SEMI-GODUNOV SCHEMES FOR MULTIPHASE FLOWS IN POROUS MEDIA

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Abstract. We describe a class of finite volume schemes for $2 \times 2$ systems of conservation laws based on a "local" decomposition of the system into a series of single conservation laws but with discontinuous coefficients. The resulting schemes are based on Godunov type solvers of the reduced equations. These schemes are very easy to implement since they do not use detailed information about the eigenstructure of the full system. We illustrate the efficiency of the schemes on a variety of numerical experiments focusing on three-phase flows in porous media and show that they are robust and approximate the flow very well, even in the presence of gravity.

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1. INTRODUCTION

We are interested in systems of conservation laws in one space dimension of the form

$$
\begin{aligned}
&u_t + f(u, v)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\
v_t + g(u, v)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\
(u, v)(x, 0) = (u_0(x), v_0(x)), & x \in \mathbb{R},
\end{aligned}
$$

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where \( u, v \) are the unknowns, whereas the initial values \( u_0, v_0 \) and the flux functions \( f, g \) are prescribed continuous differentiable functions. Such \( 2 \times 2 \) systems arise in a wide variety of applications in physics and engineering. The most frequently cited ones include the shallow-water system of geophysics, the \( p \)-system modeling isentropic gas dynamics, and systems modeling traffic flow. For detailed applications of systems of the type (1.1), refer to [13]. In this paper, we are interested in \( 2 \times 2 \) systems that arise when modeling three-phase flows in porous media.

Using vector notation, (1.1) reads

\[
U_t + F(U)_x = 0 \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+,
\]

with \( U = (u, v) \) and \( F(U) = (f, g) \). The Jacobian matrix for (1.1) reads

\[
A = \partial F = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}.
\]

Let us denote the eigenvalues of the above matrix by \( \lambda_1 \) and \( \lambda_2 \). The system is hyperbolic if both eigenvalues are real and is strictly hyperbolic if both eigenvalues are real as well as distinct. If the eigenvalues fail to be real in some parts of the phase space, then the system is of the mixed type and contains elliptic regions.

It is well known that in general, the solutions of (1.2) can contain discontinuities known as shock waves and have to be interpreted in the weak sense, i.e., a weak solution to (1.2) is a pair of functions \( U = (u, v) \) such that

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} (U \varphi_t + F(U) \varphi_x) dx dt + \int_{\mathbb{R}} U(x, 0) \varphi(x, 0) dx = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+_0).
\]

There is a comprehensive theory regarding systems that are strictly hyperbolic, but there are no general results for non-strictly hyperbolic systems, nor for systems of mixed (hyperbolic/elliptic) type.

Our main aim in this paper is to present a numerical method for non-strictly hyperbolic systems of the form (1.1). Such systems arise when modeling three-phase flows in porous media, and details of the model are given in section 2. In this case, the system fails to be strictly hyperbolic and can have coinciding eigenvalues. Furthermore, the characteristic speeds need not even be real. This leads to very complicated solutions compared to strictly hyperbolic problems. Therefore it is difficult to use many standard numerical methods designed for strictly hyperbolic problems. This is particularly so for methods based on the (approximate) solution of Riemann problems.

Most standard numerical methods for (1.1) are of the finite volume type and involve discrete versions of the conservation laws on control volumes. It is standard to use the Godunov type numerical fluxes, i.e., the approximate flux functions are based on solving Riemann problems either exactly or approximately at the cell interfaces. Such schemes resolve the solution well. But, they involve solving Riemann problems for the equation (1.1). Therefore such methods are not well suited to non-strictly hyperbolic systems and systems of the mixed type. To the best of our knowledge, there is no known solution of the Riemann problem for the three phase flows in porous media. If such a solution exists, it is likely to be very complicated and consists of many different waves. Also, the complicated algebraic expressions for the flux functions make the construction of a Roe matrix very difficult. More details about both Godunov type can be found in [13].
An alternative to the above types of fluxes are the so-called semi-Godunov schemes, see [9, 10]. These schemes are based on a “local” reduction of the system (1.1) into a series of single conservation laws with discontinuous coefficients by treating the unknown $v$ as a coefficient in the evolution of $u$ (locally in time) and vice versa. This leads to single conservation laws of the form,
\[
\begin{align*}
\frac{w_t + h(k(x, t), w)_x}{w(x, 0) = w_0(x),} \quad & (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
& x \in \mathbb{R},
\end{align*}
\]
where $k(x, t)$ is a discontinuous coefficient in both space and time. Equations of the above form (1.3) have been widely studied in recent years. A very incomplete list includes [1, 3, 6, 11, 12, 15, 17, 18]. We use Godunov type- schemes developed for equations of the type (1.3) in order to update (1.1) locally in time.

As in [9, 10], these schemes are termed semi-Godunov schemes as they are based on Riemann solvers for the single conservation laws with discontinuous coefficients rather than Riemann solutions of the full $2 \times 2$ system (1.1). They can also be interpreted as HLL solvers with speeds and wave-strengths derived from the solvers for (1.3). These schemes are as simple to implement as central schemes for (1.1) and have the same order of resolution as Godunov type of schemes. We present the efficiency of these schemes for simulating flows in porous media in a series of numerical experiments. In [9, 10], we developed semi-Godunov schemes for a special case of (1.1) when the system is triangular i.e $f(u, v) = f(u)$. In this special case, we were able to exploit the triangular structure of the system and the fact that the evolution of $u$ did not depend on $v$. This led to a very natural decoupling of the system where $u$ was solved independently of $v$ and acted as a parameter in the evolution of $v$. We extend the approach of [9, 10] to the full system (1.1) in this paper. For the full system (1.1), the evolution of $u$ depends on $v$ and vice versa. Hence, there is no “natural” decoupling of the equations. Even then, we are able to decouple the equations “locally” in time and obtain suitable numerical schemes based on a reduction to conservation laws with discontinuous coefficients. There are some new issues which come up due to the structure of the system and they are addressed in section 3.

This paper is organized as follows: in section 2, we present the three-phase flow model in some detail and outline some of the properties of the system. The semi-Godunov schemes are described in section 3 and some of their properties are highlighted. We conclude with a detailed set of numerical experiments in section 4.

