MONOTONE DIFFERENCE APPROXIMATIONS OF BV SOLUTIONS TO DEGENERATE CONVECTION-DIFFUSION EQUATIONS

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Abstract. We consider consistent, conservative-form, monotone difference schemes for nonlinear convection-diffusion equations in one space dimension. Since we allow the diffusion term to be strongly degenerate, solutions can be discontinuous and, in general, are not uniquely determined by their data. Here we choose to work with weak solutions that belong to the BV (in space and time) class and, in addition, satisfy an entropy condition. A recent result of Wu and Yin [Northeastern Math J., 5 (1989), pp. 395–422] states that these so-called BV entropy weak solutions are unique. The class of equations under consideration is very large and contains, to mention only a few, the heat equation, the porous medium equation, the two phase flow equation, and hyperbolic conservation laws. The difference schemes are shown to converge to the unique BV entropy weak solution of the problem. In view of the classical theory for monotone difference approximations of conservation laws, the main difficulty in obtaining a similar convergence theory in the present context is to show that the (strongly degenerate) discrete diffusion term is sufficiently smooth. We provide the necessary regularity estimates by deriving and carefully analyzing a linear difference equation satisfied by the numerical flux of the difference schemes. Finally, we make some concluding remarks about monotone difference schemes for multidimensional equations.

Key words. degenerate convection-diffusion equations, BV solutions, entropy condition, monotone finite difference schemes, convergence

AMS subject classifications. 65M12, 35K65, 35L65

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1. Introduction. We are interested in monotone finite difference schemes for nonlinear, possibly strongly degenerate, convection-diffusion problems of form

\[
\begin{align*}
\partial_t u + \partial_x f(u) &= \partial_x (k(u) \partial_x u), & (x, t) \in Q_T \equiv \mathbb{R} \times (0, T), & k(u) \geq 0, \\
u(x, 0) &= u_0(x), & x \in \mathbb{R},
\end{align*}
\]

where the initial condition \(u_0(x)\), the convection flux \(f(u)\), and the diffusion flux \(k(u) \geq 0\) are given sufficiently regular functions. Convection-diffusion equations arise in a variety of applications, among others, turbulence, traffic flow, financial modeling, front propagation, two phase flow in oil reservoirs, and in models describing certain sedimentation processes.

When (1) is nondegenerate, i.e., \(k(u) > 0\), it is well known that (1) admits a unique classical solution [22]. This contrasts with the degenerate case where \(k(u)\) may vanish for some values of \(u\). A simple example of a degenerate equation is the porous medium equation

\[
\partial_t u = \partial_x^2 (u^m), \quad m > 1,
\]

which degenerates at \(u = 0\). In general, a manifestation of the degeneracy in (2) is the finite speed of propagation of disturbances; i.e., if at some fixed time the solution
$u$ has compact support, then it will continue to have compact support for all later times. The transition from a region where $u > 0$ to one where $u = 0$ (this region is a free boundary of the set $\{u > 0\}$) is not smooth and it is therefore necessary to deal with (continuous) weak solutions rather than classical solutions. Several interesting facts about the free boundary are known. For example, there is no waiting time, i.e., the free boundary begins to move immediately; see [10] for details. We refer to the book [25] for a nice overview of the theory of degenerate equations.

An essential condition for uniqueness of weak solutions in the class of bounded and measurable functions is that the function

$$K(u) = \int_0^u k(\xi) \, d\xi$$

is strictly increasing in $u$, which is also sufficient for the existence of continuous solutions; see Zhao [33]. A sufficient condition for $K(u)$ to be strictly increasing is that

$$\text{meas}\{u : k(u) = 0\} = 0,$$

which does not rule out the possibility that $k(u)$ has an infinite number of zero points. Accordingly, we refer to the problem (1) as degenerate if the condition (3) holds.

If the condition (3) is not satisfied, i.e., if there exists at least one interval $[\alpha, \beta]$ such that

$$k(u) = 0 \quad \forall u \in [\alpha, \beta],$$

we say that the parabolic problem (1) is strongly degenerate. A simple example of a strongly degenerate equation is a hyperbolic conservation law,

$$\partial_t u + \partial_x f(u) = 0.$$  

(4)

Strongly degenerate equations will, in general, possess discontinuous solutions. Furthermore, discontinuous weak solutions are not uniquely determined by their data. In fact, an additional condition is needed to single out the physically relevant weak solution of the problem. We call a bounded measurable function $u(x,t)$ an entropy weak solution if

(a) $\partial_t |u - c| + \partial_x \left[ \text{sgn}(u - c)(f(u) - f(c)) \right] + \partial_x |K(u) - K(c)| \leq 0$ (weakly).

Letting $c \to \pm \infty$ in (a), it is clear that entropy weak solutions are also weak solutions. It is not difficult to construct an entropy weak solution of (1) (even in several space dimensions); see Volpert and Hudjaev [30]. However, the main open question seems to be the uniqueness of such solutions, even in one space dimension. On the other hand, uniqueness of weak solutions for the purely parabolic case (no convection term) in the class of bounded integrable functions has been proved by Brezis and Crandall [2], while uniqueness of entropy weak solutions for conservation laws is a classical result due to Kruzkov [18]. Since a general uniqueness result for mixed hyperbolic-parabolic equations is lacking, we have chosen to seek solutions in the smaller class containing the BV entropy weak solutions. We call a bounded measurable function $u(x,t)$ a BV entropy weak solution if

(b) $u(x,t) \in BV(Q_T)$ and $K(u)$ is Hölder continuous,
(c) $\partial_t |u - c| + \partial_x [\text{sgn}(u - c)(f(u) - f(c) - \partial_x K(u))] \leq 0$ (weakly).

What makes this class interesting is that a uniqueness result for solutions in the sense of (b) and (c) has been proved recently by Wu and Yin [31]; see section 2 for a precise statement of their result. Their proof depends heavily on the theory of BV functions of several variables and geometric measures. Here one should note that the jump conditions proposed by Volpert and Hudjaev [30] are in general not correct, and thus the uniqueness proof presented there is incomplete; see [31] for more details. The theory developed in [31] has also been used to treat various boundary value problems; see [4, 32]. Particularly interesting is the problem analyzed by Bürger and Wendland [3, 4], which is used to model the settling and consolidation of a flocculated suspension under the influence of gravity (a certain sedimentation process).

It seems to be a common opinion that by adding a “diffusion” term to a conservation law, one obtains an equation that is (in some sense) “easier” than the conservation law itself. This is indeed true if the diffusion term is nondegenerate. However, if the diffusion term is allowed to strongly degenerate, the solution of the convection-diffusion equation has a more complex structure than the solution of the conservation law. The following example demonstrates this. Let $f(u) = u^2$ (Burgers’ flux) and let $k(u)$ be the continuous function given by

$$k(u) = \begin{cases} 
0 & \text{for } u \in [0, 0.5], \\
2.5u - 1.25 & \text{for } u \in (0.5, 0.6), \\
0.25 & \text{for } u \in [0.6, 1.0].
\end{cases}$$

(5)

Note that $k(u)$ degenerates on the interval $[0, 0.5]$. In Figure 1 we have plotted the solution of the conservation law (4) and the solution of (1) at time $T = 0.15$.

An interesting observation is that the solution of (1) has a “new” increasing jump (shock), despite the fact that $f$ is convex. Thus the solution is not bounded in the $\text{Lip}^*$ norm, as opposed to the solution of the conservation law. We refer to Tadmor [27] (and the references therein) for a discussion of the $\text{Lip}^*$ norm and the importance of this norm in the theory of conservation laws. Moreover, while the speed of a jump in the conservation law solution is determined solely by $f(u)$ through the Rankine–Hugoniot condition, the speed of a jump in the solution of (1) is in general determined by the jumps in both $f(u)$ and $\partial_x K(u)$; see section 2 for precise statements.
of the jump conditions for (1). Finally, let us mention that techniques developed by
Kruzkov [18] (stability) and later Kuznetsov [19] (error estimates) do not apply to
entropy weak solutions of (1).

The analysis of numerical schemes for problems such as (1) has so far mainly been
cconcerned with one or two point degenerate equations and often only the "convection-
free" case. We refer to [9, 13, 15, 16, 21, 23, 24] for analysis of some finite element and
difference schemes within this context. In this paper we present a general convergence
theory for a class of difference schemes which also applies to strongly degenerate
convection-diffusion equations.

Selecting a mesh size $\Delta x > 0$, a time step $\Delta t > 0$, and an integer $N$ so that
$N \Delta t = T$, the value of our difference approximation at $(j \Delta x, n \Delta t)$ will be denoted by
$U^n_j$. Capital letters $U, V$, etc. will always denote functions on the mesh \{ $j \Delta x : j \in \mathbb{Z}$ \}.
To simplify the notation, we introduce the finite difference operators

$$D_- U_j = \frac{1}{\Delta x} (U_j - U_{j-1}), \quad D_+ U_j = \frac{1}{\Delta x} (U_{j+1} - U_j).$$

A novel feature of our difference schemes is that they will be based on differencing
the conservative-form equation

$$\partial_t u + \partial_x (f(u) - \partial_x K(u)) = 0.$$  \hfill (6)

We consider consistent, conservative, monotone finite difference schemes of the form

$$\frac{U^{n+1}_j - U^n_j}{\Delta t} + D_- (F(U^n_{j-p+1}, \ldots, U^n_{j+p}) - D_+ K(U^n_j)) = 0, \quad p \geq 1,$$  \hfill (7)

where $F : \mathbb{R}^{2p} \to \mathbb{R}$ is the convective numerical flux. The main purpose of this
paper is to show that (7) converges to the unique generalized (in the sense of (b)
and (c)) solution of the strongly degenerate problem (1). Combining the arguments
developed in this paper with the Crandall and Liggett theory [6], it is possible to give
an elegant treatment of implicit schemes as well; see [12] for details. To put this work
in a proper perspective, let us make some comments about the hyperbolic case (4).
Harten, Hyman, and Lax [14] proved that if the monotone difference approximations
converge as $\Delta x, \Delta t \to 0$, they converge to the unique entropy weak solution of the
conservation law. Kuznetsov [19] proved that monotone schemes for conservation laws
converge to the entropy solution in several space dimensions and provided suitable
error estimates. Later, Crandall and Majda [7] proved a similar result without the
error estimates. Sanders [26] proved convergence (with error estimates) for certain
three-point monotone schemes with variable spatial differencing.

