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SENSITIVITY ANALYSIS IN A MULTIDIMENSIONAL DEMOGRAPHIC PROJECTION MODEL WITH A TWO-SEX ALGORITHM*

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Formulas are derived for the effect that a slight change in the occurrence/exposure rate of the multidimensional projection model has on the elements of the population vector. The projection model classifies the population by sex, age, and marital status. The model includes a two-sex algorithm in order to ensure consistency between numbers of male and female marriages, number of divorces for the two sexes, and new widows (widowers) and deceased spouses. The sensitivity functions and elasticities are applied to data from the Netherlands for the period 1980–1984. The results indicate that marriage market mechanisms, in particular competition and substitution effects, are reasonably well modelled.

KEY WORDS: Elasticity, nuptiality model, marriage market, sensitivity analysis, multidimensional projection model, two-sex algorithm.
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1. INTRODUCTION

The use of multidimensional population projection models makes it possible to express transitions which individuals experience, as they pass from one state to another, in terms of age- and sex-specific transition rates. These rates, for example, refer to the transition from being alive to being dead, from being single to being

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married, or from being employed to being unemployed. Due to the uncertainty in the estimation of such rates, it is useful to analyze the impact of a small change in an age- and sex-specific transition rate on the model results, particularly when the model is used for projection purposes.

There are two methods applicable to this so-called sensitivity analysis. The first one is the empirical method, which is simply the computation of the population projection under the original and the changed transition rates. The differences in the results are an indication of the impact of the changing rates. This method is used, for example, by Gordijn and Heida (1979), Laaksonen (1980), and Cruijsen and Van Hoorn (1983). The second is the analytical method. This method is based on deriving sensitivity functions from the mathematical formulation of the model used. These sensitivity functions are an indication of the impact of changing a transition rate in terms of the model parameters. Theoretical aspects of the analytical method are presented by Arthur (1984) and Willekens (1977). Arthur particularly analyzed the effects of simultaneous changes in different parameters, such as changing age-specific mortality patterns.

This paper will concentrate on the analytical method. A special case of the multidimensional model, for instance as described by Willekens and Drewe (1984), will be analyzed: the nuptiality model with a two-sex algorithm. The analysis will be limited to an investigation of the effect of a change of a single parameter over a single unit projection period.

The outline of this paper is as follows. Section 2 gives a description of the multidimensional nuptiality model. It also deals with the two-sex problem and the consistency algorithm. In Section 3, sensitivity functions for the multidimensional nuptiality model are derived by applying matrix differentiation techniques. An illustration with data from the Netherlands is presented in Section 4. The focus will be on marriage market mechanisms.

2. THE MULTIDIMENSIONAL MODEL

The model analyzed in this paper is a multidimensional nuptiality model. In this type of model, individuals are classified by age, sex, and marital status. Four marital states are distinguished: single (never married), married, divorced, and widowed. Several types of transitions between these marital states are possible, namely first marriage, remarriage, divorce, and transition to widowhood. These events are the so-called internal events. They affect the structure of the model population, without affecting its size. However, the external events, like migration and death, allow transitions to and from the model population. Due to the nature of the internal events, there should be some consistency between male and female events. For example, the total number of marriages of males should equal the total number of marriages of females. This consistency problem is known as the two-sex problem in nuptiality models (Keilman, 1985a). The solution we will follow here is to adjust the initial transition rates in such a way that consistency is attained. The transition rates are adjusted by applying a so-called two-sex algorithm or consistency algorithm. For a general description of the consistency algorithm applied here, see Van Imhoff (1992) and, with respect to nuptiality models, Keilman (1985a and 1985b).

The multidimensional nuptiality model includes stock and flow variables. The number of persons in each population category is denoted by the symbol ${}_sK_i(x,t)$, indicating the number of persons of sex s, age x, and marital status i at time t. Throughout this paper, it will be assumed that the length of the unit projection period equals the length of the age intervals. The length of the age intervals is set equal to 1. The model comprises the following flow variables:

${}_sE_{ij}(x,t)$	the number of events experienced by persons of sex s and age x at time t from marital status i to marital status j in the period $(t,t+1)$,
${}_sE_{id}(x,t)$	the number of persons of sex s and age x at time t that die in marital status i in the period $(t,t+1)$,
${}_sE_{io}(x,t)$	the number of emigrants of sex s and age x at time t with marital status i in the period $(t,t+1)$,
${}_sO_{oj}(x,t)$	the number of immigrants of sex s and age x at time t with marital status j in the period $(t,t+1)$.

All flow variables are defined using a period-cohort observation criterion (see Willekens and Drewe, 1984).

The projected population at time $t+1$ equals the original population at time t plus the number of persons entering marital status i (for example, by marriage or immigration) minus the number of persons leaving status i (for example, by divorce, emigration, or death):

$$
\begin{aligned}
{}_sK_i(x+1,t+1) = {}_sK_i(x,t) - \sum_{j\neq i} {}_sE_{ij}(x,t) + \sum_{k\neq i} {}_sE_{ki}(x,t) \\
- {}_sE_{id}(x,t) - {}_sE_{io}(x,t) + {}_sO_{oi}(x,t)
\end{aligned}
\tag{1}
$$

Equation (1) does not apply to the first age group (the children born during the projection interval) and the last open-ended age group. However, nuptiality behaviour, particularly marriage and divorce, is only important in the intermediate age groups. Thus, throughout the rest of the paper, only the intermediate age groups will be dealt with. The projection model for the other age groups, and an analysis of the sensitivity functions are given elsewhere (Willekens and Drewe, 1984; Ekamper, 1990).

