partial z} = \frac{1}{2} \frac{\partial}{\partial x} |A|^2 \quad \text{at } z = 0, \]  

\[ \frac{\partial \tilde{\phi}}{\partial z} = 0 \quad \text{at } z = -h. \]  

The uniform Stokes wave correct to the fourth order is an exact solution of (10) – (13) given by

\[ A = A_0 e^{-(i/2)A_0^2} \quad \text{and} \quad \tilde{\phi} = 0, \]  

where \( A_0 \) is real. Its stability can be investigated by assuming small perturbations in amplitude and phase

\[ A = A_0(1 + a' + i\theta') e^{-(i/2)A_0^2}, \]  

having the plane wave solution

\[ (a' \theta' \tilde{\phi})^T = e^{i(\lambda x + \mu y - \omega t)} + c.c. \]
The dispersion relation for the perturbation is given by

$$\Omega = \frac{1}{2} \lambda + \frac{1}{16} \lambda^3 - \frac{3}{8} \mu \mu + \frac{3}{8} A^2_0 \lambda \pm \sqrt{R},$$  \hfill (17)

where the radicand is

$$R = \left( \frac{1}{8} \lambda^2 - \frac{1}{4} \mu^2 \right) \left( \frac{1}{8} \lambda^2 - \frac{1}{4} \mu^2 - A^2_0 \lambda^2 + A^2_0 (\lambda^2/k) \coth (Kh) \right) + \frac{1}{16} A^4_0 \mu^2,$$  \hfill (18)

and

$$K = \sqrt{\lambda^2 + \mu^2}. \hfill (19)$$

The last term in (18) is insignificant within the fourth order $O(\varepsilon^4)$, and has often been neglected in previous works. However, the actual behavior of the truncated system (10)–(13) is described by the full expression in (17) and (18). In order to better understand numerical solutions of the MNLS equation, the last term in (18) must be considered. Its effect is to reduce the extent of the unstable region, as shown in Fig. 1. However, the figure also shows that the primary instability region still extends outside the constraint $|\Delta k|/k = O(\varepsilon)$, and may render this theory somewhat unattractive for studies of the three-dimensional evolution of Stokes waves.

3. The new equation for broader bandwidth

We improve the resolution in bandwidth by the new assumptions

$$ka = O(\varepsilon), \quad |\Delta k|/k = O(\varepsilon^{1/2}), \quad (kh)^{-1} = O(\varepsilon^{1/2}). \hfill (20)$$

The previous harmonic expansions for the velocity potential (6) and surface displacement (7) can still be used, but now $\phi, \zeta, A, A_2, \ldots, B, B_2, \ldots$ are functions of the new slightly faster modulation variables $\varepsilon^{1/2} x$ and $\varepsilon^{1/2} t$. Additionally, $\tilde{\phi}$ now depends on the new slightly faster vertical variable $\varepsilon^{1/2} z$.

We insist on keeping the same accuracy in nonlinearity as in the MNLS equation. Since none of the fourth-order contributions to the MNLS equation is quartically nonlinear, it suffices to consider the new equation only up to

![Fig. 1. Primary instability region for infinite depth and $A_0 = 0.2$: (−) based on the full radicand (18); (−−) based on the radicand (18) without the last term; (•) most unstable side band perturbation.](image-url)
order $O(\epsilon^{3.5})$. Carrying the perturbation analysis through in a similar fashion, the coupled evolution equations for $A$ and $\phi$ are summarized here, while the detailed expressions for the harmonics are given in Appendix A.

\[
\frac{\partial A}{\partial t} + \frac{1}{2} \frac{\partial A}{\partial x} + i \frac{\partial^2 A}{\partial x^2} - i \frac{\partial^2 A}{\partial y^2} - \frac{1}{16} \frac{\partial^3 A}{\partial x^3} + \frac{3}{8} \frac{\partial^3 A}{\partial x \partial y^2} \\
- \frac{5i}{128} \frac{\partial^4 A}{\partial x^4} + \frac{15i}{32} \frac{\partial^4 A}{\partial x^2 \partial y^2} - \frac{3i}{32} \frac{\partial^4 A}{\partial y^4} + i \frac{1}{2} |A|^2 A + \frac{7}{256} \frac{\partial^5 A}{\partial x^5} - \frac{35}{64} \frac{\partial^5 A}{\partial x^3 \partial y^2} \\
+ \frac{21}{64} \frac{\partial^5 A}{\partial x \partial y^4} + \frac{3}{2} |A|^2 \frac{\partial A}{\partial x} - \frac{1}{4} A^2 \frac{\partial A}{\partial x} + i \frac{\partial \phi}{\partial x} A = 0 \quad \text{at } z = 0,
\]