The class of functions in which we seek solutions in this paper (see (b) and (c)
above) is significantly smaller than the class of entropy weak solutions (see (a) above).
From this point of view, we stress that it is nontrivial to show that the monotone
difference schemes produce solutions contained in this class. To complement this claim,
solutions constructed by viscous operator splitting [12] are not in this class. The reason
being that, in general, it is impossible to prove that the discrete diffusion term is
Hölder continuous. A source of inspiration is the convergence theory developed by
Crandall and Majda [8]. However, compared with their theory, the main difficulty
in the present context is indeed to show that the discrete diffusion term is Hölder
continuous. We obtain the necessary regularity estimates on the discrete diffusion
term by analyzing a certain linear difference equation which governs the behavior of
the numerical flux of the schemes; see Lemmas 3.4 and 3.6.
For completeness, let us give an example of a (three-point) monotone scheme. For a monotone flux \( f \), the upwind scheme is defined by

\[
F(U^n_j, U^n_{j+1}) = f(U^n_j) \quad \text{if} \quad f' \geq 0, \quad F(U^n_j, U^n_{j+1}) = f(U^n_{j+1}) \quad \text{if} \quad f' < 0.
\]

More generally, for a nonmonotone flux \( f \), the generalized upwind scheme of Engquist and Osher is defined by

\[
F(U^n_j, U^n_{j+1}) = f^+(U^n_j) + f^-(U^n_{j+1}),
\]

where

\[
f^+(u) = f(0) + \int_0^u \max(f'(s), 0) \, ds, \quad f^-(u) = \int_0^u \min(f'(s), 0) \, ds.
\]

A simple calculation reveals that the upwind scheme and the generalized upwind scheme for (1) are monotone provided the following CFL condition holds:

\[
\max |f'| \frac{\Delta t}{\Delta x} + 2 \max |k| \frac{\Delta t}{\Delta x^2} \leq 1.
\]

The monotone schemes devised in this paper are based on differencing the conservation-form equation (6) and not the equation in its original form. Of course, one can devise schemes based on differencing (1) directly, yielding, for example, schemes of the form

\[
\frac{U^{n+1}_j - U^n_j}{\Delta t} + D_-(F(U^n_{j-p}, \ldots, U^n_{j+p}) - k(U^n_{j+1/2})D_+ U^n_j) = 0,
\]

where \( U^n_{j+1/2} = \frac{1}{2} (U^n_j + U^n_{j+1}) \). Although it is possible to prove that (9) converges to a limit, we have not been able to show that this limit satisfies an entropy condition. In fact, we do not believe that (9) will converge to the physically correct (entropy) solution in the case where \( k \) is zero on an interval (strong degeneracy). To support this view, we now present a simple numerical example with fluxes \( \tilde{f}(u) = \frac{1}{4} u^2 \) and \( \tilde{k}(u) = 4k(u) \), where \( k \) is given in (5). In Figure 2 we have plotted the initial function and the solutions produced (using very small discretization parameters) by the schemes (7) and (9) at three different times. In these calculations the upwind flux (8) was used as the convective numerical flux. Clearly, the nonconservative scheme (9) produces a wrong solution. Moreover, the “difference” between this solution and the correct solution produced by (7) seems to increase with time. We are currently investigating this phenomenon and will come back to it in a separate report.

The rest of this paper is organized as follows. In section 2 we give a brief survey of the known mathematical theory of nonlinear strongly degenerate parabolic equations, while in section 3 we present the convergence analysis of the monotone schemes (7). Finally, in section 4 we briefly discuss monotone difference schemes for multidimensional equations.

2. Mathematical preliminaries. We shall here briefly recall the known mathematical theory of nonlinear strongly degenerate parabolic equations. Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) (\( d > 1 \)). The space \( BV(\Omega) \) of functions of bounded variation consists of all \( L^1_{\text{loc}}(\Omega) \) functions \( u(y) \) whose first-order partial derivatives \( \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_d} \) are represented by locally finite Borel measures. The total variation \( |u|_{BV(\Omega)} \) is by definition the sum of the total masses of these Borel measures. Moreover, \( BV(\Omega) \)
is a Banach space when equipped with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |u|_{BV(\Omega)}$. It is well known that the inclusion $BV(\Omega) \subset L^{d/(d-1)}(\Omega)$ holds for $d > 1$ and that $BV(\Omega) \subset L^\infty(\Omega)$ for $d = 1$. Furthermore, $BV(\Omega)$ is compactly imbedded into the spaces $L^q(\Omega)$ for $1 \leq q < d/(d-1)$. Finally, we will also need the Hölder space $C^{1+\frac{1}{2}}(\bar{Q}_T)$ consisting of bounded functions $z(x,t)$ on $\mathbb{R} \times [0,T]$ that satisfy

$$|z(y,\tau) - z(x,t)| \leq L(|y-x| + \sqrt{|\tau-t|}) \quad \forall x,y,t,\tau$$

for some constant $L > 0$ (not depending on $x,y,t,\tau$).

Here we seek generalized solutions to the problem (1) in the following sense.

**Definition 2.1.** A bounded measurable function $u(x,t)$ is said to be a BV entropy weak solution of the initial value problem (1) provided the following two conditions hold:

1. $u(x,t) \in BV(Q_T)$ and $K(u) \in C^{1+\frac{1}{2}}(\bar{Q}_T)$.
2. For all nonnegative $\phi \in C^\infty_c(Q_T)$ with $\phi|_{t=T} = 0$ and any $c \in \mathbb{R}$, the following entropy inequality holds:

$$\int_{Q_T} \left( (u - c)\partial_t \phi + \text{sgn}(u-c)(f(u) - f(c) - \partial_x K(u))\partial_x \phi \right) dt \, dx$$

$$+ \int_{\mathbb{R}} |u_0 - c\phi(x,0)| \, dx \geq 0. \quad (10)$$

**Remark.** First, in the context of hyperbolic equations ($k \equiv 0$), the entropy condition (10) coincides with the celebrated entropy condition due to Volpert [29]; see also Kruzkov [18]. Secondly, note that $K(u) \in C^{1+\frac{1}{2}}(\bar{Q}_T)$ implies that the weak derivative $\partial_x K(u)$ is in $L^\infty(Q_T)$, which in turn implies that $\partial_x K(u) \in L^1_{loc}(Q_T)$. Hence, solutions in the sense of Definition 2.1 are also solutions in the sense of Wu and Yin [31].

As pointed out earlier, entropy weak solutions of (1) can in general be discontinuous. The jump conditions take the following (correct!) form.

**Theorem 2.2 (see [31]).** Let $\Gamma_u$ be the set of jumps, i.e., $(x_0,t_0) \in \Gamma_u$ if and only if there exists a unit vector $\nu = (\nu_1,\nu_2)$ such that the approximate limits of $u$ at $(x_0,t_0)$ from the sides of the half-planes $(t-t_0)\nu_1 + (x-x_0)\nu_2 < 0$ and $(t-t_0)\nu_1 + (x-x_0)\nu_2 > 0$, denoted by $u^-(x_0,t_0)$ and $u^+(x_0,t_0)$, respectively, exist and are not equal. Similarly, let $u^+(x,t)$ and $u^-(x,t)$ denote the left and right approximate limits.
of \( u(\cdot, t) \), respectively. Introduce the notations \( \text{sgn}^+ := \text{sgn} \) and \( \text{sgn}^- := \text{sgn}^+ - 1 \), and let \( \text{int}(a, b) \) denote the closed interval bounded by \( a \) and \( b \). Finally, let \( H_1 \) denote the one-dimensional Hausdorff measure. Then for \( H_1 \) almost everywhere on \( \Gamma_u \), we have

\[
\begin{align*}
(11) & \quad k(c) = 0 \quad \forall c \in \text{int}(u^-, u^+), \quad \nu_x \neq 0; \\
(12) & \quad (u^+ - u^-)\nu + (f(u^+) - f(u^-))\nu_x - (\partial_x K(u)^r - \partial_x K(u)^l)|\nu_x| = 0.
\end{align*}
\]

For all \( c \in \mathbb{R} \),

\[
|u^+ - c|\nu_t + \text{sgn}(u^+ - c)[f(u^+) - f(c) - (\partial_x K(u)^r\text{sgn}^+\nu_x - \partial_x K(u)^l\text{sgn}^-\nu_x)]\nu_x \\
\leq |u^- - c|\nu_t + \text{sgn}(u^- - c)[f(u^-) - f(c) - (\partial_x K(u)^r\text{sgn}^+\nu_x - \partial_x K(u)^l\text{sgn}^-\nu_x)]\nu_x.
\]

**Remark.** Note the difference between \( u^\pm \) and \( u^r, u^l \). The approximate limits \( u^\pm \) are well defined for \( u \in BV(Q_T) \) while the limits \( u^r, u^l \) exist under the weaker assumption \( u(\cdot, t) \in BV(\mathbb{R}) \). For more details about the relation between \( u^\pm \) and \( u^r, u^l \) see the proof of Corollary 2.5. Observe also that outside \( \Gamma_u \) the solution \( u \) is at least Lipschitz continuous.

These jump conditions are essential ingredients in the proof of the following \( L^1 \) stability theorem.