The occurrence/exposure rates ${}_sm_{ij}(x,t)$ are obtained by dividing the number of events by the number of person-years ${}_sL_i(x,t)$. The rate for internal events is:

$$
{}_sm_{ij}(x,t) = \frac{{}_sE_{ij}(x,t)}{{}_sL_i(x,t)} \tag{2}
$$

the mortality rate is:

$$
{}_sm_{id}(x,t) = \frac{{}_sE_{id}(x,t)}{{}_sL_i(x,t)} \tag{3}
$$

and the emigration rate is:

$$
{}_sm_{io}(x,t) = \frac{{}_sE_{io}(x,t)}{{}_sL_i(x,t)}. \tag{4}
$$

In general, the number of person-years is equal to the total length of all life lines. Since the individual life lines are often not known (for example, when aggregate population statistics are used for the estimation of the occurrence/exposure rates), an assumption has to be made. Here it is assumed that population is a linear function of time. That is the case when the events are distributed uniformly over the projection interval. The number of person-years lived in marital status i then equals:

$$ {}_sL_i(x,t) = \tfrac{1}{2}[{}_sK_i(x,t) + {}_sK_i(x+1,t+1)]. \quad (5)$$

Models using this linear hypothesis are called linear models. Exponential models assume that occurrence/exposure rates are constant over $(t,t+1)$, see e.g. Van Imhoff (1990). This leads to a population which is an exponential function of time. Hoem and Funck Jensen (1982) and Gill and Keilman (1990) examine the advantages and disadvantages of both linear and exponential models.

Substitution of equations (2) to (5) into equation (1) yields:

$$\begin{aligned} {}_sK_i&(x+1,t+1) \\ &= \frac{1 - \frac{1}{2}({}_sm_{id}(x,t) + \sum_{j\neq i} {}_sm_{ij}(x,t) + {}_sm_{io}(x,t) - \sum_{k\neq i} {}_sm_{ki}(x,t))}{1 + \frac{1}{2}({}_sm_{id}(x,t) + \sum_{j\neq i} {}_sm_{ij}(x,t) + {}_sm_{io}(x,t) - \sum_{k\neq i} {}_sm_{ki}(x,t))} {}_sK_i(x,t) \\ &+ \frac{1}{1 + \frac{1}{2}({}_sm_{id}(x,t) + \sum_{j\neq i} {}_sm_{ij}(x,t) + {}_sm_{io}(x,t) - \sum_{k\neq i} {}_sm_{ki}(x,t))} {}_sO_{oi}(x,t). \end{aligned} \quad (6)$$

This equation only applies to marital status i. However, the equation can be extended to all marital states by a matrix equation:

$$\begin{aligned} {}_s\mathbf{K}(x+1,t+1) &= [\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(x,t)]^{-1}[\mathbf{I} - \tfrac{1}{2}{}_s\mathbf{M}(x,t)]_s\mathbf{K}(x,t) \\ &+ [\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(x,t)]^{-1}{}_s\mathbf{O}_o(x,t). \end{aligned} \quad (7)$$

Both the vectors ${}_s\mathbf{K}(x,t)$ and ${}_s\mathbf{O}_o(x,t)$ are column vectors containing four elements, one for each marital status. The matrix $\mathbf{I}$ is the identity matrix. The matrix ${}_s\mathbf{M}(x,t)$ can be considered a multidimensional generalization of the unidimensional occurrence/exposure rates defined by equations (2) to (4). The matrix has the following shape:

$$ {}_s\mathbf{M}(x,t) = \begin{bmatrix} {}_sm_{11}(x,t) & 0 & 0 & 0 \\ -{}_sm_{12}(x,t) & {}_sm_{22}(x,t) & -{}_sm_{32}(x,t) & -{}_sm_{42}(x,t) \\ 0 & -{}_sm_{23}(x,t) & {}_sm_{33}(x,t) & 0 \\ 0 & -{}_sm_{24}(x,t) & 0 & {}_sm_{44}(x,t) \end{bmatrix} \quad (8)$$

with ${}_sm_{ii}(x,t) = {}_sm_{id}(x,t) + \sum_{j\neq i} {}_sm_{ij}(x,t) + {}_sm_{io}(x,t)$ and 1 = single, 2 = married, 3 = divorced, 4 = widowed, d = dead, and o = emigrated. The rates for external events only appear on the main diagonal. The rates for internal events appear off the diagonal as well. The matrix ${}_s\mathbf{M}(x,t)$ is in fact a summation of five submatrices, one for each type of the events death, emigration, marriage, divorce, and transition to widowhood. This leads to the following equation:

$$ {}_s\mathbf{M}(x,t) = \sum_j {}_s\mathbf{M}_j(x,t) \tag{9} $$

where j denotes the type of event.

For every matrix ${}_s\mathbf{M}_j(x,t)$, there is a corresponding matrix ${}_s\mathbf{E}_j(x,t)$ of the numbers of events of type j. Since the vector of person-years can be written as

$$ {}_s\mathbf{L}(x,t) = [\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(x,t)]^{-1}[{}_s\mathbf{K}(x,t) + \tfrac{1}{2}{}_s\mathbf{O}_o(x,t)] \tag{10} $$

we see that

$$ {}_s\mathbf{E}_j(x,t) = {}_s\mathbf{M}_j(x,t)\mathrm{diag}[{}_s\mathbf{L}(x,t)] = {}_s\mathbf{M}_j(x,t){}_s\dot{\mathbf{L}}(x,t) \tag{11} $$

where $\mathrm{diag}[{}_s\mathbf{L}(x,t)]$ or ${}_s\dot{\mathbf{L}}(x,t)$ is a diagonal matrix with the elements of vector ${}_s\mathbf{L}(x,t)$ on the main diagonal.

The total number of events is then equal to the sum of the number of events of type j:

$$ {}_s\mathbf{E}(x,t) = {}_s\mathbf{M}(x,t)\mathrm{diag}\{[\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(x,t)]^{-1}[{}_s\mathbf{K}(x,t) + \tfrac{1}{2}{}_s\mathbf{O}_o(x,t)]\}. \tag{12} $$

Now the multidimensional model has to be extended with a consistency algorithm. The aim of such an algorithm is to ensure consistency between numbers of male and female marriages, between divorces of both sexes, and between new widows (widowers) on the one hand, and death of married males (females) on the other hand. Many algorithms have been proposed in the literature. For an evaluation of a number of these algorithms, see Keilman (1985b). The particular algorithm which we will analyze here possesses a number of attractive features related to marriage market mechanisms, as we will demonstrate in Section 4. Principally, it consists of three steps:

1. estimation of the initial model parameters ${}_s\mathbf{M}(x,t)$ and projection of initial events ${}_s\mathbf{E}(x,t)$,
2. adjustment of the events by an adjustment factor ${}_s\lambda_j(t)$ for each type of event j,
3. calculation of the adjusted model parameters ${}_s\mathbf{M}^*(x,t)$.