(21)

\[
\nabla^2 \phi = 0 \quad \text{for } -h < z < 0,
\]

(22)

\[
\frac{\partial \phi}{\partial z} = \frac{1}{2} \frac{\partial}{\partial x} |A|^2 \quad \text{at } z = 0,
\]

(23)

\[
\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -h.
\]

(24)

The uniform Stokes wave solution is still given by (14). Subject to the small perturbation in amplitude and phase (15) and (16), the dispersion relation for the perturbation is given by

\[
\Omega = P \pm \sqrt{Q \left(Q - A_0^2 + A_0^2 \frac{\lambda^2}{K} \coth(Kh)\right) + \frac{1}{16} A_0^4 \lambda^2},
\]

(25)

where

\[
P = \frac{1}{2} \lambda + \frac{1}{16} \lambda^3 - \frac{3}{8} \lambda \mu^2 + \frac{7}{256} \lambda^5 - \frac{35}{64} \lambda^3 \mu^2 + \frac{21}{64} \lambda^3 \mu^4 + \frac{1}{2} A_0^2 \lambda,
\]

(26)

\[
Q = \frac{1}{8} \lambda^2 - \frac{1}{4} \mu^2 + \frac{5}{128} \lambda^4 - \frac{15}{32} \lambda^2 \mu^2 + \frac{3}{32} \mu^4,
\]

(27)

and $K$ is still given by (19).

A prominent feature of the new equation is that the neutral stability curves in the limit $A_0 = 0$ are no longer straight lines, as shown in Fig. 2. Recall that both the nonlinear Schrödinger equation and the MNLS equation have neutral stability along the intersecting straight lines $\frac{1}{2} \lambda^2 - \frac{1}{4} \mu^2 = 0$ when $A_0 = 0$. In the new theory the

Fig. 2. Neutral stability curves in the limit $A_0 = 0$. (---) new result; (—) nonlinear Schrödinger equation and MNLS; (···) exact curve for infinite depth [14].
corresponding neutral stability curves are given by \( Q = 0 \), which approximate the exact curves more closely for moderate values of \( K \) and large depth. Recall that the exact neutral stability curves for the first member of the class 1 instabilities are given by

\[
\sqrt{k_1 \tanh(k_1 h)} + \sqrt{k_2 \tanh(k_2 h)} = 2\sqrt{\tanh h},
\]

where \( k_1 = (1 + \lambda, \mu) \) and \( k_2 = (1 - \lambda, -\mu) \), see for instance [12]. The special case of infinite depth corresponds to the “figure eight” of Phillips [14].

For comparison with the exact instability computations of McLean [11] we note that his wave steepness, which we denote by \((ka)_M\), is given in terms of half the crest to trough height of the wave. The crest to trough height of the Stokes wave solution (14) with third-order accuracy can be deduced from the expressions given in Appendix A,

\[
(ka)_M = \frac{1}{2} (\zeta_{\text{crest}} - \zeta_{\text{trough}}) = A_0 + \frac{1}{2} A_0^3.
\]

With third-order accuracy, the appropriate amplitude \( A_0 \) for comparison is then

\[
A_0 = (ka)_M - \frac{1}{2} (ka)_M^3.
\]

In Figs. 3 and 4 we show the instability regions for infinite depth with \((ka)_M = 0.1\) and \(0.2\) respectively. The results of the new equation are compared with the MNLS equation and with the exact results of McLean [11]. The most unstable side band perturbations are indicated for the new equation and the exact theory of McLean [11]. The new equation is seen to predict an instability region in better agreement with the exact results.

In Fig. 5 we show the instability region for depth \( h = 6 \) with \( A_0 = 0.1 \). The result of the new equation is compared with MNLS. The most unstable side band perturbation wave vector is indicated for the new equation.

McLean [12] performed exact instability computations for finite depth \( h = 2 \) and less. Our present theory for deep water should not be used for the small depths considered by him. If a broader-banded theory for finite depth is desired, the equations of Brinch-Nielsen and Jonsson [3] should be extended in a similar manner as we have done in the present theory for deep water.

