RENORMALIZED SOLUTIONS OF AN ANISOTROPIC REACTION-DIFFUSION-ADVECTION SYSTEM WITH $L^1$ DATA

MOSTAFA BENDAHMANE AND KENNETH H. KARLSEN

Abstract. We prove existence of a renormalized solution to a system of nonlinear partial differential equations with anisotropic diffusivities and transport effects, supplemented with initial and Dirichlet boundary conditions. The data are assumed to be merely integrable. This system models the spread of an epidemic disease through a heterogeneous habitat.

Contents
1. Introduction and statement of main result 1
2. Preliminaries 4
   2.1. Functional spaces 4
   2.2. Truncation/renormalization functions 5
   2.3. Renormalized solutions 6
   2.4. Chain rule 8
3. Properties of renormalized solutions 8
4. Proof of Theorem 1.1 12
   4.1. Approximate problems 12
   4.2. Basic convergence results 13
   4.3. Strong convergence of truncations 17
   4.4. Concluding the proof of Theorem 1.1 23
5. Uniqueness for a related scalar equation 24
References 30

1. Introdution and statement of main result

We consider the propagation of an epidemic disease in a simple population $p = u_1 + u_2$, where $u_1 = u_1(t)$ and $u_2 = u_2(t)$ are the respective densities of susceptible (those who can catch the disease) and infected individuals (those who have the disease and can transmit it) at time $t$. When no spatial considerations are involved, the propagation of Feline Immunodeficiency Virus (F.I.V.) within a population of
cats is governed by the following system of ordinary differential equations:

\[
\begin{align*}
    u_1'(t) &= -\sigma(u_1, u_2) + bp(t) - (m + kp(t))u_1(t), & u_1(0) > 0, \\
    u_2'(t) &= \sigma(u_1, u_2) - \alpha u_2(t) - (m + kp(t))u_2(t), & u_2(0) > 0,
\end{align*}
\]

where \( b \) is the (linear) natural birth rate, \( m \) is natural death rate, and \( k > 0 \) is a positive constant, yielding a density dependent death rate \( \delta(p) = m + kp \), and \( \alpha \) is the disease induced death rate in the infected class. For \( b - m > 0 \), \( K_p = \frac{b - m}{k} \) is the carrying capacity. If \( \alpha = 0 \), system (1.1) reduces to the logistic equation

\[
P'(t) = (b - m - kp(t))p(t), \quad p(0) = p_0 > 0,
\]

and \( p(t) \to K_p \) as \( t \to \infty \). More details concerning the propagation of F.I.V. may be found in [11] and the references cited therein (see also [20]).

The loss of individuals from the susceptible class into the latently infected class is modeled by the incidence function, which we denote by \( \sigma(u_1, v_1) \). There are two common choices of the incidence function: proportionate mixing and mass action. In the case of mass action,

\[
\sigma(u_1, u_2) = \sigma_1 u_1 u_2,
\]

with \( \sigma_1 > 0 \), while a proportionate mixing term has the form

\[
\sigma(u_1, u_2) = \sigma_2 \frac{u_1 u_2}{p},
\]

with \( \sigma_2 > 0 \).

We consider a bounded open spatial domain \( \Omega \subset \mathbb{R}^N \) \( (N \geq 2) \), with a Lipschitz boundary denoted by \( \partial \Omega \). Fixing a final time \( T > 0 \), we set \( Q_T = (0, T) \times \Omega \).

In this paper we are concerned with spatial densities, and the total populations of our subclasses are then given by

\[
U_1(t) = \int_{\Omega} u_1(t, x) \, dx, \quad U_2(t) = \int_{\Omega} u_2(t, x) \, dx.
\]

The total population is given by

\[
P(t) = \int_{\Omega} (u_1 + u_2)(t, x) \, dx = \int_{\Omega} p(t, x) \, dx.
\]

A prototype of a nonlinear system that governs the spreading of F.I.V. through a cat population in a heterogeneous spatial domain is the following reaction-diffusion-advection system [3]:

\[
\partial_t u_i - \sum_{l=1}^{N} \partial_x \left( \beta_{i,l}(t, x) |\partial_x u_i|^{p_i-2} \partial_x u_i \right) - \text{div}_x (u_i K_i(t, x)) = F_i(t, x, u_1, u_2) \quad \text{in } Q_T,
\]

(1.4)

together with a Dirichlet boundary condition

\[
u_i = 0 \quad \text{on } (0, T) \times \partial \Omega,
\]

(1.5)
corresponding to a nonfavorable domain on \( (0, T) \times \partial \Omega \), and an initial distribution

\[
u_i(0, x) = u_{i,0}(x), \quad x \in \Omega,
\]

(1.6)

for \( i = 1, 2 \).

Quite often in this paper we will not state explicitly that the indices \( i \) and \( l \) run over \( 1, 2 \) and \( 1, \ldots, N \), respectively, whenever this should be clear from the context.
In (1.4), the nonlinearities $F_1$ and $F_2$ take the form
\begin{equation}
F_1(t, x, u_1, u_2) = -r_1(t, x, u_1, u_2) - \sigma(t, x, u_1, u_2)
+ b(t, x)(u_1 + u_2) - m(t, x)u_1 + f_1(t, x),
\end{equation}
\begin{equation}
F_2(t, x, u_1, u_2) = -r_2(t, x, u_1, u_2) + \sigma(t, x, u_1, u_2)
- (m(t, x) + \alpha(t, x))u_2 + f_2(t, x).
\end{equation}

The functions $\beta_{i,l} : Q_T \to \mathbb{R}$ satisfy
\begin{equation}
\beta_{i,l} \in L^\infty(Q_T), \quad \beta_{i,l} \geq \beta_0 > 0 \text{ for a.e. } (t, x) \in Q_T.
\end{equation}

The transport vectors $K_i : Q_T \to \mathbb{R}^N$ satisfy
\begin{equation}
K_i \in L^\infty(Q_T; \mathbb{R}^N), \quad \text{div}_x K_i = 0 \text{ in } D'(Q_T).
\end{equation}

In passing, we mention that the zero divergence assumption is made just for simplicity, and in the case of a bounded divergence the proofs would still go through.

The functions $m, b, \alpha$ are defined on $Q_T$ with values in $\mathbb{R}_+ := [0, \infty)$ and satisfy
\begin{equation}
m, b, \alpha \in L^\infty(Q_T).
\end{equation}

The density dependent mortality rates have the form
\begin{equation}
\begin{cases}
r_1(t, x, u_1, u_2) = k_1(t, x) u_1 (u_1 + u_2)^{p_1 - 1}, \\
r_2(t, x, u_1, u_2) = k_2(t, x) u_2 (u_1 + u_2)^{p_2 - 1},
\end{cases}
\end{equation}
for $(t, x) \in Q_T$ and $u_1, u_2 \in [0, \infty)$, where the exponents $p_i$ satisfy
\begin{equation}
p_i \geq \max \left( \frac{p_{i,l}}{p_{i,l} - 1}, p_{i,l} \right) > 1, \quad l = 1, \ldots, N,
\end{equation}
and the functions $k_i : Q_T \to \mathbb{R}_+$ satisfy
\begin{equation}
k_i \in L^\infty(Q_T), \quad k_i(t, x) \geq k_0 > 0 \text{ for a.e. } (t, x) \in Q_T.
\end{equation}

The incidence function $\sigma(t, x, u_1, u_2)$ is assumed to be nonnegative, measurable in $(t, x) \in Q_T$ for all $u_1, u_2 \in [0, \infty)$, continuous in $u_1, u_2 \in [0, \infty)$ for a.e. $(t, x) \in Q_T$, and it should satisfy the following growth condition:
\begin{equation}
\begin{cases}
\text{there exist two bounded functions } L, M : Q_T \to [0, \infty) \\
\text{and two numbers } s_1 \geq 1 \text{ and } 1 \leq s_2 < p_2, \text{ such that} \\
0 \leq \sigma(t, x, u_1, u_2) \leq L(t, x)u_1^{s_1}u_2^{s_2} + M(t, x) \\
\text{for a.e. } (t, x) \in Q_T \text{ and for all } u_1, u_2 \in [0, \infty).
\end{cases}
\end{equation}

In addition, we assume the “nonnegativity condition”
\begin{equation}
\begin{cases}
\sigma(t, x, 0, u_2) = 0, \quad \text{if } u_2 \geq 0, \\
\sigma(t, x, u_1, 0) \geq 0, \quad \text{if } u_1 \geq 0.
\end{cases}
\end{equation}

We remark that the examples of proportionate mixing (1.2) and mass action (1.3) are covered by our assumptions. Indeed, take $s_1 = s_2 = 1$, $L = M \equiv 1$ and assume $p_1, p_2 > 1$ (for example the choice $p_1 = p_2 = 2$ is consistent with (1.1)).

Recently, Bendahmane, Langlais, and Saad [3] have shown for a variant of the above system that there exists a weak solution. Although the notion of a weak solution makes sense and one can indeed prove the existence of such a solution, it well known that one cannot in general expect weak solutions to be unique when the data are merely in $L^1$, see the counterexamples in [23, 26]. This motivates the study of the system (1.4), (1.5), (1.6) in the framework of renormalized solutions,
which is precisely what we set out to do in this paper. Our main result is the following theorem:

**Theorem 1.1.** Suppose the conditions in (1.8)-(1.15) hold, $u_0 = (u_{1,0}, u_{2,0}) \in L^1_\infty(\Omega; \mathbb{R}^2)$, and $f = (f_1, f_2) \in L^1_+ (Q_T; \mathbb{R}^2)$. Then there exists at least one nonnegative renormalized solution $u = (u_1, u_2)$ of the problem (1.4), (1.5), (1.6).

Under additional restrictions on the exponents $p_{i,l}$, it is (as usual) possible to recover some Sobolev regularity (below the natural Sobolev exponents) of the renormalized solution $u = (u_1, u_2)$ constructed in Theorem 1.1, see Lemma 3.2 in Section 3 for details.

We recall that the notion of renormalized solutions was introduced by DiPerna and Lions [13] in their study of the Boltzmann equation. This notion was then adapted to the study of some nonlinear isotropic elliptic problems with Dirichlet boundary conditions by Boccardo, Giachetti, Diaz, and Murat [9] and Lions and Murat (see Lions book on the Navier-Stokes equations [19]). For the corresponding isotropic parabolic equations with $L^1$ data, existence and uniqueness of renormalized solutions is established in Blanchard and Murat [5], see also Rakotoson [24, 25] and Lions [19] for some time dependent problems motivated by the Navier-Stokes equations. For more recent results, see the papers [6, 7, 22]. We also refer to the papers cited so far for a more complete account on the history of renormalized solutions and a long list of relevant references. Finally, let us mention that an equivalent notion of solutions, called entropy solutions, was introduced independently by Bénilan et al. [4], see also [1].

To prove uniqueness of a renormalized solution to a system like the one above is in general difficult due to the non-Lipschitz/non-monotone character of the nonlinear terms. Of course, this is also true for isotropic/anisotropic scalar equations with non-Lipschitz/non-monotone nonlinear terms. Nevertheless, we will prove a uniqueness result for a related scalar equation with a nondecreasing nonlinearity. This result extends some of the existing uniqueness results for isotropic parabolic equations with $L^1$ data to a class of anisotropic parabolic equations. Our uniqueness proof as well the proof of Theorem 1.1 is strongly inspired by the work [5].

The remaining part of this paper is organized as follows: In Section 2 we introduce some notations/functional spaces and the notion of renormalized solutions. Some properties of renormalized solutions are derived in Section 3. Theorem 1.1 is proved in Section 4. Finally, in Section 5 we prove uniqueness of renormalized solutions for a related scalar equation with a monotone nonlinearity.

## 2. Preliminaries

### 2.1. Functional spaces.

We use $\mathcal{D}(Q_T)$ to denote the infinitely differentiable functions on $Q_T$ with compact support. The distributions on $Q_T$ are denoted by $\mathcal{D}'(Q_T)$. We let $L^p_+(Q_T)$ denote the nonnegative functions in $L^p(Q_T)$. If $X$ is a Banach space and $1 \leq p \leq \infty$, then $L^p(0,T; X)$ denotes the space of measurable functions $u: (0,T) \to X$ for which $\|u(\cdot)\|_X$ belongs to $L^p(0,T)$.

Anisotropic Sobolev spaces were introduced and studied by Nikolskii [21] and Troisi [28], and later by Trudinger [29] in the framework of Orlicz spaces. We need the anisotropic Sobolev space

$$W^{1,p,l}(\Omega) = \{ u \in W^{1,1}(\Omega) \mid \partial_{x_i} u \in L^p(\Omega) \}.$$
This is a Banach space under the norm
\[ \|u\|_{W^{1,p}(\Omega)} = \|u\|_{W^{1,1}(\Omega)} + \|\partial_x u\|_{L^p(\Omega)} \]

We let
\[ W^{1,p}_{0}(\Omega) = \{ u \in W^{1,p}(\Omega) \mid u = 0 \text{ on } (0,T) \times \partial \Omega \}. \]

The following theorem contains the anisotropic Sobolev inequality.

**Theorem 2.1** (Troisi [28]). Suppose \( u \in \cap_{i=1}^N W^{1,p_i}_{0}(\Omega) \), and set
\[ \frac{1}{p} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}, \quad r = \begin{cases} \frac{N}{p} N, & \text{if } \frac{p}{p'} < N, \\ \text{any number from } [1, \infty), & \text{if } \frac{p}{p'} \geq N. \end{cases} \]

Then there exists a constant \( C \), depending on \( N, p_1, \ldots, p_N \), \( p < N \) and also on \( r \) and \( |\Omega| \) if \( \frac{p}{p'} \geq N \), such that
\[ \|u\|_{L^r(\Omega)} \leq C \prod_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^\frac{p}{N} . \]

Applying the inequality between geometric and arithmetic means to (2.1) gives
\[ \|u\|_{L^r(\Omega)} \leq \frac{C}{N} \sum_{i=1}^N \|\partial_x u\|_{L^{p_i}(\Omega)}, \]
so that in particular there is a continuous embedding of \( \cap_{i=1}^N W^{1,p_i}_{0}(\Omega) \) into \( L^q(\Omega) \) for all \( q \in [1, p^*] \), whenever \( \frac{p}{p'} < N \).

### 2.2. Truncation/renormalization functions.

For any given \( \gamma > 0 \), we define the truncation function \( T_\gamma : \mathbb{R} \to \mathbb{R} \) by
\[ T_\gamma(z) = \begin{cases} -\gamma, & \text{if } z \leq -\gamma, \\ z, & \text{if } |z| < \gamma, \\ \gamma, & \text{if } z \geq \gamma. \end{cases} \]

Moreover, we will need the following associated functions (renormalizations)
\[ \Phi_\gamma(r) = \int_0^r T_\gamma(z) \, dz, \quad \phi_{\gamma,c}(r) = T_{\gamma+c}(r) - T_\gamma(r), \]
\[ \Psi_{\gamma,c}(r) = \int_0^r \phi_{\gamma,c}(z) \, dz = \Phi_{\gamma+c}(r) - \Phi_\gamma(r), \]
for any \( \gamma, c > 0 \). Notice that \( T_\gamma \) and \( \phi_{\gamma,c} \) are Lipschitz functions satisfying \( 0 \leq \|\phi_{\gamma,c}(r)\| \leq c \) and \( |\Psi_{\gamma,c}(r)| \leq c|r| \).