Consistency between the total (that is, summed over all ages, and over the three non-married states) number of marrying males and the total number of marrying females, is achieved by taking the harmonic mean of the two initially projected numbers of marriages. Multiplication of the initial number of marrying persons by the adjustment factor gives the adjusted number of marrying persons. For a given event, the adjustment factor is applied to every single age group and previous marital status (see Keilman, 1985a). For events of type j corresponding to consistency algorithm j:

$$ {}_s\mathbf{E}_j^*(x,t) = {}_s\lambda_j(t){}_s\mathbf{E}_j(x,t) = {}_s\lambda_j(t){}_s\mathbf{M}_j(x,t){}_s\dot{\mathbf{L}}(x,t) \tag{13} $$

and for the sum of all types of events:

$$ {}_s\mathbf{E}^*(x,t) = \sum_j [{}_s\lambda_j(t){}_s\mathbf{M}_j(x,t)]{}_s\dot{\mathbf{L}}(x,t). \tag{14} $$

The consistency with respect to the transition to widowhood is related to the mortality of married persons of the other sex. Therefore it is necessary to treat the mortality of married persons separately from the mortality of the other marital states. Instead of the five types of events mentioned before, the six following types of events j can be distinguished: mortality of single, divorced, and widowed persons (d), emigration (o), mortality of married persons (1), marriage (2), divorce (3), and transition to widowhood (4).

Four consistency relations can be distinguished:

1. female transition to widowhood: ${}_mE_{2d}(t) = {}_fE_{24}(t)$
2. marriage: ${}_mE_{12}(t) + {}_mE_{32}(t) + {}_mE_{42}(t) = {}_fE_{12}(t) + {}_fE_{32}(t) + {}_fE_{42}(t)$
3. divorce: ${}_mE_{23}(t) = {}_fE_{23}(t)$
4. male transition to widowhood: ${}_mE_{24}(t) = {}_fE_{2d}(t)$

where ${}_sE_{ij}(t)$ is the total number of transitions from state i to state j in $(t, t+1)$. These four consistency relations all can be written as:

$$\sum_x [\mathbf{A}_j\, {}_m\mathbf{M}_j(x,t)\, {}_m\mathbf{L}(x,t)] = \sum_x [\mathbf{A}_j\, {}_f\mathbf{M}_j(x,t)\, {}_f\mathbf{L}(x,t)] \tag{15}$$

where $\mathbf{A}_j$ is:

$$\begin{aligned} \mathbf{A}_j &= [1\ 0\ 1\ 1] \quad \text{if} \quad j = 2 \quad \text{and} \\ \mathbf{A}_j &= [0\ 1\ 0\ 0] \quad \text{if} \quad j = 1,\ 3,\ \text{or } 4 \end{aligned} \tag{16}$$

Now it is possible to derive an expression for the adjustment factors ${}_s\lambda_j(t)$ in terms of the initial parameters ${}_s\mathbf{M}(x,t)$. The consistent number of events are found by taking the harmonic mean of the initial numbers of male and female events. The corresponding adjustment factor is the ratio between consistent and initial events. Hence, the adjustment factor for event j is:[1]

$${}_m\lambda_j(t) = 2\frac{\sum_x [\mathbf{A}_j\, {}_f\mathbf{M}_j(x,t)\, {}_f\mathbf{L}(x,t)]}{\sum_s \sum_x [\mathbf{A}_j\, {}_s\mathbf{M}_j(x,t)\, {}_s\mathbf{L}(x,t)]} = \frac{2\, {}_fE_j(t)}{{}_mE_j(t) + {}_fE_j(t)} \qquad \text{for} \quad j = 1, \ldots, 4. \tag{17}$$

Because of the harmonic mean, the adjustment factor for females is:

$${}_f\lambda_j(t) = 2 - {}_m\lambda_j(t). \tag{18}$$

In equations (17) and (18) the indices m and f are interchangeable.

1

$${}_m\lambda_j = \frac{{}_mE_j^*}{{}_mE_j}$$

and

$${}_mE_j^* = {}_fE_j^* = 2\left[\frac{1}{{}_mE_j} + \frac{1}{{}_fE_j}\right]^{-1} = \frac{2\, {}_mE_j\, {}_fE_j}{{}_mE_j + {}_fE_j}$$

substitution yields:

$${}_m\lambda_j = \frac{2\, {}_fE_j}{{}_mE_j + {}_fE_j}.$$

The main equation of the nuptiality model is equation (7). This equation can easily be extended with the consistency algorithm by substituting ${}_s\mathbf{M}(x,t)$ with ${}_s\mathbf{M}^*(x,t)$. As the sensitivity analysis will be carried out for changes in initial model parameters, equation (7) must be formulated in terms of the initial matrix ${}_s\mathbf{M}(x,t)$. A simple relation between ${}_s\mathbf{M}(x,t)$ and ${}_s\mathbf{M}^*(x,t)$ does not exist. Therefore, we write:

$$
\begin{aligned}
{}_s\mathbf{K}(x+1,t+1) &= {}_s\mathbf{K}(x,t) + {}_s\mathbf{O}_o(x,t) - {}_s\mathbf{E}^*(x,t)\iota \\
&= {}_s\mathbf{K}(x,t) + {}_s\mathbf{O}_o(x,t) - \sum_j [{}_s\lambda_j(t){}_s\mathbf{M}_j(x,t){}_s\mathbf{L}(x,t)] \qquad (19)
\end{aligned}
$$

where ι is a column vector consisting of ones only.

In words: the projected population equals the original population vector plus the vector of immigrants minus the vector of adjusted other types of events (which can have both positive and negative elements). In the next section, equation (19) will be used for deriving sensitivity functions.

