Modelling a "rogue wave"- speculations or a realistic possibility?

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1 Introduction.

There is growing evidence that wave records may under certain conditions have occurrences of extreme waves in excess of those predicted by the Rayleigh distribution (e.g.Skourup et.al.(1996)). The occurrences of dangerous wave conditions in coastal waters may possibly be explained by focussing (or caustics) due to refraction by bottom topography or current gradients, and even reflection from land. Well documented in that respect are the giant waves sometimes found in the Aguhlas current on the eastern coast of South Africa (Lavrenov (1998))

It seems, however, that this kind of freak- or rogue waves exist even in the open ocean far away from strong current gradients (Sand et.al. (1990), Skourup et.al.(1996)). In the following I shall concentrate on this latter case.

Skourup et.al.(1996) analyzed more than 12 years of wave records from the central north sea (the Gorm field). They used the following criteria to select candidates for their rogue wave collection: Single waves with cam heights, $a_c > 1.1H_s$,[1] or wave heigts larger than $2H_s$, where H_s is the significant wave height of the surrounding 20 min. wave record. They find the expected extreme value of the ratio a_c/H_s to be approximately 1.8, which is outside the range of Gaussian waves.

Warren et.al. (1998) analyzed some other North Sea data. Comparisons were made with the modified Rayleigh distribution of Tung & Huang (1985), which take into account second order nonlinear effects. For the case of deep water waves (see their figure 11) the data is not easily reconciled with the theoretical distribution.

Ratios of $a_c/H_s > 2$ and $H_{\max}/H_s > 2.5$ has been reported (Kjeldsen (1984), Sand et.al. (1990)). Although years of wave data from numerous buoys have been analyzed, the number of freak wave events recorded are still modest. The chances that such a wave hits a buoy is even lower than was previously expected, as pointed out by Magnusson et.al. (1999).

Interest in these waves is not only because of our rather limited knowledge of their statistical probability of occurrence. We need to know more about their dynamics, what they look like, how long they last and so on.

2 The freak character of extreme waves.

The rogue wave is often described as a freak, "the one out of nowhere". The paper by Skourup et.al.(1996) sheds some light on the form of a rogue wave event. They found that the expected ratio between the crest height and the corresponding wave

[1] The probability $P(a_c > 1.1H_s)$ for such an event to happen according to the Rayleigh distribution is roughly $\simeq 6 \cdot 10^{-5}$.

height of their rogue waves was approximately 0.7. This large ratio can clearly not be explained by the nonlinear crest asymmetry of Stokes waves, as pointed out by the authors. It is, however, rather easily explained as the effect of a very short group as demonstrated in Figure 1 by comparing groups of different lengths.

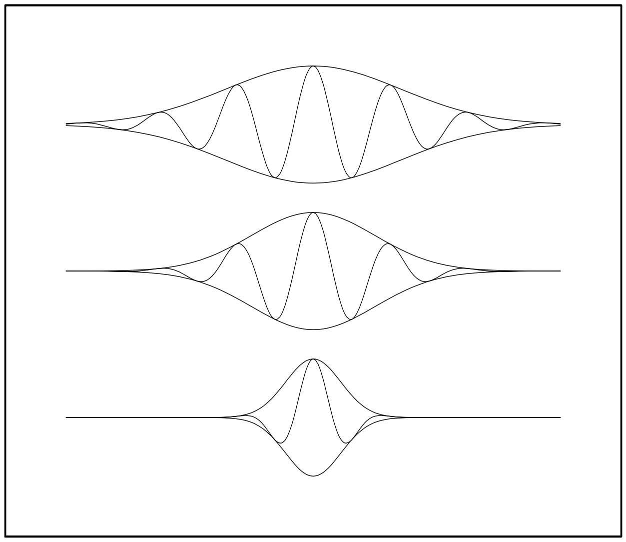

Figure 1: Shows the ratio a_c/H to increase with decreasing group length.

Boccotti (1981) (see also Phillips et.al. (1993)) have investigated the expected configuration in space and time surrounding extremely high crests in a random Gaussian wave field. The most likely configuration was found to have approximately the form of the auto-correlation function for the wave field. A real test of this result need more data than is presently available. The findings of Phillips et.al. (1993) seem to indicate that the variability of the extreme event configuration is rather large.