**Theorem 2.3 (see [31]).** Let \( u_1 \) and \( u_2 \) be \( BV \) entropy weak solutions of (1) with integrable initial data \( u_{0,1} \) and \( u_{0,2} \), respectively. Then for any \( t > 0 \), we have

\[
\int_{\mathbb{R}} |u_1(x, t) - u_2(x, t)| \, dx \leq \int_{\mathbb{R}} |u_{0,1}(x) - u_{0,2}(x)| \, dx.
\]

The uniqueness of \( BV \) entropy weak solutions of the problem (1) is an immediate consequence of the above theorem. Cockburn and Gripenberg [7] have used the theory of Crandall and Liggett [6] to construct semigroup solutions of multidimensional degenerate convection-diffusion equations. Furthermore, they have proved that these semigroup solutions depend continuously on the nonlinear fluxes of the problem (see below). Now observe that since “parabolic regularizations” are smooth, the semigroup solution of (1) coincides with the viscosity solution of (1). Moreover, it turns out that the viscosity solution of (1) is also a solution in the sense of Definition 2.1. (This follows from [30] and Theorem 3.11 in this paper.) Hence, the semigroup solution coincides with the unique \( BV \) entropy weak solution in the case of one-dimensional equations and we have the following theorem.

**Theorem 2.4 (see [7]).** Let \( u_1, u_2 \) be \( BV \) entropy weak solutions of (1) with integrable initial data \( u_{0,1}, u_{0,2} \), convective fluxes \( f_1, f_2 \), and diffusive fluxes \( k_1, k_2 \), respectively. Furthermore, suppose that \( m \leq u_{0,1}, u_{0,2} \leq M \), and put \( C = \min \{ |u_{0,1}|_{BV(\mathbb{R})}, |u_{0,2}|_{BV(\mathbb{R})} \} \). Then for any \( t > 0 \), we have

\[
\int_{\mathbb{R}} |u_1(x, t) - u_2(x, t)| \, dx \leq \int_{\mathbb{R}} |u_{0,1}(x) - u_{0,2}(x)| \, dx \\
+C \left( t \sup_{u \in [m, M]} \left| f_1'(u) - f_2'(u) \right| + 4\sqrt{t} \sup_{u \in [m, M]} \left| \sqrt{k_1(u)} - \sqrt{k_2(u)} \right| \right).
\]

Finally, we note that the jump conditions in Theorem 2.2 can be stated more instructively as follows.

**Corollary 2.5.** Assume that \( k(c) = 0 \) for \( c \in [u_+, u^-] \) for some \( u_+, u^- \in [m, M] \). Let \( u(x, t) \) be a piecewise smooth \( BV \) entropy weak solution of (1) and let \( \Gamma_u \) be a
smooth discontinuity curve of \( u(x,t) \). A jump between two values \( u^l \) and \( u^r \) of the solution \( u(x,t) \), which we refer to as a shock, can occur only for \( u^l, u^r \in [u_*, u^*] \). This shock must satisfy the following two conditions:

1. The shock speed \( s \) is given by
   \[
   s = \frac{f(u^r) - f(u^l) - (\partial_x K(u^r) - \partial_x K(u^l))}{u^r - u^l}.
   \]

2. For all \( c \in \text{int}(u^l, u^r) \), the following entropy condition holds:
   \[
   \frac{f(u^r) - f(c) - \partial_x K(u^r)}{u^r - c} \leq s \leq \frac{f(u^l) - f(c) - \partial_x K(u^l)}{u^l - c}.
   \]

Proof. The fact that a jump can only occur in the interval of degeneracy \( [u_*, u^*] \) is a direct consequence of (11). For \( u \in L^\infty(Q_T) \cap BV(Q_T) \) it can be shown that the following relation between \( u^+, u^-, w^+ \), and \( u^l \) holds \( H_1 \) almost everywhere on \( \Gamma_u^w = \{(x,t) \in \Gamma_u : \nu_x \neq 0\} \),

\[
\begin{align*}
 u^+(x,t) &= u^l(x,t) \text{sgn}^+\nu_x - u^l(x,t) \text{sgn}^-\nu_x, \\
 u^-(x,t) &= u^l(x,t) \text{sgn}^+\nu_x - u^r(x,t) \text{sgn}^+\nu_x.
\end{align*}
\]

These identities are nontrivial and we refer to [31] for a proof. Invoking (16) it is not difficult to see that (12) can be written on the form

\[
(u^+ - u^l)\nu_t + (f(u^+)-f(u^l))\nu_x - (w^+_u - w^+_l)\nu_x = 0.
\]

Let \( s = -\frac{\nu_t}{\nu_x} \), then (14) follows.

Since \( |\nu_x| = (\text{sgn}^+\nu_x + \text{sgn}^-\nu_x)\nu_x \), (12) can be written as

\[
\begin{align*}
(u^+ - u^-)\nu_t + (f(u^+)-f(u^-))\nu_x - (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x \\
&+ (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x = 0,
\end{align*}
\]

where \( w^+_u = \partial_x K(u^r) \) and \( w^+_l = \partial_x K(u^l) \). For \( c \in \text{int}(u^-, u^+) = \text{int}(u^l, u^r) \), we have the relation \( \text{sgn}(u^+ - c) = -\text{sgn}(u^- - c) \). In light of this and (17), we now use (13) and perform the following calculation:

\[
\text{sgn}(u^+ - c)[(u^+ - c)\nu_t + (f(u^+)-f(c))\nu_x - (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x]
\]

\[
\leq -\text{sgn}(u^+ - c)[(u^- - c)\nu_t + (f(u^-)-f(c))\nu_x - (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x]
\]

\[
= -\text{sgn}(u^+ - c)[(u^- - u^+)\nu_t + (f(u^-)-f(u^+))\nu_x + (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x \\
- (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x]
\]

\[
= -\text{sgn}(u^+ - c)[(u^- - c)\nu_t + (f(u^-)-f(c))\nu_x - (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x \\
+ (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x - (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x]
\]

Hence

\[
\text{sgn}(u^+ - c)[(u^+ - c)\nu_t + (f(u^+)-f(c))\nu_x - (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)\nu_x] \leq 0.
\]

Dividing by \( |u^+ - c| \) yields

\[
\frac{\nu_t + (f(u^+)-f(c)) - (w^+_u \text{sgn}^+\nu_x - w^+_l \text{sgn}^-\nu_x)}{u^+ - c} \nu_x \leq 0
\]
or
\[
\frac{\left[ f(u^+) - (w_u^+ \text{sgn}\nu_x - w_u^- \text{sgn}\nu_x) \right] - f(c)}{u^+ - c} \nu_x \leq -\nu_t.
\] (18)

Similarly, we can show that
\[
-\nu_t \leq \frac{\left[ f(u^-) - (w_u^+ \text{sgn}\nu_x - w_u^- \text{sgn}\nu_x) \right] - f(c)}{u^- - c} \nu_x.
\] (19)

Combining (18) and (19) we have for \( c \in \text{int}(u^l, u^r) \) that
\[
\frac{\left[ f(u^+) - (w_u^+ \text{sgn}\nu_x - w_u^- \text{sgn}\nu_x) \right] - f(c)}{u^+ - c} \nu_x
\leq -\nu_t \leq \frac{\left[ f(u^-) - (w_u^+ \text{sgn}\nu_x - w_u^- \text{sgn}\nu_x) \right] - f(c)}{u^- - c} \nu_x.
\] (20)

Finally, in view of (16), we see that (20) is equivalent to (15). Hence the proof is completed.

Remark. The jump conditions (14) and (15) represent a generalization of the Rankine–Hugoniot condition and Oleinik’s entropy condition for conservation laws, respectively. Note that in general the limits \( \partial_x K(u^l) \) and \( \partial_x K(u^r) \) are unknown a priori, which implies that the propagation of a shock cannot be predicted a priori. This contrasts with what is known from the theory of hyperbolic conservation laws (the Rankine–Hugoniot condition).

3. Convergence analysis. In this section we analyze the monotone difference schemes. Implicit versions of these schemes are analyzed in [12]. In the following we treat the case where \( u_0 \) has compact support and \( f, K \) are locally \( C^1 \). Then towards the end of this section we will briefly discuss the general case where \( u_0 \) is not necessarily compactly supported and \( f, K \) are locally Lipschitz continuous. If not otherwise stated, we will always assume, without loss of generality, that \( f(0) = 0 \). The function space that contains \( u_0 \) will be taken as
\[
B(f, K) = \{ z \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) : |f(z) - \partial_x K(z)|_{BV(\mathbb{R})} < \infty \}.
\] (21)

Letting \( F(U; j) \) denote the convective numerical flux, i.e,
\[
F(U; j) := F(U_{j-p+1}, \ldots, U_{j+p}),
\]
the schemes under consideration take the form
\[
\begin{aligned}
\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} + D_{-} \left( F(U_{j}^{n}; j) - D_{+} K(U_{j}^{n}) \right) = 0, & \quad (j, n) \in \mathbb{Z} \times \{0, \ldots, N - 1\}, \\
U_{j}^{0} = \frac{1}{\Delta x} \int_{j \Delta x}^{(j+1)\Delta x} u_{0}(x) \, dx, & \quad j \in \mathbb{Z}.
\end{aligned}
\] (22)

To make the schemes (22) consistent with the convection-diffusion equation (1) it is sufficient to require that
\[
F(u, \ldots, u) = f(u).
\]
The assumption of monotonicity guarantees that (22), when viewed as an algorithm of the form (suppressing the $\Delta x$ and $\Delta t$ dependency)

\[ U_{j}^{n+1} = S(U_{j-p+1}^{n}, \ldots, U_{j+p}^{n}) =: S(U^{n}; j), \]

has the property that $S$ is a nondecreasing function of all its arguments.