Pick any positive \( C^\infty(\mathbb{R}) \) function \( s(\cdot) \) such that \( s(z) = 1 \) if \( |z| \leq 1 \), \( s(z) = 0 \) if \( |z| \geq 2 \), and \( 0 \leq s(z) \leq 1 \) for all \( z \in \mathbb{R} \). For any \( n \geq 2 \), define the function \( S_n(r) \) by \( S_n(r) = \int_0^r s_n(z) \, dz \), where
\[ s_n(z) = \begin{cases} 1, & \text{if } |z| \leq n - 1, \\ s(z - (n-1)\text{sign}(z)), & \text{if } |z| \geq n - 1, \end{cases} \]
where \( \text{sign}(z) \) denotes the sign of \( z \). For each integer \( n \geq 2 \), the function \( S_n \) satisfies
\[ \begin{cases} S_n(r) = S_n(T_{n+1}(r)), \quad \|S'_n\|_{L^\infty(\mathbb{R})} \leq \|s\|_{L^\infty(\mathbb{R})}, \\ \text{supp } S'_n \subset [-n+1, n+1], \quad \text{supp } S''_n \subset [-n+1, -n] \cup [n, n+1]. \end{cases} \]
2.3. Renormalized solutions. We shall use the following definition of renormalized solutions for the reaction-diffusion-advection system (1.4)-(1.5)-(1.6):

**Definition 2.1.** A renormalized solution of (1.4), (1.5), (1.6) is a pair of functions \( u = (u_1, u_2) \), with \( u_i \in L^\infty(0, T; L^1(\Omega) \cap L^p(Q_T)) \) for \( i = 1, 2 \), satisfying the following conditions for any \( \gamma > 0 \):

\[
T_\gamma(u_i) \in \mathcal{N}_{l=1}^N L^{p_i,l}(0, T; W^{1,1}_{0,l}(\Omega)), \quad i = 1, 2.
\]

For any real number \( c > 0 \),

\[
\lim_{n \to \infty} \sum_{l=1}^N \int_{\{n \leq |u_i| \leq n + c\}} |\partial_{x_l} u_i|^{p_i,l} \, dx \, dt = 0, \quad i = 1, 2.
\]

For any renormalization \( S \in C^\infty(\mathbb{R}) \) such that \( \text{supp} S' \subset [-M, M] \) for some \( M > 0 \),

\[
\partial_t S(u_i) - \sum_{l=1}^N \partial_{x_l} \left( S'(u_i) \beta_{i,l}(t, x)|\partial_{x_l} u_i|^{p_i,l-2} \partial_{x_l} u_i \right) - \text{div}_x \left( S'(u_i) u_i K_i(t, x) \right)
\]

\[
+ \sum_{l=1}^N S''(u_i) \beta_{i,l}(t, x)|\partial_{x_l} u_i|^{p_i,l} + S''(u_i) u_i K_i(t, x) \cdot \nabla u_i
\]

\[
= F_i(t, x, u_1, u_2) S'(u_i) \quad \text{in} \ D'(Q_T).
\]

The initial function is satisfied in the following sense:

\[
S(u_i)|_{t=0} = S(u_{i,0}) \quad \text{a.e. in} \ \Omega.
\]

**Remark.** Let \( u = (u_1, u_2) \) be a renormalized solution. Since \( u_i \in W^{1,1}(\Omega) \), the Stampacchia theorem [15] tells us that

\[
\partial_{x_l} T_\gamma(u_i) = 1_{\{|u_i| < \gamma\}} \partial_{x_l} u_i, \quad l = 1, \ldots, N,
\]

where \( 1_{\{|u_i| < \gamma\}} \) denotes the characteristic function of the measurable set \( \{|u_i| < \gamma\} \subset Q_T, i = 1, 2 \).

**Remark.** One can easily check that all the terms in (2.4) make sense. In particular, in view of (1.11) and (1.14),

\[
r_i(t, x, u_1, u_2) S'(u_i), \sigma(t, x, u_1, u_2) S'(u_i) \in L^1(Q_T),
\]

which implies that \( F_i(t, x, u_1, u_2) S'(u_i) \in L^1(Q_T), i = 1, 2 \).

**Remark.** Let \( u = (u_1, u_2) \) be a renormalized solution. By the Young and Hölder inequalities, (1.12), and the boundness of \( u_i \) in \( L^p(Q_T) \) for \( i = 1, 2 \), there exist constants \( c, c', c'' > 0 \) such that
\[ \left| \int \int_{\{n \leq |u_i| \leq n+1\}} u_i K_i \cdot \nabla u_i \, dx \, dt \right| \]

\[ \leq c \sum_{l=1}^{N} \int \int_{\{n \leq |u_i| \leq n+1\}} |u_i|^{\frac{p_{i,l}}{p_{i,l} - 1}} \, dx \, dt \]

\[ + c' \sum_{l=1}^{N} \int \int_{\{n \leq |u_i| \leq n+1\}} |\partial_{x_l} u_i|^{p_{i,l}} \, dx \, dt \]

\[ \leq c'' \sum_{l=1}^{N} \left\{ \left\{ n \leq |u_i| \leq n+1 \right\} \right\}^{1-\frac{p_{i,l}}{p_{i,l} - 1}} \left( \int \int_{\{n \leq |u_i| \leq n+1\}} |u_i|^{p_{i,l}} \, dx \, dt \right)^{\frac{p_{i,l}}{p_{i,l} - 1}} \]

\[ + c' \sum_{l=1}^{N} \int \int_{\{n \leq |u_i| \leq n+1\}} |\partial_{x_l} u_i|^{p_{i,l}} \, dx \, dt \]

As \( u_i \in L^1(Q_T) \) and because of (2.3), this implies that

\[ \lim_{n \to \infty} \left| \int \int_{\{n \leq |u_i| \leq n+1\}} u_i K_i \cdot \nabla u_i \, dx \, dt \right| = 0, \quad i = 1, 2. \]

**Lemma 2.2.** Let \( p_l > 1, l = 1, \ldots, N \). Suppose

\[ u \in \cap_{l=1}^{N} L^{p_l}(0, T; W_{0}^{1,p_{i,l}}(\Omega)), \]

\[ \partial_t u \in \sum_{l=1}^{N} L^{p'_l}(0, T; (W_{0}^{1,p_{i,l}}(\Omega))^*) + L^1(Q_T), \]

where \( p'_l \) denotes the conjugate exponent of \( p_l \): \( \frac{1}{p_l} + \frac{1}{p'_l} = 1 \). Then \( u \in C(0, T; L^1(\Omega)) \).

**Proof.** Let \( p_{\min} = \min(p_1, \ldots, p_N) > 1 \). Then \( u \) belongs to \( L^{p_{\min}}(0, T; W_{0}^{1,p_{\min}}(\Omega)) \) and \( \partial_t u \) belongs to \( L^{p'_{\min}}(0, T; W_{0}^{-1,p_{\min}}(\Omega)) + L^1(Q_T) \), and thus we can apply the result of Porretta [22]. \( \square \)

Let \( u = (u_1, u_2) \) be a renormalized solution. Then we have

\[ S(u_i) \in \cap_{l=1}^{N} L^{p_{i,l}}(0, T; W_{0}^{1,p_{i,l}}(\Omega)), \]

\[ \partial_t S(u_i) \in \sum_{l=1}^{N} L^{p'_{i,l}}(0, T; (W_{0}^{1,p_{i,l}}(\Omega))^*) + L^1(Q_T), \]

so that we can apply Lemma 2.2 to conclude that \( S(u_i) \in C(0, T; L^1(\Omega)) \). Thus the initial condition (2.5) makes sense.
Moreover, we can use as test functions in (2.4) not only functions in $D$. In this paper, we denote by $\mathcal{K}$ the set of all functions $u \in L^1(\Omega)$. For technical reasons, we need to extend the incidence function $\sigma$ so that it becomes defined for all $(t, x, u_1, u_2) \in Q_T$. We do this by setting

$$\sigma(t, x, u_1, u_2) = \begin{cases} \sigma(t, x, u_1, 0), & \text{if } u_1 \geq 0, u_2 < 0, \\ \sigma(t, x, 0, u_2), & \text{if } u_1 < 0, u_2 \geq 0, \\ \sigma(t, x, 0, 0), & \text{if } u_1 < 0, u_2 < 0. \end{cases}$$

The functions $r_1, r_2$ in (1.11) are also extended to the whole of $Q_T$. We do this by setting

$$\begin{align*}
\frac{r_1(t, x, u_1, u_2)}{u_1} &= k_1(t, x) u_1 |u_1 + u_2|^{p-1}, \\
\frac{r_2(t, x, u_1, u_2)}{u_2} &= k_2(t, x) u_2 |u_1 + u_2|^{p-1},
\end{align*}$$

Remark. In this paper, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between

$$\sum_{i=1}^N (W_0^{1,p_i,l}(\Omega))^* + L^1(\Omega) \cap \cap_{i=1}^N W_0^{1,p_i,l}(\Omega) \cap L^\infty(\Omega).$$

2.4 Chain rule. We will need the following chain rule lemma. The proof of this lemma is standard, so we omit it (see for example [10] for a similar result in an isotropic context).

Lemma 2.3. Suppose $p_1 > 1$, $l = 1, \ldots, N$, and $u \in \cap_{i=1}^N L^{p_i}(0, T; W_0^{1,p_i,l}(\Omega)) \cap C(0, T; L^1(\Omega))$, where $\partial_t u$ belongs to the space $\sum_{i=1}^N L^{p_i}(0, T; (W_0^{1,p_i,l}(\Omega))^*) + L^1(\Omega)$ and $u_{|t=0} = u_0$. Then for every bounded Lipschitz continuous function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ and for every function $\phi \in D([0, T] \times \Omega)$ there holds

$$\begin{align*}
&- \int_0^t \langle \partial_t u, h(u) \phi \rangle \, dt = \int_0^t \int_\Omega \partial_t \phi \left( \int_0^u h(r) \, dr \right) \, dx \, dt \\
&\quad + \int_\Omega \phi(0, x) \left( \int_0^{u_0} h(r) \, dr \right) \, dx - \int_\Omega \phi(s, x) \left( \int_0^{u(s, x)} h(r) \, dr \right) \, dx,
\end{align*}$$

for any $s \in (0, T)$.
In what follows, we replace the expressions for the functions $F_1, F_2$ in (1.7) by the following expressions:

\[
F_1(t, x, u_1, u_2) = -r_1(t, x, u_1, u_2) - \sigma(t, x, u_1, u_2) + (b(t, x) - \lambda)u_1 \\
+ b(t, x)u_2 - m(t, x)u_1 + f_1(t, x),
\]

\[
F_2(t, x, u_1, u_2) = -r_2(t, x, u_1, u_2) + \sigma(t, x, u_1, u_2) \\
- (m(t, x) + \alpha(t, x) + \lambda)u_2 + f_2(t, x),
\]

where $\lambda > 0$ is a constant satisfying

\[
(3.1) \quad \lambda - b(t, x) \geq 0 \text{ for a.e. } (t, x) \in Q_T.
\]

At the expense of changing the coefficients in (1.4) in a non-essential way, we can always make this assumption about $F_1, F_2$. Indeed, define $v = (v_1, v_2)$ by setting $u_i = e^{\lambda t}v_i$, $i = 1, 2$. Then $v$ satisfies (1.4) with $b, \alpha, \beta_i, r_i$, and $F_i$ replaced by $-\lambda + b, \alpha + \lambda, \beta_i^t, R_i$, and $H_i$, respectively, for $i = 1, 2$ and $l = 1, \ldots, N$. Moreover, $\beta_{i,j}^t(t, x) = e^{(p_{i,j} - 1)\lambda t}\beta_{i,j}(t, x)$, $R_i(t, x, u_1, v_2) = e^{(p_{i,j} - 1)\lambda t}R_i(t, x, u_1, v_1)$, and

\[
H_1(t, x, v_1, v_2) = -R_1(t, x, v_1, v_2) - e^{-\lambda \sigma(t, x, e^{\lambda t}v_1, e^{\lambda t}v_2)} + (b(t, x) - \lambda)v_1 \\
+ b(t, x)v_2 - m(t, x)v_1 + e^{-\lambda}f_1(t, x),
\]

\[
H_2(t, x, v_1, v_2) = -R_2(t, x, v_1, v_2) + e^{-\lambda \sigma(t, x, e^{\lambda t}v_1, e^{\lambda t}v_2)} \\
- (m(t, x) + \alpha(t, x) + \lambda)v_2 + e^{-\lambda}f_2(t, x).
\]

**Lemma 3.1.** A renormalized solution $u = (u_1, u_2)$ is nonnegative.

**Proof.** In (2.8) with $i = 2$, we take $S = S_n$ ($S_n$ is defined in Subsection 2.2) and $\varphi_2 = -T_\gamma(u_2)$, where $u_2^- = \max(0, -u_2)$. Regarding the first term in (2.8), we use the chain rule (Lemma 2.3) to write

\[
\lim_{n \to \infty} \int_0^T \int_\Omega h_i S_n(u_2, \varphi_2) \, dt = \frac{d}{dt} \int_\Omega \Phi_\gamma(u_2^-) \, dx,
\]

where $\Phi_\gamma$ is defined in Subsection 2.2.

Let us study the remaining terms in (2.8). Using the nonnegativity of $S_n$, the definition of $\sigma$, and the fact that $r_2$ has the same sign as $u_2$, we have

\[
\int_0^T \int_\Omega \sigma(u_1, u_2)S_n'(u_2)T_\gamma(u_2^-) \, dx \, dt \geq 0,
\]

\[
\int_0^T \int_\Omega r_2(u_1, u_2)S_n'(u_2)T_\gamma(u_2^-) \, dx \, dt \leq 0.
\]

By the choice of $\lambda$ in (3.1) and the definition of $S_n$, we find

\[
(3.2) \quad \lim_{n \to \infty} \int_0^T \int_\Omega (m(t, x) + \alpha(t, x) + \lambda)S_n'(u_2)u_2^- T_\gamma(u_2^-) \, dx \, dt \geq 0.
\]
By the divergence theorem, (1.9), and (1.5)
\[
\lim_{n \to \infty} -\int_0^T \int_\Omega S_n'(u_2)u_2K_2(t,x) \cdot \nabla T_\gamma(u_2^-) \, dx \, dt
\]
\[= \lim_{n \to \infty} \int_0^T \int_\Omega S_n'(T_\gamma(u_2^-))T_\gamma(u_2^-)K_2(t,x) \cdot \nabla T_\gamma(u_2^-) \, dx \, dt \]
\[= \lim_{n \to \infty} \int_0^T \int_\Omega \text{div}_x \left( K_2(t,x) \int_0^{T_\gamma(u_2^-)} S_n'(r) \, dr \right) \, dx \, dt = 0. \tag{3.3} \]

Observe that
\[
\sum_{l=1}^N \int_0^T \int_\Omega S_n''(u_2)T_\gamma(u_2^-)\beta_{2,l}(t,x)|\partial_{x_l} u_2|^{p_2,l} \, dx \, dt
\leq C_{\gamma} \sum_{l=1}^N \int_{\gamma \leq |u_2| \leq \gamma + 1} |\partial_{x_l} u_2|^{p_2,l} \, dx \, dt \to 0 \quad \text{as } n \to \infty, \quad \text{due to (2.3)},
\]
\[
\left| \int_0^T \int_\Omega S_n''(u_2)T_\gamma(u_2^-)u_2K_2(t,x) \cdot \nabla u_2 \, dx \, dt \right|
\leq C_{\gamma} \int_{\gamma \leq |u_2| \leq \gamma + 1} |u_2K_2(t,x) \cdot \nabla u_2| \, dx \, dt \to 0 \quad \text{as } n \to \infty, \quad \text{due to (2.7)}. \]

In view of above calculations and the nonnegativity of \(f_2\), letting \(n \to \infty\) in (2.8) with \(i = 2\), \(S = S_n\), \(\varphi_2 = -T_\gamma(u_2^-)\) yields
\[
\frac{d}{dt} \int_\Omega \Phi_\gamma(u_2^-) \, dx + \beta_0 \sum_{l=1}^N \int_{|u_2^-| \leq \gamma} |\partial_{x_l} u_2|^{p_2,l} \, dx \leq 0,
\]
and thus \(\frac{d}{dt} \int_\Omega \Phi_\gamma(u_2^-) \, dx \leq 0\). Since \(u_{2,0}\) is nonnegative, we deduce that \(u_2^- = 0\).

