A MARKOV CHAIN APPROXIMATION SCHEME FOR A SINGULAR INVESTMENT-CONSUMPTION PROBLEM WITH LÉVY DRIVEN STOCK PRICES

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1. Introduction

We consider an optimal investment-consumption problem for a small investor whose wealth is divided between a riskless asset (a bank account) and a risky asset (a stock) with log-returns following a Lévy process. The investor preferences, in contrast to the standard von Neumann-Morgenstern time-additive preferences, allow for cumulative consumption patterns with possible jumps/singular sample paths, and they incorporate the notion of local substitution. The dynamic programming equation of this singular stochastic control problem is a degenerate elliptic integro-differential variational inequality (a free boundary problem). This problem has been thoroughly investigated from an analytical point of view in Benth, Karlsen, and Reikvam [7, 8] (see also [5, 6, 9] for some related problems). Herein we present a Markov chain approximation scheme for solving the investment-consumption problem. A feature of the suggested numerical scheme is that it is based on a simplified dynamic programming equation obtained by approximating the original Lévy process by a more simple and tractable (Lévy) process which can be written as an independent sum of a drift, a Brownian component, and a finite number of compound Poisson processes. This approximation reduces the integral operator in the dynamic programming equation to a finite series operator. The convergence analysis of the numerical scheme is based on the theory of viscosity solutions and the details can be found in our forthcoming paper [15].

2. The Financial Market

2.1. The exponential Lévy model. We consider a financial market operating in the conditions of uncertainty described by a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped by a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual assumptions of right-continuity and completeness.

The financial market consists of a bond paying a constant interest rate \(r > 0\) and a stock with price dynamics \(S = (S_t)_{t \geq 0}\) given by

\[
S_t = S_0 \exp(L_t), \quad S_0 > 0,
\]

where \(L = (L_t)_{t \geq 0}\) is a Lévy process with Lévy-Khintchine decomposition

\[
L_t = bt + \sigma B_t + \int_0^t \int_{\{|z| < 1\}} z \tilde{N}(ds,dz) + \int_0^t \int_{\{|z| \geq 1\}} z N(ds,dz),
\]

where \(b \in \mathbb{R}, \sigma \geq 0, B\) a Brownian motion, and \(N(dt,dz)\) is the jump measure of \(L\) with a deterministic compensator of the form \(\nu(dz) \times dt. \ \nu(dz)\), called the Lévy measure of \(L\), is a positive measure on \(\mathbb{R}\) such that

\[
\nu(\{0\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}} (|z|^2 \wedge 1) \nu(dz) < \infty.
\]

The triplet \(\mathcal{T}(L) \equiv (b, \sigma^2, \nu(dz))\), is the characteristic triplet of the process \(L\), and completely determines its law. Moreover, the characteristic function of \(L\) is of the form \(\mathbb{E}[\exp(i\theta L_t)] = \)
exp(tΨ(θ)), where the function Ψ(θ), called the characteristic exponent, has the Lévy-Khintchine representation:

\[
\Psi_{b,\sigma,\nu}(\theta) = ib\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left( \exp(it\theta z) - 1 - i\theta h(z) \right) \nu(dz).
\]

Using Itô’s formula we get

\[
S_t = S_0 \mathcal{E}(R_t), \quad S_0 > 0,
\]

where \( \mathcal{E} \) denotes the stochastic exponential and

\[
R_t = \left( b + \frac{\sigma^2}{2} \right) t + \sigma B_t + \int_0^t \int_{\{|z|<1\}} z \tilde{N}(ds,dz) + \int_0^t \int_{\mathbb{R}} \left( e^z - 1 - z \mathbf{1}_{\{|z|<1\}} \right) \nu(ds,dz).
\]

We shall make the standing assumption that the Lévy measure satisfies the following integrability condition

\[
\int_{\{|z|\geq 1\}} (e^z - 1) \nu(dz) < \infty.
\]