For later use, recall that the $L^\infty(\mathbb{Z})$ norm, the $L^1(\mathbb{Z})$ norm and the $BV(\mathbb{Z})$ semi norm of a lattice function $U$ are defined, respectively, as follows:

\[
\|U\|_{L^\infty(\mathbb{Z})} = \sup_{j \in \mathbb{Z}} |U_j|,
\]
\[
\|U\|_{L^1(\mathbb{Z})} = \sum_{j \in \mathbb{Z}} |U_j|,
\]
\[
|U|_{BV(\mathbb{Z})} = \sum_{j \in \mathbb{Z}} |U_j - U_{j-1}| = \|D_- U\|_{L^1(\mathbb{Z})}.
\]

If not specified, $i, j$ will always denote integers from $\mathbb{Z}$; $m, n, l$ integers from $\{0, \ldots, N\}, x, y, c$ real numbers from $\mathbb{R}$, and $t, \tau$ real numbers from $[0, T]$. Furthermore, $C$ will denote a generic positive constant that can depend on the data of the problem but not on $\Delta x, \Delta t$.

We shall need the following lemma due to Crandall and Tartar [8].

**Lemma 3.1 (see [8]).** Let $(\Omega, \mu)$ be a measure space. If the operator $T : L^1(\Omega) \to L^1(\Omega)$ satisfies $\int_{\Omega} T(u) \, d\mu = \int_{\Omega} u \, d\mu$, then $T$ is a contraction on $L^1(\Omega)$ if and only if $T$ is monotone.

We shall also need the following lemma, which is due to Lucier [20].

**Lemma 3.2 (see [20]).** Any contraction $T$ from $L^1(\mathbb{Z})$ or $L^1(\mathbb{R})$ to itself that preserves the integral and commutes with translations satisfies the minimum principle and the maximum principle, i.e.,

\[
\liminf T(u)(x) \geq \liminf u(x), \quad \limsup T(u)(x) \leq \limsup u(x).
\]

In a series of lemmas we will provide uniform (in $\Delta x, \Delta t, T$) a priori estimates on the difference approximations. The first lemma gives the classical $L^\infty$ and $BV$ (space) estimates.

**Lemma 3.3.** We have

\[ \|U^n\|_{L^\infty(\mathbb{Z})} \leq \|U^0\|_{L^\infty(\mathbb{Z})}, \quad |U^n|_{BV(\mathbb{Z})} \leq |U^0|_{BV(\mathbb{Z})}. \]

**Proof.** Recall that we can rewrite the difference approximation (22) as $U^{n+1} = S(U^n)$, where $S : L^1(\mathbb{Z}) \to L^1(\mathbb{Z})$ maps sequences $U = \{U_j\}$ to sequences according to the formula

\[
S(U; j) = U_j - \Delta t D_-(F(U^n; j) - D_+ K(U^n))
\]

Since the difference approximation has compact support, we get $\sum_{j \in \mathbb{Z}} S(U; j) = \sum_{j \in \mathbb{Z}} U_j$. Thanks to Lemmas 3.1 and 3.2, the lemma now follows since $S$ is monotone and obviously commutes with translations.

The next lemma (see also Lemma 3.6), which eventually will lead to the desired regularity properties possessed by the diffusion term $K(u)$, plays a key role in our convergence analysis. It has no counterpart in the theory of monotone schemes for conservation laws as developed by Harten, Hyman, and Lax [14] and later by Crandall.
Multiplying the difference equation (28) evaluated at (31) by differentiating (6) with respect to $x$, we find that

$$\partial_t v + a(x,t)\partial_x v = \partial_x (b(x,t)\partial_x v), \quad v(x,t) = \int_{-\infty}^{x} \partial_t u(\xi, t) \, d\xi,$$

where $a = f'(u)$ and $b = k(u)$. This is a nondegenerate linear parabolic equation with smooth bounded coefficients, which has a unique smooth solution $v(x,t)$ satisfying

$$\|v(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|v(\cdot, 0)\|_{L^\infty(\mathbb{R})}, \quad \|v(\cdot, t)\|_{BV(\mathbb{R})} \leq \|v(\cdot, 0)\|_{BV(\mathbb{R})}.$$  

Thus, since $v = f(u) - \partial_x K(u)$, we get uniform $L^\infty(\mathbb{R})$ and $BV(\mathbb{R})$ estimates on $\partial_x K(u(\cdot, t))$. However, this is merely formalism since the solution to (1), in general, only exists in a weak sense, but these calculations clearly motivate similar results for the finite difference approximations (see also Theorem 3.11).

**Lemma 3.4.** We have

$$\|F(U^n; j) - D_+ K(U^n_j)\|_{L^\infty(Z)} \leq \|F(U^0; j) - D_+ K(U^0_j)\|_{L^\infty(Z)};$$

$$\|F(U^n; j) - D_+ K(U^n_j)\|_{BV(Z)} \leq \|F(U^0; j) - D_+ K(U^0_j)\|_{BV(Z)}.$$

**Proof.** To make the calculations more transparent and the notation simpler, we are going to write out the proof of this lemma only for the three-point schemes

$$\frac{U^n_{j+1} - U^n_j}{\Delta t} + D_+ (F(U^n_j, U^n_{j+1}) - D_+ K(U^n_j)) = 0.$$  

The proof in the general case (22) is similar to the three-point case, but the notation is messier. Let $F_u$ and $F_v$ denote the partial derivatives of $F = F(u, v)$ with respect to the first argument and the second argument, respectively. A simple calculation will reveal that (28) is monotone provided the following conditions hold:

$$F_u(r_1, r_2) + \frac{1}{\Delta x} k(r_3) \geq 0, \quad \frac{1}{\Delta x} k(r_3) - F_v(r_1, r_2) \geq 0 \quad \forall (r_1, r_2, r_3),$$

$$1 - \frac{\Delta t}{\Delta x} (F_u(r_1, r_2) - F_v(r_3, r_1)) - 2 \frac{\Delta t}{\Delta x^2} k(r_4) \geq 0 \quad \forall (r_1, r_2, r_3, r_4).$$

Note that a sufficient condition for (29) to hold is that $F_u(r_1, r_2) \geq 0$ and $F_v(r_1, r_2) \leq 0$ for all $(r_1, r_2)$. Let us begin with proving (26). To this end, we define the quantity

$$V^n_j = \Delta x \sum_{i=-\infty}^{j} \left( \frac{U^n_i - U^{n-1}_i}{\Delta t} \right).$$

Multiplying the difference equation (28) evaluated at $i \Delta x$ by $\Delta x$ and subsequently summing over $i = -\infty, \ldots, j$, we get the relation

$$V^{n+1}_j = -(F(U^n_j, U^{n+1}_j) - D_+ K(U^n_j)).$$

Next we derive an equation for the quantity $\{V^n_j\}$. For this purpose consider the difference equation (28) evaluated at $i \Delta x$ and subtract the corresponding equation at
time $n\Delta t$, yielding
\[
\frac{(U_i^{n+1} - U_i^n)}{\Delta t} + D_-( (F(U_i^n, U_{i+1}^n) - F(U_i^{n-1}, U_{i+1}^{n-1})) - D_+ (K(U_i^n) - K(U_i^{n-1})) = 0.
\]
Multiplying this equality by $\Delta x$ and then summing over $i = -\infty, \ldots, j$, yields
\[
\frac{(V_j^{n+1} - V_j^n)}{\Delta t} + (F(U^n_j, U_{j+1}^n) - F(U^{n-1}_j, U_{j+1}^{n-1})) - D_+ (K(U^n_j) - K(U^{n-1}_j)) = 0.
\]
Observe that
\[
\frac{U_j^n - U_j^{n-1}}{\Delta t} = \frac{1}{\Delta x} \left[ \Delta x \sum_{i=-\infty}^{j} \left( \frac{U_i^n - U_i^{n-1}}{\Delta t} \right) - \Delta x \sum_{i=-\infty}^{j-1} \left( \frac{U_i^n - U_i^{n-1}}{\Delta t} \right) \right] = D_-V_j^n.
\]
Having this identity in mind, we can write
\[
F(U_j^n, U_{j+1}^n) - F(U_j^{n-1}, U_{j+1}^{n-1})
\]
\[
\equiv (F(U_j^n, U_{j+1}^n) - F(U_j^{n-1}, U_{j+1}^n)) + (F(U_j^{n-1}, U_{j+1}^n) - F(U_j^{n-1}, U_{j+1}^{n-1}))
\]
\[
= F_u(a_{u,j}^n, U_j^{n-1}) (U_j^n - U_j^{n-1}) + F_v(U_j^{n-1}, \tilde{\alpha}_{j+1}^n) (U_j^{n-1} - U_j^{n-1})
\]
\[
= \Delta t a_{u,j}^n D_-V_j^n + \Delta t a_{v,j}^n D_-V_j^{n+1},
\]
where
\[
(a_{u,j}^n = F_u(\alpha_{u,j}^n, U_{j+1}^n), \quad a_{v,j}^n = F_v(U_j^{n-1}, \tilde{\alpha}_{j+1}^n), \quad \alpha_{u,j}^n, \tilde{\alpha}_{j+1}^n \in \text{int}(U_j^{n-1}, U_j^n).
\]
Similarly, we can write
\[
K(U_j^n) - K(U_j^{n-1}) = k(\beta_j^n) (U_j^n - U_j^{n-1}) = \Delta t b_j^n D_-V_j^n,
\]
where
\[
b_j^n = k(\beta_j^n), \quad \beta_j^n \in \text{int}(U_j^{n-1}, U_j^n).
\]
Summing up, we see that the sequence $\{V_j^n\}$ satisfies the linear difference equation
\[
\frac{V_j^{n+1} - V_j^n}{\Delta t} + (a_{u,j}^n D_-V_j^n + a_{v,j}^n D_-V_j^{n+1}) = D_+ (b_j^n D_-V_j^n).
\]
We will now show that the solution of (35) satisfies a maximum principle. To this end, observe that (35) can be written as
\[
V_j^{n+1} = A_j^n V_{j-1}^n + B_j^n V_j^n + C_j^n V_j^{n+1},
\]
where
\[
A_j^n = \left[ \frac{\Delta t}{\Delta x} a_{u,j}^n + \frac{\Delta t}{\Delta x^2} b_j^n \right],
\]
\[
B_j^n = \left[ 1 - \frac{\Delta t}{\Delta x} (a_{u,j}^n - a_{v,j}^n) - \frac{\Delta t}{\Delta x^2} (b_j^n + b_{j+1}^n) \right],
\]
\[
C_j^n = \left[ \frac{\Delta t}{\Delta x} b_{j+1}^n - \frac{\Delta t}{\Delta x} a_{v,j}^n \right].
\]
Since (29) and (30) are assumed to hold,

\[ A^n_j, B^n_j, C^n_j \geq 0, \quad A^n_j + B^n_j + C^n_j \equiv 1. \]

Consequently, we obtain from (36) that

\[ \sup_{j \in \mathbb{Z}} |V_j^{n+1}| \leq \sup_{j \in \mathbb{Z}} |V_j^n| \leq \cdots \leq \sup_{j \in \mathbb{Z}} |V_j^1|. \]

In view of the relation (32), we can immediately conclude that (26) is true.