Similarly, we take \(\varphi_1 = T_\gamma(u_1^-)\) and \(S = S_n\) in (2.8) with \(i = 1\). Since \(f_1\) and \(u_2\) are nonnegative, and by the choice of \(\lambda\) in (3.1), we find as before \(\int_\Omega \Phi_\gamma(u_1^-) \, dx \leq 0\). Since the initial function \(u_{1,0}\) is nonnegative, we conclude \(u_1^- = 0\). \(\square\)

**Lemma 3.2.** Let \(u = (u_1, u_2)\) be a renormalized solution. Then there exist constants \(c_1, c_2, c_3, c_4 > 0\) such that
\[
\|u_1 + u_2\|_{L^\infty(0,T;L^1(\Omega))} + \|\sigma(u_1, u_2)\|_{L^1(\Omega)} \leq c_1, \tag{3.4}
\]
\[
\|u_1\|_{L^{p_1}(\Omega)} + \|u_2\|_{L^{p_2}(\Omega)} \leq c_2, \tag{3.5}
\]
\[
\|\partial_{x_l} T_\gamma(u_l)\|_{L^{p_1,l}(\Omega)} \leq c_3 \gamma, \quad l = 1, \ldots, N, \quad i = 1, 2, \tag{3.6}
\]
\[
\sup_{\gamma > 0} \sum_{l=1}^N \int_{\gamma \leq |u_l| \leq \gamma + 1} |\partial_{x_l} u_i|^{p_l} \, dx \, dt \leq c_4, \quad i = 1, 2. \tag{3.7}
\]
Let \( p_i \) be defined in (1.11) and let \( \overline{p}_i \) denote the harmonic mean of \( p_{i,1}, \ldots, p_{i,N} \), i.e.,

\[
\frac{1}{\overline{p}_i} = \frac{1}{N} \sum_{l=1}^{N} \frac{1}{p_{i,l}}.
\]

Suppose \( 2 - \frac{1}{N+1} < p_{i,l} < \frac{\overline{p}_i(N+1)}{N} \) and \( \overline{p}_i \leq N + \frac{1}{N+1} \). Then, for \( 1 \leq q_{i,l} < p_{i,l} \overline{p}_i (\overline{p}_i - N)^{-1} \), \( l = 1, \ldots, N \), there holds

\[
(3.8) \quad u_i \in \cap_{l=1}^{N} L^{q_{i,l}} (0, T; W_0^{1,q_{i,l}}(\Omega)), \quad i = 1, 2,
\]

**Proof.** In the proof we make repeated use of the chain rule (Lemma 2.3). Moreover, all test functions/renormalizations used in the proof are defined in Subsection 2.2.

Proof of (3.4). We take \( S = S_n \) and choose \( \varphi_i = \frac{1}{n} T_\gamma ( u_i ) \) in (2.8) with \( i = 1, 2 \). Using Lemma 3.1, the choice of \( \lambda \) in (3.1), the nonnegativity of \( \alpha = (u_1, u_2) \), and letting first \( n \to \infty \) and second \( \gamma \to 0 \), it follows by adding together the resulting equations that

\[
\| (u_1 + u_2)(t, \cdot) \|_{L^1(\Omega)} \\
\leq \| u_{1,0} \|_{L^1(\Omega)} + \| u_{2,0} \|_{L^1(\Omega)} + \int_0^t \| (f_1 + f_2)(\tau, \cdot) \|_{L^1(\Omega)} d\tau,
\]

for \( t \in (0, T) \). In the course of deriving this bound, we have used, as in (3.2) and (3.3), that

\[
\lim_{n \to \infty} \left[ \int_0^T \int_\Omega \lambda_1 S_\gamma' (u_i) u_i T_\gamma (u_i) dx \, dt + \int_0^T \int_\Omega S_\gamma' (u_i) u_i K_i \cdot \nabla T_\gamma (u_i) dx \, dt \right] \geq 0,
\]

where \( \lambda_1 = \lambda - b \) and \( \lambda_2 = \lambda + n + a \). Moreover, by the definitions of \( S_n, T_\gamma, \) and (2.7),

\[
\left| \int_0^T \int_\Omega S''_\gamma (u_i) T_\gamma (u_i) u_i K_i \cdot \nabla u_i dx \, dt \right| \leq c \gamma \int_{\{ |u_i| \leq n+1 \}} |u_i K_i \cdot \nabla u_i | dx \, dt \\
\to 0 \quad \text{as} \quad n \to \infty.
\]

Next, we take \( S = S_n \) and \( \varphi_1 = \frac{1}{n} T_\gamma ( u_i ) \) in (2.8), and proceed as in the first part of the proof of (3.4). From Lemma 3.1 and by the choice of \( \lambda \) in (3.1), we deduce upon letting first \( n \to \infty \) and second \( \gamma \to 0 \)

\[
\| \sigma (u_1, u_2) \|_{L^1(Q_T)} \leq \left( \| u_{0,1} \|_{L^1(\Omega)} + \| f_1 \|_{L^1(Q_T)} + \| b \|_{L^\infty(Q_T)c_1 T} \right).
\]

This concludes the proof of (3.4).

**Proof of (3.5).** We use \( S = S_n \) and \( \varphi_1 = \frac{1}{n} T_\gamma ( u_i ) \) in (2.8) with \( i = 1, 2 \). After letting first \( n \to \infty \) and second \( \gamma \to 0 \), we find

\[
\int_0^T \int_\Omega (k_1(t, x)u_1 + u_2)^{p_{i,1} - 1} \leq \| u_{0,1} \|_{L^1(\Omega)} + \| u_{0,2} \|_{L^1(\Omega)} + \| f_1 + f_2 \|_{L^1(Q_T)}.
\]

Then, by (1.13), (3.5) follows.
Proof of (3.6). We take $S = S_n$ and $\varphi_1 = T_\gamma(u_1)$ in (2.8). From the choice of $\lambda$ in (3.1), Lemma 3.1, and letting $n \to \infty$, we obtain
\[
\sum_{i=1}^{N} \int_0^T \int_{\Omega} |\partial_{x_i} T_\gamma(u_1)|^{p_{i,1}} \, dx \, dt \leq \gamma \left( \|f_1\|_{L^1(Q_T)} + \|u_{0,1}\|_{L^1(\Omega)} + \|b\|_{L^\infty(Q_T)} c_1 T \right),
\]
which yields (3.6) for $u_1$. In the same way we deduce (3.6) for $u_2$.

Proof of (3.7). Let $\gamma > 0$. Take $S = S_n$ and $\varphi_1 = \phi_\gamma(u_1)$ in (2.8). From the choice of $\lambda$ in (3.1), Lemma 3.1, and letting $n \to \infty$, we obtain
\[
\sum_{i=1}^{N} \int \int_{\{\gamma \leq |u_1| \leq \gamma + 1\}} |\partial_{x_i} u_1|^{p_{i,1}} \, dx \, dt \leq \left( \|f_1\|_{L^1(Q_T)} + \|u_{0,1}\|_{L^1(\Omega)} + \|b\|_{L^\infty(Q_T)} c_1 T \right),
\]
which yields (3.7) for $u_1$. In the same way we deduce (3.7) for $u_2$.

Proof of (3.8). By using the anisotropic Sobolev inequality (Theorem 2.1) and a standard interpolation step, together with (3.4) and (3.7), we obtain
\[
\|\partial_{x_i} u_i\|_{L^{\infty}(Q_T)} \leq c, \quad \|u_i\|_{L^{\infty}(Q_T)} \leq c, \quad i = 1, 2,
\]
where $\gamma_i$ satisfies $\frac{1}{\gamma_i} = \frac{1}{N} \sum_{l=1}^{N} \frac{1}{\gamma_{i,l}}$. We refer [3] and the references therein for more details.

In view of the regularity proved in Lemma 3.2, it is clear that a renormalized solution is also a weak solution. Provided $f$ and $u_0$ are regular enough, the converse is also true.

Lemma 3.3. Suppose $f_i \in \sum_{i=1}^{N} L^{p_{i,1}}(0, T; (W^{1,p_{i,1}}_0(\Omega)))^*$ for $i = 1, 2$ and $u_0 \in L^2(Q_T; \mathbb{R}^2)$. Then a weak solution $u$ is also a renormalized solution.

Proof. By a weak solution $u = (u_1, u_2)$ we mean that for $i = 1, 2$
\[
u_i \in \cap_{i=1}^{N} L^{p_{i,1}}(0, T; W^{1,p_{i,1},1}_0(\Omega)) \cap L^p(Q_T) \cap C(0, T; L^2(\Omega))
\]
and
\[
\int_0^T \langle \partial_t u_i, \varphi_i \rangle \, dt + \sum_{i=1}^{N} \int_0^T \int_{\Omega} \beta_{i,2}(t, x) |\partial_{x_2} u_i|^{p_{i,1}-2} \partial_{x_2} u_i \partial_{x_2} \varphi_i \, dx \, dt + \int_0^T \int_{\Omega} u_i K_1 \cdot \nabla \varphi_i \, dx \, dt = \int_0^T \int_{\Omega} F_i(t, x, u_1, u_2) \varphi_i \, dx \, dt,
\]
for any $\varphi_i \in \cap_{i=1}^{N} L^{p_{i,1}}(0, T; W^{1,p_{i,1},1}_0(\Omega)) \cap L^\infty(Q_T)$, $i = 1, 2$. Taking $\varphi_i = S'(u_i) \psi_i$ in (3.10), where $S \in C^\infty(\mathbb{R})$, supp $S' \subset [-M, M]$ for some $M > 0$, and $\psi_i \in \mathcal{D}(Q_T)$, and using the chain rule, we deduce easily that $u_i$ satisfies the renormalized equation (2.8). Condition (2.3) is obtained by choosing $\varphi_i = \phi_{n,c}(u_i)$ ($\phi_{n,c}$ is defined in Subsection 2.2) in (3.10), using the chain rule, and then letting $n \to \infty$. The remaining conditions for being a renormalized solution hold trivially.

4. Proof of Theorem 1.1

4.1. Approximate problems. We introduce smooth approximations of our data $f = (f_1, f_2)$ and $u_0 = (u_{0,1}, u_{0,2})$. Pick functions $f_\varepsilon = (f_{1,\varepsilon}, f_{2,\varepsilon})$ and $u_{0,\varepsilon} =$
Lemma 4.2. For a subsequence as $u_{0,i}$, $u_{0,2,i} \to u_i$ a.e. in $Q_T$ and strongly in $L^{p_i}(Q_T)$ for any $1 \leq q_i < p_i$.

\begin{equation}
\{ f_{i,\varepsilon} \in D(Q_T), \quad u_{0,i,\varepsilon} \in D(\Omega),
\end{equation}

\begin{equation}
\| f_{i,\varepsilon} \|_{L^1(Q_T)} \leq \| f_i \|_{L^1(Q_T)}, \quad f_{i,\varepsilon} \to f_i \text{ in } L^1(Q_T) \text{ as } \varepsilon \to 0,
\end{equation}

\begin{equation}
\| u_{0,i,\varepsilon} \|_{L^1(\Omega)} \leq \| u_{0,i} \|_{L^1(\Omega)}, \quad u_{0,i,\varepsilon} \to u_{0,i} \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0.
\end{equation}

Then classical results, see, e.g., [18, 16], provide us with the existence of functions $u_{i,\varepsilon} \in \cap_{i=1}^N L^{p_i}(0, T; W_0^{1,p_i,i}(\Omega)) \cap L^p(Q_T) \cap C([0, T]; L^2(\Omega))$.

Moreover, each $u_{\varepsilon} = (u_{i,\varepsilon}, u_{2,\varepsilon})$ is a weak solution of (1.4), (1.5), (1.6) with $u_0$ and $f$ replaced by $u_{0,i,\varepsilon}$ and $f_{i,\varepsilon}$, respectively:

\begin{equation}
\int_0^T (\partial_t u_{i,\varepsilon}, \varphi_i) \, dt + \sum_{i=1}^N \int_0^T \int_{\Omega} \beta_i(t, x)|\partial_{x_i} u_{i,\varepsilon}|^{p_i-2}\partial_{x_i} u_{i,\varepsilon} \partial_{x_i} \varphi_i \, dx \, dt
\end{equation}

\begin{equation}
+ \int_0^T \int_{\Omega} u_{i,\varepsilon} K_i \cdot \nabla \varphi_i \, dx \, dt = \int_0^T \int_{\Omega} F_i(t, x, u_{i,\varepsilon}, u_{2,\varepsilon}) \varphi_i \, dx \, dt,
\end{equation}

for all $\varphi_i \in \cap_{i=1}^N L^{p_i}(0, T; W_0^{1,p_i,i}(\Omega)) \cap L^\infty(Q_T)$, $i = 1, 2$.

4.2. Basic convergence results.

Lemma 4.1. The weak solution $u_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})$ is nonnegative. Moreover, the estimates in Lemma 3.2 hold with $u$ replaced by $u_\varepsilon$, and all the constants are independent of $\varepsilon$.

Proof. See the proofs of Lemmas 3.1 and 3.2. \qed

The a priori estimates in Lemma 4.1 imply the following basic convergences:

Lemma 4.2. For a subsequence as $\varepsilon \to 0$ and $i = 1, 2$,

\begin{equation}
u_{i,\varepsilon} \rightharpoonup u_i \text{ a.e. in } Q_T \text{ and strongly in } L^{p_i}(Q_T) \text{ for any } 1 \leq q_i < p_i,
\end{equation}

\begin{equation}u_{i,\varepsilon} \to u_i \text{ strongly in } C(0, T; L^1(\Omega)),
\end{equation}

\begin{equation}T_\gamma(u_{i,\varepsilon}) \rightharpoonup T_\gamma(u_i) \text{ weakly in } L^{p_i}(0, T; W_0^{1,p_i,i}(\Omega)),
\end{equation}

for $l = 1, \ldots, N$ and any $\gamma > 0$.

Proof. For $M > 0$, choose any nondecreasing smooth function $S_M : \mathbb{R} \to \mathbb{R}$ satisfying

\begin{equation}S_M(z) = \begin{cases}
z, & \text{if } |z| \leq \frac{M}{2}, \\
M \text{sign}(z), & \text{if } |z| \geq M.
\end{cases}
\end{equation}

Clearly, $S_M' \subset [-M, M]$. Thanks to Lemma 4.1,

\begin{equation}(S_M(u_{i,\varepsilon}))_{0<\varepsilon\leq 1} \text{ is bounded in } L^{p_i}(0, T; W_0^{1,q_i}(\Omega)),
\end{equation}

where $q_i = \min_{1 \leq i \leq N} p_i$ for $i = 1, 2$. Moreover, from the renormalized formulation of (4.1) (see (4.51)) below, we can prove that

\begin{equation}(\partial_t S_M(u_{i,\varepsilon}))_{0<\varepsilon\leq 1} \text{ is bounded in } L^{p_i}(0, T; (W_0^{1,q_i}(\Omega))^*) + L^1(Q_T).
\end{equation}
Therefore, possibly at the cost of extracting subsequences (see, e.g., Corollary 4 in [27]), we can assume that for any \( M > 0 \)
\[
\text{(4.5)} \quad SF_{M}(u_{i,\varepsilon}) \rightarrow SF_{M}(u_{i}) \quad \text{a.e. in } Q_{T} \text{ and in } L^{\infty}(Q_{T}).
\]
It is standard (see for example [1, 4]) to prove that (4.5) implies \( u_{i,\varepsilon} \rightarrow u_{i} \) in measure, at least along a subsequence, and thus along another subsequence \( u_{i,\varepsilon} \rightarrow u_{i} \) a.e. in \( Q_{T} \). In view of this and Lemma 4.1, we conclude that (4.2) holds.