We remark that if the new equation had been truncated at order \( O(\varepsilon^3) \), the primary instability region would be of infinite extent and asymptotic to the solid curve in Fig. 2. This is rather similar to the behavior of the original nonlinear Schrödinger equation, which has an instability region of infinite extent asymptotic to the dashed line Fig. 2. Hence the highest order \( O(\varepsilon^{3.5}) \) nonlinear terms found by Dysthe [6] are necessary to limit the primary instability region.
4. Bifurcation of the most unstable perturbation

It is well known that for Stokes waves on sufficiently deep water, the most unstable perturbation wave vectors are collinear with the carrier wave vector, while on sufficiently shallow water they are oblique to the carrier wave vector. This can be seen in Figs. 3–5, where the most unstable perturbation wave vectors have been indicated. The fact that this qualitative change depends both on the depth and on the steepness was observed by Brinch-Nielsen and Jonsson [3], but the actual dependence has not been found. We have calculated the parametric boundary between these two qualitatively different behaviors in terms of the dimensionless amplitude $A_0$ and the dimensionless depth $h$, based on the dispersion relations (17) and (25). The results are presented in Fig. 6. It is seen that the presence of a deep bottom can make a qualitative difference in the evolution of surface waves if the steepness is sufficiently small. The good agreement between the new broader-banded equation and the MNLS equation is anticipated since the most unstable perturbations are not broad-banded.
5. Conclusions

The addition of a few new linear terms to the modified nonlinear Schrödinger equation of Dysthe [6] has improved the resolution in bandwidth significantly. With the new equation, the extent of the instability region for a Stokes wave has been reduced, and compares quantitatively better with exact results. The bandwidth of the theory has also been widened. As a result, the instability region for a Stokes wave is now significantly better contained within the asymptotic validity of the theory.

The problem of energy leakage described by Martin and Yuen [10], which was partially solved by the work of Dysthe [6], should now have been eliminated. By the enhanced resolution in bandwidth, it is tempting to believe that the new equation may satisfy the major objection against using band-limited evolution equations for numerical computations on three-dimensional weakly nonlinear slowly modulated water-surface waves. There might however be new numerical stability problems introduced by the presence of fourth- and fifth-order derivatives.

We hope that the new equation has sufficient bandwidth to be useful for realistic ocean wave problems. Numerical integration is currently being pursued by us, and will be reported in the future.

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Appendix A. Details of the perturbation expansion

We here summarize the dimensionless expressions for the variables $A_2$, $A_3$, $B$, $B_2$, $B_3$ and $\zeta$ in terms of the basic unknowns $A$ and $\tilde{\phi}$ for the perturbation expansion in Section 3. These expressions are correct to order $O(\varepsilon^3)$. All variables are evaluated at $z = 0$.

$$B = iA + \frac{1}{2} \frac{\partial A}{\partial x} + i \frac{\partial^2 A}{\partial x^2} - \frac{i}{4} \frac{\partial^2 A}{\partial y^2} - \frac{1}{16} \frac{\partial^3 A}{\partial x^3} - \frac{3}{8} \frac{\partial^3 A}{\partial x \partial y^2}$$
\[-\frac{5i}{128} \frac{\partial^4 A}{\partial x^4} + \frac{15i}{32} \frac{\partial^4 A}{\partial x^2 \partial y^2} - \frac{3i}{32} \frac{\partial^4 A}{\partial y^4} + \frac{i}{8} |A|^2 A, \quad (A.1)\]

\[A_2 = \frac{i}{2} A \frac{\partial^2 A}{\partial y^2} - \frac{i}{2} \left( \frac{\partial A}{\partial y} \right)^2, \quad (A.2)\]

\[B_2 = -\frac{1}{2} A^2 + i A \frac{\partial A}{\partial x} + \frac{1}{8} A \frac{\partial^2 A}{\partial x^2} + \frac{3}{8} \left( \frac{\partial A}{\partial x} \right)^2 - \frac{1}{4} A \frac{\partial^2 A}{\partial y^2} + \frac{3}{4} \left( \frac{\partial A}{\partial y} \right)^2, \quad (A.3)\]

\[A_3 = 0, \quad (A.4)\]

\[B_3 = -\frac{3}{8} i A^3, \quad (A.5)\]

\[\bar{\zeta} = -\frac{\partial \bar{\phi}}{\partial t} - \frac{1}{16} \frac{\partial^2 |A|^2}{\partial x^2} - \frac{1}{8} \frac{\partial^2 |A|^2}{\partial y^2}. \quad (A.6)\]

References