3. THE SENSITIVITY FUNCTIONS

The objective of sensitivity analysis is to measure the effect of a change in a model parameter on a model variable. A projection model like the one described in Section 2 generally has three types of parameters: the occurrence/exposure rates for marriage, mortality, and so on; the fertility rates; and the number of immigrants. This section will deal with the occurrence/exposure rates only, that is, the elements of matrix ${}_s\mathbf{M}(x,t)$. The model variable used is the population vector ${}_s\mathbf{K}(x+1,t+1)$. Let us call this model variable $\mathbf{Y}$ and the vector of model parameters $\mathbf{X}$. A single element of $\mathbf{X}$ is denoted by x. Let $\mathbf{Y}$ be a function of $\mathbf{X}$, that is $\mathbf{Y} = \mathbf{F}(\mathbf{X})$. The sensitivity function then is the partial derivative $\partial\mathbf{Y}/\partial x$.[2]

The matrix ${}_s\mathbf{M}(x,t)$ consists of three types of elements: ${}_sm_{ij}(x,t)$, ${}_sm_{id}(x,t)$, and ${}_sm_{io}(x,t)$. They will all be denoted by ${}_sm_{ik}(x,t)$, with $k = d$ (mortality), o (emigration), 2 (marriage), 3 (divorce), and 4 (transition to widowhood). We shall first consider the derivative of the stock vector ${}_s\mathbf{K}(x+1,t+1)$ with respect to a rate ${}_sm_{ik}(x,t)$ for the same sex s and the same birth cohort $(t-x-1)$. Next we compute the derivative with respect to a rate for a general birth cohort not equal to $(t-x-1)$, and finally the derivative for rates of the opposite sex (and all birth cohorts).

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[2]Let $\mathbf{X}$ be an (n by m) matrix, and let x be the (i,j)th element of this matrix. The derivative of $\mathbf{X}$ with respect to x is written as $\partial\mathbf{X}/\partial x$, and it is defined as the (n by m) matrix $\mathbf{J}$, which has the value 1 for its (i,j)th element, and zeroes elsewhere. For the derivative of an inverted matrix $\mathbf{X}^{-1}$, we have

$$\frac{\partial[\mathbf{X}^{-1}]}{\partial x} = -\mathbf{X}^{-1}\cdot\mathbf{J}\cdot\mathbf{X}^{-1}.$$

A more general treatment of the required matrix differentiation rules is given by Willekens (1977), and in the references to that article.

Differentiation of ${}_s\mathbf{K}(x+1,t+1)$ with respect to ${}_sm_{ik}(x,t)$ for age a yields (see expression (19)):

$$\frac{\partial_s\mathbf{K}(x+1,t+1)}{\partial_s m_{ik}(a,t)} = -\sum_j \left[\frac{\partial_s\lambda_j(t)}{\partial_s m_{ik}(a,t)} {}_s\mathbf{M}_j(x,t)_s\mathbf{L}(x,t) + {}_s\lambda_j(t)\frac{\partial_s\mathbf{M}_j(x,t)}{\partial_s m_{ik}(a,t)}\mathbf{L}(x,t) + {}_s\lambda_j(t)_s\mathbf{M}_j(x,t)\frac{\partial_s\mathbf{L}(x,t)}{\partial m_{ik}(a,t)} \right] \tag{20}$$

We first compute the derivative of ${}_s\mathbf{M}_j(x,t)$ with respect to ${}_sm_{ik}(a,t)$. In order to do so, we introduce two Kronecker delta functions. Because the four consistency relations in the model are independent, every transition from state i to state k, (ik), only appears in one consistency relation j, and therefore we define:

$$\delta_{j,ik} = 1 \quad \text{if} \quad (ik) \text{ in } j, \quad \text{and} \quad \delta_{j,ik} = 0 \quad \text{otherwise.}$$

Furthermore, we write

$$\delta_{a,x} = 1 \quad \text{if} \quad a = x, \quad \text{and} \quad \delta_{a,x} = 0 \quad \text{if} \quad a \neq x.$$

If the element ${}_sm_{ik}(a,t)$ appears in the matrix ${}_s\mathbf{M}_j(x,t)$, the derivative is different from zero. In this case, the derivative is equal to a matrix containing zeros, except for the element in the ith column and the jth row (-1), and the element in the ith column and the ith row $(+1)$.

We now can write:

$$\frac{\partial_s\mathbf{M}_j(x,t)}{\partial_s m_{ik}(a,t)} = \delta_{a,x}\delta_{j,ik}\,{}_s\mathbf{J}_{ik} \tag{21}$$

where

$${}_s\mathbf{J}_{ik} = \frac{\partial_s\mathbf{M}(x,t)}{\partial_s m_{ik}(x,t)}. \tag{22}$$

For differentiation with respect to the element ${}_sm_{12}(x,t)$, for example, the derivative is:

$$\frac{\partial_s\mathbf{M}_2(x,t)}{\partial_s m_{12}(x,t)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

This result is somewhat different from the result found by Willekens (1977). Willekens presents a matrix $\mathbf{J}_{ik}$ containing only one element not equal to zero. This is correct for external events such as mortality, but not for internal events.