Note that (H1) is a necessary and sufficient condition for the stock price given by (1) to possess first moments. This means that we exclude exponential Lévy models driven by \( \alpha \)-stable Lévy motions. In this case we set

\[
\hat{b} = b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} \left( e^z - 1 - z \mathbf{1}_{\{|z|<1\}} \right) \nu(dz),
\]

and write

\[
R = \hat{b} t + \sigma B_t + \int_0^t \int_{\mathbb{R}} \left( e^z - 1 \right) \tilde{N}(ds,dz).
\]

Under assumption (H1), the moment generating function \( \mathbb{E}[\theta L_t] \) exists for all \( \theta \in [0,1] \) and

\[
\mathbb{E}[\theta L_t] = \exp \left( t\chi(\theta) \right),
\]

where the function \( \chi \), referred to as the cumulant generating function, is given by

\[
\chi(\theta) = b\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left( e^{\theta z} - 1 - \theta z \mathbf{1}_{\{|z|<1\}} \right) \nu(dz).
\]

For a general reference on Lévy processes we refer to Bertoin [10], Sato [27], and Jacod and Shiryaev [20].

2.2. Some specific Lévy models. The standard model of stock prices is the geometric Brownian motion. This model assumes that the stock returns are normally distributed. However, the normal distribution poorly fits the stock returns. One of the first to report some fundamental deviation of stock returns distribution from the normal distribution, was Mandelbrot [24]. He observed that returns distributions are leptokurtic, and have longer and fatter tails than the normal distribution. He suggested an exponential Lévy model, where the Lévy process is a symmetric \( \alpha \)-stable Lévy motion with \( \alpha < 2 \). More recently, other exponential Lévy models have been suggested. In Madan and Seneta [23] the variance gamma (VG) model, which is a normal gamma Lévy process, is studied as a model for share market returns. In Eberlein and Keller [14] the hyperbolic (HYP) Lévy model is proposed as a model for German stock prices, and it is shown to give an extremely good fit. In Barndorff-Nielsen [4] the normal inverse Gaussian (NIG) Lévy model is suggested, and in Rydberg [26] it is shown to perform well in modelling German stock prices. The last two models belong to the class of generalized hyperbolic (GH) Lévy models. These models are characterized by independent increments which belong to the class of GH distributions. This class of distributions, and in particular its two corresponding subclasses, NIG distributions and HYP distributions, has proved to provide an excellent fit to empirically observed log-return distributions. The class of stock price dynamics (1) considered herein compromises all models mentioned above, except from the Mandelbrot [24] model. We shall now survey some of these models.
Brownian motion with drift and Poisson jumps. In this class of models, broadly referred to as jump-diffusion models, jumps are added to the Brownian motion as an additional orthogonal compound Poisson process:

\[ L_t = bt + \sigma B_t + \sum_{k=1}^{N_t} Z_k, \]

where \( B = (B_t)_{t \geq 0} \) is a standard Brownian motion, \( N = (N_t)_{t \geq 0} \) a homogeneous Poisson process with intensity \( \lambda > 0 \), and \((Z_k)_{k \geq 1}\) i.i.d. random variables with a distribution \( Q(dz) \). Furthermore, the processes \( B, N \), and \((Z_k)_{k \geq 1}\) are jointly independent. In this case \( L \) has the characteristic triplet

\[ \mathcal{T} \equiv \left( b + \lambda \int h(z) Q(dz), \sigma^2, \lambda Q(dz) \right). \]

A model of this type is Merton’s jump-diffusion model \([25]\), where the heights of the jumps \( Z_1, Z_2, \ldots \) are assumed to be normally distributed: \( Q(dz) = \exp \left( -\frac{(z-\alpha)^2}{2\sigma^2} \right) / (2\pi \sigma^2)^{1/2} dz \).