To prove (4.3) we use \( \frac{1}{\gamma}T_{\gamma}(u_{i,\varepsilon} - u_{i,\varepsilon}') \) as a test function in the weak formulations (4.1) for \( u_{i,\varepsilon} \) and \( u_{i,\varepsilon}' \), then take the difference of the two resulting equations, and finally send \( \gamma \) to zero. In this way we obtain that \( (u_{i,\varepsilon})_{0<\varepsilon \leq 1} \) is a Cauchy sequence in \( C(0,T;L^{1}(\Omega)) \).

We know from Lemma 4.1 that \( \partial_{t}T_{\gamma}(u_{i,\varepsilon}) \) is bounded in \( L^{p_{i},\varepsilon}(Q_{T}) \). From this and (4.2), it follows that (4.4) holds. \( \square \)

Next we study the convergences of the nonlinear terms \( \sigma, r_{1}, r_{2} \).

**Lemma 4.3.** For a subsequence as \( \varepsilon \rightarrow 0 \),
\[
\text{(4.6)} \quad \sigma(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) \rightarrow \sigma(t,x,u_{1},u_{2}) \quad \text{a.e. in } Q_{T} \text{ and strongly in } L^{1}(Q_{T}),
\]
\[
\text{(4.7)} \quad r_{i}(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) \rightarrow r_{i}(t,x,u_{1},u_{2}) \quad \text{a.e. in } Q_{T} \text{ and strongly in } L^{1}(Q_{T}),
\]
for \( i = 1,2 \). Consequently,
\[
\text{(4.8)} \quad F_{i}(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) \rightarrow F_{i}(t,x,u_{1},u_{2}) \quad \text{a.e. in } Q_{T} \text{ and strongly in } L^{1}(Q_{T}),
\]
for \( i = 1,2 \).

**Proof.** We start by proving for \( i = 1,2 \)
\[
\text{(4.9)} \quad \lim_{\gamma \rightarrow \infty} \left( \sup_{0<\varepsilon \leq 1} \int_{\{|u_{i,\varepsilon}| > \gamma\}} \left( |\sigma(t,x,u_{1,\varepsilon},u_{2,\varepsilon})| + |r_{i}(t,x,u_{1,\varepsilon},u_{2,\varepsilon})| \right) dx \right) = 0.
\]
Note that
\[
\text{(4.10)} \quad \int_{\{|u_{1,\varepsilon}| > \gamma\}} r_{1}(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) dx dt + \int_{\{|u_{1,\varepsilon}| > \gamma\}} \sigma(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) dx dt \\
\quad \leq \frac{1}{\gamma} \int_{0}^{T} \int_{\Omega} T_{\gamma}(u_{1,\varepsilon}) \left( r_{1}(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) + \sigma(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) \right) dx dt.
\]
Using \( \varphi_{1} = T_{\gamma}(u_{1,\varepsilon}) \) in (4.1), we obtain
\[
\int_{\Omega} \Phi_{\gamma}(u_{1,\varepsilon})(t,x) dx + \sum_{i=1}^{N} \int_{0}^{T} \int_{\Omega} \beta_{i}(t,x) |\partial_{x_{i}}T_{\gamma}(u_{1,\varepsilon})|^{p_{i}} dx dt \\
\quad + \int_{0}^{T} \int_{\Omega} u_{1,\varepsilon} \mathbf{K} \cdot \nabla T_{\gamma}(u_{1,\varepsilon}) dx dt + \int_{0}^{T} \int_{\Omega} r_{1}(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) T_{\gamma}(u_{1,\varepsilon}) dx dt \\
\quad + \int_{0}^{T} \int_{\Omega} \sigma(t,x,u_{1,\varepsilon},u_{2,\varepsilon}) T_{\gamma}(u_{1,\varepsilon}) dx dt \\
\quad \leq \int_{\Omega} \Phi_{\gamma}(u_{0,1,\varepsilon}) dx + \int_{0}^{T} \int_{\Omega} |f_{1,\varepsilon} T_{\gamma}(u_{1,\varepsilon})| dx dt + \int_{0}^{T} \int_{\Omega} |b(t,x) u_{2,\varepsilon} T_{\gamma}(u_{1,\varepsilon})| dx dt.
\]
By the nonnegativity of $\Phi_\gamma$ and the divergence theorem, we deduce from this (4.11)

\[ 0 \leq \int_0^T \int_\Omega r_1(t, x, u_{1, \varepsilon}, u_{2, \varepsilon}) T_\gamma(u_{1, \varepsilon}) \, dx \, dt + \int_0^T \int_\Omega \sigma(t, x, u_{1, \varepsilon}, u_{2, \varepsilon}) T_\gamma(u_{1, \varepsilon}) \, dx \, dt \]

\[ \leq \int_\Omega \Phi_\gamma(u_{0, 1, \varepsilon}) \, dx + \int_0^T \int_\Omega |f_1, \varepsilon T_\gamma(u_{1, \varepsilon})| \, dx \, dt \]

\[ + \int_0^T \int_\Omega |b(t, x) u_{2, \varepsilon} T_\gamma(u_{1, \varepsilon})| \, dx \, dt. \]

For any $M > 0$, we have

\[ 0 \leq \Phi_\gamma(s) \leq M^2 + \gamma |s| 1_{\{|s| > M\}}, \]

\[ 0 \leq |T_\gamma(s)| \leq M + \gamma 1_{\{|s| > M\}}. \]

Using (4.12) in (4.11), combining the result with (4.10), we deduce

\[ \int_{\{|u_{1, \varepsilon}| \geq \gamma\}} r_1(t, x, u_{1, \varepsilon}, u_{2, \varepsilon}) \, dx \, dt + \int_{\{|u_{1, \varepsilon}| \geq \gamma\}} |\sigma(t, x, u_{1, \varepsilon}, u_{2, \varepsilon})| \, dx \, dt \]

\[ \leq \frac{M^2}{\gamma} \|u_{0, 1, \varepsilon}\|_{L^1(\Omega)} + \int_{\{|u_{0, 1, \varepsilon}| > M\}} |u_{0, 1, \varepsilon}| \, dx + \frac{M}{\gamma} \|f_1, \varepsilon\|_{L^1(Q_T)} \]

\[ + \int_{\{|u_{1, \varepsilon}| > M\}} |f_1, \varepsilon| \, dx \, dt + \|b\|_{L^\infty(Q_T)} \int_{\{|u_{1, \varepsilon}| > M\}} |u_{2, \varepsilon}| \, dx \, dt. \]

We know $(u_{0, 1, \varepsilon})_{0 < \varepsilon \leq 1}$, $(f_1, \varepsilon)_{0 < \varepsilon \leq 1}$ and $(u_{2, \varepsilon})_{0 < \varepsilon \leq 1}$ are converging in $L^1(\Omega)$, $L^1(Q_T)$, and $L^1(Q_T)$, respectively, so that we can make the first, third, and fifth terms on the right-hand side arbitrarily small by making $\gamma$ sufficiently large (uniformly in $\varepsilon$).

On the other hand, since

\[ \text{meas}\{|u_{1, \varepsilon}| > M\} \leq \frac{1}{M} \|u_{1, \varepsilon}\|_{L^1(\Omega)} \to 0 \quad \text{as} \quad M \text{ tends to} \ 0, \]

we can make the second, fourth, and sixth terms on the right-hand side arbitrarily small by choosing $M$ sufficiently large (uniformly in $\varepsilon$). This completes the proof of (4.9).

Thanks to (4.2), $\sigma(t, x, u_{1, \varepsilon}, u_{2, \varepsilon}) \to \sigma(t, x, u_1, u_2)$ a.e. in $Q_T$. Hence, by Vitali’s theorem (see, e.g., [14]), $(\sigma(t, x, u_{1, \varepsilon}, u_{2, \varepsilon}))_{0 < \varepsilon \leq 1}$ is strongly convergent in $L^1(Q_T)$ if we show that this sequence lies in a weakly compact subset of $L^1(Q_T)$, which in turn follows if we can prove that it is equiintegrable. Let $B$ be any measurable set in $Q_T$, and write

\[ \int_B |\sigma(t, x, u_{1, \varepsilon}, u_{2, \varepsilon})| \, dx \, dt \]

\[ \leq \int_{B \cap \{|u_{1, \varepsilon}| \leq \gamma\}} |\sigma(t, x, u_{1, \varepsilon}, u_{2, \varepsilon})| \, dx \, dt + \int_{B \cap \{|u_{1, \varepsilon}| > \gamma\}} |\sigma(t, x, u_{1, \varepsilon}, u_{2, \varepsilon})| \, dx \, dt. \]

By (4.9), the second term on the right-hand side tends to zero (uniformly in $\varepsilon$) as $\gamma \to \infty$. By assumption (1.14),

\[ \int_{B \cap \{|u_{1, \varepsilon}| \leq \gamma\}} |\sigma(t, x, u_{1, \varepsilon}, u_{2, \varepsilon})| \, dx \, dt \]
\[ \leq \|L\|_{L^\infty(Q_T)} \int_B \gamma^{n_1 u_{2,\varepsilon}^2} \, dx \, dt + \text{meas}(B) \|M\|_{L^\infty(Q_T)}, \]

and thanks to (4.2) we thus conclude that
\[
\lim_{|B| \to 0} \sup_{0 < \varepsilon \leq 1} \int_{B \cap \{u_{1,\varepsilon} \leq \gamma\}} |\sigma(t, x, u_{1,\varepsilon}, u_{2,\varepsilon})| \, dx \, dt = 0.
\]

This proves that the sequence \((\sigma(t, x, u_{1,\varepsilon}, u_{2,\varepsilon}))_{0 < \varepsilon \leq 1}\) is equiintegrable, and thus (4.6) follows. Similarly, we can prove that the sequences \((r_i(t, x, u_{1,\varepsilon}, u_{2,\varepsilon}))_{0 < \varepsilon \leq 1}, i = 1, 2,\) are equiintegrable, thereby yielding (4.7). This also concludes the proof of (4.8).

**Lemma 4.4.** For \(i = 1, 2,\)
\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \sum_{l=1}^N \int_0^T \int_{\{n \leq |u_{1,\varepsilon}| \leq n + c\}} |\partial_{xi} u_{1,\varepsilon}|^{p_{i,1}} \, dx \, dt = 0,
\]
for constant \(c > 0.\)

**Proof.** Let \(\phi_{n,c}\) and \(\Psi_{n,c}\) be as defined in Subsection 2.2. Via the chain rule, substituting \(\varphi_1 = \phi_{n,c}(u_{1,\varepsilon})\) into (4.1) yields
\[
\int_{\Omega} \Psi_{n,c}(u_{1,\varepsilon})(T, x) \, dx + \sum_{l=1}^N \int_0^T \int_{\Omega} \beta_{1,l}(t, x) |\partial_{xi} u_{1,\varepsilon}|^{p_{1,i}-2} \partial_{x_i} u_{1,\varepsilon} \partial_{x_l} \phi_{n,c}(u_{1,\varepsilon}) \, dx \, dt \\
+ \int_0^T \int_{\Omega} u_{1,\varepsilon} K_1 \cdot \nabla \phi_{n,c}(u_{1,\varepsilon}) \, dx \, dt = \int_0^T \int_{\Omega} F_1(t, x, u_{1,\varepsilon}, u_{2,\varepsilon}) \phi_{n,c}(u_{1,\varepsilon}) \, dx \, dt \\
+ \int_{\Omega} \Psi_{n,c}(u_{1,\varepsilon})(0, x) \, dx.
\]

Since \(\Psi_{n,c}\) is nonnegative and by the choice of \(\lambda,\) we deduce from this and (4.2), (4.6) that
\[
\beta_0 \lim_{\varepsilon \to 0} \sum_{l=1}^N \int_0^T \int_{\{n \leq |u_{1,\varepsilon}| \leq n + c\}} |\partial_{xi} u_{1,\varepsilon}|^{p_{i,1}} \, dx \, dt
\]

\[
\leq \int_0^T \int_{\Omega} F_1(t, x, u_1, u_2) \phi_{n,c}(u_1) \, dx \, dt + \int_{\Omega} \Psi_{n,c}(u_0,1) \, dx.
\]

Since \(\partial_{xi} \phi_{n,c}(u_{1,\varepsilon}) = 1_{\{n \leq |u_{1,\varepsilon}| \leq n + c\}} \partial_{xi} u_{1,\varepsilon}\) a.e. in \(Q_T,\) we deduce from (4.13)
\[
\beta_0 \lim_{\varepsilon \to 0} \sum_{l=1}^N \int_0^T \int_{\{n \leq |u_{1,\varepsilon}| \leq n + c\}} |\partial_{xi} u_{1,\varepsilon}|^{p_{i,1}} \, dx \, dt
\]

\[
\leq \int_0^T \int_{\Omega} F_1(t, x, u_1, u_2) \phi_{n,c}(u_1) \, dx \, dt + \int_{\Omega} \Psi_{n,c}(u_0,1) \, dx.
\]

It shows that \(\phi_{n,c}(u_1)\) is bounded in \(\cap_{i=1}^N L^{p_{1,i}}(0, T; W^{1,p_{1,i},1}_0(\Omega))\) independently of \(n.\) Since \(u_1 \in L^\infty(0, T; L^1(\Omega)),\) we obtain the a.e. convergence of \(\phi_{n,c}(u_1)\) to 0 as \(n \to \infty.\) Consequently \(\int_0^T \int_{\Omega} F_1(t, x, u_1, u_2) \phi_{n,c}(u_1) \, dx \, dt \to 0\) as \(n \to \infty.\) Moreover, \(\Psi_{n,c}(u_{0,1}) \to 0\) a.e. in \(\Omega\) as \(n \to \infty\) and \(|\Psi_{n,c}(u_{0,1})| \leq c|u_{0,1}|\) a.e. in \(\Omega.\) Since \(u_{0,1} \in L^1(\Omega),\) Lebesgue’s convergence theorem shows that \(\int_{\Omega} \Psi_{n,c}(u_{0,1}) \, dx \to 0\) as \(n \to \infty.\)
Consequently, passing to the limit \( n \to \infty \) in (4.14) yields the desired result for \( i = 1 \). Along the same lines we reach the result for \( i = 2 \). □

4.3. **Strong convergence of truncations.** The main obstacle in proving that \( u_\varepsilon \) converges to a renormalized solution \( u \) is that in the limiting process in the renormalized formulation there can be concentration effects, that is, singular (Dirac) measures can be produced. Indeed, equipped with the convergences in Lemma 4.2 we can only conclude that up to some “defect measure” \( \nu_i \)

\[
\sum_{l=1}^{N} S''(u_{i,\varepsilon}) |\partial_x u_{i,\varepsilon}|^{p_{i,l}} \to \sum_{l=1}^{N} S''(u_{i}) |\partial_x u_{i}|^{p_{i,l}} + \nu_i,
\]

in the sense of measures, \( i = 1, 2 \). The limit \( u \) can be a renormalized solution only if \( \nu_i \equiv 0 \). The results of this subsection will show that the defect measure \( \nu_i \) in fact vanishes, since it will be proved that \( T_{\gamma}(u_{i,\varepsilon}) \) converges strongly in \( L^{p_{i,l}}(0, T; W^{1,p_{i,l},l}_0(\Omega)) \), \( l = 1, \ldots, N \). The arguments used in this subsection follow closely those in Blanchard, Murat, and Redwane [6] (see also Porretta [22] and Dall’Aglio and Orsina [12]).

We start by recalling a suitable time-regularization procedure, which was first introduced by Landes [17], and subsequently employed by several authors to solve nonlinear time dependent problems with \( L^1 \) or measure data (see [12, 8, 22, 6]).