Next, we prove that the solution of (35) has bounded variation on \( \mathbb{Z} \). Introduce the quantity \( Z_j^n = V_j^n - V_{j-1}^n \) and observe that

\[ \frac{Z_j^{n+1} - Z_j^n}{\Delta t} + D_-(a^n_{u,j} Z_j^n + a^n_{v,j} Z_{j+1}^n) = D_- (b^n_j Z_j^n). \]

Similarly to (36), we can write this equation as

\[ Z_j^{n+1} = \tilde{A}_j^n Z_{j-1}^n + \tilde{B}_j^n Z_j^n + \tilde{C}_j^n Z_{j+1}^n, \]

where

\[
\begin{align*}
\tilde{A}_j^n &= \left[ \frac{\Delta t}{\Delta x} a^n_{u,j-1} + \frac{\Delta t}{\Delta x^2} b^n_{j-1} \right], \\
\tilde{B}_j^n &= \left[ 1 - \frac{\Delta t}{\Delta x} (a^n_{u,j} - a^n_{v,j-1}) - 2 \frac{\Delta t}{\Delta x^2} b^n_j \right], \\
\tilde{C}_j^n &= \left[ \frac{\Delta t}{\Delta x^2} b^n_{j+1} - \frac{\Delta t}{\Delta x} a^n_{v,j} \right].
\end{align*}
\]

Since (29) and (30) are again assumed to hold,

\[ \tilde{A}_j^n, \tilde{B}_j^n, \tilde{C}_j^n \geq 0, \quad \tilde{A}_{j+1}^n + \tilde{B}_j^n + \tilde{C}_{j-1}^n \equiv 1. \]

We can thus derive from (37) that

\[ \sum_{j \in \mathbb{Z}} |Z_j^{n+1}| \leq \sum_{j \in \mathbb{Z}} (\tilde{A}_j^n + \tilde{B}_j^n + \tilde{C}_{j-1}^n) |Z_j^n| \equiv \sum_{j \in \mathbb{Z}} |Z_j^n| \leq \cdots \leq \sum_{j \in \mathbb{Z}} |Z_j^1|, \]

which immediately implies (27). This concludes the proof of the lemma. \( \square \)

A direct consequence of the previous lemma is that the difference approximations are \( L^1 \) Lipschitz continuous in the time variable (and thus in \( BV \) in both space and time).

**Lemma 3.5.** We have

\[ \| U^m - U^n \|_{L^1(\mathbb{Z})} \leq \| F(U^n; j) - D_+ K(U^n_j) \|_{BV(\mathbb{Z})} \frac{\Delta t}{\Delta x} |m - n|. \]

**Proof.** Suppose that \( m > n \). Using (22), we readily calculate that

\[
\sum_{j \in \mathbb{Z}} |U_j^m - U_j^n| \leq \sum_{i=n}^{m-1} \sum_{j \in \mathbb{Z}} |U_{j+1}^i - U_j^i| \leq \Delta t \sum_{i=n}^{m-1} \sum_{j \in \mathbb{Z}} |D_- (F(U^i; j) - D_+ K(U^i_j))| \\
\leq \| F(U^n; j) - D_+ K(U^n_j) \|_{BV(\mathbb{Z})} \frac{\Delta t}{\Delta x} |m - n|,
\]

where the \( BV \) estimate (27) has been used. This concludes the proof of the lemma. \( \square \)
Let us now return to the formal discussion which led to the uniform $L^\infty$ and $BV$ estimates on $\partial_x K(u(\cdot, t))$ in the case of nondegeneracy. As we will see, it is possible to use the $BV$ estimate to derive a result concerning also the continuity of $K(u)$ with respect to the time variable. To this end, we shall employ a technique introduced by Kruzkov [17] to derive a modulus of continuity in time from a known modulus of continuity in space of certain parabolic equations. Let $\phi(x)$ be a test function on $\mathbb{R}$. Multiplying (25) by $\phi$, integrating the result in space, and subsequently doing integrating by parts on one of the terms, yields

$$\left| \int_\mathbb{R} \phi(x) \partial_x v(x,t) \, dx \right| \leq \left( \|a\|_{L^\infty(Q_T)} \|\phi\|_{L^\infty(\mathbb{R})} + \|b\|_{L^\infty(Q_T)} \|\phi'\|_{L^\infty(\mathbb{R})} \right) \int_\mathbb{R} |\partial_x v(x,t)| \, dx.$$

From this estimate we get the following weak continuity result:

$$v \left| \int_\mathbb{R} \phi(x)(v(x,\tau) - v(x,t)) \, dx \right| = \mathcal{O}(1) (\|\phi\|_{L^\infty(\mathbb{R})} + \|\phi'\|_{L^\infty(\mathbb{R})}) |\tau-t|,$$

where we have taken into account that $|v(\cdot, t)|_{BV(\mathbb{R})} \leq |v(\cdot, 0)|_{BV(\mathbb{R})} < \infty$ and the uniform boundedness of $a = a(x,t)$, $b = b(x,t)$. It is not difficult, using a suitable approximation of $\text{sgn}(v(\cdot, \tau) - v(\cdot, t))$ (see the proof of Lemma 3.6 below), to conclude from this that

$$\|v(\cdot, \tau) - v(\cdot, t)\|_{L^1(\mathbb{R})} = \mathcal{O}(1) \sqrt{|\tau-t|}.$$

Now, since $u$ is $L^1$ Lipschitz continuous in the time variable and $v = f(u) - \partial_x K(u)$, it follows that

$$\|\partial_x K(u(\cdot, \tau)) - \partial_x K(u(\cdot, t))\|_{L^1(\mathbb{R})} = \mathcal{O}(1) \sqrt{|\tau-t|}.$$

Observe that

$$K(u(x,t)) = \int_{-\infty}^x \partial_x K(u(\xi,t)) \, d\xi,$$

which implies the desired Hölder result:

$$\|K(u(\cdot, \tau)) - K(u(\cdot, t))\|_{L^\infty(\mathbb{R})} \leq \|\partial_x K(u(\cdot, \tau)) - \partial_x K(u(\cdot, t))\|_{L^1(\mathbb{R})} = \mathcal{O}(1) \sqrt{|\tau-t|}.$$

Again this is merely formalism since the solution of (1) is in general nonsmooth. However, our next lemma states that a Hölder estimate on the discrete diffusion term is indeed true.

**Lemma 3.6.** We have

$$\left| K(U_i^m) - K(U_j^n) \right| \leq C \left( |(i-j)| \Delta x + \sqrt{|m-n|} \Delta t \right). \quad (39)$$

**Proof.** We will write out the proof of this lemma only for three-point schemes, for which the proof is essentially to apply a discrete version of Kruzkov’s technique [17] to the parabolic difference equation (35). The proof in the general case (22) is similar to the three-point case, and it is therefore omitted. First, notice that

$$\left| K(U_i^m) - K(U_j^n) \right| \leq \left| K(U_i^m) - K(U_j^m) \right| + \left| K(U_j^m) - K(U_j^n) \right| =: I_1 + I_2.$$