Generalized hyperbolic Lévy motions. The class of GH distributions, introduced by Barndorff-Nielsen \([3]\), can be characterized as normal variance-mean mixtures, where the mixture distribution is a generalized inverse Gaussian (GIG) distribution. This class of distributions includes many interesting subclasses, and limiting cases like the NIG, HYP, VG, Student-\( t \), and normal distributions. All of them have been used to model financial returns.

The density of a GH distribution depends on five parameters \((\lambda, \alpha, \beta, \delta, \mu)\), with domain of variation \( \lambda \in \mathbb{R}, \alpha > 0, \beta \in (-\alpha, \alpha), \delta > 0, \mu \in \mathbb{R} \), and with the following interpretation: \( \alpha \) is a steepness parameter (the larger \( \alpha \), the steeper density), \( \beta \) is a parameter of asymmetry (if \( \beta = 0 \) the density is symmetric around the mean), \( \delta \) is a scale parameter, and \( \mu \) is a location parameter. The special case of \( \lambda = -\frac{1}{2} \) gives a NIG distribution. For \( \lambda = \frac{1}{2} \) we get the HYP distribution.

Generalized hyperbolic distributions are infinitely divisible (ID). This follows readily from the fact that the GIG distributions are ID \([2]\). Therefore, every member of this family generates a Lévy process \((L_t)_{t \geq 0}\), i.e. a process with stationary independent increments such that \( L_0 = 0 \) and \( \mathcal{L}(L_1) \), the distribution of \( L_1 \), has a GH distribution. We can choose a càdlàg version, and call this process the generalized hyperbolic Lévy process.

The characterization of the GH as a normal variance-mean mixture gives us the form of the moment generating function as

\[ M_{\text{GH}}(\theta) = \exp(\mu \theta) M_{\text{GIG}} \left( \frac{1}{2} (2\beta \theta + \theta^2) \right), \]

from which we may derive the expression of the first moment

\[ \mathbb{E}[\text{GH}] = \mu + \frac{\beta \delta^2}{\zeta} K_{\lambda+1}(\zeta) / K_\lambda(\zeta). \]

The characteristic triplet of a GH \((\lambda, \alpha, \beta, \delta, \mu)\) distribution (w.r.t. to the “truncation” function \( z \mapsto z \)) is given by

\[ \mathcal{T}_{\text{GH}} = \left( \mathbb{E}[\text{GH}], 0, \nu^{\text{GH}} \right). \]

The Lévy measure \( \nu^{\text{GH}}(dz) \) is absolutely continuous w.r.t. the Lebesgue measure \( dz \), and its density is given by (the fairly complicated) representation:

\[ d\nu^{\text{GH}}(dz) = \begin{cases} \frac{e^{\beta z}}{|z|} \left( \int_0^\infty \frac{\exp(-\sqrt{2y} + \alpha^2 |z|)}{\pi^2 y (J_\lambda^z(\delta \sqrt{2y}) + Y_\lambda^z(\delta \sqrt{2y}))} dy + \lambda e^{-\alpha |z|} \right), & \lambda \geq 0, \\ \frac{e^{\beta z}}{|z|} \left( \int_0^\infty \frac{\exp(-\sqrt{2y} + \alpha^2 |z|)}{\pi^2 y (J_{-\lambda}^z(\delta \sqrt{2y}) + Y_{-\lambda}^z(\delta \sqrt{2y}))} dy, & \lambda < 0, \end{cases} \]
where $J_\lambda$ and $Y_\lambda$ are the Bessel functions of the first and the second kind with index $\lambda$, respectively. For the NIG distribution this expression is simplified to

$$
\nu^{(\text{NIG})}(dz) = \frac{\delta\alpha}{\pi|z|} \exp(\beta z) K_1(\alpha|z|) \, dz,
$$

where $K_1$ is the modified Bessel function of the third kind [see 4, Eq. 3.15]. For the HYP distribution we have

$$
\nu^{(\text{HYP})}(dz) = \frac{e^{\beta z}|z|}{z} \left( \int_0^\infty \exp\left(-\sqrt{2y} + \alpha^2 z\right) \frac{\exp\left(-\sqrt{2y} + \alpha^2 z\right)}{\pi^2 y \left(J_1^2(\delta \sqrt{2y}) + Y_1^2(\delta \sqrt{2y})\right)} \, dy + e^{-\alpha|z|} \right) \, dz.
$$