We shall apply this time regularization to \( T_{\gamma}(u_{i}) \), where \( u = (u_1, u_2) \) is a renormalized solution and \( \gamma > 0 \). We denote this regularized function by \( (T_{\gamma}(u_{i}))_{\mu} \), with \( \mu > 0 \). One can easily check that there exists a unique solution \( (T_{\gamma}(u_{i}))_{\mu} \in C_{\text{loc}}(\mathbb{R}^+) L^{p_{i,l}}(0, T; W^{1,p_{i,l},l}_0(\Omega)) \cap L^\infty(Q_T) \) of the equation

\[
\partial_t (T_{\gamma}(u_{i}))_{\mu} + \mu(T_{\gamma}(u_{i}))_{\mu} - T_{\gamma}(u_{i}) = 0 \quad \text{in} \ D'(Q_T),
\]

supplemented with the initial condition

\[
(T_{\gamma}(u_{i}))_{\mu}|_{t=0} = w_{0,i}^\mu \quad \text{in} \ \Omega,
\]

where \( w_{0}^{\mu} = (w_{0,1}^{\mu}, w_{0,2}^{\mu}) \) is a sequence of functions such that

\[
w_{0,i}^{\mu} \in C_{\text{loc}}(\mathbb{R}^+) W^{1,p_{i,l},l}_0(\Omega) \cap L^\infty(\Omega), \ |w_{0,i}^{\mu}|_{L^\infty(\Omega)} \leq \gamma,
\]

\[
w_{0,i}^{\mu} \to T_{\gamma}(u_{0,i}) \quad \text{a.e. in} \ \Omega \quad \text{as} \ \mu \to \infty,
\]

\[
\frac{1}{\mu} |w_{0,i}^{\mu}|_{W^{1,p_{i,l},l}_0(\Omega)} \to 0 \quad \text{as} \ \mu \to \infty, \ l = 1, \ldots, N.
\]

Following [17] we can easily prove

\[
\partial_t (T_{\gamma}(u_{i}))_{\mu} \in C_{\text{loc}}(\mathbb{R}^+) L^{p_{i,l}}(0, T; W^{1,p_{i,l},l}_0(\Omega)), \ |(T_{\gamma}(u_{i}))_{\mu}|_{L^\infty(Q_T)} \leq \gamma,
\]

\[
(T_{\gamma}(u_{i}))_{\mu} \to T_{\gamma}(u_{i}) \quad \text{a.e. in} \ Q_T, \ \text{weakly-}* \text{ in} \ L^\infty(Q_T), \ \text{and}
\]

\[
\text{strongly in} \ L^{p_{i,l}}(0, T; W^{1,p_{i,l},l}_0(\Omega)), \ l = 1, \ldots, N, \ \text{as} \ \mu \to \infty.
\]

To continue our proof of Theorem 1.1 we need the following lemma, whose proof is very similar to that in [6] and we include it just for the convenience of the reader.

**Lemma 4.5.** Fix \( \gamma > 0 \). Let \( S \in \mathcal{D}(\mathbb{R}) \) be a nonincreasing function such that \( S(r) = r \) for \( |r| \leq \gamma \) and \( \text{supp} \ S' \subseteq [-M, M] \) for some \( M > 0 \). Then

\[
\lim_{\mu \to \infty} \lim_{\varepsilon \to 0} \int_0^T \int_0^t \left( \partial_s S(u_{i,s}) T_{\gamma}(u_{i,s}) - (T_{\gamma}(u_{i}))_{\mu} \right) dt \, ds \geq 0, \quad i = 1, 2.
\]
where $\langle \cdot, \cdot \rangle$ denotes the duality between $\sum_{i=1}^{N} (W_{0}^{1,p_{i},l}(\Omega))^{*} + L^1(\Omega)$ and $\cap_{i=1}^{N} W_{0}^{1,p_{i},l}(\Omega) \cap L^{\infty}(\Omega)$.

Proof. By the compact support of $S'$,

$$S(u_{i,\varepsilon}), S(u_{i}) \in \cap_{i=1}^{N} L^{p_{i},l}(0,T; W_{0}^{1,p_{i},l}(\Omega)) \cap L^{\infty}(\Omega).$$

Thanks to this and (2.8), it follows that

$$\partial_{t} S(u_{i,\varepsilon}) \in \sum_{i=1}^{N} L^{p_{i},l}(0,T; (W_{0}^{1,p_{i},l}(\Omega))^{*}) + L^{1}(\Omega).$$

Clearly, by our assumption on $S$,

$$T_{\gamma}(S(u_{i,\varepsilon})) = T_{\gamma}(u_{i,\varepsilon}), \quad T_{\gamma}(S(u_{i})) = T_{\gamma}(u_{i}), \quad \begin{cases} \partial_{t} S(u_{i,\varepsilon}) \in \sum_{i=1}^{N} L^{p_{i},l}(0,T; (W_{0}^{1,p_{i},l}(\Omega))^{*}) + L^{1}(\Omega). \\ (T_{\gamma}(S(u_{i})))_{\mu} = (T_{\gamma}(u_{i}))_{\mu}, \end{cases} \quad \text{a.e. in } Q_{T}.$$

Equipped with the fact that $\partial_{t}(T_{\gamma}(S(u_{i})))_{\mu} \in \cap_{i=1}^{N} L^{p_{i},l}(0,T; W_{0}^{1,p_{i},l}(\Omega))$ and (4.20), we calculate as follows for $i = 1,2$:

$$\int_{0}^{T} \int_{0}^{t} \partial_{t} S(u_{i,\varepsilon}), T_{\gamma}(u_{i,\varepsilon}) - (T_{\gamma}(u_{i}))_{\mu} \, ds \, dt$$

$$= \int_{0}^{T} \int_{0}^{t} \partial_{t} S(u_{i,\varepsilon}), T_{\gamma}(S(u_{i})) - (T_{\gamma}(S(u_{i})))_{\mu} \, ds \, dt$$

$$= \int_{0}^{T} \int_{0}^{t} \partial_{t} (S(u_{i,\varepsilon}) - (T_{\gamma}(S(u_{i})))_{\mu}), S(u_{i,\varepsilon}) - (T_{\gamma}(S(u_{i})))_{\mu} \, ds \, dt$$

$$- \int_{0}^{T} \int_{0}^{t} \partial_{t} S(u_{i,\varepsilon}), S(u_{i,\varepsilon}) - T_{\gamma}(S(u_{i,\varepsilon})) \, dt \, ds$$

$$+ \int_{0}^{T} \int_{0}^{t} \partial_{t} T_{\gamma}(S(u_{i})))_{\mu}(S(u_{i,\varepsilon}) - (T_{\gamma}(S(u_{i})))_{\mu}) \, dx \, ds \, dt.$$

Using the chain rule (see Lemma 2.3), we get from (4.21)

$$\int_{0}^{T} \int_{0}^{t} \partial_{t} S(u_{i,\varepsilon}), T_{\gamma}(u_{i,\varepsilon}) - (T_{\gamma}(u_{i}))_{\mu} \, dt \, ds$$

$$= \frac{1}{2} \int_{0}^{T} \int_{\Omega} |S(u_{i,\varepsilon}) - (T_{\gamma}(S(u_{i})))_{\mu}|^{2} \, dx \, dt$$

$$- \frac{T}{2} \int_{\Omega} |S(u_{i,\varepsilon}) - (T_{\gamma}(S(u_{i})))_{\mu}|^{2}(t = 0) \, dx$$

$$- \frac{1}{2} \int_{0}^{T} \int_{\Omega} |S(u_{i,\varepsilon}) - T_{\gamma}(S(u_{i,\varepsilon}))|^{2} \, dx \, dt$$

$$+ \frac{T}{2} \int_{\Omega} |S(u_{i,\varepsilon}) - T_{\gamma}(S(u_{i}))|^{2}(t = 0) \, dx$$

$$+ \int_{0}^{T} \int_{0}^{t} \partial_{t} T_{\gamma}(S(u_{i})))_{\mu}(S(u_{i,\varepsilon}) - (T_{\gamma}(S(u_{i})))_{\mu}) \, dx \, ds \, dt.$$

By the boundedness of $S$ and (4.2),

$$S(u_{i,\varepsilon}) \to S(u_{i}) \text{ strongly in } L^{2}(Q_{T}) \text{ and weakly-}$*$ in } L^{\infty}(Q_{T}) \text{ as } \varepsilon \to 0.
Moreover, using the strong convergence of \( u_{0,\varepsilon} \rightarrow u_0 \) in \( L^1(\Omega) \), it follows from the initial condition for \( u_{i,\varepsilon} \) that

\[
S(u_{i,\varepsilon})|_{t=0} = S(u_{0,\varepsilon}) \rightarrow S(u_0) \text{ in } L^2(\Omega) \text{ as } \varepsilon \rightarrow 0.
\]

Sending \( \varepsilon \rightarrow 0 \) in (4.22), using (4.23) and (4.24), we find

\[
\lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^t \langle \partial_t S(u_{i,\varepsilon}), T_\gamma(u_{i,\varepsilon}) - (T_\gamma(u_i))_\mu \rangle \, ds \, dt = \frac{1}{2} \int_0^T \int_\Omega |S(u_i) - (T_\gamma(S(u_i)))_\mu|^2 \, dx \, dt
\]

\[
- \frac{T}{2} \int_\Omega |S(u_{0,i}) - (T_\gamma(S(u_i)))_\mu(t = 0)|^2 \, dx
\]

\[
- \frac{1}{2} \int_0^T \int_\Omega \partial_t |S(u_i)|^2 \, dx \, dt + \frac{T}{2} \int_\Omega |S(u_{0,i}) - T_\gamma(S(u_i))|^2 \, dx
\]

\[
+ \int_0^T \int_\Omega \partial_t |T_\gamma(S(u_i))|_\mu(S(u_i) - (T_\gamma(S(u_i)))_\mu) \, dx \, ds \, dt.
\]

By (4.17) and (4.18),

\[
(T_\gamma(S(u_i)))_\mu \rightarrow T_\gamma(S(u_i)) \text{ in } L^2(Q_T) \text{ as } \mu \rightarrow \infty
\]

and

\[
(T_\gamma(S(u_i)))_\mu(t = 0) \rightarrow T_\gamma(S(u_{0,i})) \text{ in } L^2(\Omega) \text{ as } \mu \rightarrow \infty.
\]

Using this and (4.15), we obtain from (4.25)

\[
\liminf_{\mu \rightarrow \infty} \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_0^t \langle \partial_t S(u_{i,\varepsilon}), (T_\gamma(u_{i,\varepsilon}) - (T_\gamma(u_i))_\mu) \rangle \, ds \, dt
\]

\[
= \liminf_{\mu \rightarrow \infty} \int_\Omega \int_0^T \int_0^t \langle T_\gamma(S(u_i)) - (T_\gamma(S(u_i)))_\mu|S(u_i) - (T_\gamma(S(u_i)))_\mu| \rangle \, dx \, ds \, dt,
\]

and, by (4.18) and (4.20), the integrand is nonnegative. \( \square \)

**Proposition 4.6.** Fix any truncation level \( \gamma > 0 \). As \( \varepsilon \rightarrow 0 \), we have for \( i = 1,2 \)

\[
\lim_{\mu \rightarrow \infty} \sup_{\varepsilon \rightarrow 0} \sum_{i=1}^N \int_0^T \int_0^t \int_\Omega |\partial_x (T_\gamma(u_{i,\varepsilon})))|^{p_i} \, dx \, ds \, dt
\]

\[
\leq \sum_{i=1}^N \int_0^T \int_0^t \int_\Omega |\partial_x (T_\gamma(u_i)))|^{p_i} \, dx \, ds \, dt
\]

and

\[
T_\gamma(u_{i,\varepsilon}) \rightarrow T_\gamma(u_i) \text{ strongly in } L^{p_i}(0,T;W^{1,p_i}_0(\Omega)), l = 1, \ldots, N.
\]

**Proof.** We start by showing how (4.28) can be deduced from (4.27).

**Proof of (4.28).** Fix any \( i = 1,2 \). We recall the following well known inequalities, which hold for any two real numbers \( a, b \) and \( p > 1 \):

\[
|a|^p - |b|^p - (a - b)^p \geq c(p) \begin{cases} |a - b|^p, & \text{if } p \geq 2 \\ \frac{|a - b|^2}{(|a| + |b|)^{p-2}}, & \text{if } 1 < p < 2 \end{cases} \geq 0,
\]

\[
(a - |b|^p - (a - b)^p \geq c(p) \begin{cases} |a - b|^p, & \text{if } p \geq 2 \\ \frac{|a - b|^2}{(|a| + |b|)^{p-2}}, & \text{if } 1 < p < 2 \end{cases} \geq 0,
\]

\[
\sum_{i=1}^N \int_0^T \int_0^t \int_\Omega |\partial_x (T_\gamma(u_{i,\varepsilon})))|^{p_i} \, dx \, ds \, dt
\]

\[
\leq \sum_{i=1}^N \int_0^T \int_0^t \int_\Omega |\partial_x (T_\gamma(u_i)))|^{p_i} \, dx \, ds \, dt
\]

and

\[
T_\gamma(u_{i,\varepsilon}) \rightarrow T_\gamma(u_i) \text{ strongly in } L^{p_i}(0,T;W^{1,p_i}_0(\Omega)), l = 1, \ldots, N.
\]
where $c(p) = 2^{2-p}$ when $p > 2$ and $c(p) = p - 1$ when $1 < p < 2$.

When $p_{i,l} \geq 2$ for some $l = 1, \ldots, N$, by (4.29) we have
\[
c(p_{i,l}) \int_0^T \int_0^t \int_\Omega \beta_{i,l}(t,x) |\partial_{x_l} T_\gamma(u_{i,e}) - \partial_{x_l} T_\gamma(u_i)|^{p_{i,l}} \, dx \, ds \, dt
\leq \sum_{l=1}^N \int_0^T \int_0^t \int_\Omega \beta_{i,l}(t,x)
\times \left( |\partial_{x_l} T_\gamma(u_{i,e})|^{p_{i,l}-2} |\partial_{x_l} T_\gamma(u_i)| - |\partial_{x_l} T_\gamma(u_i)|^{p_{i,l}-2} |\partial_{x_l} T_\gamma(u_i)| \right) dx \, ds \, dt =: E(\varepsilon).
\]

When $1 < p_{i,l} < 2$ for some $l = 1, \ldots, N$, we employ (4.29) as follows:
\[
\int_0^T \int_0^t \int_\Omega \beta_{i,l}(t,x) |\partial_{x_l} T_\gamma(u_{i,e}) - \partial_{x_l} T_\gamma(u_i)|^{p_{i,l}} \, dx \, ds \, dt
\leq \left( \int_0^T \int_0^t \int_\Omega \beta_{i,l}(t,x) \frac{|\partial_{x_l} T_\gamma(u_{i,e}) - \partial_{x_l} T_\gamma(u_i)|^2}{(|\partial_{x_l} T_\gamma(u_{i,e})| + |\partial_{x_l} T_\gamma(u_i)|)^{2-p_{i,l}}} \, dx \, ds \, dt \right)^{p_{i,l}}
\times \left( \int_0^T \int_0^t \int_\Omega \beta_{i,l}(t,x) \frac{|\partial_{x_l} T_\gamma(u_{i,e})| + |\partial_{x_l} T_\gamma(u_i)|}{|\partial_{x_l} T_\gamma(u_{i,e})| + |\partial_{x_l} T_\gamma(u_i)|} dx \, ds \, dt \right)^{2-p_{i,l}/2}
\leq (c(p_{i,l}))^{-p_{i,l}/2}(E(\varepsilon))^{p_{i,l}/2}
\times \left( \int_0^T \int_0^t \int_\Omega \beta_{i,l}(t,x) \frac{|\partial_{x_l} T_\gamma(u_{i,e})| + |\partial_{x_l} T_\gamma(u_i)|}{|\partial_{x_l} T_\gamma(u_{i,e})| + |\partial_{x_l} T_\gamma(u_i)|} dx \, ds \, dt \right)^{2-p_{i,l}/2}.
\]

Since $T_\gamma(u_{i,e})$ is bounded in $\cap_{l=1}^N L^{p_{i,l}}(0,T;W^{1,p_{i,l}}_0(\Omega))$ and using (4.4) and (4.27), it follows that $E(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, sending $\varepsilon \to 0$ in (4.30) and (4.31) yields
\[
\lim_{\varepsilon \to 0} \int_0^T \int_0^t \int_\Omega \beta_{i,l}(t,x) |\partial_{x_l} T_\gamma(u_{i,e}) - \partial_{x_l} T_\gamma(u_i)|^{p_{i,l}} \, dx \, ds \, dt = 0,
\]
for $l = 1, \ldots, N$, which proves (4.28).