In view of Lemma 3.4, $\|D_x K(U^m)\|_{L^\infty(\mathbb{Z})} = \mathcal{O}(1)$ and therefore $I_1 = \mathcal{O}(1) |(i-j)| \Delta x$.
Next, we wish to bound $I_2$. To this end, let $\phi(x)$ be a test function, put $\phi_j = \phi(j\Delta x)$, and let $m < n$. Using the difference equation (35) and summation by parts, we get

$$\Delta x \sum_{j \in \mathbb{Z}} \phi_j (V_j^m - V_j^n) = \Delta x \left| \sum_{l=n}^{m-1} \sum_{j \in \mathbb{Z}} \phi_j (V_j^{l+1} - V_j^l) \right|$$

$$\leq \Delta x \Delta t \left| \sum_{l=n}^{m-1} \sum_{j \in \mathbb{Z}} \phi_j (a_{u,j} D_- V_j^l + a_{v,j} D_- V_{j+1}^l) \right| + \Delta x \Delta t \left| \sum_{l=n}^{m-1} \sum_{j \in \mathbb{Z}} D_- \phi_j (b_j D_- V_j^l) \right|$$

$$\leq \left( \| \phi \|_{L^\infty(\mathbb{R})} (\sup_{j,l} |a_{u,j}| + \sup_{j,l} |a_{v,j}|) + \| \phi' \|_{L^\infty(\mathbb{R})} \sup_{j,l} |b_j| \right) \sup \| V^l \|_{BV(\mathbb{Z})} \Delta t (m-n)$$

$$= \mathcal{O}(1) \left( \| \phi \|_{L^\infty(\mathbb{R})} + \| \phi' \|_{L^\infty(\mathbb{R})} \right) \Delta t (m-n),$$

since $a_{u,j}, a_{v,j}$ and $|V^l|_{BV(\mathbb{Z})}$ are all uniformly bounded quantities. Next, introduce the function

$$\beta(x) = \begin{cases} 
\text{sgn} \left( \sum_{j \in \mathbb{Z}} (V_j^m - V_j^n) \chi_j(x) \right) & \text{for } |x| \leq J - \rho, \\
0 & \text{for } |x| > J - \rho,
\end{cases}$$

where $\chi_j$ denotes the characteristic function of $[j\Delta x, (j+1)\Delta x)$ and $J \in \mathbb{Z}$. Let $\omega_\rho(x)$ be a standard $C_C^\infty$-mollifier given by $\omega_\rho(x) = \frac{1}{\rho} \omega(\frac{x}{\rho})$, where

$$\omega(x) \in C_C^\infty(\mathbb{R}), \quad \omega(x) \geq 0, \quad \omega(x) = 0 \quad \text{for } |x| \geq 1, \quad \int_\mathbb{R} \omega(x) dx = 1.$$
where we have used that $|E|_{BV(\mathbb{Z})} < \infty$. Next, using (40) with $\phi_j = \beta^n(j \Delta x)$, we have

$$Q_2 = \mathcal{O}(1)(m - n)\Delta t / \rho.$$ 

Hence, for some constants $C_1$ and $C_2$ not depending on $\Delta x$, $\Delta t$, it follows that

$$\Delta x \sum_{j=-J}^J |V_j^m - V_j^n| \leq C_1 \rho + C_2 (m - n)\Delta t / \rho.$$ 

Choosing $\rho = \sqrt{(m - n)\Delta t}$ and letting $J \to \infty$, we obtain

$$\Delta x \sum_{j \in \mathbb{Z}} |V_j^m - V_j^n| = \mathcal{O}(1) \sqrt{(m - n)\Delta t}.$$ 

On the other hand, from the relation (32) and Lemma 3.5, we also have

$$\Delta x \sum_{j \in \mathbb{Z}} |V_j^m - V_j^n| = \mathcal{O}(1) \Delta x \sum_{j \in \mathbb{Z}} |U_j^m - U_j^n| + \Delta x \sum_{j \in \mathbb{Z}} |D_+ K(U_j^m) - D_+ K(U_j^n)|$$

$$= \mathcal{O}(1)(m - n)\Delta t + \Delta x \sum_{j \in \mathbb{Z}} |D_+ K(U_j^m) - D_+ K(U_j^n)|.$$ 

We thus conclude that

$$\Delta x \sum_{j \in \mathbb{Z}} |D_+ K(U_j^m) - D_+ K(U_j^n)| = \mathcal{O}(1) \sqrt{(m - n)\Delta t}.$$ 

From this estimate the desired Hölder estimate in time follows:

$$I_2 = |K(U_j^m) - K(U_j^n)| = \Delta x \left| \sum_{i=-\infty}^j D_+ K(U_i^m) - \sum_{i=-\infty}^j D_+ K(U_i^n) \right|$$

$$\leq \Delta x \sum_{i \in \mathbb{Z}} |D_+ K(U_i^m) - D_+ K(U_i^n)| = \mathcal{O}(1) \sqrt{(m - n)\Delta t}.$$ 

This concludes the proof of (39). $\Box$

In what follows, we use the standard notations $u \vee v = \max(u, v)$ and $u \wedge v = \min(u, v)$.

**Lemma 3.7.** The following cell entropy inequality holds

$$\frac{|U_j^{n+1} - c| - |U_j^n - c|}{\Delta t} + D_-(F(U^n \vee c; j) - F(U^n \wedge c; j) - D_+ |K(U_j^n) - K(c)|) \leq 0.$$ 

**Proof.** Crandall and Majda [7] showed how to naturally get a cell entropy inequality in the purely hyperbolic case; see also [14]. As we will see, this construction applies to the mixed hyperbolic-parabolic case as well. First, a direct calculation yields the equality

$$|U_j^n - c| - \Delta t D_-(F(U^n \vee c; j) - F(U^n \wedge c; j) - D_+ |K(U_j^n) - K(c)|)$$

$$= S(U^n \vee c; j) - S(U^n \wedge c; j),$$

(41)
where $S$ is defined by (23). Next, by monotonicity of the scheme (22),
\[
S(U^n \lor c, j) - S(U^n \land c, j) \geq S(U^n, j) \lor c - S(U^n, j) \land c = |U^n_{j+1} - c|,
\]
which inserted into (41) produces the desired cell entropy inequality. \□

Let $u_\Delta$ (where $\Delta = (\Delta x, \Delta t)$) be the interpolant of degree one associated with the discrete data points $\{U^n_j\}$; i.e., $u_\Delta$ interpolates at the vertices of each rectangle
\[
R^n_j = [j\Delta x, (j + 1)\Delta x] \times [n\Delta t, (n + 1)\Delta t].
\]

Note that $u_\Delta$ is continuous everywhere, differentiable almost everywhere, and inside each rectangle $R^n_j$ it is explicitly given by the formula
\[
\begin{aligned}
  u_\Delta(x, t) &= U^n_j + (U^n_{j+1} - U^n_j) \left( \frac{x - j\Delta x}{\Delta x} \right) + (U^n_{j+1} - U^n_j) \left( \frac{t - n\Delta t}{\Delta t} \right) \\
  &\quad+ (U^n_{j+1} - U^n_{j+1} - U^n_j) \left( \frac{x - j\Delta x}{\Delta x} \right) \left( \frac{t - n\Delta t}{\Delta t} \right).
\end{aligned}
\]

We have the following compactness results.

**Lemma 3.8.** Let $\{\Delta_j\}$ be a sequence of discretization parameters tending to zero. Then there exists a subsequence $\{\Delta_j\}$ such that $\{u_{\Delta_j}\}$ converges in $L^1_{\text{loc}}(Q_T)$ and pointwise almost everywhere in $Q_T$ to a limit $u$ as $j \to \infty$,
\[
u \in L^\infty(Q_T) \cap BV(Q_T).
\]

Furthermore, $\{K(u_{\Delta_j})\}$ converges uniformly on compact sets $K \subset Q_T$ to $K(u)$ as $j \to \infty$,
\[
K(u) \in C^{1,\frac{1}{2}}(\overline{Q_T}).
\]

**Proof.** From (42) and Lemma 3.1, we get that $u_{\Delta}$ is uniformly bounded by $\|u_0\|_{L^\infty(\mathbb{R})}$. Using Lemma 3.3, we get that
\[
\begin{aligned}
\int_Q \int_{R^n_j} |\partial_x u_{\Delta}| \, dt \, dx &\leq \sum_{j,n} \int_{R^n_j} \frac{1}{\Delta x} \left( 1 - \frac{t - n\Delta t}{\Delta t} \right) |U^n_{j+1} - U^n_j| \, dt \, dx \\
&\quad+ \sum_{j,n} \int_{R^n_j} \frac{1}{\Delta x} \left( \frac{t - n\Delta t}{\Delta t} \right) |U^n_{j+1} - U^n_{j+1}| \, dt \, dx \\
&\leq \frac{\Delta t}{2} \sum_{n,j} |U^n_{j+1} - U^n_j| + \frac{\Delta x}{2} \sum_{n,j} |U^n_{j+1} - U^n_{j+1}| \leq T|U^0|_{BV(\mathbb{Z})}.
\end{aligned}
\]

Similarly, from (42) and Lemma 3.5, we also obtain that
\[
\begin{aligned}
\int_Q \int_{R^n_j} |\partial_t u_{\Delta}| \, dt \, dx &\leq \sum_{j,n} \int_{R^n_j} \frac{1}{\Delta t} \left( 1 - \frac{x - j\Delta x}{\Delta x} \right) |U^n_{j+1} - U^n_j| \, dt \, dx \\
&\quad+ \sum_{j,n} \int_{R^n_j} \frac{1}{\Delta t} \left( \frac{x - j\Delta x}{\Delta x} \right) |U^n_{j+1} - U^n_{j+1}| \, dt \, dx \\
&\leq \frac{\Delta x}{2} \sum_{n,j} |U^n_{j+1} - U^n_j| + \frac{\Delta x}{2} \sum_{n,j} |U^n_{j+1} - U^n_{j+1}| \\
&\leq T|F(U^0, j) - D+K(U^0)|_{BV(\mathbb{Z})}.
\end{aligned}
\]
Consequently, there is a finite constant $C = C(T) > 0$ (independent of $\Delta$) such that
\begin{equation}
\|u_\Delta\|_{L^\infty(Q_T)} \leq C, \quad |u_\Delta|_{BV(Q_T)} \leq C.
\end{equation}

These estimates show that $\{u_\Delta\}$ is bounded in $BV(K)$ for any compact set $K \subset Q_T$. Since $BV(K)$ is compactly imbedded into the space $L^1(K)$, it is possible to select a subsequence that converges in $L^1(K)$ and pointwise almost everywhere in $K$.