Next we calculate the derivative of ${}_s\lambda_j(t)$ with respect to ${}_sm_{ik}(a,t)$. According to equation (17), ${}_s\lambda_j(t)$ contains the number of person-years of every age group. Thus a small change in an element of ${}_s\mathbf{M}(x,t)$, for any age x, changes the value ${}_s\lambda_j(t)$. Differentiation of the right-hand side of equation (17) yields the following sensitivity function:

$$\frac{\partial_s\lambda_j(t)}{\partial_s m_{ik}(a,t)} = -2\frac{{}_sE_j(t)}{({}_mE_j(t)+{}_fE_j(t))^2}\frac{\partial_sE_j(t)}{\partial_s m_{ik}(a,t)} = -2\frac{{}_s\lambda_j(t)}{{}_mE_j(t)+{}_fE_j(t)}\frac{\partial_sE_j(t)}{\partial_s m_{ik}(a,t)} \tag{24}$$

Differentiation of ${}_sE_j(t)$ yields:

$$\frac{\partial_s E_j(t)}{\partial_s m_{ik}(a,t)} = \mathbf{A}_j \sum_x \left[\frac{\partial_s \mathbf{M}_j(x,t)}{\partial_s m_{ik}(a,t)} {}_s\mathbf{L}(x,t) + {}_s\mathbf{M}_j(x,t)\frac{\partial_s \mathbf{L}(x,t)}{\partial_s m_{ik}(a,t)}\right]$$

$$= \mathbf{A}_j \sum_x \left[\delta_{a,x}\delta_{j,ik}\,{}_s\mathbf{J}_{ik}\,{}_s\mathbf{L}(x,t) + {}_s\mathbf{M}_j(x,t)\frac{\partial_s \mathbf{L}(x,t)}{\partial_s m_{ik}(a,t)}\right]. \quad (25)$$

The derivative of ${}_s\mathbf{L}(x,t)$ in equation (25) is (see expression (10)):

$$\frac{\partial_s \mathbf{L}(x,t)}{\partial_s m_{ik}(a,t)} = -\delta_{a,x}\tfrac{1}{2}[\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(x,t)]^{-1}\sum_i \left(\frac{\partial_s \mathbf{M}_i(x,t)}{\partial_s m_{ik}(a,t)}\right)$$

$$\cdot [\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(x,t)]^{-1}[{}_s\mathbf{K}(x,t) + \tfrac{1}{2}{}_s\mathbf{O}_o(x,t)]$$

$$= -\tfrac{1}{2}\delta_{a,x}[\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(x,t)]^{-1}{}_s\mathbf{J}_{ik}\,{}_s\mathbf{L}(x,t). \quad (26)$$

Substitution of equation (26) in equation (25) yields:

$$\frac{\partial_s E_j(t)}{\partial_s m_{ik}(a,t)} = \mathbf{A}_j\delta_{j,ik}\,{}_s\mathbf{J}_{ik}\,{}_s\mathbf{L}(a,t) - \mathbf{A}_j\,{}_s\mathbf{M}_j(a,t)\tfrac{1}{2}[\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(a,t)]^{-1}{}_s\mathbf{J}_{ik}\,{}_s\mathbf{L}(a,t)$$

$$= \mathbf{A}_j(\delta_{j,ik}\mathbf{I} - \tfrac{1}{2}{}_s\mathbf{M}_j(a,t)[\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(a,t)]^{-1})_s\mathbf{J}_{ik}\,{}_s\mathbf{L}(a,t). \quad (27)$$

The final result for ${}_s\lambda_j(t)$ is:

$$\frac{\partial_s \lambda_j(t)}{\partial_s m_{ik}(a,t)} = -\frac{{}_s\lambda_j(t)}{{}_mE_j(t) + {}_fE_j(t)}\mathbf{A}_j(\delta_{j,ik}\mathbf{I} - \tfrac{1}{2}{}_s\mathbf{M}_j(a,t)[\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(a,t)]^{-1})_s\mathbf{J}_{ik}\,{}_s\mathbf{L}(a,t). \quad (28)$$

As ${}_s\lambda_j(t)$ is a scalar, the derivative of ${}_s\lambda_j(t)$ with respect to ${}_sm_{ik}(x,t)$ is also a scalar. To simplify matters, the derivative will be written as:

$$\frac{\partial_s \lambda_j(t)}{\partial_s m_{ik}(a,t)} = {}_sh_j(t). \quad (29)$$

In this equation the value of ${}_sh_j(t)$ depends on k as well: see expression (28). From equation (18) it will be clear that differentiation of ${}_s\lambda_j(t)$ with respect to an occurrence/exposure rate of the opposite sex leads simply to:

$$\frac{\partial_f \lambda_j(t)}{\partial_m m_{ik}(a,t)} = {}_fh_j(t) = -{}_mh_j(t). \quad (30)$$

In this equation the indices m and f are interchangeable.

Substitution of equations (21), (26), and (29) in equation (20) yields, finally:

$$\frac{\partial_s \mathbf{K}(x+1,t+1)}{\partial_s m_{ik}(a,t)} = -\sum_j [{}_sh_j(t)_s\mathbf{E}_j(x,t)\iota] - \delta_{a,x}\,{}_s\lambda_k(t)_s\mathbf{J}_{ik}\,{}_s\mathbf{L}(x,t)$$

$$+ \frac{1}{2}\delta_{a,x}\sum_j [{}_s\lambda_j(t)_s\mathbf{M}_j(x,t)][\mathbf{I} + \tfrac{1}{2}{}_s\mathbf{M}(x,t)]^{-1}{}_s\mathbf{J}_{ik}\,{}_s\mathbf{L}(x,t) \quad (31)$$

This sensitivity function applies to the general case of the effect of a small change of an occurrence/exposure rate for age a on the population of the same birth cohort (which has age $a+1$ after one unit of time) or any other birth cohort. For the other birth cohorts, expression (31) still holds, but the Kronecker $\delta_{a,x}$ equals zero. Thus we see that:

$$\frac{\partial_s \mathbf{K}(x+1,t+1)}{\partial_s m_{ik}(a,t)} = -\sum_j [{}_sh_j(t)_s\mathbf{E}_j(x,t)\imath] \qquad \text{for} \quad x \neq a. \tag{32}$$

A slight change in the rate for age a also has an effect on the population in other age groups, by the adjustment factor ${}_s\lambda_j(t)$ and its derivative ${}_sh_j(t)$. Not only does such a change affect the other ages, but there is also an effect on all ages of the opposite sex. Considering a slight change in the occurrence/exposure rate of men, the sensitivity function is:

$$\frac{\partial_f \mathbf{K}(x+1,t+1)}{\partial_m m_{ik}(a,t)} = -\sum_j [{}_mh_j(t)_f\mathbf{E}_j(x,t)\imath] = \sum_j [{}_fh_j(t)_f\mathbf{E}_j(x,t)\imath]. \tag{33}$$

Thus a one-unit increase in the marriage rate of males aged a has a positive effect on the number of married females of all ages (since ${}_fh_j(t) > 0$ for $j = k$). This effect is just as large as a one-unit *decrease* in the marriage rate of females aged a (except for married females aged a, for whom the effect of such a decrease is negative). The mechanism described here is a consequence of the fact that the consistency algorithm takes into account the competition between marriage candidates of different ages in the marriage market.