2. Three-phase flow models

Simulation of a variety of oil recovery processes involve models of three-phase flow in porous media. Often the three-phases of interest are oil, gas, and water moving in a porous rock. Examples include primary production below bubble point with moving water, water flooding in the presence of free gas, gas flooding and water-alternating gas injections. These models also describe many problems of environmental interest like contamination of the phreatic zone by non-aqueous phase liquids, and geological carbon dioxide sequestration. As a model we consider incompressible, immiscible three-phase flow in a one-dimensional homogeneous and isotropic reservoir (see, e.g., [4]). The oil, water, and gas saturations are denoted by $S_o$, $S_w$ and $S_g$ respectively.
We present the models in a one dimensional setting, where \( x \) is a spatial coordinate and \( t \) denotes time. The mass conservation equation for phase \( l = w, o, g \) reads
\[
\phi(S_l)_t + (U_l)_x = 0,
\]
where \( \phi \) is the porosity of the medium and \( U_l \) is the Darcy velocity or flow rate corresponding to each phase \( l = w, o, g \). By Darcy’s law, the flow rate is given by
\[
U_l = -k \lambda_l \left( \frac{\partial P_l}{\partial x} - G \rho_l \right) \quad l = w, o, g,
\]
where \( k \) denotes the absolute permeability of the medium, \( \lambda_l \) is the mobility (relative permeability divided by viscosity) of phase \( l \), \( P_l \) is the pressure of phase \( l \), and \( G \) is proportional to the gravitational constant. We assume that the flow is incompressible i.e., the total flow rate
\[
q = \sum_{l=w,o,g} U_l
\]
is a constant. For the sake of simplicity, we assume that the differences in the capillary pressures between the phases are zero. This assumption is reasonable when the total flow rate is high (the flow is convection dominated).

By adding the mass conservation equations (2.1) and using the above assumptions, we arrive at the following \( 2 \times 2 \) system of conservation laws:
\[
\begin{align*}
(S_g)_t + (F_g(S_g, S_w, S_o))_x &= 0, \\
(S_w)_t + (F_w(S_g, S_w, S_o))_x &= 0, \\
S_g + S_w + S_o &= 1.
\end{align*}
\]
(2.2)
where the fluxes are given by,
\[
F_g(S_g, S_w, S_o) = q \frac{\lambda_w}{\lambda_t} + \frac{k}{\lambda_t} \lambda_w \lambda_g (\rho_w - \rho_g) G + \frac{k}{\lambda_t} \lambda_o \lambda_g (\rho_o - \rho_g) G,
\]
\[
F_w(S_g, S_w, S_o) = q \frac{\lambda_w}{\lambda_t} + \frac{k}{\lambda_t} \lambda_w \lambda_g (\rho_o - \rho_w) G + \frac{k}{\lambda_t} \lambda_o \lambda_g (\rho_w - \rho_g) G.
\]
Here
\[
\lambda_t = \lambda_o + \lambda_g + \lambda_w
\]
is called the total mobility and \( \rho_l \) is the density of the phase \( l \). It is generally assumed that the permeabilities satisfy the following assumptions.

A.1 \( \lambda_g(0, S_w) = 0, \lambda_w(S_g, 0) = 0 \) and \( \lambda_o(S_g, 1 - S_g) = 0 \) for all \( S_g, S_w \in \Omega \).
A.2 \( \frac{\partial \lambda_w}{\partial S_g}(S_g, S_w) > 0 \) for all \( S_g, S_w \in \Omega \).
A.3 \( \frac{\partial \lambda_o}{\partial S_g}(S_g, S_w) > 0 \) for all \( S_g, S_w \in \Omega \).
A.4 \( \frac{\partial \lambda_w}{\partial S_w}(S_g, S_w) < 0 \) and \( \frac{\partial \lambda_o}{\partial S_w}(S_g, S_w) < 0 \) for all \( S_g, S_w \in \Omega \).

If these assumptions hold, then the equations (2.2) posses a invariant region given by \( \Omega = \{ S_g, S_w : 0 \leq S_g, S_w, 1 - S_g - S_w \leq 1 \} \), in the sense that if the initial data are in \( \Omega \) then the solution remains in \( \Omega \) for all \( t > 0 \).

Assumption A.1 simply states that the permeability of each phase is zero if the corresponding phase saturation is zero and other assumptions follow from the fact that the relative permeabilities of each phase increases when the phase saturation increases provided that one of the other two phases is at constant saturation. One
can easily check that the above assumptions imply that $F_g(., S_w)$ and $F_w(S_g, .)$ are non-convex.

For many realistic permeability functions, the system \((2.2)\) is not strictly hyperbolic and can even have elliptic regions in the phase space, see [16, 8]. Also note that the flux functions are complicated algebraically. Some typical flux functions will be mentioned in section 4. Since the analysis of these equations gets to be very complicated in this generality, one often uses the following Stone’s assumption,

$$A.5 \quad \lambda_g(S_g, S_w) = \lambda_g(S_g) \quad \text{and} \quad \lambda_w(S_g, S_w) = \lambda_w(S_w).$$

Stone’s assumption states that the relative permeabilities of gas and water depend only on their corresponding phase saturation. This is motivated by the fact that water is wetting both in connection with gas and oil and gas is not wetting either in contact with oil or with water. We will also need the following lemma later,

**Lemma 2.1.** Assume that there is no effect of gravity i.e., $G = 0$ or the densities are equal, and that the permeabilities satisfy assumptions $A.1$ to $A.5$ and denote $A(S_g, S_w)$ as the flux Jacobian of \((2.2)\), then following holds

(i.) The eigenvalues of $A(S_g, S_w)$ are positive if they are real.

(ii.) The diagonal entries of $A(S_g, S_w)$ are positive.