**Proof of (4.27).** We substitute $\varphi_i = S_n(u_{i,e})V_{\varepsilon,\mu,i}$ in (4.1) and integrate over $(0,t)$, where
\[
V_{\varepsilon,\mu,i} = T_\gamma(u_{i,e}) - (T_\gamma(u_i))_\mu.
\]
The result is
\[
\int_0^T \int_0^t \langle \partial_t S_n(u_{i,e}), V_{\varepsilon,\mu,i} \rangle \, ds \, dt + J^{1.1}_{\varepsilon,\mu,n} + J^{1.2}_{\varepsilon,\mu,n} + J^{1.3}_{\varepsilon,\mu,n} + J^{1.4}_{\varepsilon,\mu,n} = J^{1.5}_{\varepsilon,\mu,n},
\]
where
\[
J^{1.1}_{\varepsilon,\mu,n} = \sum_{l=1}^N \int_0^T \int_0^t \int_\Omega S_n^l(u_{i,e}) \beta_{i,l}(t,x) |\partial_{x_l} u_{i,e}|^{p_{i,l}-2} \partial_{x_l} u_{i,e} \partial_{x_l} V_{\varepsilon,\mu,i} \, dx \, ds \, dt,
\]
where \( C \) is a constant independent of \( n \).

From the definitions of \( S_n \) (see Subsection 2.2) and \( V_{\varepsilon, \mu, i} \), we can use Lemma 4.5 with \( S = S_n \) to conclude that for any \( n \geq \gamma \) there holds

\[
\lim_{\mu \to \infty} \liminf_{\varepsilon \to 0} \int_0^T \int_0^t \langle \partial_t S_n(u_{i, \varepsilon}), V_{\varepsilon, \mu, i} \rangle \, ds \, dt \geq 0.
\]

By the definition of \( V_{\varepsilon, \mu, i} \), (4.2), and Lemma 4.2, we deduce for any \( \mu > 0 \)

\[
\begin{align*}
V_{\varepsilon, \mu, i} &\to T_\gamma(u_i) - (T_\gamma(u_i))_\mu \text{ weakly in } \cap_{t=1}^N L^{p_{i,\varepsilon}}(0, T; W_0^{1,p_{i,\varepsilon}}(\Omega)) \text{ as } \varepsilon \to 0, \\
\|V_{\varepsilon, \mu, i}\|_{L^\infty(Q_T)} &\leq 2\gamma \text{ for any } \varepsilon > 0, \\
V_{\varepsilon, \mu, i} &\to T_\gamma(u_i) - (T_\gamma(u_i))_\mu \text{ a.e. in } Q_T \text{ and weakly-* in } L^\infty(Q_T) \text{ as } \varepsilon \to 0.
\end{align*}
\]

Limit of \( J_{\varepsilon, \mu, n}^0 \). As \( \text{supp} S_n^0 \subset [-n+1, -n] \cup [n, n+1] \), we have for any \( n \geq 2 \) and any \( \mu > 0 \)

\[
|J_{\varepsilon, \mu, n}^0| \leq T \|S_n^0(u_{i, \varepsilon})\|_{L^\infty(\mathbb{R})} \|V_{\varepsilon, \mu, i}\|_{L^\infty(\mathbb{R})} 
\times \sum_{i=1}^N \int_{\{n \leq |u_{i, \varepsilon}| \leq n+1\}} \beta_{i, \varepsilon}(t, x) |\partial_{x_i} u_{i, \varepsilon}|^{p_{i, \varepsilon}} \, dx \, dt,
\]

and from (4.40) we deduce for any \( n \geq 2 \)

\[
\limsup_{\mu \to \infty} \limsup_{\varepsilon \to 0} |J_{\varepsilon, \mu, n}^0| \leq C \limsup_{\mu \to \infty} \limsup_{\varepsilon \to 0} \sum_{i=1}^N \int_{\{n \leq |u_{i, \varepsilon}| \leq n+1\}} |\partial_{x_i} u_{i, \varepsilon}|^{p_{i, \varepsilon}} \, dx \, dt,
\]

where \( C \) is a constant independent of \( n \). Now, by Lemma 4.4, we obtain from (4.41)

\[
\lim_{n \to \infty} \limsup_{\mu \to \infty} \limsup_{\varepsilon \to 0} |J_{\varepsilon, \mu, n}^0| = 0.
\]

Limit of \( J_{\varepsilon, \mu, n}^1 \). We have

\[
S_n^0(u_{i, \varepsilon}) u_{i, \varepsilon} K_i(t, x) \cdot \nabla V_{\varepsilon, \mu, i} = S_n^0(u_{i, \varepsilon}) T_{n+1}(u_{i, \varepsilon}) K_i(t, x) \cdot \nabla V_{\varepsilon, \mu, i} \text{ a.e. in } Q_T.
\]

and, by (4.2),

\[
S_n^0(u_{i, \varepsilon}) T_{n+1}(u_{i, \varepsilon}) K_i(t, x) \to S_n^0(u_{i}) T_{n+1}(u_{i}) K_i(t, x),
\]

(4.35)

\[
J_{\varepsilon, \mu, n}^{i, 2} = \sum_{i=1}^N \int_0^T \int_0^t \int_{\Omega} S_{n}^{0}(u_{i, \varepsilon}) \beta_{i, t}(t, x) |\partial_{x_i} u_{i, \varepsilon}|^{p_{i, \varepsilon}} V_{\varepsilon, \mu, i} \, dx \, dt,
\]

(4.36)

\[
J_{\varepsilon, \mu, n}^{i, 3} = \int_0^T \int_0^t \int_{\Omega} S_{n}^{0}(u_{i, \varepsilon}) u_{i, \varepsilon} K_{i}(t, x) \cdot \nabla V_{\varepsilon, \mu, i} \, dx \, dt,
\]

(4.37)

\[
J_{\varepsilon, \mu, n}^{i, 4} = \int_0^T \int_0^t \int_{\Omega} S_{n}^{0}(u_{i, \varepsilon}) K_{i}(t, x) \cdot \nabla u_{i, \varepsilon} V_{\varepsilon, \mu, i} \, dx \, dt,
\]

(4.38)

\[
J_{\varepsilon, \mu, n}^{i, 5} = \int_0^T \int_0^t \int_{\Omega} F_{i}(t, x, u_{1, \varepsilon}, u_{2, \varepsilon}) S_{n}^{0}(u_{i, \varepsilon}) V_{\varepsilon, \mu, i} \, dx \, dt.
\]
a.e. in $Q_T$ and weakly-* in $L^\infty(Q_T)$ as $\varepsilon \to 0$. From (4.40), (4.42), and (4.43), we find that for any $\mu > 0$

(4.44)

$$\lim_{\varepsilon \to 0} J_{\varepsilon,\mu,n}^{(.3)} = \int_0^T \int_0^t \int_\Omega S''_n(u_i) T_{n+1}(u_i) K_i(t,x) \cdot (\nabla T_{\gamma}(u_i) - \nabla (T_{\gamma}(u_i))_\mu) \, dx \, ds \, dt.$$  

Using (4.18) when passing to the limit in (4.44), we find that

$$\lim_{\mu \to \infty} \lim_{\varepsilon \to 0} J_{\varepsilon,\mu,n}^{(.3)} = 0,$$

for any $n > 2$.

**Limit of $J_{\varepsilon,\mu,n}$**

From the definition of $S_n$, we have

$$S''_n(u_i) u_i K_i(t,x) \cdot \nabla u_i V_{\varepsilon,\mu,i} = S''_n(u_i) T_{n+1}(u_i) K_i(t,x) \cdot \nabla T_{n+1}(u_i) V_{\varepsilon,\mu,i},$$

e.a. in $Q_T$, and from (4.2), (4.40), and Lemma 4.2, we obtain for any $\mu > 0$

(4.45)

$$\lim_{\varepsilon \to 0} J_{\varepsilon,\mu,n}^{(5)} = \int_0^T \int_0^t \int_\Omega S''_n(u_i) T_{n+1}(u_i) K_i(t,x) \cdot \nabla T_{n+1}(u_i) (T_{\gamma}(u_i))_\mu - (T_{\gamma}(u_i))_\mu) \, dx \, ds \, dt.$$  

Hence using (4.18) and passing to the limit as $\mu \to \infty$ in (4.45), we finally obtain for any $n \geq 2$

$$\lim_{\mu \to \infty} \lim_{\varepsilon \to 0} J_{\varepsilon,\mu,n}^{(5)} = 0.$$

**Limit of $J_{\varepsilon,\mu,n}$**

In view of (4.6) and (4.40), Lebesgue’s convergence theorem implies that for any $\mu > 0$ and any $n \geq 2$

(4.46)

$$\lim_{\varepsilon \to 0} J_{\varepsilon,\mu,n}^{(5)} = \int_0^T \int_0^t \int_\Omega F_i(t,x,u_1,u_2) S''_n(u_i) (T_{\gamma}(u_i))_\mu - (T_{\gamma}(u_i))_\mu) \, dx \, ds \, dt.$$  

We use (4.18) and that $F_i(t,x,u_1,u_2) S''_n(u_i) \in L^1$ when passing to the limit $\mu \to \infty$ in (4.46). This yields for any fixed $n \geq 2$

$$\lim_{n \to \infty} \lim_{\mu \to \infty} \lim_{\varepsilon \to 0} J_{\varepsilon,\mu,n}^{(5)} = 0.$$  

Now we can pass to the limit sup in (4.33) when $\varepsilon \to 0$, $\mu \to \infty$, and $n \to \infty$, respectively. The result is that for any $\gamma > 0$

(4.47)

$$\lim_{n \to \infty} \lim_{\mu \to \infty} \lim_{\varepsilon \to 0} \sum_{l=1}^N \int_0^T \int_0^t \int_\Omega S''_n(u_i,\varepsilon) \beta_{i,l}(t,x) \times |\partial_{x_l} u_i,\varepsilon|^{p_i} |\partial_{x_l} u_i,\varepsilon|^{p_i} |\partial_{x_l} u_i,\varepsilon|^{p_i} \, dx \, ds \, dt \leq 0.$$  

By the definition of $S_n$ (see Subsection 2.2), we have for any $n \geq \gamma$

$$S''_n(u_i,\varepsilon) \beta_{i,l}(t,x) |\partial_{x_l} u_i,\varepsilon|^{p_i} |\partial_{x_l} u_i,\varepsilon|^{p_i} T_{\gamma}(u_i,\varepsilon) = \beta_{i,l}(t,x) |\partial_{x_l} u_i,\varepsilon|^{p_i} |\partial_{x_l} u_i,\varepsilon|^{p_i} |\partial_{x_l} u_i,\varepsilon|^{p_i} T_{\gamma}(u_i,\varepsilon).$$

Using this and the definition of $V_{\varepsilon,\mu,i}$, we deduce from (4.47)

$$\lim_{n \to \infty} \lim_{\mu \to \infty} \lim_{\varepsilon \to 0} \sum_{l=1}^N \int_0^T \int_0^t \int_\Omega \beta_{i,l}(t,x) |\partial_{x_l} u_i,\varepsilon|^{p_i} |\partial_{x_l} u_i,\varepsilon|^{p_i} T_{\gamma}(u_i,\varepsilon) \, dx \, ds \, dt \leq \lim_{n \to \infty} \lim_{\mu \to \infty} \lim_{\varepsilon \to 0}$$
\[
\sum_{i=1}^{N} \int_{0}^{T} \int_{\Omega} S'_{n}(u_{i,\varepsilon}) \beta_{i,l}(t, x) |\partial_{x_{i}} u_{i,\varepsilon}|^{p_{i,l}-2} \partial_{x_{i}} u_{i,\varepsilon} \partial_{x_{i}}(T_{\gamma}(u_{i})) \mu \, dx \, ds \, dt.
\]

By the definition of \( S_{n} \), we obtain
\[
S'_{n}(u_{i,\varepsilon}) |\partial_{x_{i}} u_{i,\varepsilon}|^{p_{i,l}-2} \partial_{x_{i}} u_{i,\varepsilon} = S''_{n}(u_{i,\varepsilon}) |\partial_{x_{i}} T_{n+1}(u_{i,\varepsilon})|^{p_{i,l}-2} \partial_{x_{i}} T_{n+1}(u_{i,\varepsilon}) \quad \text{a.e. in } Q_{T}.
\]

Using the a.e. convergence of \( S'_{n}(u_{i,\varepsilon}) \) to \( S'_{n}(u_{i}) \) as \( \varepsilon \to 0 \), the boundedness of \( S''_{n} \), and the weak convergence of \( T_{n+1}(u_{i,\varepsilon}) \) to \( T_{n+1}(u_{i}) \) in \( L^{p_{i,l}}(0, T; W_{0}^{1,p_{i,l}}(\Omega)) \), \( l = 1, \ldots, N \), as \( \varepsilon \to 0 \), together with an application of the usual Minty argument, we obtain
\[
S'_{n}(u_{i,\varepsilon}) |\partial_{x_{i}} u_{i,\varepsilon}|^{p_{i,l}-2} \partial_{x_{i}} u_{i,\varepsilon} \to S'_{n}(u_{i}) |\partial_{x_{i}} T_{n+1}(u_{i})|^{p_{i,l}-2} \partial_{x_{i}} T_{n+1}(u_{i})
\]
weakly in \( L^{p_{i,l}}(Q_{T}) \). Hence, recalling that
\[
(T_{\gamma}(u_{i}))_{\mu} \to T_{\gamma}(u_{i}) \quad \text{strongly in } \cap_{l=1}^{N} L^{p_{i,l}}(0, T; W_{0}^{1,p_{i,l}}(\Omega)) \quad \text{as } \mu \to \infty,
\]
we deduce for any \( n \geq \gamma \)
\[
\lim_{\mu \to -\infty} \lim_{\varepsilon \to 0} \left( \sum_{i=1}^{N} \int_{0}^{T} \int_{\Omega} S'_{n}(u_{i,\varepsilon}) \beta_{i,l}(t, x) |\partial_{x_{i}} u_{i,\varepsilon}|^{p_{i,l}-2} \partial_{x_{i}} u_{i,\varepsilon} \partial_{x_{i}}(T_{\gamma}(u_{i})) \mu \, dx \, ds \, dt \right)
\]
\[
= \sum_{i=1}^{N} \int_{0}^{T} \int_{\Omega} S'_{n}(u_{i}) \beta_{i,l}(t, x) |\partial_{x_{i}} T_{n+1}(u_{i})|^{p_{i,l}-2} \partial_{x_{i}} T_{n+1}(u_{i}) \partial_{x_{i}}(T_{\gamma}(u_{i})) \mu \, dx \, ds \, dt.
\]