The expression of the $M_{GH}$ given by (8), in conjunction with the convolution properties of the GIG distributions, imply that the family of the NIG distributions is the only member of the class of GH distributions which is closed under convolution. The limiting case of VG is also closed under convolution.

By (8), the $L(L_1)$ has moments of any order, and we therefore have the trivial decomposition

$$
L_t = (L_t - \mathbb{E}[L_t]) + t \mathbb{E}[L_1].
$$

The martingale $(L_t - \mathbb{E}[L_t])$ is a pure jump martingale (has no continuous part) with infinitely many small jumps in every time interval. In terms of the random measure of $L$, (12) can be written as

$$
L_t = \int_0^t \int_{\mathbb{R}} z \left(N(ds,dz) - f(z) \, ds \, dz\right) + t \mathbb{E}[L_1],
$$

where $f(z)$ is the density of the Lévy measure $\nu^{(GH)}(dz)$.

An alternative to the canonical representation (13) is the representation of the process by subordination, as a random change of Brownian motion. This means that the calendaristic time is changed to a business time or operational time, reflecting, for instance, the volume of trade at an exchange. Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion, and let $(\tau_t)_{t \geq 0}$ be a Lévy process generated by a GIG distribution (with parameters $(\lambda, \delta, \sqrt{\alpha^2 - \beta^2})$). Since the increments of $\tau$ are GIG distributed, and hence can only be positive, the process $\tau$ is increasing. Therefore, it can be interpreted as a time that increases with a randomly varying speed. The process

$$
L_t = bt + \beta \tau_t + B_{\tau_t}
$$

is then a generalized hyperbolic Lévy process (with parameters $(\lambda, \alpha, \beta, \delta, \mu)$).

**CGMY Lévy model.** This model, defined in Carr et al. [11], assumes that the martingale component of the movement in the logarithm of prices is given by a CGMY Lévy process. A CGMY process is a pure jump Lévy process with Lévy measure given by the four-parameter density

$$
d\nu^{(\text{CGMY})}/dz = \begin{cases} 
C \exp\left(-G|z|\right) / |z|^{1+\gamma}, & \text{for } z < 0, \\
C \exp\left(-M|z|\right) / |z|^{1+\gamma}, & \text{for } z > 0.
\end{cases}
$$

The range of the constant parameters is $C, G, M > 0$ and $Y < 2$. These parameters, as shown in [11], have the following interpretation: $C$ may be viewed as a measure of the overall level of activity, $G$ and $M$ respectively control the rate of exponential decay on the right and left of the Lévy density, leading to skewed distributions when they are unequal. As shown in Carr et al. [11, Theorem 2], the path behavior of the CGMY process is determined by the parameter $Y$: the paths have finite variation if and only if $Y \in [0, 1)$, and they have infinite variation if and only if $Y \in [1, 2)$.

**The variance gamma Lévy process.** The VG Lévy process can be defined as a Brownian motion with drift, time changed by a gamma process:

$$
L_t = \theta \gamma_t^1 \nu + \sigma B(\gamma_t^1 \nu),
$$

where $\nu$ and $\gamma_t$ are random variables with gamma distribution. The VG Lévy process can be defined as a Brownian motion with drift, time changed by a gamma process:

$$
L_t = \theta \gamma_t^1 \nu + \sigma B(\gamma_t^1 \nu),
$$
where $B$ is a standard Brownian motion, $\theta$ a constant, and $\gamma^1, \nu$ is a gamma process with mean rate unity, and variance rate $\nu$. Moreover $B$ and $\gamma^1, \nu$ are independent. It is the special case of a CGMY Lévy process [see 11, Sec. 2.2], with parameter identification

$$Y = 0, \quad C = \frac{1}{\nu}, \quad G = \frac{1}{\eta_n}, \quad M = \frac{1}{\eta_p}.$$  