**Proof.** For the proof of (i.), see [8], Theorem 5.3. For the proof of (ii.), we calculate the diagonal entries of the matrix $A$ (using the Stone’s assumption $A.5$) as

$$\frac{\partial F_g}{\partial S_g} = (\lambda_o + \lambda_w) \frac{\partial \lambda_g}{\partial S_g} - \lambda_g \frac{\partial \lambda_o}{\partial S_g} \geq 0$$

$$\frac{\partial F_w}{\partial S_w} = (\lambda_o + \lambda_g) \frac{\partial \lambda_w}{\partial S_w} - \lambda_w \frac{\partial \lambda_o}{\partial S_w} \geq 0.$$  

The sign is derived by using assumptions $A.2, A.3$ and $A.4$. \[\square\]

In many situations, the mobility of the gaseous phase is much larger than that of the other phases. This means that the flux of gas is largely independent of whether the other phase is oil or water. As a consequence

$$F_g(S_g, S_w, S_o) \approx \tilde{F}(S_g, 1 - S_g) = \hat{F}(S_g).$$

Assuming that this is an equality, system \((2.2)\) reduces to the following system

\begin{align*}
(S_g)_t + (\tilde{F}_g(S_g))_x &= 0 \\
(S_w)_t + (F_w(S_g, S_w))_x &= 0.
\end{align*}

The above equation is a special case of \((1.1)\), and is called a triangular system. The Jacobian of this system is always hyperbolic but not strictly hyperbolic. Numerical schemes and detailed analysis of these systems are described in [9, 10].

We remark that a one dimensional model like the one that we are using is a good starting point for developing numerical schemes for the full three dimensional model where one can use the one dimensional numerical fluxes in directions normal to volume interfaces or along streamlines.

3. Semi-Godunov schemes

As stated in the introduction, we will describe two classes of semi-Godunov schemes for \((1.1)\). The schemes are based on the local in time decompostion of the system \((1.1)\) into single conservation laws with discontinuous coefficients.
3.1. Staggered semi-Godunov (SSG) scheme. We start with a description of the staggered versions of Semi-Godunov schemes. Let \( \Delta x \) be mesh size, for simplicity we assume a uniform mesh. Set \( x_j = j\Delta x \) and \( j = \ldots, -3/2, -1, -1/2, 0, 1/2, 1, 3/2, 2, \ldots \) and let \( \Delta t \) be the uniform time step and \( t^n = n\Delta t \) for \( n = 0, 1, 2, \ldots \). We assume that the time step and the mesh size satisfy the following CFL condition,
\[
2\lambda M \leq 1
\]
where
\[
\lambda = \frac{\Delta t}{\Delta x}, \quad M = \max \{ \max |f_u|, \max |g_v| \}.
\]
Let \( I_j \) and \( I^n \) denote the intervals
\[
I_j = [x_{j-1/2}, x_{j+1/2}), \quad I^n = [t^n, t^{n+1}).
\]
Set
\[
\chi^n_j(x, t) = \chi_{I^n_j}(x)\chi_{I^n}(t),
\]
where \( \chi_{\Omega} \) denotes the characteristic function of a set \( \Omega \). Define \( U^0_j = (u^0_{j+1/2}, v^0_j) \) as
\[
u^0_{j+1/2} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1/2}} u_0(x) \, dx, \quad v^0_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} v_0(x) \, dx.
\]
Given given \( U^n_j = (u^n_{j+1/2}, v^n_j) \), we shall determine \( U^{n+1}_j = (u^{n+1}_{j+1/2}, v^{n+1}_j) \). In this scheme, the unknowns \( u \) and \( v \) are discretized on staggered meshes and the resulting schemes will be termed as Staggered Semi-Godunov (SSG) schemes. We define approximate solutions as
\[
u^{\Delta x}(x, t) = \sum_{n, j+1/2} \chi^n_{j+1/2}(x, t) u^n_{j+1/2},
\]
\[
u^{\Delta x}(x, t) = \sum_{n, j} \chi^n_j(x, t) v^n_j.
\]
The next step is to consider the evolution equation for \( u \) and treat \( v \) as a coefficient ("locally in time") in this equation, thus we will replace \( v \) by \( v^{\Delta x}(x, t^n) \) in (1.1) in order to get the following conservation law with the discontinuous coefficient \( v^{\Delta x} \):
\[
u_t^{\Delta x} + f(u^{\Delta x}, v^n) = 0, \quad u^{\Delta x}(x, t^n) = \begin{cases} u^n_{j+1/2}, & x < x_j + 1 \\ u^n_{j+3/2}, & x > x_j + 1. \end{cases}
\]
Since waves from different Riemann problems at \( t^n \) do not interact by the CFL-condition, we can use a Godunov scheme to determine \( u^{n+1}_{j+1/2} \). We evolve the solution of the Riemann problem until \( t = t^{n+1} \). At time \( t = t^{n+1} \), we define \( u^{n+1}_{j+1/2} \) by averaging over grid cells \( I_{j+1/2} \):
\[
u^{n+1}_{j+1/2} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1/2}} u^{\Delta x}(x, t^{n+1}) \, dx.
\]
This gives the formula,
\[
u^{n+1}_{j+1/2} = u^n_{j+1/2} - \lambda \left( f^G(u^n_{j+1/2}, u^n_{j+3/2}, v^n) - f^G(u^n_{j-1/2}, u^n_{j+1/2}, v^n) \right),
\]
where \( f^G(a, b, v) \) is the standard Godunov flux corresponding to the flux function \( u \mapsto f(u, v) \). We emphasize that the flux \( f^G \) depends on \( u \) as the unknowns and \( v \).
as a parameter. The standard Godunov flux $h^G$ corresponding to any continuous function $h$ is given by

$$
(3.5) \quad h^G(a, b) = \begin{cases} 
\min_{\theta \in [a, b]} h(\theta), & \text{if } a \leq b, \\
\max_{\theta \in [a, b]} h(\theta), & \text{otherwise}.
\end{cases}
$$

Note that in the schemes proposed in [9][10], the evolution of $u$ did not depend on $v$ making the form of the above flux much simpler than the above numerical flux. The next step is to consider the evolution equation for $v$ and treat $u$ as a coefficient (“locally in time”) in this equation, thus we will replace $u$ by $u^\Delta x(x, t^n)$ in (1.1) in order to get the following conservation law with the discontinuous coefficient $u^\Delta x$:

$$
(3.6) \quad v_t^\Delta x + g \left( u^n_{j+1/2}, v^\Delta x \right)_x = 0, \quad v^\Delta x(x, t^n) = \begin{cases} 
v^n_j, & x < x_{j+1/2} \\
v^n_{j+1}, & x > x_{j+1/2}.
\end{cases}
$$

We evolve the solution of the Riemann problem until $t = t^{n+1}$. At time $t = t^{n+1}$, we define $v^{n+1}_j$ by averaging over grid cells $I_j$:

$$
(3.7) \quad v^{n+1}_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} v^\Delta x(x, t^{n+1}) \, dx.
$$