Furthermore, using a standard diagonal process, we can construct a sequence that converges in $L^1_{loc}(Q_T)$ and pointwise almost everywhere in $Q_T$ to a limit $u$,

$$u \in L^\infty(Q_T) \cap BV(Q_T).$$

Next, we analyze the sequence $\{K(u_\Delta)\}$. To this end, let $w_\Delta = K(u_\Delta)$. Thanks to (42), (24), and (39), $w_\Delta$ is continuous everywhere, uniformly bounded and satisfies
\begin{equation}
|w_\Delta(i \Delta x, m \Delta t) - w_\Delta(j \Delta x, n \Delta t)| = O(1) \left( |(i - j) \Delta x| + \sqrt{|(m - n) \Delta t|} \right).
\end{equation}

Let $(x, t)$ and $(y, \tau)$ be some given coordinates and choose integers $(j, n)$ and $(i, m)$ such that $(x, t) \in R^m_j$ and $(y, \tau) \in R^n_i$. (Here $R^m_j$ and $R^n_i$ may coincide.) Then
\begin{align*}
|w_\Delta(y, \tau) - w_\Delta(x, t)| &\leq |w_\Delta(y, \tau) - w_\Delta(i \Delta x, m \Delta t)| + |w_\Delta(i \Delta x, m \Delta t) - w_\Delta(j \Delta x, n \Delta t)| \\
&= |w_\Delta(j \Delta x, n \Delta t) - w_\Delta(x, t)| =: I_1 + I_2 + I_3.
\end{align*}

From (44) we know that $I_2 = O(1) \left( |(i - j) \Delta x| + \sqrt{|(m - n) \Delta t|} \right)$. Since the interpolant $u_\Delta$ does not introduce new extrema and $K(u)$ is nondecreasing in $u$, we get $I_1 + I_3 = O(1) \left( \Delta x + \sqrt{\Delta t} \right)$. Consequently, we have arrived at
\begin{equation}
|w_\Delta(y, \tau) - w_\Delta(x, t)| \leq C \left( |y - x| + \sqrt{\tau - t} + \Delta x + \sqrt{\Delta t} \right),
\end{equation}
where $C > 0$ is a finite constant not depending on $\Delta, x, y, t, \tau$.

Now, by repeating the proof of the Ascoli–Arzelà compactness theorem, we deduce the existence of a subsequence of $\{w_\Delta\}$ converging uniformly on each compactum $K \subset Q_T$ to a limit $w$,

$$w \in C^{1, \frac{1}{2}}(\overline{Q_T}).$$

Let $\{\Delta_j\}$ be a sequence of discretization parameters tending to zero such that $u_{\Delta_j} \to u$ and $w_{\Delta_j} \to w$ as $j \to \infty$ (such a sequence can certainly be found in view of the previous discussion). Since $u_{\Delta_j}$ converges pointwise almost everywhere to $u$ and $w$ is continuous,

$$w = K(u).$$

This concludes the proof of the lemma. \hfill \Box

In view of Lemma 3.8 and Theorem 2.3, we can assume that the sequences $\{u_\Delta\}$ and $\{K(u_\Delta)\}$ themselves converge to $u$ and $K(u)$, respectively. We continue by showing that the limit $u$ satisfies the entropy condition (10). Let $\phi(x, t)$ be a suitable test function and put $\phi^n_j = \phi(j \Delta x, n \Delta t)$. Multiplying the cell entropy inequality in Lemma 3.7 by $\phi^n_j \Delta x$, summing over all $j, n$ and applying summation by parts, we get
\begin{align}
\Delta x \Delta t \sum_{j \in \mathbb{Z}} \sum_{n=0}^{N-1} \left( U^{n+1}_j - c \right) \phi^n_{j+1} - \phi^n_j \\
+ (F(U^n \vee c; j) - F(U^n \wedge c; j)) D_+ \phi^n_j \\
+ |K(U^n_j) - K(c)| D_- \phi^n_j \\
+ \Delta x \sum_{j \in \mathbb{Z}} |U^n_j - c| \phi^n_j \geq 0.
\end{align}
Using (24) and that $F$ is consistent with $f$, we can obviously write
\[
\Delta x \sum_{j \in \mathbb{Z}} (F(U^n \vee c; j) - F(U^n \wedge c; j)) D_+ \phi^n_j = \Delta x \sum_{j \in \mathbb{Z}} \text{sgn}(U^n_j - c)(f(U^n_j) - f(c)) D_+ \phi^n_j + O(\Delta x),
\]
and hence replace (45) by
\[
\int_Q \left( |u_\Delta - c| \partial_t \phi + \text{sgn}(u_\Delta - c)(f(u_\Delta) - f(c)) \partial_x \phi + |K(u_\Delta) - K(c)| \partial_x^2 \phi \right) dt \, dx
\]
\[
+ \int_{\mathbb{R}} |u_0 - c| \phi(x,0) \, dx \geq -C(\Delta x + \Delta t),
\]
where $C = C(T) > 0$ is a constant not depending on $\Delta$. Finally, after passing to the limit in (46) and then doing integration by parts, we get that the limit $u$ satisfies (10). This completes our discussion when $u_0$ has compact support and $f, K$ are $C^1$ locally.

For $u_0 \in B(f, K)$ not necessarily compactly supported and $f, K$ merely locally Lipschitz continuous, we approximate $u_0$ by a compactly supported function $u_0^p$ and $f, k$ by a smoother function $f^p, k^p$, compute the difference approximation of the resulting problem and then let $p \to \infty$ and $\Delta t, \Delta x \to 0$ (see [7] for more details in the hyperbolic case).

Remark. We can also do integration by parts in (46) (before passing to the limit), so that $I(\Delta) := \int_Q |K(u_\Delta) - K(c)| \partial_x^2 \phi \, dt \, dx$ becomes $-\int_Q \text{sgn}(u_\Delta - c) \partial_x K(u_\Delta) \partial_x \phi \, dt \, dx$. Since $\partial_x K(u_\Delta) \rightarrow \partial_x K(u)$ in $L^\infty(Q_T)$,
\[
\lim_{\Delta \to 0} I(\Delta) = -\int_Q \text{sgn}(u - c) \partial_x K(u) \, dt \, dx.
\]

We are now ready to state our main result.

Theorem 3.9. Suppose that $f, K$ are locally Lipschitz continuous and let $u_0 \in B(f, K)$ (see (21)). Then the (whole) sequence $\{u_\Delta\}$, which is built from (22) and (42), converges in $L^1_{\text{loc}}(Q_T)$ and pointwise almost everywhere in $Q_T$ to a BV entropy weak solution of the the initial value problem
\[
\partial_t u + \partial_x f(u) = \partial_x (k(u) \partial_x u), \quad u(x,0) = u_0(x), \quad (x,t) \in Q_T, \quad k(u) \geq 0.
\]
Furthermore, the sequence $\{K(u_\Delta)\}$ converges uniformly on compacta $K \subset Q_T$ to $K(u)$.

We let $C(0,T;L^1(\mathbb{R}))$ denote the usual Bochner space consisting of all continuous functions $u : [0,T] \to L^1(\mathbb{R})$ for which the norm $\|u\|_{C(0,T;L^1(\mathbb{R}))} = \sup_{t \in [0,T]} \|u(t)\|_{L^1(\mathbb{R})}$ is finite. A closer inspection of the arguments leading to Theorem 3.9 will reveal that $\{U_\Delta(t)\}$ converges in $C(0,T;L^1_{\text{loc}}(\mathbb{R}))$ to the unique BV entropy weak solution $u(t)$, with $u(0) = u_0$, of the initial value problem (1). A reexamination of the proofs leading to Theorem 3.9 shows that we have proved the following result on existence and properties of solutions of (1).

Corollary 3.10. Let $f$ and $K$ be locally Lipschitz continuous. Then for any initial function $u_0 \in B(f, K)$ (see (21)) there exists a BV entropy weak solution $u \in C(0,T;L^1_{\text{loc}}(\mathbb{R}))$ of the initial value problem (1). Denoting this solution by $S_t u_0$, we have the following properties:
Then $k$ nondegenerate diffusion coefficients $\{\}$

We can again use the Ascoli–Arzelà theorem to produce a subsequence $\{\}$ satisfies an entropy condition. To this end, let us consider the multidimensional problem of the form (41) and 3.6, there is a constant $C > 0$. In view of the formal discussion before Lemmas 3.4 and 3.6, there is a constant $C > 0$, which is independent of $\varepsilon$, such that

$$|K(u, \tau) - K(u, \tau)| \leq C(|y - x| + \sqrt{\tau - t} + \varepsilon).$$

We can again use the Ascoli–Arzelà theorem to produce a subsequence $\{K(u^j)\}$ which converges uniformly on compact sets $\mathcal{K} \subset \mathcal{Q}_T$ to $K(u) \in C^{1, \frac{1}{2}}(\mathcal{Q}_T)$ as $j \to \infty$. The fact that $K(u)$ is Lipschitz continuous in the space variable was first proved by Tassa [28]. This regularity is optimal as demonstrated by an example due to Barenblatt and Zeldovich [1]; see [28] for more details. We have taken the (continuous) analysis in [28] a step further by showing that $K(u)$ is Hölder continuous in the time variable. A direct consequence is that the viscosity solution of (1) is also a solution of (1) in the sense of Definition 2.1.

Summing up, we have proven the following theorem, which generalizes the regularity result of Tassa [28].

**Theorem 3.11 (Viscosity solutions).** Let $u$ denote the viscosity solution of (1). Then $K(u)$ is contained in the Hölder space $C^{1, \frac{1}{2}}(\mathcal{Q}_T)$.