4. AN APPLICATION

This section presents an illustration of the previously presented sensitivity functions. The data apply to The Netherlands, cover the period January 1, 1980 until January 1, 1985, and stem from The Netherlands Central Bureau of Statistics (NCBS). The initial population is the population on January 1, 1980, by sex, five-year age group, and marital status. The mortality, nuptiality, and migration data refer to the five-year period mentioned. The data make it possible to 'project' the population on January 1, 1985, using occurrence/exposure rates (for all age-sex combinations) computed from the data as model parameters. The most interesting part of the analysis, however, is the impact of changes of the model parameters on the model variables. The concept of elasticity (E) will be used for measuring the impact of the changing rates. Elasticity is defined as follows:

$$E[\mathbf{Y},x] = \frac{\left(\frac{\partial \mathbf{Y}}{\mathbf{Y}}\right)}{\left(\frac{\partial x}{x}\right)} = \frac{\partial \mathbf{Y}}{\partial x}\cdot\frac{x}{\mathbf{Y}} \tag{34}$$

where $\mathbf{Y}$ is the model variable and x is the model parameter. The elasticity is equal to the relative change in variable $\mathbf{Y}$ divided by the relative change in parameter x. For example, if $E[\mathbf{Y},x] = 0.01$, this means that if the value of x increases by 1%, the value of Y will increase by 0.01%. The term $\partial Y/Y$ in equation (34) denotes element-by-element division.

The sensitivity analysis in this section will focus on the effect of a slight change in the marriage rate of single persons. Of course, nuptiality behaviour is not important at every age. Therefore, the results of the sensitivity analysis presented will be limited to the age groups 15–39. Empirical results for the other age groups, as well as for fertility, mortality, and migration, are given by Ekamper (1990).

The following data are required to make an initial projection: the population in 1980 (Table 1), the number of immigrants during the period 1980–1984 (Table 2), and the occurrence/exposure rates for the period 1980–1984 (Table 3). The results of the projection is the population in 1985 (Table 4). The total number of initial events and the adjustment factors calculated for the period 1980–1984 are some important by-products of the projection (see Tables 5 and 6). These results apply to the initial situation with observed parameters. But what would happen if, for example, the initial marriage rate of single men aged 20–24 was subjected to a slight increase? Obviously the number of single men aged 25–29 in 1985 would decrease, and the number of married men aged 25–29 would increase. The impact of a slight increase in the marriage rate on the projected population in 1985 by sex, some specific age groups, and marital status is presented in Table 7. Table 7 clarifies that a one percent increase in the initial marriage rate of single men aged 20–24 in 1980 (the initial rate is 0.51) decreases the number of single men of the same birth cohort after five years by 0.44%, and increases the number of married men of the same birth cohort by 0.42%. By the adjustment factors for consistency, the effect of the increase in the marriage rate is distributed across the other age groups, marital statuses, and sex. The strongest relative effect is that for the divorced and widowed males and single females of the same birth cohort. In general, this indirect effect of a changing parameter is most important for the age cohort directly involved and the closest other age groups (see also Figure 1). For the opposite sex, the most important impact, as mentioned, is on the number of single females aged 25–29, being −0.28% (see Figure 2). According to Table 8, a one percent increase in the marriage rate of men aged 25–29 leads to a 0.54% decrease in the number of single men aged 30–34 in 1985, and to a 0.17% increase in the number of married men. The impact on the female population is roughly two to three times smaller than that in Table 7, with a maximum effect of −0.13% on the number of single females aged 25–29. Although the model presented in Section 2 does not fully meet the demands of competition and substitution (see Keilman, 1985a and 1985b), the results presented here indicate that these demands are reasonably well fulfilled, at least qualitatively. The competition effect means that the marriage chances of a male aged 20–24, for example, will very likely decrease if there is an increase in the supply of eligible males aged 25–29 or 30–34. The substitution effect means that an extra supply of single males aged 25–29, for example, will lead to a stronger decrease in the number of marrying males aged 20–24, than a similar extra supply of single males aged 30–34. Due to the one percent increase in the marriage rate of men aged 20–24 in 1980, the number of married men that are one age group younger (aged 20–24 in 1985) decreases by 0.22%. Similar negative effects can be noted for married men in other age groups. This is in accordance with the competition demand. The same relative

TABLE 1
Population on January 1, 1980, by Marital Status, Sex, and Age

age group	males				females			
	single	married	divorced	widowed	single	married	divorced	widowed
15-19	636344	1632	19	6	595113	16365	169	174
20-24	494784	109548	2075	52	326673	251926	5819	361
25-29	206480	367974	125[illegible]	313	108706	431057	18474	1338
30-34	101038	490385	21697	787	51605	493440	25726	3046
all ages	3293231	3420055	133851	147143	2897709	3414959	179854	604212

Source: NCBS.

TABLE 2
Number of Immigrants in the Period 1980–1984 by Marital Status, Sex, and Age on January 1, 1980

age group	males				females			
	single	married	divorced	widowed	single	married	divorced	widowed
15-19	22095	2041	35	1	17839	10441	185	20
20-24	21538	9377	372	21	12690	15561	828	79
25-29	11061	13796	1202	50	5242	15983	1284	108
30-34	4083	13717	1525	50	1963	13682	1199	136
all ages	129876	65357	6718	900	102807	81488	6055	3895

Source: NCBS.