Here $\eta_p, -\eta_n$ are the roots of the equation

$$x^2 - \theta \nu x - \sigma^2 \nu / 2 = 0.$$  

The Lévy measure of the variance gamma distribution $VG(\sigma, \theta, \nu, \mu)$ is given by

$$\nu(VG)(dz) = \exp\left(\frac{\theta z}{\sigma^2}\right) \frac{\nu |z| \exp\left(-\sqrt{2\nu} + \frac{\theta^2}{\sigma^2} \sigma |z|\right)}{\nu} dz,$$

[see 22, Eq. 14.]. This measure has infinite mass, and hence, a VG Lévy process has infinitely many jumps in any time interval. Since the Lévy measure of the variance gamma integrates the function $z \mapsto |z|$, a VG Lévy process has paths of finite variation.

### 3. Problem Formulation

We consider a small investor, whose actions cannot affect market prices and who wants to divide her wealth between the bond and the stock so as to maximize her utility. Let $\pi_t$ be the fraction of her wealth invested in the stock at time $t$, $C_t$ her cumulative consumption up to time $t$, and assume that there are no transaction costs in the market.

#### 3.1. Preferences.

The investor preferences over consumption patterns are represented by a non-time-additive utility functional

$$J(C) = \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(Y_t) dt\right],$$

where

- $\delta > 0$ is a discount factor;
- $U$ is the utility function defined over the average past consumption process $Y = (Y_t)_{t \geq 0}$, which reads

$$Y_t = Y_0 e^{-\beta t} + \beta e^{-\beta t} \int_0^t e^{\beta s} dC_s,$$

almost surely for all $t \geq 0$, with $Y_0 > 0$;
- $C = (C_t)_{t \geq 0}$ is a cumulative consumption process, assumed to be nonnegative, adapted, càdlàg, and nondecreasing, with initial condition $C_0^- = 0$;
- $\beta$ is a weighting factor.

Note that $Y$ is an exponentially weighted average of past consumption. Higher values of $\beta$ imply higher emphasis on the recent past consumption and less emphasis on the distant past consumption. The process $Y$ represent the level of satisfaction of the agent.

Starting with the economically sensible requirement that preferences should exhibit the notion of local substitution, Hindy and Huang [18] show that utility functionals which directly depend on the rate of consumption, unless they are linear, fail to incorporate the notion of local substitution. Hindy and Huang present (18) as an alternative model of preferences, and show that they incorporate the notion of substitution.

#### 3.2. Admissibility.

Denote by $X^{x, \pi, C}_t = (W^{\pi, C}_t, Y^{\pi, C}_t)_{t \geq 0}$ the two-dimensional process consisting of the level of the wealth and satisfaction of the investor, endowed with an initial wealth $x_1$ and an initial level of satisfaction $x_2$, and following an investment-consumption strategy $(\pi, C)$ at time $t$. In accordance with the model set forth in (3), (5) and (19), we then have

$$X^{\pi, C}_t = X_0^{\pi, C} + \beta C_{t^-} + \int_0^t b \left(X^{\pi, C}_s, \pi_s\right) ds + \int_0^t \sigma \left(X^{\pi, C}_s, \pi_s\right) dB_s + \int_0^t \int_{\mathbb{R}} f \left(X^{\pi, C}_t, \pi_{t^-}, z\right) \tilde{N}(ds, dz),$$
where we have introduced the notations:
\[ b : (x, \pi) \in \mathbb{R}^2 \times [0, 1] \mapsto b(x, \pi) = x_1 (r + (\hat{b} - r) \pi) e_1 - \beta x_2 e_2 \in \mathbb{R}^2, \]
\[ \sigma : (x, \pi) \in \mathbb{R}^2 \times [0, 1] \mapsto \sigma(x, \pi) = \text{diag} (x_1 \pi \sigma, 0) \in \mathbb{R}^{2 \times 2}, \]
\[ j : \mathbb{R}^2 \times [0, 1] \times \mathbb{R} \mapsto j(x, \pi, z) = x_1 \pi (e^z - 1) e_1 \in \mathbb{R}^2, \]
\[ \beta = (-1, \beta) \in \mathbb{R}^2, \text{ with } (e_1, e_2) \text{ being the unit vectors in } \mathbb{R}^2. \]