In this section we briefly discuss the multidimensional case. The main problem is that a general uniqueness result for multidimensional strongly degenerate convection-diffusion equations is lacking. Nevertheless, it is at least possible to show that a subsequence of monotone difference approximations must converge to a $L^\infty(\mathcal{Q}_T) \cap BV(\mathcal{Q}_T)$ function which satisfies an entropy condition. To this end, let us consider the multidimensional problem

$$\begin{align*}
\partial_t u + \nabla \cdot f(u) &= \Delta K(u), \\
u(x, 0) &= u_0(x),
\end{align*}$$

where $K(u)$ is the solution operators associated with the two equations

$$\begin{align*}
\partial_t u + \partial_x f_1(u) &= \partial_x(k_1(u)\partial_x u), \\
\partial_t u + \partial_x f_2(u) &= \partial_x(k_2(u)\partial_x u),
\end{align*}$$

respectively, then the following comparison result holds (see Theorem 2.4 and [5]):

$$\begin{align*}
\|S_1 u_0 - S_2 u_0\|_{L^1(\mathcal{K})} &\leq C \left( t \|f_1 - f_2\|_{L^\infty(\mathcal{K})} + 4\sqrt{t} \|\sqrt{f_1} - \sqrt{f_2}\|_{L^\infty(\mathcal{K})} \right),
\end{align*}$$

where $m \leq u_0 \leq M$ and $C = \|u_0\|_{BV(\mathcal{K})}$.

Finally, we will make a remark concerning the viscosity solution of (1). For any $\varepsilon > 0$, let $u_\varepsilon(x, t)$ denote the classical solution of the parabolic problem (1) with a nondegenerate diffusion coefficient $k_\varepsilon(u) = k(u) + \varepsilon$. Moreover, let

$$u(x, t) = \lim_{\varepsilon \to 0} u_\varepsilon(x, t)$$

denote the viscosity solution of the strongly degenerate problem (1); see [30]. In view of the formal discussion before Lemmas 3.4 and 3.6, there is a constant $C > 0$, which is independent of $\varepsilon$, such that

$$|K(u_\varepsilon(y, \tau)) - K(u_\varepsilon(x, t))| \leq C(|y - x| + \sqrt{\tau - t} + \varepsilon).$$

We can again use the Ascoli–Arzelà theorem to produce a subsequence $\{K(u^j)\}$ which converges uniformly on compact sets $\mathcal{K} \subset \mathcal{Q}_T$ to $K(u) \in C^{1, \frac{1}{2}}(\mathcal{Q}_T)$ as $j \to \infty$. The fact that $K(u)$ is Lipschitz continuous in the space variable was first proved by Tassa [28]. This regularity is optimal as demonstrated by an example due to Barenblatt and Zeldovich [1]; see [28] for more details. We have taken the (continuous) analysis in [28] a step further by showing that $K(u)$ is Hölder continuous in the time variable. A direct consequence is that the viscosity solution of (1) is also a solution of (1) in the sense of Definition 2.1.

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$$\begin{align*}
\partial_t u + \nabla \cdot f(u) &= \Delta K(u), \\
u(x, 0) &= u_0(x),
\end{align*}$$

where $K(u)$ is the solution operators associated with the two equations

$$\begin{align*}
\partial_t u + \partial_x f_1(u) &= \partial_x(k_1(u)\partial_x u), \\
\partial_t u + \partial_x f_2(u) &= \partial_x(k_2(u)\partial_x u),
\end{align*}$$

respectively, then the following comparison result holds (see Theorem 2.4 and [5]):

$$\begin{align*}
\|S_1 u_0 - S_2 u_0\|_{L^1(\mathcal{K})} &\leq C \left( t \|f_1 - f_2\|_{L^\infty(\mathcal{K})} + 4\sqrt{t} \|\sqrt{f_1} - \sqrt{f_2}\|_{L^\infty(\mathcal{K})} \right),
\end{align*}$$

where $m \leq u_0 \leq M$ and $C = \|u_0\|_{BV(\mathcal{K})}$.

Finally, we will make a remark concerning the viscosity solution of (1). For any $\varepsilon > 0$, let $u_\varepsilon(x, t)$ denote the classical solution of the parabolic problem (1) with a nondegenerate diffusion coefficient $k_\varepsilon(u) = k(u) + \varepsilon$. Moreover, let

$$u(x, t) = \lim_{\varepsilon \to 0} u_\varepsilon(x, t)$$

denote the viscosity solution of the strongly degenerate problem (1); see [30]. In view of the formal discussion before Lemmas 3.4 and 3.6, there is a constant $C > 0$, which is independent of $\varepsilon$, such that

$$|K(u_\varepsilon(y, \tau)) - K(u_\varepsilon(x, t))| \leq C(|y - x| + \sqrt{\tau - t} + \varepsilon).$$

We can again use the Ascoli–Arzelà theorem to produce a subsequence $\{K(u^j)\}$ which converges uniformly on compact sets $\mathcal{K} \subset \mathcal{Q}_T$ to $K(u) \in C^{1, \frac{1}{2}}(\mathcal{Q}_T)$ as $j \to \infty$. The fact that $K(u)$ is Lipschitz continuous in the space variable was first proved by Tassa [28]. This regularity is optimal as demonstrated by an example due to Barenblatt and Zeldovich [1]; see [28] for more details. We have taken the (continuous) analysis in [28] a step further by showing that $K(u)$ is Hölder continuous in the time variable. A direct consequence is that the viscosity solution of (1) is also a solution of (1) in the sense of Definition 2.1.

Summing up, we have proven the following theorem, which generalizes the regularity result of Tassa [28].

**Theorem 3.11 (Viscosity solutions).** Let $u$ denote the viscosity solution of (1). Then $K(u)$ is contained in the Hölder space $C^{1, \frac{1}{2}}(\mathcal{Q}_T)$.
where \( d \geq 1 \). A bounded measurable function \( u(x,t) \) is said to be an entropy weak solution of (47) if for all nonnegative \( \phi \in C^\infty_0(Q_T) \) with \( \phi|_{t=T} = 0 \) and any \( c \in \mathbb{R} \),

\[
\iint_{Q_T} \left( |u - c| \partial_t \phi + \text{sgn}(u - c)(f(u) - f(c)) \cdot \nabla \phi + |K(u) - K(c)| \Delta \phi \right) dt \, dx + \int_{\mathbb{R}^d} |u_0 - c| \phi(x,0) \, dx \geq 0.
\]

Remark. Observe that in the multidimensional case it is not possible to use the regularity \( u(x,t) \in BV(Q_T) \) to conclude that each \( \partial_x jK(u) \) is a finite measure on \( Q_T \), although \( \Delta K(u) \) is. This fact prevents one from deriving the analogue of Theorem 2.2 (the jump conditions), and thus Theorem 2.3 (\( L^1 \) stability), in the multidimensional case, even though the one-dimensional \( BV_x \) theory [31] can be easily extended to the multidimensional case.

The associated multidimensional difference equation can again be written in the form (see [7] for further (similar) details)

\[
U^{n+1} = S(U^n),
\]

where \( S \) is a monotone operator under a suitable CFL condition. Hence, in view of Lemma 3.1 and 3.2, we can again conclude that Lemma 3.3 holds. In particular, it follows that the approximate solution is of bounded variation in the space variable.

Next, we want to conclude that our approximation is \( L^1 \) Lipschitz in the time variable. Recall that in one dimension this property is a direct consequence of the regularity estimate of the discrete total flux of the difference scheme (Lemma 3.4). Such an estimate is not available for the multidimensional case. However, Lemma 3.1 ensures that (48) is \( L^1 \) contractive, i.e.,

\[
\|S(U) - S(V)\|_{L^1(\mathbb{Z}^d)} \leq \|U - V\|_{L^1(\mathbb{Z}^d)}.
\]

Following [7], this is sufficient to guarantee that the monotone difference approximation is \( L^1 \) Lipschitz continuous in the time variable. This can be seen from the following calculation (see [7] for further (similar) details):

\[
\|U^m - U^n\|_{L^1(\mathbb{Z}^d)} = \|S(U^{m-1}) - S(U^{n-1})\|_{L^1(\mathbb{Z}^d)}
= \|S^{m-1}(U^0) - S^{n-1}(U^0)\|_{L^1(\mathbb{Z}^d)}
\leq \sum_{i=1}^{n-m} \|S^{n-(i+1)}(SU^0) - S^{n-(i+1)}(U^0)\|_{L^1(\mathbb{Z}^d)}
\leq \|S(U^0) - U^0\|_{L^1(\mathbb{Z}^d)} |m - n| = \mathcal{O}(1)|m - n|.
\]

Moreover, using the monotonicity, it is not difficult to show that our scheme satisfies a multidimensional version of the cell entropy inequality of Lemma 3.7. Hence we conclude with the following theorem.

**Theorem 4.1.** There exists a subsequence of monotone difference approximations which converges to a limit function \( u \in L^\infty(Q_T) \cap BV(Q_T) \). Furthermore, the limit function \( u(x,t) \) is an entropy weak solution of the multidimensional initial value problem (47).

Remark. Note that in the one-dimensional case the estimate (49) together with the finite difference scheme implies that

\[
\|F(U^n;j) - D_+K(U^n;j)\|_{BV(\mathbb{Z})} \leq \|F(U^0;j) - D_+K(U^0;j)\|_{BV(\mathbb{Z})}.
\]
This approach thus yields an alternative way to show estimate (27) without deriving an equation for the total flux. It is easy to see that then we also have the estimate
\[ \left\| F(U^n_j) - D_j K(U^n_j) \right\|_{L^\infty(\Omega)} \leq \left\| F(U^0_j) - D_j K(U^0_j) \right\|_{BV(\Omega)}, \]
which implies that the discrete diffusion term is Lipschitz continuous in the space variable. Note that this estimate is not as sharp as (26). Finally, recall that we in section 3 proved that the discrete diffusion term was Hölder continuous in time by means of the linear equation for the total flux; see Lemma 3.6 and its proof. In the present situation it is not obvious how to get this continuity in time.

In closing, we should emphasize that this paper is intended as a preliminary theoretical thrust at the numerical approximation of nonclassical solutions of degenerate parabolic equations, and we utilize discrete approximations which are “too crude” for practical applications. Having said this, we are currently looking into the issue of devising higher-order difference schemes for degenerate parabolic equations. In addition, we are currently analyzing split and unsplit finite difference schemes for degenerate equations with source terms.

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REFERENCES