This gives the formula,

$$
v^{n+1}_j = v^n_j - \lambda \left( g^G \left( u^n_{j+1/2}, v^n_j, v^n_{j+1} \right) - g^G \left( u^n_{j-1/2}, v^n_{j-1}, v^n_j \right) \right),
$$

where $g^G(u, a, b)$ is the standard Godunov flux (3.5) corresponding to the flux function $v \mapsto g(u, v)$. Collecting the updates for $u$ and $v$, we get the following finite volume scheme:

$$
(3.8) \quad \begin{align*}
u^{n+1}_{j+1/2} &= u^n_{j+1/2} - \lambda \left( f^G \left( u^n_{j+1/2}, u^n_{j+3/2}, v^n_{j+1} \right) - f^G \left( u^n_{j-1/2}, u^n_{j+1/2}, v^n_{j} \right) \right), \\
v^{n+1}_j &= v^n_j - \lambda \left( g^G \left( v^{n+1}_{j+1/2}, v^n_j, v^n_{j+1} \right) - g^G \left( v^n_{j-1/2}, v^n_{j-1}, v^n_j \right) \right).
\end{align*}
$$

The resulting numerical fluxes are consistent and the scheme is conservative.

**Remark 3.1.** The CFL condition in the definition of the above scheme is given in terms of $f_u$ and $g_v$, i.e. the diagonal entries of the Jacobian matrix. This results from the fact the equations are decoupled and the maximum velocity in (3.3) is given by the maximum of $f_u$ and the maximum velocity in (3.6) is given by maximum of $g_v$. In other words, if we rewrite (3.8) as an HLL-solver, then the outer wave speeds are bounded by the maximum of $f_u, g_v$ over the phase space. This is slightly non-standard compared to the usual CFL condition in terms of the eigenvalues of the Jacobian Matrix and results from the decoupling of the equation into single conservation laws. The choice is based on heuristic arguments and we are unable to come out with a theoretical justification for it. Some additional comments about the CFL condition is made in the section on numerical experiments.

**Remark 3.2.** In the case of three-phase flows without gravity i.e., $u$ and $v$ are the gas and water saturations and $f$ and $g$ are defined by (2.3) with the gravity constant $G = 0$. Furthermore, the fluxes satisfy the assumptions [A.1] to [A.5] then from Lemma 2.1, we know that both eigenvalues of the Jacobian are positive. Also
from Lemma 2.1, we have that $f_u$ and $g_v$ are non-negative. In this case, the SSG-scheme takes a particularly simple form given by

$$u_j^{n+1} = u_j^{n+1/2} - \lambda \left( f(u_j^{n+1/2}, v_j^{n+1}) - f(u_j^{n-1/2}, v_j^n) \right)$$

$$v_j^{n+1} = v_j^n - \lambda \left( g(u_j^{n+1/2}, v_j^n) - g(u_j^{n-1/2}, v_j^n) \right).$$

The above form is very similar to the upwind scheme in this case as all the information at each cell interface is taken from the left of the interface. Since both eigenvalues are positive, the upwind scheme can be used in this case. Thus, in this case, the SSG-scheme can be thought of as an upwind scheme.

3.2. Aligned semi-Godunov (ASG) scheme. Unlike the SSG-scheme, for the ASG-scheme we align the discretizations of both the unknowns. Define $U_j^0 = (u_j^0, v_j^0)$ by

$$u_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x) \, dx, \quad v_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} v_0(x) \, dx.$$

We define an approximate solutions $u^{\Delta x}, v^{\Delta x}$ on $\mathbb{R} \times \mathbb{R}^+$ by

$$u^{\Delta x}(x, t) = \sum_{n,j} \lambda_j^n(x, t) u_j^n.$$

$$v^{\Delta x}(x, t) = \sum_{n,j} \lambda_j^n(x, t) v_j^n.$$

As in the SSG-scheme, we use $v^{\Delta x}(x, t)$ and define the evolution of $u$ by the solution of the conservation law (3.3). Hence, at the time level $t^n$, we solve the following local Riemann problem at each interface $x_{j+1/2}$:

$$u_t^{\Delta x} + f(u^{\Delta x}, v_j^n)_x = 0, \quad \text{if } x < x_{j+1/2},$$

$$u_t^{\Delta x} + f(u^{\Delta x}, v_{j+1}^n)_x = 0, \quad \text{if } x > x_{j+1/2},$$

$$u^{\Delta x}(x, t^n) = \begin{cases} u_j^n, & x < x_{j+1/2}; \\ u_{j+1}^n, & x > x_{j+1/2}. \end{cases}$$

As we have aligned the discretization of both the unknowns, we end up with a Riemann problem corresponding to a single conservation law with a discontinuous coefficient. The Riemann problem (3.11) can be solved (see, e.g., [2], [3]), and an explicit formula for the (Godunov type) numerical flux has been obtained in [1], [15] for a large class of flux functions.

We define $u_j^{n+1}$ by averaging, and obtain

$$u_j^{n+1} = u_j^n - \lambda \left( f_R^A((u_j^n, v_j^n), (v_j^n, v_{j+1}^n)) - f_R^A((u_j^n, u_{j+1}^n), (v_j^n, v_{j+1}^n)) \right),$$

where $f_R^A(a, b)(k, l)$ is the Godunov numerical flux corresponding to the Riemann problem with left flux function $f(\cdot, k)$, right flux function $f(\cdot, l)$ and Riemann data $a$ (left) and $b$ (right). As mentioned above, explicit formulas for $f_R^A$ can be given in many cases. For example, if $u \to f(u, v)$ has at most one minimum and no maxima for every $v$, then the flux is given by

$$f_R^A(a, b, (k, l)) = \max \{ f(\max(a, \theta_k), k), f(\min(\theta_l, b), l) \},$$

where $\theta_k, \theta_l$ are the minimum points of $f(\cdot, k)$ and $f(\cdot, l)$ respectively. Explicit formulas in other (non-convex) cases are given in [2], [15].
The update in $v$ is obtained in an identical manner by replacing $u$ in the second equation of (3.1) with $u^t(x, t^n)$ and we obtain the following update formulas for both $u$ and $v$ given by

$$
u_j^{n+1} = v_j^n - \lambda \left(f_R(u_j^n, u_{j+1}^n, v_j^n, v_{j+1}^n) - g_R(u_{j-1}^n, u_j^n, v_j^n, v_{j+1}^n)\right),$$

$$u_j^{n+1} = u_j^n - \lambda \left(f_A(u_j^n, u_{j+1}^n, v_j^n, v_{j+1}^n) - g_A(u_{j-1}^n, u_j^n, v_j^n, v_{j+1}^n)\right),$$

where $g_R(k, l)(a, b)$ is the Godunov numerical flux corresponding to the Riemann problem with left flux function $g(k, \cdot)$, and has similar explicit formulas like $f_R$. The aligned scheme was first proposed for the special case of a triangular system of conservation laws by the authors in [9].