TABLE 3
Initial Occurrence/Exposure Rates for the Period 1980–1984 by Marital Status Before the Event, Sex, Marital Status After the Event, and Age on January 1, 1980

age group	marital status	males				females			
		single	married	divorced	widowed	single	married	divorced	widowed
15-19	single	0.1367	0.0000	0.0000	0.0000	0.3357	0.0000	0.0000	0.0000
	married	-0.1122	0.1087	-0.5200	-0.6630	-0.3121	0.1263	-0.6058	-0.2938
	divorced	0.0000	-0.0694	0.6231	0.0000	0.0000	-0.0780	0.6471	0.0000
	widowed	0.0000	-0.0015	0.0000	0.8287	0.0000	-0.0029	0.0000	0.4929
20-24	single	0.5575	0.0000	0.0000	0.0000	0.6867	0.0000	0.0000	0.0000
	married	-0.5115	0.1097	-0.5264	-0.6948	-0.6413	0.1176	-0.5645	-0.3706
	divorced	0.0000	-0.0759	0.5886	0.0000	0.0000	-0.0806	0.6182	0.0000
	widowed	0.0000	-0.0011	0.0000	0.7613	0.0000	-0.0028	0.0000	0.4393
25-29	single	0.4900	0.0000	0.0000	0.0000	0.4181	0.0000	0.0000	0.0000
	married	-0.4279	0.1120	-0.4956	-0.6645	-0.3630	0.1121	-0.4150	-0.2169
	divorced	0.0000	-0.0757	0.5578	0.0000	0.0000	-0.0740	0.4628	0.0000
	widowed	0.0000	-0.0016	0.0000	0.7340	0.0000	-0.0037	0.0000	0.2629
30-34	single	0.2475	0.0000	0.0000	0.0000	0.2091	0.0000	0.0000	0.0000
	married	-0.1935	0.1030	-0.4212	-0.5731	-0.1662	0.0970	-0.2949	-0.1040
	divorced	0.0000	-0.0668	0.4837	0.0000	0.0000	-0.0630	0.3304	0.0000
	widowed	0.0000	-0.0024	0.0000	0.6323	0.0000	-0.0056	0.0000	0.1373

TABLE 4
Projected Population on January 1, 1985, by Marital Status, Sex, and Age

age group	males				females			
	single	married	divorced	widowed	single	married	divorced	widowed
20-24	575949	67999	1914	41	437022	178251	5790	254
25-29	297009	301480	13683	211	168281	398049	23375	1050
30-34	134494	420885	31707	642	75281	444309	38476	2538
35-39	82514	484861	41100	1333	43572	479267	45395	5336
all ages	3374291	3416750	217384	149665	2942834	3430517	271325	662454

TABLE 5
Initial Number of Events (For All Age Groups) in the Period 1980–1984 by Marital Status, Sex, and Type of Event

type of event	males				females			
	single	married	divorced	widowed	single	married	divorced	widowed
mortality	38092	203347	11811	68684	36451	79389	9470	140510
marriage	369078		51865	8133	372921		46493	5099
divorce		145583				148384		
widowhood		79161				203276		
emigration	91912	65887	6757	880	73262	63155	5343	3328

TABLE 6
Adjustment Factors by Sex and Type of Event

type of event	males	females
mortality while married	0.9998	0.9986
marriage	0.9947	1.0053
divorce	1.0095	0.9905
transition to widowhood	1.0014	1.0002

TABLE 7
Elasticities of the Population in 1985 Due to a Slight Increase in the Marriage Rate of Single Men Aged 20–24 Years in 1980, by Marital Status, Sex, and Age

age group	males				females			
	single	married	divorced	widowed	single	married	divorced	widowed
20-24	0.0258	-0.2194	0.0261	0.0714	-0.0814	0.2008	-0.0369	-0.0331
25-29	-0.4404	0.4180	0.4251	0.4368	-0.2772	0.1202	-0.0485	-0.0437
30-34	0.1704	-0.0587	0.0545	0.0847	-0.1502	0.0306	-0.0574	-0.0307
35-39	0.0628	-0.0167	0.0688	0.0840	-0.0581	0.0103	-0.0508	-0.0152

TABLE 8
Elasticities of the Population in 1985 Due to a Slight Increase in the Marriage Rate of Single Men Aged 25–29 in 1980, by Marital Status, Sex, and Age

age group	males				females			
	single	married	divorced	widowed	single	married	divorced	widowed
20-24	0.0118	-0.1006	0.0120	0.0326	-0.0373	0.0921	-0.0170	-0.0151
25-29	0.0799	-0.0794	0.0163	0.0373	-0.1271	0.0551	-0.0223	-0.0200
30-34	-0.5394	0.1671	0.1027	0.1132	-0.0689	0.0140	-0.0263	-0.0141
35-39	0.0288	-0.0077	0.0316	0.0384	-0.0266	0.0047	-0.0233	-0.0070

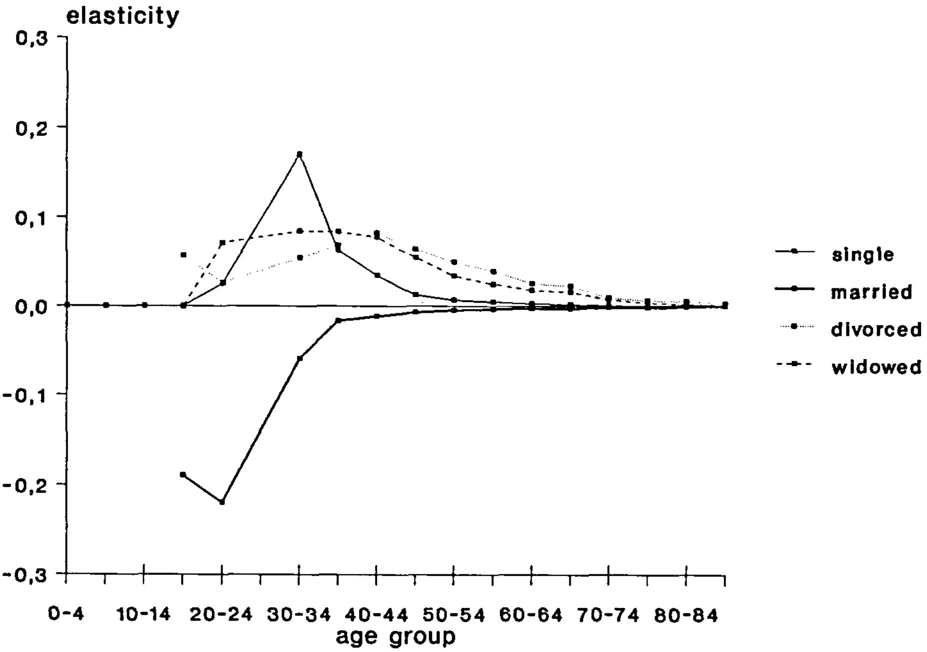

FIGURE 1. Effect of a slight increase in the marriage rate of single men, aged 20–24 on January 1, 1980, on the number of men by age group (except age group 25–29) and marital status on January 1, 1985.