3 The physics of rogue wave events.

What about the physics behind rogue waves? Clearly they represent a very high concentration of wave energy compared to the average[2]. A number of mechanisms are known that produce large waves from moderately small ones by focusing the energy. Basically there are three types of effects:

Spatial focussing.

This is due to refraction by bottom topography or current gradients and is a well known reason for dangerous waves in coastal waters. An example of the effect of current refraction is the giant waves reported in the Agulhas current off the African south-east coast (see e.g. Lavrenov (1998)). Far offshore on the open ocean with only very small current velocities (less than 20cm/s say) it would seem that these effects are negligible. White and Fornberg (1998) have pointed out, however, that even small random current fluctuations with rms values of the order 10cm/s can give focussing provided their scale is sufficiently large (of the order of 10km). Thus

[2]For a wave with $a_c = 1.5H_s$ the consentration is roughly a factor 18, if the energy density in the rogue wave is estimated by $\rho g a_c^2/2$.

they maintain that even the very weak refraction found in the open ocean may produce ”hotspots” of wave energy. In their numerical ray-tracing calculations the incoming wave field is unidirectional, and they get caustics after some distance into the fluctuating wave field. Further ”downstream” the rays appear rather random.

From experience with similar refraction calculations (Trulsen et.al. 1990) we suggest that even a small directional distribution of the incoming wave field will ”smear out” the caustics and thus reduce the effect of weak refraction to minor fluctuations in energy density. This effect is illustrated by the refraction calculations shown in Figure 2. In Figure2 (a) three neighboring rays (note the different scales on the two axis) enter in the same direction, and a caustic forms after some distance into the weak current area. In Figure 2(b) each of the three original rays have got two companions starting in slightly different directions ($\backsim 1^o, 2^o$). The location of the caustic is seen to depend strongly on the angular direction of the incoming waves.

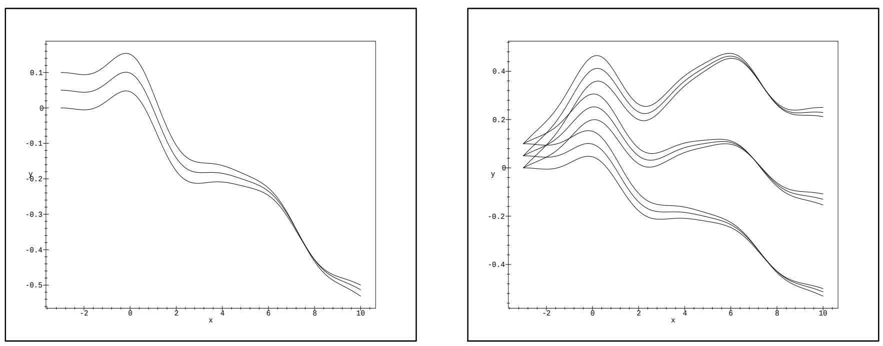

Figure 2: (a) Caustic formed by three initially parallel rays. (b) Shows how the caustic move sideways by slight changes (1^o, 2^o) of the initial ray direction.

It is demonstrated elsewhere (Dysthe 2001) that the curvature $\varkappa$ of the refracted rays is given by the simple formula

$$\varkappa = \frac{\zeta}{v_g} + O(\mu^2) \tag{1}$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the vertical component of vorticity of the (horizontal) current velocity $\mathbf{u} = (u, v)$ and v_g is the group velocity, provided that $\mu \equiv |\mathbf{u}| / v_g \ll 1$. For a zero mean, random vorticity field which is statistically homogeneous and isotropic it is shown in the appendix that rays moving through such an area of the ocean experience an angular diffusion i.e. $\langle \Delta\theta^2 \rangle = 4Ds$ where s is the arclength and $\Delta\theta$ the angular deviation from some reference point on a ray, and the diffusion coefficient is given by