Both the schemes are not based on a characteristic decomposition of the full system (3.1) and are as easy to implement as central schemes.

In the special case of three-phase flows without gravity, we have that the ASG-scheme is identical to the upwind scheme, and thus also this scheme can be viewed as a generalization of the upwind scheme.

### 3.3. Alternative time discretizations.

For a fixed time $t$, the semi-discrete form of the SSG-scheme (3.8) is given by

$$u_{j+1/2}(t) = -\frac{1}{\Delta x} \left(f^G(u_{j+1/2}(t), u_{j+3/2}(t), v_{j+1}(t)) - f^G(u_{j-1/2}(t), u_{j+1/2}(t), v_j(t))\right),$$

$$v_j(t) = -\frac{1}{\Delta x} \left(g^G(u_{j+1/2}(t), v_j(t), v_{j+1}(t)) - g^G(u_{j-1/2}(t), v_{j-1}(t), v_j(t))\right).$$

We can replace the Godunov fluxes $f^G$ and $g^G$ by the corresponding Engquist-Osher fluxes for $u \mapsto f(u, v)$ and $v \mapsto g(u, v)$ to obtain “Engquist-Osher versions” of our schemes. Numerical tests indicate that these versions perform similarly to the Godunov versions.

Note that by taking a Forward Euler time discretization, we can deduce (3.8) from (3.14). We can also use alternative time discretization strategies. Two of such methods that we can consider are:

a) **Runge-Kutta time discretization**: Instead of using a forward Euler time discretization, we can use the second order strong stability preserving (SSP) Runge-Kutta discretization of [7] in order to increase the stability region of the time discretization. This time discretization takes the form

$$u_{j+1/2}^{n} = u_{j+1/2}^n - \lambda \left( f^G(u_{j+1/2}^n, u_{j+3/2}^n, v_{j+1}^n) - f^G(u_{j-1/2}^n, u_{j+1/2}^n, v_j^n) \right),$$

$$v_j^* = v_j^n - \lambda \left( g^G(u_{j+1/2}^n, v_{j+1}^n, v_{j+3/2}^n) - g^G(u_{j-1/2}^n, v_j^n, v_{j+1}^n) \right),$$

$$u_{j+1/2}^{n+1} = u_{j+1/2}^n - \lambda \left( f^G(u_{j+1/2}^*, u_{j+3/2}^*, v_{j+1}^*) - f^G(u_{j-1/2}^*, u_{j+1/2}^*, v_j^*) \right),$$

$$v_j^{**} = v_j^* - \lambda \left( g^G(u_{j+1/2}^*, v_{j+1}^*, v_{j+3/2}^*) - g^G(u_{j-1/2}^*, v_j^*, v_{j+1}^*) \right),$$

$$v_j^{n+1} = \frac{1}{2}(v_j^n + v_j^{**}).$$

This time discretization is second order in time but since the spatial discretization is at most first order accurate, so we expect the overall discretization to be first order.
The second order time-discretization is considered as it increases the stability of the scheme.

b) \textit{Picard type iterative discretization}: Another possibility of discretizing in time is to take the following Picard type two stage discretization (3.16)
\[ u^*_{j+1/2} = u_j^{n+1} - \lambda \left( f^G \left( u_{j+1/2}^{n}, u_{j+3/2}^{n}, v_j^{n+1} \right) - f^G \left( u_{j-1/2}^{n}, u_{j+1/2}^{n}, v_j^{n} \right) \right), \]
\[ v_{j+1/2}^* = v_{j+1/2}^{n} - \lambda \left( g^G \left( u_{j+1/2}^{n}, v_{j+1}^{n} \right) - g^G \left( u_{j-1/2}^{n}, v_{j-1}^{n} \right) \right), \]
\[ u_{j+1/2}^{n+1} = u_j^{n+1} - \lambda \left( f^G \left( u_{j+1/2}^{n}, u_{j+3/2}^{n}, v_j^{n+1} \right) - f^G \left( u_{j-1/2}^{n}, u_{j+1/2}^{n}, v_j^{n} \right) \right), \]
\[ v_{j+1/2}^{n+1} = v_{j+1/2}^{n} - \lambda \left( g^G \left( v_{j+1/2}^{n+1}, v_{j+1}^{n} \right) - g^G \left( v_{j-1/2}^{n+1}, v_{j-1}^{n} \right) \right). \]

Note that (3.16) is an analogue of the Picard type iterations for solving ordinary differential equations. It is motivated by the fact that we replace \( v \) in the first equation of (1.1) by \( v^{\Delta t}(x, t^n) \) and the above iteration gives a better estimate on the coefficient. Exactly, the same method can be applied when we view \( u \) as a coefficient in the second equation of (1.2).

\section{4. Numerical experiments}

In this section, we will report the performance of the semi-Godunov schemes of the previous section in a series of numerical experiments. We start with some model linear equations and illustrate how the schemes perform in these cases. It will also help us identify conditions under which the schemes are stable. After that, we will consider three-phase flows in a porous medium, both with and without the effects of gravity.