increase in the marriage rate of men aged 25–29 in 1980 decreases the number of married men aged 20–24 by only 0.10%. In accordance with the substitution effect, one would expect these changed rates to have opposite effects on the number of married men aged 35–39. However, a one percent higher marriage rate of men aged 20–24 decreases the number of married men aged 30–34 by 0.02%, whereas the same increase in the marriage rate of men aged 25–29 leads to a drop in the number of married men aged 35–39 by only 0.01%, instead of decreasing it by more than 0.02%. Keilman (1985b) demonstrated that a change in the marriage rate for age x has proper substitution effects for ages y and z, provided that ages x, y, and z are all located either to the left or to the right of the modal age m of the schedule

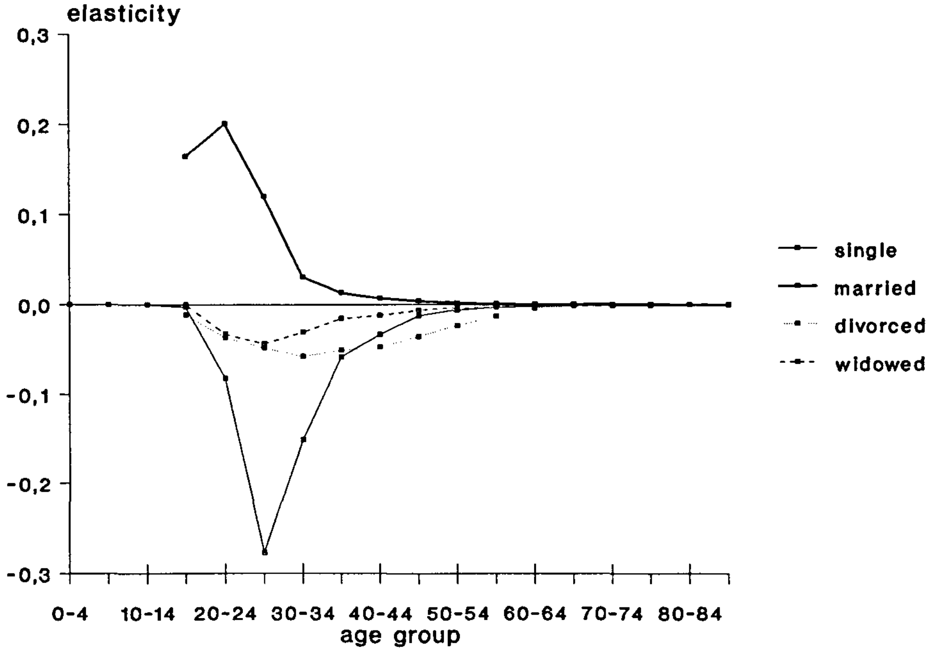

FIGURE 2. Effect of a slight increase in the marriage rate of single men, aged 20–24 on January 1, 1980, on the number of women by age group and marital status on January 1, 1985.

TABLE 9
Elasticities of the Population in 1985 Due to a Slight Increase in the Marriage Rate of Single Women Aged 15–19 in 1980, by Marital Status, Sex, and Age

age group	males				females			
	single	married	divorced	widowed	single	married	divorced	widowed
20-24	-0.0195	0.1654	-0.0199	-0.0539	-0.2566	0.5974	0.6868	0.6390
25-29	-0.1313	0.1306	-0.0269	-0.0616	0.2089	-0.0906	0.0367	0.0328
30-34	-0.1284	0.0442	-0.0412	-0.0638	0.1132	-0.0231	0.0434	0.0231
35-39	-0.0473	0.0126	-0.0519	-0.0633	0.0438	-0.0077	0.0384	0.0114

TABLE 10
Elasticities of the Population in 1985 Due to a Slight Increase in the Marriage Rate of Single Women Aged 20–24 in 1980, by Marital Status, Sex, and Age

age group	males				females			
	single	married	divorced	widowed	single	married	divorced	widowed
20-24	-0.0224	0.1905	-0.0220	-0.0620	0.0707	-0.1744	0.0314	0.0286
25-29	-0.1512	0.1504	-0.0302	-0.0708	-0.7006	0.2836	0.2408	0.2079
30-34	-0.1480	0.0509	-0.0469	-0.0735	0.1304	-0.0265	0.0494	0.0266
35-39	-0.0545	0.0145	-0.0593	-0.0729	0.0504	-0.0089	0.0438	0.0132

of age-specific marriage rates. In the former case, a further condition is that $z < y < x < m$, and in the latter case $m < x < y < z$. In other situations, substitution is not guaranteed. But the elasticities are so slight that these violations of the substitution demand are of minor importance.

Tables 9 and 10 present results for changes in marriage rates of single women aged 15–19 and 20–24. A one percent increase in the marriage rate of women aged 15–19 decreases the number of single women of the same birth cohort in 1985 by 0.26%. This increases the number of married women aged 20–24 in 1985 by 0.60%. A one percent increase in the marriage rate of single women aged 20–24 in 1980, however, will decrease the number of single women aged 25–29 in 1985 by 0.70%, and increase the number of married women by 0.28%. The increase of the marriage rate of women aged 20–24 has a stronger effect on the decrease in the number of married women aged 30–34, than an increase of the marriage rate of women aged 25–29. The same applies to married women aged 35–39. This again supports the characteristics of the competition and substitution effect.

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