$$D = \frac{1}{2} \frac{\langle \zeta^2 \rangle}{v_g^2} L_{int} \tag{2}$$

The integral length scale L_{int} is given in terms of the normalized vorticity auto-correlation function $R(|\mathbf{x} - \mathbf{y}|) = \langle \zeta(\mathbf{x})\zeta(\mathbf{y}) \rangle / \langle \zeta^2 \rangle$ as $L_{int} = \int_0^\infty R(x)dx$. As an example we use parameters roughly corresponding to those used in Figure 6a of White & Fornberg (1998) with $\langle \zeta^2 \rangle^{1/2} = 2 \cdot 10^{-5} s^{-1}$, $v_g = 8m/s$ and $L_{int} = 3km$. We then get $D \simeq 0.94 \cdot 10^{-5} km^{-1}$. To produce a rms angular spread of $\pm 5^o$ to initially mono-directional rays will then require a propagation distance of

$\simeq 270km$, which seems in reasonable agreement with their result (Figure 6a). For swell this tendency towards angular spreading will counteract the tendency towards "directional filtering" due to distance from the storm area that created it. The initial mono directional wave field of White & Fornberg therefore appears rather unrealistic.

Temporal-spatial focusing.

This is the result of dispersion and a chirped spatial distribution of frequencies. The effect is used in a well-known technique for producing short groups of large waves at a given position in a wave tank. It is done by producing a long and chirped wave group (with steadily decreasing frequency) by the wave maker. With proper design of the frequency chirp, dispersion brings this group to contract to a few wavelengths at a given position. This type of focussing has been suggested by Pelinovsky et.al. (2000) (see also their article in this book) as a possible explanation for freak waves. They show (using the KdV equation for shallow water waves) that if a given chirped wavetrain produces strong focusing in the absence of other waves, it will still do so (although somewhat weaker) when a random wave field is added. If the amplitude of the deterministic chirped wavetrain is below the rms value of the random waves it will remain "invisible" until it focuses.

For the temporal-spatial focussing to work, however, a spatial ordering of frequencies in a chirped wavetrain is needed. So far the question of how such a situation may develop spontaneously has not been answered.

Nonlinear focusing.

The so-called Benjamin Feir (BF) instability of regular wavetrains is well-known. Henderson, Peregrine and Dold (1999) have investigated what they call steep wave events (SWE) by simulating the evolution of a periodically perturbed regular wavetrain. Due to the BF instability the wavetrain breaks up into periodic groups. Within each group a further focusing takes place producing a very large wave having a steepness roughly 3 times the initial steepness of the wavetrain. For narrow band waves centered around the wave number k, the lowest order evolution equation accounting for both nonlinearity and dispersion is the Non-Linear Schroedinger equation (NLS). The surface elevation ζ can be represented as

$$\zeta = A(x,t)e^{i(kx-\omega(k)t)} + A_2(x,t)e^{i2(kx-\omega(k)t)} + ..c.c$$

and a similar expression for the velocity potential. Here the complex amplitude functions A and A_2 (of the first and second harmonic) are slowly varying in space and time compared to the wavelength and wave period. The ratio $A_2/A = O(\epsilon)$ where ϵ is a typical wave steepness. To lowest significant order the amplitude function A of the first harmonic satisfies the NLS equation

$$i(A_t + v_g A_x) - \frac{\omega}{8k^2}A_{xx} = \frac{\omega k^2}{2}A\left|A\right|^2$$

By the transformation to non-dimensional variables

$$\begin{aligned} x &\to k(x - v_g t) \\ t &\to \frac{1}{2}\omega t \\ A &\to \frac{k^2}{\sqrt{2}\omega}A^* \end{aligned}$$

this equation attains the canonical form

$$iA_t + A_{xx} + 2A\left|A\right|^2 = 0 \tag{3}$$

There are two types of solutions of (3) associated with a group of large waves. The first is the envelope soliton solution

$$\frac{e^{it}}{\cosh(x)}$$

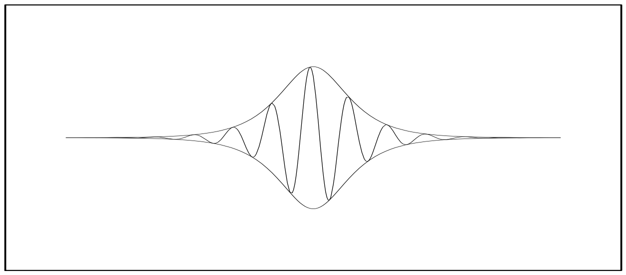

Figure 3: Envelope soliton.