Observe that for any \( n \geq \gamma \)
\[
|\partial_{x_{i}} T_{n+1}(u_{i,\varepsilon})|^{p_{i,l}-2} \partial_{x_{i}} T_{n+1}(u_{i,\varepsilon}) 1_{\{|u_{i,\varepsilon}| \leq \gamma\}} = |\partial_{x_{i}} T_{\gamma}(u_{i,\varepsilon})|^{p_{i,l}-2} \partial_{x_{i}} T_{\gamma}(u_{i,\varepsilon}) 1_{\{|u_{i,\varepsilon}| \leq \gamma\}} \quad \text{a.e. in } Q_{T}.
\]

Sending \( \varepsilon \to 0 \) in (4.49), we deduce for any \( n \geq \gamma \)
\[
|\partial_{x_{i}} T_{n+1}(u_{i})|^{p_{i,l}-2} \partial_{x_{i}} T_{n+1}(u_{i}) 1_{\{|u_{i}| \leq \gamma\}} = |\partial_{x_{i}} T_{\gamma}(u_{i})|^{p_{i,l}-2} \partial_{x_{i}} T_{\gamma}(u_{i}) 1_{\{|u_{i}| \leq \gamma\}} \quad \text{a.e. in } A_{1,\gamma},
\]
where \( A_{1,\gamma} = \{(t, x) \in Q_{T} \mid |u_{i}(t, x)| \leq \gamma\} \) By (4.50), we conclude that
\[
|\partial_{x_{i}} T_{n+1}(u_{i})|^{p_{i,l}-2} \partial_{x_{i}} T_{n+1}(u_{i}) \partial_{x_{i}} T_{\gamma}(u_{i}) = |\partial_{x_{i}} T_{\gamma}(u_{i})|^{p_{i,l}-2} \partial_{x_{i}} T_{\gamma}(u_{i}) \quad \text{a.e. in } Q_{T},
\]
for any \( n \geq \gamma \), which should be inserted into (4.48). The proof of Proposition 4.6 is completed. \( \square \)

4.4. Concluding the proof of Theorem 1.1. Let \( S \in C^{\infty}(R) \) be such that \( \text{supp } S' \subset [-M, M] \) for some \( M > 0 \). Pointwise multiplication of (1.4) by \( S'(u_{i,\varepsilon}) \) yields
\[
\partial_{t} S(u_{i,\varepsilon}) - \sum_{l=1}^{N} \partial_{x_{l}} \left( S'(u_{i,\varepsilon}) \beta_{i,l}(t, x) |\partial_{x_{l}} u_{i,\varepsilon}|^{p_{i,l}-2} \partial_{x_{l}} u_{i,\varepsilon} \right)
\]
\[
- \text{div}_{x} \left( S'(u_{i,\varepsilon}) u_{i,\varepsilon} K_{i}(t, x) \right) + \sum_{l=1}^{N} \partial_{x_{l}} S''(u_{i,\varepsilon}) \beta_{i,l}(t, x) |\partial_{x_{l}} u_{i,\varepsilon}|^{p_{i,l}}
\]
\[
+ S''(u_{i,\varepsilon}) u_{i,\varepsilon} K_{i}(t, x) \cdot \nabla u_{i,\varepsilon} = F_{i}(t, x, u_{1,\varepsilon}, u_{2,\varepsilon}) S'(u_{i,\varepsilon}) \quad \text{in } \mathcal{D}'(Q_{T})
\]
for $i = 1, 2$. In following we pass to the limit $\varepsilon \to 0$ (in the sense of distributions) in each of the terms in (4.51). Thanks to (4.2) and (4.6), it is easy to pass to the limit in the first, third, and sixth terms.

Let us consider the second term. For $l = 1, \ldots, N$,
\[
S'(u_{i,\varepsilon}) \beta_{i,l}(t,x) |\partial_{x_i} u_{i,\varepsilon}|^{p_i-1} \partial_{x_i} u_{i,\varepsilon} = S'(u_{i,\varepsilon}) \beta_{i,l}(t,x) |\partial_{x_i} T_M(u_{i,\varepsilon})|^{p_i-1} \partial_{x_i} T_M(u_{i,\varepsilon}),
\]
and, because of (4.28),
\[
S'(u_{i,\varepsilon}) \partial_{x_i} T_M(u_{i,\varepsilon})|^{p_i-2} \partial_{x_i} T_M(u_{i,\varepsilon}) \to S'(u_{i}) \partial_{x_i} T_M(u_{i})|^{p_i-2} \partial_{x_i} T_M(u_{i})
\]
strongly in $L^{p_i,1}(Q_T)$. Since
\[
S'(u_{i}) \partial_{x_i} T_M(u_{i})|^{p_i-2} \partial_{x_i} T_M(u_{i}) = S'(u_{i}) \partial_{x_i} u_{i}|^{p_i-2} \partial_{x_i} u_{i},
\]
this concludes passing to the limit in the second term in (4.51).

Let us now study the third term. Clearly, since $\text{supp} S'' \subset [-M, M]$, we have for each $l = 1, \ldots, N$
\[
S''(u_{i,\varepsilon}) |\partial_{x_i} u_{i,\varepsilon}|^{p_i} = S''(u_{i,\varepsilon}) |\partial_{x_i} T_M(u_{i,\varepsilon})|^{p_i}\ a.e.\ in\ Q_T.
\]
Thanks to (4.28), the fact that $S''(u_{i,\varepsilon})$ converges to $S''(u_{i})$ a.e. in $Q_T$, and the boundedness of $S''$, the following convergence is true:
\[
S''(u_{i}) |\partial_{x_i} T_M(u_{i,\varepsilon})|^{p_i} \to S''(u_{i}) |\partial_{x_i} T_M(u_{i})|^{p_i}\ strongly\ in\ L^1(Q_T)\ as\ \varepsilon \to 0.
\]
This concludes the treatment of the third term in (4.51), as $S''(u_{i}) |\partial_{x_i} T_M(u_{i})|^{p_i} = S''(u_{i}) |\partial_{x_i} u_{i}|^{p_i}$.

Finally, let us consider the fifth term. In view of (4.2), (4.28), the boundedness/compact support of $S''(\cdot)$, the boundedness of $K$, it follows that
\[
S''(u_{i,\varepsilon}) u_{i,\varepsilon} K_{i,l} \partial_{x_i} u_{i,\varepsilon} \to T_M(u_{i}) K_{i,l} \partial_{x_i} S'(u_{i})\ strongly\ in\ L^{p_i,1}(Q_T),
\]
and $T_M(u_{i}) K_{i,l} \partial_{x_i} S'(u_{i})$ is identified with $S''(u_{i}) u_{i} K_{i,l} \partial_{x_i} u_{i}$, for $l = 1, \ldots, N$.

5. Uniqueness for a related scalar equation

The purpose of this section is to adapt some of the existing uniqueness results for isotropic parabolic equations with $L^1$ data (see [5, 6] and the references therein) to anisotropic parabolic equations. The related scalar problem reads
\[
\begin{cases}
\partial_t u - \sum_{l=1}^{N} \partial_{x_l} \left( \beta_l(t,x) |\partial_{x_l} u|^{p_l-2} \partial_{x_l} u \right) - \text{div}_x \left( u K(t,x) \right) \\
= -h(t,x,u) + f \\n\text{in} \ Q_T, \\
u = 0 \ \text{on} \ (0,T) \times \partial \Omega, \quad u|_{t=0} = u_0 \ \text{in} \ \Omega,
\end{cases}
\]
where we make the following assumptions:

\begin{equation}
\begin{aligned}
\begin{cases}
\mu_l > 1 & \beta_l \in L^\infty(Q_T), \quad \beta_l \geq \beta_0 > 0 \text{ for a.e. } (t,x) \in Q_T, \quad l = 1, \ldots, N, \\
K \in L^\infty(Q_T; \mathbb{R}^N), & \text{div}_x K = 0 \text{ in } D'(Q_T), \quad f \in L^1(Q_T), \quad u_0 \in L^1(\Omega), \\
h(t,x,s): Q_T \times \mathbb{R} \text{ is measurable in } (t,x) \in Q_T \text{ for every } s \in \mathbb{R} \\
\text{and continuous in } s \in \mathbb{R} \text{ for a.e. } (t,x) \in Q_T. \\
(h(t,x,s) - h(t,x,s')) (s - s') \geq 0 \text{ for a.e. } (t,x) \in Q_T \text{ and all } s, s' \in \mathbb{R}. \\
\end{cases}
\end{aligned}
\end{equation}

There exists an exponent \( p > 1 \) and a constant \( c \) such that for all \( s \in \mathbb{R} \) \( h(t,x,s) s \geq c |s|^{p+1} \).

For all \( \gamma > 0 \) the functions \( \sup_{|s| \leq \gamma} |h(t,x,s)| \) are locally integrable on \( Q_T \).

As an explicit example of a nonlinearity \( h \) satisfying (5.2) let us state

\[ h(t,x,u) = c(t,x) |u|^{p-1} u, \]

where \( c \in L^\infty(Q_T) \) and \( p > 1 \).

**Definition 5.1.** A renormalized solution of (5.1) is a function \( u \in L^\infty(0,T; L^1(\Omega)) \cap L^p(\Omega) \) satisfying the following four conditions:

- For all \( \gamma > 0 \)
  \begin{equation}
  T_{\gamma}(u) \in \cap_{i=1}^N L^{p_i}(0,T; W^{1,p_i}_0(\Omega)).
  \end{equation}

- For any real number \( c > 0 \)
  \begin{equation}
  \lim_{n \to -\infty} \sum_{i=1}^N \int\int_{\{n \leq |u| \leq n+c\}} |\partial_{x_i} u|^{p_i} \, dx \, dt = 0.
  \end{equation}

- For any function \( S \in C^\infty(\mathbb{R}) \) such that \( S' \subset [-M,M] \) for some \( M > 0 \),
  \begin{equation}
  \begin{aligned}
  \partial_t S(u) - \sum_{l=1}^N \partial_{x_l} \left( S'(u) \beta_l(t,x) |\partial_{x_l} u|^{p_l-2} \partial_{x_l} u \right) - \text{div}_x \left( S'(u) u K(t,x) \right) \\
  + \sum_{l=1}^N S''(u) \beta_l(t,x) |\partial_{x_l} u|^{p_l} + S''(u) u K(t,x) \cdot \nabla u \\
  = (-h(t,x,u) + f) S'(u) \quad \text{in } D'(Q_T),
  \end{aligned}
  \end{equation}

- \( S(u)|_{t=0} = S(u_0) \) a.e. in \( \Omega \).

**Theorem 5.1.** Suppose the conditions in (5.2) are fulfilled. Then the renormalized solution of problem (5.1) is unique.

To prove Theorem 5.1 we need the following lemma.

**Lemma 5.2.** Any renormalized solution \( u \) of (5.1) satisfies for any \( 0 < \theta < 1 \) and \( \mu > 1 \)

\begin{equation}
\frac{1}{\theta} \sum_{l=1}^N \int_{\mu-\theta \leq |u| \leq \mu+\theta} \beta_l(t,x) |\partial_{x_l} u|^{p_l} \, dx \, dt \leq \varepsilon(\mu),
\end{equation}

for some positive function \( \varepsilon(\cdot) \) that is independent of \( \theta \) and satisfies \( \lim_{\mu \to \infty} \varepsilon(\mu) = 0 \).
For $\mu > 0$, define the function $S_\mu$ in $W^{2,\infty}(\mathbb{R})$ defined by $S_\mu(0) = 0$ and

$$S_\mu'(r) = \begin{cases} 1, & \text{for } |r| \leq \mu, \\ \mu + 1 - |r|, & \text{for } \mu \leq |r| \leq \mu + 1, \\ 0, & \text{for } |r| \geq \mu + 1. \end{cases}$$ (5.8)

We observe that $\text{supp} \ S_{\mu + 1} \subset [-\mu - 2, \mu + 2]$, and thus we can take $S = S_{\mu + 1}$ in (5.5) to obtain

$$\partial_t S_{\mu + 1}(u) - \sum_{l=1}^{N} \partial_{x_l} \left( S_{\mu + 1}'(u) \beta_l(t, x) |\partial_{x_l} u|^{p_l - 2} \partial_{x_l} u \right) - \text{div}_x \left( S_{\mu + 1}'(u) u K(t, x) \right)$$

$$+ \sum_{l=1}^{N} S_{\mu + 1}''(u) \beta_l(t, x) |\partial_{x_l} u|^{p_l} + S_{\mu + 1}'(u) u K(t, x) \cdot \nabla u$$

$$= (-h(t, x, u) + f) S_{\mu + 1}'(u) \text{ in } D'(Q_T).$$ (5.9)

Note that for any $\mu > 1$ and $0 < \theta < 1$, the function $T_{\mu, \theta}(u) = \frac{1}{2} (T_{\mu + \theta}(u) - T_{\mu - \theta}(u))$ belongs to $\cap_{l=1}^{N} L^{p_l}(0, T; W^{1, p_l}_0(\Omega)) \cap L^\infty(Q_T)$ and supp $T_{\mu, \theta}' \subset [-\mu - \theta, -\mu + \theta] \cup [\mu - \theta, \mu + \theta]$. Thus $T_{\mu, \theta}$ can be used as a test function in (5.5) and we obtain

$$\int_0^T \langle \partial_t S_{\mu + 1}'(u), T_{\mu, \theta}(u) \rangle \, dt$$

$$+ \sum_{l=1}^{N} \int_0^T \int_{\Omega} S_{\mu + 1}''(u) \beta_l(t, x) |\partial_{x_l} u|^{p_l - 2} \partial_{x_l} u \partial_{x_l} T_{\mu, \theta}(u) \, dx \, dt$$

$$+ \int_0^T \int_{\Omega} S_{\mu + 1}'(u) u K(t, x) \cdot \nabla T_{\mu, \theta}(u) \, dx \, dt$$

$$+ \sum_{l=1}^{N} \int_0^T \int_{\Omega} S_{\mu + 1}''(u) \beta_l(t, x) |\partial_{x_l} u|^{p_l} T_{\mu, \theta}(u) \, dx \, dt$$

$$+ \int_0^T \int_{\Omega} S_{\mu + 1}'(u) u K(t, x) \cdot \nabla u T_{\mu, \theta}(u) \, dx \, dt$$

$$= \int_0^T \int_{\Omega} S_{\mu + 1}'(u)(-h(t, x, u) + f) T_{\mu, \theta}(u) \, dx \, dt.$$ (5.10)

Since supp $S_{\mu + 1}' \subset [-\mu - 2, \mu + 2]$, then $u$ may be replaced by $T_{\mu + 2}(u)$ in (5.10).