\subsection*{4.1. Linear hyperbolic systems.}

\subsection*{4.1.1. Numerical experiment 1.} We consider (1.1) with the following fluxes
\[ f(u, v) = 2u + v, \quad \text{and} \quad g(u, v) = 3u + 2v, \]
and Riemann initial data,
\[ u(x, 0) = \begin{cases} 1.0 & x < 0 \\ 2.0 & x \geq 0 \end{cases} \quad \text{and} \quad v(x, 0) = \begin{cases} 2.5 & x < 0 \\ 1.5 & x > 0. \end{cases} \]

In this linear case, exact solution is given by,
\[ u(x, t) = \begin{cases} 1.0 & x < \lambda_1 t \\ 1.8 & \lambda_1 t < x < \lambda_2 t \\ 2.0 & x \geq \lambda_2 t \end{cases} \quad \text{and} \quad v(x, t) = \begin{cases} 2.5 & x < \lambda_1 t \\ 1.1 & \lambda_1 t < x < \lambda_2 t \\ 1.5 & x \geq \lambda_2 t. \end{cases} \]

Where the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are approximately equal to 0.26 and 3.73 respectively. Note that in this case, both the eigenvalues as well as the diagonal entries of the flux Jacobian i.e \( f_u \) and \( g_v \) are all positive. The numerical results with the ASG-scheme (3.13) and the SSG-scheme (3.8) (along with the mesh parameters) are shown in Figure \[\square\]. In this case, the ASG-scheme reduces to the upwind scheme (hence also the Godunov scheme for this linear system) and resolves the discontinuities quite well even on a coarse mesh. The SSG-scheme also resolves the discontinuities quite well. In fact, it resolves the rightmost discontinuity with better accuracy than the ASG-scheme but it has a small undershoot at the right
Figure 1. Solutions computed for numerical experiment 1 with the ASG-scheme and the SSG-scheme with $\Delta x = 0.1$ and $\Delta x = 0.01$ and $CFL = 0.5$ at time $t = 0.375$. Left: $u$ and Right: $v$, the horizontal axis is $x/t$.

Figure 2. Solutions computed for numerical experiment 1 with the SSG-scheme, SSG-RK scheme (3.15) and the SSG-Picard scheme (3.16) with $\Delta x = 0.01$ and $CFL = 0.5$ at time $t = 0.375$. Left: $u$ and Right: $v$.

Runge-Kutta type or the Picard type discretizations remove the undershoots that the SSG-scheme has and improve its numerical performance, although at the cost of adding some more diffusion. Both the Runge-Kutta and Picard type iterations are almost on top of each other in this case and there is a very slight difference between them. The total discretization in space-time is still first order. We also observed that the Picard-type discretization as well as the Runge-Kutta discretizations improve the range of $CFL$ numbers that can be taken in this case. The CFL numbers for the SSG-scheme was 0.5 in this case whereas the CFL-number of the Picard and
Runge-Kutta schemes was 1. Using the Runge-Kutta and the Picard type iterations for the ASG-scheme, we observed that the CFL numbers were similarly increased.

4.1.2. Numerical experiment 2. We consider (1.1) with the following fluxes
\[ f(u, v) = -2u + v, \quad \text{and} \quad g(u, v) = 3u + 2v, \]
and Riemann initial data,
\[ u(x, 0) = \begin{cases} 1.0 & x < 0 \\ 2.0 & x \geq 0 \end{cases} \quad v(x, 0) = \begin{cases} 2.5 & x < 0 \\ 1.5 & x > 0 \end{cases} \]
In this linear case, exact solution is given by,
\[ u(x, t) = \begin{cases} 1.0 & x < \lambda_1 t, \\ 2.1 & \lambda_1 t < x < \lambda_2 t \\ 2.0 & x \geq \lambda_2 t, \end{cases} \quad v(x, t) = \begin{cases} 2.5 & x < \lambda_1 t, \\ 2.5 & \lambda_1 t < x < \lambda_2 t \\ 1.5 & x \geq \lambda_2 t. \end{cases} \]
Where the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are equal to \(-2.6 \) and \( 2.6 \) respectively. In this case, one of the eigenvalues is positive and the other negative, similarly one of the diagonal entries of the flux-Jacobian is positive and the other negative. In this case, the upwind scheme is not defined and one has to do a characteristic decomposition in order to define a Godunov type scheme. Yet, the ASG-scheme as well as the SSG-scheme work very well in this case with sharp resolution of the discontinuities. The numerical results with both schemes are shown in Figure 3.

Note that we do not use any characteristic information about the system in defining

\[ \text{Figure 3. Solutions computed for numerical experiment 2 with} \]
the Semi-Godunov schemes and yet, they still resolve the discontinuities with the same precision as a standard Godunov type scheme will do. In particular, the aligned scheme resolves the discontinuities more sharply (similar to the Godunov scheme in this case) than the staggered scheme does and this is clearly indicated in the coarse mesh discretizations in Figure 3.
4.1.3. Numerical experiment 3. We consider (1.1) with the following fluxes
\[ f(u, v) = 2u + 2v, \quad \text{and} \quad g(u, v) = 3u + v, \]
and Riemann initial data,
\[
    u(x, 0) = \begin{cases} 
        1.0 & x < 0 \\
        2.0 & x \geq 0
    \end{cases},
\]
\[
    v(x, 0) = \begin{cases} 
        2.5 & x < 0 \\
        1.5 & x > 0
    \end{cases}.
\]
In this linear case, exact solution is given by,
\[
    u(x, t) = \begin{cases} 
        1.0 & x < \lambda_1 t, \\
        1.8 & \lambda_1 t < x < \lambda_2 t \\
        2.0 & x \geq \lambda_2 t,
    \end{cases}
\]
\[
    v(x, t) = \begin{cases} 
        2.5 & x < \lambda_1 t, \\
        1.25 & \lambda_1 t < x < \lambda_2 t \\
        1.5 & x \geq \lambda_2 t.
    \end{cases}
\]
where the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are approximately equal to \(-1\) and 4 respectively. In this case, one of the eigenvalues is positive and the other negative, but both the diagonal entries of the flux-Jacobian are positive. In this case, the decomposition into scalar equations does not even respect the characteristic directions. The ASG-scheme becomes unstable in this case and the SSG-scheme works well but with some oscillations. We tried changing the CFL number by taking a CFL condition based on the maximum eigenvalue. The modulus of the oscillations were reduced but they still persisted. This seems to justify our choice of the magnitude of the diagonal entries of Jacobian for the CFL condition. The oscillations are removed when one uses either the Runge-Kutta or the Picard type time discretizations. The numerical results with both schemes are shown in Figure 4.

**Figure 4.** Solutions computed for numerical experiment 3 with the SSG-scheme and the SSG(Picard)-scheme with \( \Delta x = 0.1 \) and \( \Delta x = 0.01 \) and \( CFL = 0.5 \) at time \( t = 0.375 \). Left: \( u \) and Right: \( v \), the horizontal axis is \( x/t \).