whose first harmonic envelope $|A|$ does not change its form (see Figure 3). The second are the so-called breather solutions, a one parameter family of solutions that can be written

$$e^{2it}\frac{\cosh(\Omega t - 2i\varphi) - \cos\varphi\cos(px)}{\cosh(\Omega t) - \cos\varphi\cos(px)} \tag{4}$$

where

$$p = 2\sin\varphi \qquad \text{and} \qquad \Omega = 2\sin(2\varphi)$$

For real φ the solution is space-periodic. It evolves from a nearly uniform wavetrain to space-periodic soliton-like groups, and back to a uniform wavetrain.

For imaginary φ the solution (4) is time-periodic, "breathing" itself up from a nearly uniform wavetrain to a soliton-like group and back to a uniform wavetrain during one period (see Dysthe & Trulsen (1999) and the references therein).

As a limiting case for these two solutions (i.e. when $\varphi \to 0$) (4) tends to the Peregrine solution (Peregrine (1983))

$$e^{2it}\left[1 - \frac{4(1+4it)}{1+4x^2+16t^2}\right] \tag{5}$$

which is illustrated in Figure 4(a) and 4(b). Henderson et.al. (1999) (see also Dysthe and Trulsen (1999)) suggest that the SWE they observe in their simulations can be approximately modelled by this breather solution. A Peregrine type breather at its maximum is shown in Figure 5. Here the second order term A_2 is taken into account and the initial uniform wavetrain had a steepness of 0.12 . While the solution (5) is in the frame of reference moving with the group velocity, Figure 5 is a breather "time-series" as it would have been observed by buoys at three slightly different horizontal locations (the envelope of the first harmonic is also shown).

It was shown by Alber (1978) that if the bandwidth exceeds some small critical value, there is no BF instability. The natural wind wave spectra seems to always exceed this critical value. Thus the nonlinear focussing as described by Henderson et. al. is not likely to work. This does not mean, however, that the form and dynamics of the SWE they observe may not have a close relation to the rogue wave phenomena.

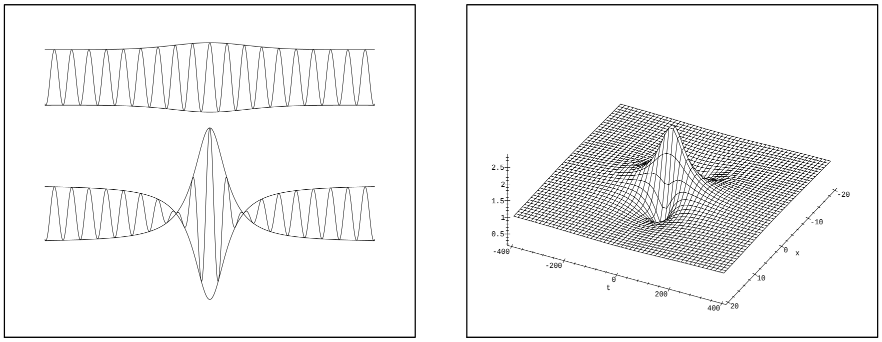

Figure 4: (a) The Peregrine breather solution (equation (5)) at two different times. (b) Space-time illustration of the Peregrine breather envelope.

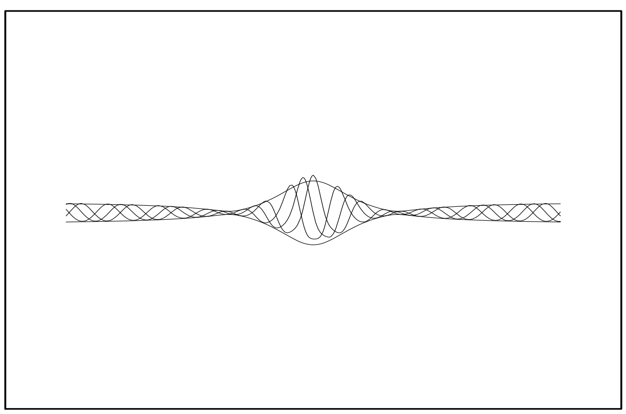

Figure 5: Time series of a passing Peregrine breather from three slightly different horizontal locations. Here the full second order expression for the elevation ζ is used. Also shown is the first harmonic envelope. The steepness of the initial wavetrain is 0.12.