With this and the divergence theorem, we have

$$\int_0^T \int_{\Omega} S_{\mu + 1}'(u) u K(t, x) \cdot \nabla T_{\mu, \theta}(u) \, dx \, dt$$

$$\int_0^T \int_{\Omega} S_{\mu + 1}'(T_{\mu + 2}(u)) T_{\mu, \theta}(T_{\mu + 2}(u)) T_{\mu + 2}(u) K(t, x) \cdot \nabla T_{\mu + 2}(u) \, dx \, dt$$

$$= \int_0^T \int_{\Omega} \text{div}_x \left( K(t, x) \int_0^{T_{\mu + 2}(u)} S_{\mu + 1}'(r) T_{\mu, \theta}(r) \, dr \right) \, dx \, dt = 0.$$ (5.11)
Similary, we have

\[ \int_0^T \int_{\Omega} S_{\mu+1}''(u) u K(t,x) \cdot \nabla u \, T_{\mu,\theta}(u) \, dx \, dt = 0. \]  

Since \( 0 < \theta < 1 \), we have \( T_{\mu,\theta}(u) = T_{\mu,\theta}(S_{\mu+1}(u)) \) and then by Lemma 2.3,

\[ \int_0^T \langle \partial_t S_{\mu+1}(u), T_{\mu,\theta}(u) \rangle \, dt = \int_{\Omega} \Phi_{\mu,\theta}(S_{\mu+1}(u))(T,x) \, dx - \int_{\Omega} \Phi_{\mu,\theta}(S_{\mu+1}(u))(0,x) \, dx, \]

where \( \Phi_{\mu,\theta}(r) = \int_0^r T_{\mu,\theta}(r) \, dr \). Now, using the definition of \( \Phi_{\mu,\theta} \) to deduce from (5.13)

\[ \int_0^T \langle \partial_t S_{\mu+1}(u), T_{\mu,\theta}(u) \rangle \, dt \geq - \int_{\{|u_0| > \mu - 1\}} |u_0| \, dx. \]

Note that by definitions of \( S_\mu \) and \( T_{\mu,\theta} \), we deduce from (5.10) and (5.14)

\[ \frac{1}{\theta} \sum_{l=1}^N \int_{\{|u| \leq \mu + \theta\}} \beta_l(t,x) |\partial_x u|^p \, dx \, dt \]

\[ \leq 2 \sum_{l=1}^N \int_{\{|u| \leq \mu + 2\}} \beta_l(t,x) |\partial_x u|^p \, dx \, dt \]

\[ + 2 \int_{\{|u| > \mu - 1\}} (|h(t,x,u)| + |f|) \, dx \, dt + \int_{\{|u_0| > \mu - 1\}} |u_0| \, dx := \varepsilon(\mu), \]

for any \( \mu > 1 \) and \( 0 < \theta < 1 \). Since \( f, h(\cdot, \cdot, u) \in L^1(Q_T) \) and \( u \) belongs to \( L^\infty(0,T; L^1(\Omega)) \), we deduce from (5.15) that (5.7) holds and \( \lim_{\mu \to +\infty} \varepsilon(\mu) = 0. \)

**Proof of Theorem 5.1.** In what follows, we let \( u, v \) be two renormalized solutions in the sense of Definition 5.1 with data \((f, u_0), (g, v_0)\), respectively.

For \( \mu, \theta > 0 \), define the function \( S_{\mu,\theta} \) in \( W^{2,\infty}(\mathbb{R}) \) defined by \( S_{\mu,\theta}(0) = 0 \) and

\[ S_{\mu,\theta}(r) = \begin{cases} 
1, & \text{for } |r| \leq \mu, \\
\frac{1}{\theta} (\mu + \theta - |r|), & \text{for } \mu \leq |r| \leq \mu + \theta, \\
0, & \text{for } |r| \geq \mu + 1.
\end{cases} \]

Note that \( \text{supp} S_{\mu,\theta}' \subset [-\mu - \theta, \mu + \theta] \) and \( S_{\mu,\theta} \) in \( W^{2,\infty}(\mathbb{R}) \), thus we can take \( S = S_{\mu,\theta} \) in (5.5). The function \( \frac{1}{\theta} T_\gamma(S_{\mu,\theta}(u) - S_{\mu,\theta}(v)) \) belongs to \( \cap_{l=1}^N L^p(0,T; W^{1,\infty}_l(\Omega)) \cap L^\infty(Q_T) \), and thus it can be used as a test function in (5.5) (after a straightforward approximation argument).

Now, we use \( \frac{1}{\theta} T_\gamma(S_{\mu,\theta}(u) - S_{\mu,\theta}(v)) \) as a test function in (5.5) for \( u \) and \( v \), and then subtract the resulting equations. We obtain upon integrating the result
over \((0, t)\) and then over \((0, T)\) the following equation:

\[
\int_0^T \int_0^t \frac{1}{\gamma} \langle \partial_t S_{\mu, \theta}(u) - \partial_t S_{\mu, \theta}(v), T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \rangle (s) \, ds \, dt \\
+ A_1^{\mu, \gamma, \theta} + A_2^{\mu, \gamma, \theta} + A_3^{\mu, \gamma, \theta} + A_4^{\mu, \gamma, \theta} \\
+ \frac{1}{\gamma} \int_0^T \int_0^t \int_\Omega \left( h(t, x, u) S_{\mu, \theta}'(u) - h(t, x, v) S_{\mu, \theta}'(v) \right) T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \, dx \, ds \, dt \\
= \frac{1}{\gamma} \int_0^T \int_0^t \int_\Omega \left( f S_{\mu, \theta}'(u) - g S_{\mu, \theta}'(v) \right) T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \, dx \, ds \, dt,
\]

where

\[
A_1^{\mu, \gamma, \theta} = \sum_{l=1}^N \frac{1}{\gamma} \int_0^T \int_0^t \int_\Omega \beta_l(t, x) \left[ S_{\mu, \theta}'(u) |\partial_x u|^{|p_l-2|} \partial_x u - S_{\mu, \theta}'(v) |\partial_x v|^{|p_l-2|} \partial_x v \right] \\
\times \partial_t T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \, dx \, ds \, dt,
\]

\[
A_2^{\mu, \gamma, \theta} = \sum_{l=1}^N \frac{1}{\gamma} \int_0^T \int_0^t \int_\Omega S_{\mu, \theta}'(u) T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \beta_l(t, x) |\partial_x u|^{|p_l|} \, dx \, ds \, dt,
\]

\[
A_3^{\mu, \gamma, \theta} = -\sum_{l=1}^N \frac{1}{\gamma} \int_0^T \int_0^t \int_\Omega S_{\mu, \theta}'(v) T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \beta_l(t, x) |\partial_x v|^{|p_l|} \, dx \, ds \, dt,
\]

\[
A_4^{\mu, \gamma, \theta} = \frac{1}{\gamma} \int_0^T \int_0^t \int_\Omega |S_{\mu, \theta}(u) - S_{\mu, \theta}(v)| K(t, x) \cdot \nabla T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \, dx \, ds \, dt.
\]

To derive this equation we used the following fact:

\[ -\text{div}_x \left( S_{\mu, \theta}'(u) K(t, x) \right) + S_{\mu, \theta}'(u) u K(t, x) \cdot \nabla u = -\text{div}_x \left( K(t, x) S_{\mu, \theta}(u) \right). \]

By Lemma 2.3,

\[
\int_0^t \langle \partial_t S_{\mu, \theta}(u) - \partial_t S_{\mu, \theta}(v), T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \rangle (s) \, ds \\
= \int_\Omega \Phi_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v))(t, x) \, dx - \int_\Omega \Phi_\gamma(S_{\mu, \theta}(u_0) - S_{\mu, \theta}(v_0))(x) \, dx,
\]

where \( \Phi_\gamma \) is defined in Subsection 2.2.

In view of (5.17) and (5.18), we deduce for any \( t \in [0, T] \),

\[
\int_0^T \int_\Omega \frac{1}{\gamma} \Phi_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \, dx \, dt + A_1^{\mu, \gamma, \theta} + A_2^{\mu, \gamma, \theta} + A_3^{\mu, \gamma, \theta} + A_4^{\mu, \gamma, \theta} \\
+ \frac{1}{\gamma} \int_0^T \int_0^t \int_\Omega \left( h(t, x, u) S_{\mu, \theta}'(u) - h(t, x, v) S_{\mu, \theta}'(v) \right) T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \, dx \, ds \, dt \\
\leq \frac{1}{\gamma} \int_0^T \int_0^t \int_\Omega \left( f S_{\mu, \theta}'(u) - g S_{\mu, \theta}'(v) \right) T_\gamma(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \, dx \, ds \, dt \\
+ T \int_\Omega \frac{1}{\gamma} \Phi_\gamma(S_{\mu, \theta}(u_0) - S_{\mu, \theta}(v_0)) \, dx.
\]
By definition of $S_{\mu,\theta}$, we have for any fixed $\mu > 0$ and when $\theta \to 0$

(5.20) \[ S'_{\mu,\theta}(z) \to 1_{\{|z| \leq \mu\}} \text{ a.e. in } Q_T \text{ and strongly in } L^r(Q_T) \text{ for any } r < \infty, \]

and (5.20), we deduce $\mu > \theta$

Next, we use (5.20) to deduce $\mu > \theta$

Along the same lines we obtain $\mu > \theta$

By definition of $\mu,\theta$

Note that for any $0 < \theta < 1$, we have supp $S'_{\mu,\theta} \subset [-\mu - 1, \mu + 1]$. Using this and (5.20), we deduce

\[
\lim_{\theta \to 0} A^{\mu,\gamma,\theta}_4
= \sum_{l=1}^{N} \frac{1}{\gamma} \int_0^T \int_0^t \int_{\Omega} \beta_l(t,x) 1_{\{|u| \leq \mu\}} |\partial_{x_l} T_{\mu+1}(u)|^{p^*_l-2} \partial_{x_l} T_{\mu+1}(u) \\
\times \partial_{x_l} T_{\gamma}(T_{\mu}(u) - T_{\mu}(v)) \, dx \, ds \, dt
\]

\[
= \sum_{l=1}^{N} \frac{1}{\gamma} \int_0^T \int_0^t \int_{\Omega} \beta_l(t,x) 1_{\{|u| \leq \mu\}} |\partial_{x_l} T_{\mu+1}(v)|^{p^*_l-2} \partial_{x_l} T_{\mu+1}(v) \\
\times \partial_{x_l} T_{\gamma}(T_{\mu}(u) - T_{\mu}(v)) \, dx \, ds \, dt
\]

\[
= \sum_{l=1}^{N} \frac{1}{\gamma} \int_0^T \int_0^t \int_{\Omega} \beta_l(t,x) \left[ |\partial_{x_l} T_{\mu}(u)|^{p^*_l-2} \partial_{x_l} T_{\mu}(u) - |\partial_{x_l} T_{\mu}(v)|^{p^*_l-2} \partial_{x_l} T_{\mu}(v) \right] \\
\times \partial_{x_l} T_{\gamma}(T_{\mu}(u) - T_{\mu}(v)) \, dx \, ds \, dt.
\]

From (4.29),

(5.21) \[ A^{\mu,\gamma,\theta}_4 \geq 0. \]

By the definition of $S_{\mu,\theta}$ and Lemma 5.2, we obtain for any $\gamma > 0$, any $0 < \theta < 1$ and $\mu > 1$

(5.22) \[ |A^{\mu,\gamma,\theta}_4| \leq \frac{C_1}{\theta} T \int_{\{|u| \leq \mu\}} |\partial_{x_l} u|^{p_l} dx dt \leq \varepsilon_1(\mu), \]

where $\varepsilon_1(\mu)$ tends to zero as $\mu \to 0$.

Along the same lines we obtain

(5.23) \[ |A^{\mu,\gamma,\theta}_5| \leq \varepsilon_2(\mu), \]

where $\varepsilon_2(\mu)$ tends to zero as $\mu \to 0$.

Next, we use (5.20) to deduce

\[
\lim_{\theta \to 0} \left| A^{\mu,\gamma,\theta}_4 \right|
= \frac{1}{\theta} \sum_{l=1}^{N} \int_0^T \int_0^t \int_{\Omega} \left| T_{\mu}(u) - T_{\mu}(v) \right| K(t,x) \cdot \nabla T_{\gamma}(T_{\mu}(u) - T_{\mu}(v)) \, dx \, ds \, dt
\]

\[
\leq C T \sum_{l=1}^{N} \int_0^T \int_{\Omega} \left| 1_{\{|T_{\mu}(u) - T_{\mu}(v)| \leq \gamma\}} \partial_{x_l}(T_{\mu}(u) - T_{\mu}(v)) \right| \, dx \, dt.
\]

Since $(T_{\mu}(u) - T_{\mu}(v)) \in \cap_{l=1}^{N} L^{p_l}(0,T;W^{1,p_1,l}_0(\Omega))$,

$1_{\{|T_{\mu}(u) - T_{\mu}(v)| \leq \gamma\}} \partial_{x_l}(T_{\mu}(u) - T_{\mu}(v)) \to 0$ strongly in $L^{p_l}(Q_T)$, $l = 1, \ldots, N$,

as $\gamma \to 0$. This implies that $\lim_{\gamma \to 0} \lim_{\theta \to 0} A^{\mu,\gamma,\theta}_4 = 0$. 


Let \( \text{sign}_0(\cdot) \) denote the usual sign function with \( \text{sign}_0(0) = 0 \). Using all the previous calculations and (5.20), together with the facts that, as \( \theta \to 0 \) and \( \gamma \to 0 \),

\[
\begin{align*}
(h(t, x, u) S_{\mu, \theta}(u) - h(t, x, v) S_{\mu, \theta}(v)) \frac{1}{\gamma} T_{\gamma}(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \\
\to (h(t, x, u) 1_{\{|u| \leq \mu\}} - h(t, x, v) 1_{\{|v| \leq \mu\}}) \text{sign}_0(T_{\mu}(u) - T_{\mu}(v))
\end{align*}
\]

strongly in \( L^1(\Omega) \), and

\[
\begin{align*}
(f S_{\mu, \theta}(u) - g S_{\mu, \theta}(v)) \frac{1}{\gamma} T_{\gamma}(S_{\mu, \theta}(u) - S_{\mu, \theta}(v)) \\
\to (f 1_{\{|u| \leq \mu\}} - g 1_{\{|v| \leq \mu\}}) \text{sign}_0(T_{\mu}(u) - T_{\mu}(v))
\end{align*}
\]

strongly in \( L^1(\Omega) \), and \( \frac{1}{\gamma} \Phi_{\gamma}(S_{\mu, \theta}(u_0) - S_{\mu, \theta}(v_0)) \to |T_{\mu}(u_0) - T_{\mu}(v_0)| \) strongly in \( L^1(\Omega) \), it follows, by sending \( \theta \to 0 \) and \( \gamma \to 0 \) in (5.19) that

\[
\begin{align*}
\int_0^T \int_\Omega |T_{\mu}(u(t, x)) - T_{\mu}(v(t, x))| \, dx \, dt \\
+ \int_0^T \int_0^t \int_\Omega \left(h(t, x, u) 1_{\{|u| \leq \mu\}} - h(t, x, v) 1_{\{|v| \leq \mu\}}\right) \\
\times \text{sign}_0(T_{\mu}(u) - T_{\mu}(v)) \, dx \, ds \, dt \\
\leq \int_0^T \int_0^t \int_\Omega (f 1_{\{|u| \leq \mu\}} - g 1_{\{|v| \leq \mu\}}) \text{sign}_0(T_{\mu}(u) - T_{\mu}(v)) \, dx \, ds \, dt \\
+ T \int_\Omega |T_{\mu}(u_0) - T_{\mu}(v_0)| \, dx.
\end{align*}
\]

(5.24)

Sending \( \mu \to \infty \) in (5.24), taking into account the monotonicity condition in (5.2), we deduce

\[
\int_0^T \int_\Omega |u(t, x) - v(t, x)| \, dx \, dt \leq T \int_0^T \int_\Omega |f - g| \, dx \, dt + T \int_\Omega |u_0(x) - v_0(x)| \, dx.
\]

Now the uniqueness assertion follows by taking \( g \equiv f, v_0 \equiv u_0 \).

\[ \square \]

**Remark.** Theorem 5.1 continues to hold when \( h(x, t, u) \) is globally Lipschitz continuous in \( u \) instead of nondecreasing.

**References**


(Mostafa Bendahmane)
Centre of Mathematics for Applications
University of Oslo
P.O. Box 1053, Blindern, N–0316 Oslo, Norway
E-mail address: mostafab@math.uio.no

(Kenneth Hvistendahl Karlsen)
Centre of Mathematics for Applications
University of Oslo
P.O. Box 1053, Blindern, N–0316 Oslo, Norway
and
Department of Scientific Computing
Simula Research Laboratory
P.O.Box 134, N–1325 Lysaker, Norway
E-mail address: kennethk@math.uio.no
URL: http://www.math.uio.no/~kennethk/