Thus summarizing the results of the experiments with linear systems, we conclude that both the ASG- and SSG- schemes work very well with low numerical diffusion whenever the signs of the eigenvalues coincide with that of the signs of the diagonal entries of the Jacobian. The numerical diffusion of the SSG-scheme is slightly greater than that of the ASG-scheme. Despite this, the SSG-scheme is more robust and even works in cases where the signs of the eigenvalues and the diagonal entries of the Jacobian are different. Furthermore, the over/under shoots that are present in the SSG-scheme are removed by either using a Runge-Kutta or
a Picard type iterative time discretization. Equipped with these conclusions, we present numerical examples from three phase flows in porous media.

4.2. Three phase flows without gravity. We start with two numerical examples modeling three phase flows without gravity.

4.2.1. Numerical experiment 4. We consider equation (2.2) with the following parameters. The relative permeabilities and the viscosities are given by,

\[
\begin{align*}
\lambda_g &= S_g^2, & \lambda_w &= S_w^2, & \lambda_o &= S_o^2, \\
\nu_g &= 1, & \nu_w &= 80, & \nu_o &= 100.
\end{align*}
\]

The absolute permeability is set to unity and the Gravity constant is set to zero. Thus, there are no gravity effects on the flow. It is easy to check that all the assumptions $A.1$ to $A.5$ are satisfied in this case. Hence the ASG-scheme reduces to the upwind scheme in this case. This example was first proposed in [9]. We test with the following Riemann initial data,

\[
\begin{align*}
u(x,0) = \begin{cases} 
1.0 & x < 0 \\
0.0 & x \geq 0
\end{cases} \quad v(x,0) = \begin{cases} 
0.0 & x < 0 \\
0.5 & x > 0
\end{cases}
\]

We show the solutions given by the ASG-scheme and the SSG-scheme in Figure 5. The exact solution in this case is a rarefaction-shock moving towards the right and

![Figure 5](image)

**Figure 5.** Solutions computed for numerical experiment 4 with the ASG-Scheme, and the SSG-scheme with $\Delta x = 0.1$ and $\Delta x = 0.01$ and $CFL = 0.5$ at time $t = 0.4$. Left: Gas-saturation and Right: Water Saturation.

both the ASG-scheme and the SSG-scheme resolve the solution quite well. The difference between results obtained by both schemes is negligible. Observe from Figure 3 that even at a very coarse mesh discretization of $\Delta x = 0.1$, both schemes resolve the rarefaction to a very good extent. Note that the SSG-scheme gives a very slight overshoot at $x = 0$ and this overshoot is remedied if one uses either the Runge-Kutta or the Picard type time discretization.
4.2.2. Numerical experiment 5. We consider (2.2) with the following parameters. The relative permeabilities and the viscosities are given by,

\[
\begin{align*}
\lambda_g &= 0.1 S_g + 0.9 S_g^2, & \lambda_w &= S_w^2, & \lambda_o &= (1 - S_w - S_g)(1 - S_w)(1 - S_g), \\
\nu_g &= 0.012, & \nu_w &= 0.35, & \nu_o &= 0.8.
\end{align*}
\]

The absolute permeability is set to unity and the gravity constant is set to zero. Thus, there are no gravity effects on the flow. Again, all the assumptions A.1 to A.5 are satisfied in this case. Hence the ASG-scheme reduces to the upwind scheme in this case. This example was proposed in [14]. We test with the following Riemann initial data,

\[
\begin{align*}
u(x, 0) &= \begin{cases} 
0 & x < 0 \\
0.4 & x \geq 0
\end{cases}, \\
u(x, 0) &= \begin{cases} 
0.6 & x < 0 \\
0.05 & x > 0.
\end{cases}
\end{align*}
\]

We show the solutions given by the ASG-scheme in Figure 6. The exact solution in this case (check [14]) is a rarefaction-shock and a shock going towards the right and the ASG-scheme resolves the solution quite well. For the sake of comparison, we also show the solution obtained with a local Lax-Friedrichs scheme.

In the above numerical experiments and several others that we conducted, we found that the Semi-Godunov schemes performed quite well and resolved shocks sharply. The resolution at compound waves was also quite good. This can be explained by the fact that they are very close to the upwind scheme in this case where there is no gravity. Next, we consider some numerical examples of three phase flow in the presence of gravity.

4.3. Three-phase flows with gravity.

4.3.1. Numerical experiment 6. We consider flows with relative permeabilities and viscosities given by (4.2) and with constant of gravity as unity. We use the following densities,

\[
\begin{align*}
\rho_g &= 0.05, & \rho_w &= 1, & \rho_o &= 0.9.
\end{align*}
\]
In this case, the flux- functions are more complicated and no longer monotone. In particular, the function \( v \mapsto g(u, v) \) is very complicated and is not monotone in \( v \) for a fixed \( u \) and has two critical points i.e one minima and one maxima for some values of \( u \). We consider this problem with the Riemann initial data given by,

\[
\begin{align*}
  u(x, 0) &= \begin{cases} 
    1.0 & x < 0 \\
    0.0 & x \geq 0 
  \end{cases} \\
  v(x, 0) &= \begin{cases} 
    0.0 & x < 0 \\
    0.75 & x > 0.
  \end{cases}
\end{align*}
\]

We show the solutions given by the SSG-scheme in Figure 7. In this case, we have shown the results with SSG-scheme. The ASG-scheme is complicated to implement in this case due to the complicated form of the explicit formulas and we do not use the ASG-scheme in this case. Since we do not know the exact solution in this case, the approximate solution is compared with the solution computed with a local Lax-Friedrichs scheme on a mesh 10 times finer than the one used for the SSG-scheme. Clearly, SSG scheme resolves the solution, particularly the compound wave very well. In this case, the situation is very similar to that in Experiment 2 and the SSG-scheme is no longer similar to the upwind scheme. Note that we have not used any characteristic information in using the SSG-scheme.