Thus it seems that all the above mechanisms for producing large waves need some special preparation or coherence to work.

Does this leave us with the old idea that the rogue waves are simple (and unlikely) constructive interference phenomena that can be explained by linear- or slightly (second order) nonlinear theory? This seems to be a rather popular assumption, and serves as a basis for the statistical estimates.

Another possibility, however, is that weak (third order) nonlinear wave interactions may play a role. Although these interactions are slow they are known to produce large waves under special conditions. The correlation they introduces between the interacting waves may change the probability of constructive interference.

I think it is fair to say that nobody knows the answer to these questions yet. To test this latter idea, a project funded by The Norvegian Research Council is presently starting up. The idea is to simulate a piece of the ocean surface of dimensions approximately 100x100 wavelengths. Starting with a wave field based on a suitably truncated empirical spectrum (like JONSWAP) we will use the numerical model described by Trulsen et.al. (2000) to follow the evolution of the wave field. The probability of seeing a freak wave event in a simulation is estimated to be more than 10^4 times higher than for a corresponding point measurement (buoy) over the same period of time.

4 Appendix

We shall now consider angular diffusion of rays moving through a random eddy field. For simplicity we shall assume this vorticity field ζ to be statistically homogeneous and isotropic with zero mean.

We denote by $\Delta\theta$ the angular change of direction over an arclength s of the ray path from some reference starting point. From the formula (1) we have to order ϵ

$$\Delta\theta = \frac{1}{v_g} \int_0^s \zeta(\mathbf{x}(s'))ds'$$

It follows that

$$\left\langle \Delta\theta^2 \right\rangle = \frac{1}{v_g^2} \int_0^s \int_0^s \left\langle \zeta(\mathbf{x}(s'))\zeta(\mathbf{x}(s'')) \right\rangle ds'ds''$$

and by the assumption of homogeneity and isotropy we have

$$\left\langle \Delta\theta^2 \right\rangle = \frac{\left\langle \zeta^2 \right\rangle}{v_g^2} \int_0^s \int_0^s R(|\mathbf{x}(s')-\mathbf{x}(s'')|)ds'ds'' \tag{6}$$

where the normalized vorticity auto-correlation function R is given by

$$R(|\mathbf{x}-\mathbf{y}|) = \frac{\langle \zeta(\mathbf{x})\zeta(\mathbf{y}) \rangle}{\langle \zeta^2 \rangle}$$

Let L be a characteristic correlation distance for ζ, i.e. $R(|\mathbf{x}-\mathbf{y}|)$ is small when $|\mathbf{x}-\mathbf{y}| > L$. If we assume that $\varkappa L << 1$, then the ray $\mathbf{x}(s)$ can be approximated by a straight line in the integral (6). Doing this we obtain for $s >> L$

$$\begin{aligned} \left\langle \Delta\theta^2 \right\rangle &= \frac{\left\langle \zeta^2 \right\rangle}{v_g^2} \int_0^s \int_0^s R(|s'-s''|)ds'ds'' \\ &\cong 2\frac{\left\langle \zeta^2 \right\rangle}{v_g^2} \int_0^\infty R(x)(s-x)dx \cong 2s\frac{\left\langle \zeta^2 \right\rangle}{v_g^2} \int_0^\infty R(x)dx \equiv 4Ds \end{aligned}$$

where D is the diffusion coefficient of angular diffusion given by

$$D = \frac{1}{2} \frac{\langle \zeta^2 \rangle}{v_g^2} L_{int}$$

and L_{int} is the integral lengthscale

$$L_{int} = \int_0^\infty R(x) dx$$

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