4.3.2. Numerical experiment 7. We consider the relative permeabilities and viscosities given in (4.1) and the densities given in (4.3). The absolute permeability and the gravity constant are set to unity. We consider the following initial data,

\[
\begin{align*}
  u(x, 0) &= \begin{cases} 
    0 & x < 0 \\
    \frac{1}{2} + \frac{1}{4} \sin(2\pi x) & x \geq 0,
  \end{cases} \\
  v(x, 0) &= 0.
\end{align*}
\]

In this case, we are modeling a reservoir containing a mixture of oil and gas initially and water is injected from the left i.e the boundary conditions at the left boundary are set as \( v(0, t) \equiv 1 \). The water is supposed to eject the oil and gas out of the reservoir. We use the SSG-scheme to simulate the reservoir and show the results in Figures 8 and 9. In this case, since the SSG-scheme and the ASG-scheme gave very similar results, we show the results obtained with the SSG-scheme only. The results in Figure 8 show the space-time evolution of the solution upto time \( t = 2 \).
Approximate solutions of example 7, with the initial values (4.4), and $\Delta x = 1/100$. Top left: gas saturation; $u$, top right: oil saturation; $1 - u - v$, bottom left: water saturation; $v$.

Figure 8.

Figure 9. Solutions computed for numerical experiment 7 with the SSG-scheme for different $\Delta x$ with $\Delta x = \frac{\Delta}{N}$ and $N = 50, 100, 200, 400$ and $2000$ respectively and $CFL = 0.5$ at time $t = 1$. Left: Gas-saturation and Right: Water Saturation.

with $\Delta x = 0.01$. Initially, a sinusoidal mixture of oil and gas is present in the reservoir and as water is injected at the left boundary, the water front pushes the oil and gas out in a non-linear way. The initial sinusoidal profiles of oil and gas start forming the well known N-wave patterns with alternating Shocks and Rarefactions away from the water front. The water front is a compound shock and results in the
formation of a region of a very high oil concentration that pushes the gas out. Note that the SSG-scheme resolves the complex flow structure quite well in this case.

We also use this numerical example to conduct a study on the order of convergence of the SSG-scheme. A reference solution is calculated with the SSG-scheme with $\Delta x = 0.0025$ and solutions computed with different $\Delta x$’s are shown in Figure 9. We also compute the error in the following norm, let $u^{\Delta x}$ and $v^{\Delta x}$ be the approximate solutions computed with SSG-scheme (3.8), the error is defined by

$$E^{\Delta x}(t) = \Delta x \left( \sum_j |u^r(x_j, t) - u^{\Delta x}(x_j, t)| + \sum_j |v^r(x_j, t) - v^{\Delta x}(x_j, t)| \right)$$

Where $u^r, v^r$ is the reference solution calculated on a fine mesh. The errors for different discretizations are shown in Table 1. From the above table, we see that

<table>
<thead>
<tr>
<th>n</th>
<th>$E^{\Delta x}$</th>
<th>EOC</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.388</td>
<td>0.926</td>
</tr>
<tr>
<td>100</td>
<td>0.204</td>
<td>0.687</td>
</tr>
<tr>
<td>200</td>
<td>0.126</td>
<td>1.382</td>
</tr>
<tr>
<td>400</td>
<td>0.048</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1. Error $E^{\Delta x}$ for the SSG-scheme in Experiment 7. We used $\Delta x = \frac{\Delta}{n}$ in the interval $[0, 5]$ with $n$ being the number of mesh points. EOC represents the Observed Order of convergence.

the average order of convergence is 1.002 which is expected as the scheme is a first order scheme. Similar convergence results were obtained in the other numerical experiments.

From the above experiments with three flows both without and with gravity, we observe that the Semi-Godunov schemes perform very well and have good resolution both at shocks and rarefactions.

5. Conclusion

In this paper, we have described a class of finite volume schemes for approximating solutions to systems of conservation laws. These schemes are based on a local decomposition of the system into a series of scalar equations but with discontinuous coefficients. The resulting schemes are based on using Godunov type solvers to approximate conservation laws with variable coefficients. The schemes are based on either aligning the discretizations of both unknowns or on staggering them.

The resulting schemes do not use any characteristic information about the full system and are hence very easy to implement. The analysis of these schemes will be taken up in a forthcoming paper. A good example of interesting systems where these schemes perform very well are three phase flows in a porous medium. In the absence of gravity and some other general assumptions, we show that the aligned version of the scheme coincides with the standard upwind scheme and the staggered scheme is a perturbation of it. Hence, these schemes can be thought of as extensions of upwind schemes to flows with gravity even though we do not explicit compute eigenvalues of the full system.

We illustrate the performance of these schemes on a variety of numerical experiments for linear hyperbolic systems and multiphase flows without and with
gravity. For linear systems, the aligned scheme works very well as long as the signs of the eigenvalues of the full system and the signs of the diagonal entries of the flux Jacobian are similar and resolves the solutions of the linear systems with high degree of accuracy. The staggered version of the scheme can be slightly more diffusive and can give over/under shoots but its performance can be improved by using either a Runge-Kutta time discretization or a Picard-type iterative scheme. Furthermore, the staggered scheme is very robust and works in situations where the aligned scheme may become unstable. More analysis needs to be done in order to determine conditions under which the schemes are stable and convergent.

Regarding three phase flows, we have conducted several numerical experiments without and with gravity and found that both the aligned and the staggered schemes work quite well and resolve the complex flow patterns to good accuracy. Since these schemes are easy to implement and do not require any detailed information about the eigenstructure, we advocate their use for calculating flows in reservoirs.

Comparing the two schemes i.e the Aligned and the Staggered scheme, we see from the numerical experiments that in some problems, the aligned scheme is less diffusive than staggered scheme. Yet, the aligned scheme does not work for linear systems when the signs of the eigenvalues are different from that of the diagonal entries of the Jacobian. On three-phase flows with gravity, the aligned scheme is difficult to implement on account of the complicated flux shapes. On the other hand, the staggered scheme is robust and is easier to implement. It can give small under/overshoots at shocks but these are removed when one uses the staggered scheme with either the Runge-kutta or the Picard type iterative discretizations.

Among further extensions, we would like to mention that it is very easy to extend these schemes to the case of heterogenous media. Similarly, we can extend these results to higher order by following standard ENO/WENO or DG reconstructions. These schemes are also easy to extend to general $n \times n$ systems of conservation laws and we have conducted numerical tests with multiphase flows, sedimentation of polydisperse suspensions and multi-lane traffic flows, shallow water and Euler equations and we observed that the schemes resolve the solution quite well. These tests will be reported in a forthcoming paper.

**References**


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