An investment-consumption strategy consists of a pair \((\pi, C)\):

1. An investment strategy \(\pi = (\pi_t)_{t \geq 0}\) which is a progressively measurable process that satisfies
   \[ \int_0^t |\pi_s W_s|^2 \, ds < +\infty \text{ almost surely for all } t \geq 0. \]
2. A consumption plan, which is a nonnegative, adapted, càdlàg and nondecreasing process, with initial value \(C_0^- = 0\) (so that we allow for an initial jump when \(C_0 > 0\)), that satisfies
   \[ \mathbb{E} [C_t] < \infty \text{ almost surely for all } t \geq 0. \]

An investment-consumption strategy \((\pi, C)\) is called admissible for an initial level of wealth and satisfaction \(x\), and we write \((\pi, C) \in \mathcal{A}_x\), if it satisfies

1. the short-selling constraint: \(\pi_t \in [0, 1]\), almost surely for all \(t \geq 0\),
2. the budget (state) constraint: \(W_t^{\pi, \pi, C} \geq 0\), almost surely for all \(t \geq 0\).

Due to the budget constraint the process \(X^{\pi, \pi, C}\), starting from an initial level \(x\), is constrained to stay in the positive quadrant of \(\mathbb{R}^2\),

\[ \mathcal{D} = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \left| x_1 > 0, x_2 > 0 \right. \right\}, \]

for any admissible strategy \((\pi, C) \in \mathcal{A}_x\).

Given her initial level of wealth and satisfaction \(x\) at time 0, the investor problem is to find a (the) policy \((\pi, C) \in \mathcal{A}_x\) that solves the portfolio-consumption problem

\[ V(x) = \sup_{(\pi, C) \in \mathcal{A}_x} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} U(Y_t) \, dt \right]. \]

Define the integro-differential operator \(\mathcal{B}^\pi\) by

\[ \mathcal{B}^\pi v(x) := \int_\mathbb{R} \left( v(x + j(x, \pi, z)) - v(x) - \langle j(x, \pi, z), D_v(x) \rangle \right) v(z) \, dz, \]

where \(D_v\) denotes the gradient of a scalar function \(v\), and \(\langle \cdot, \cdot \rangle\) denotes the Euclidean inner product. Define the second order differential operator \(\mathcal{L}^\pi\) by

\[ \mathcal{L}^\pi v(x) := \langle b(x, \pi), D_v(x) \rangle + \frac{1}{2} \text{tr} \left( \sigma \sigma'(x, \pi) D^2_v(x) \right), \]

where \(D^2_v\) denotes the Hessian matrix of a scalar function \(v\), and \('\) the transpose of a matrix. Let \(C^p(\mathcal{D})\) denote the usual space of \(p\)-times continuously differentiable functions on \(\mathcal{D}\). For \(p > 0\), \(\gamma > 0\), and \(\varphi \in C^p(\mathcal{D})\), we write \(\varphi \in C^p_\gamma(\mathcal{D})\) whenever \(\varphi\) satisfies the growth condition

\[ \sup_{x \in \mathcal{D}} \left( |\varphi(x)| / (1 + x_1 + x_2)^\gamma \right) < \infty. \]

The domain of the integral operator defined in (21) is the class of functions \(C^2_\gamma(\mathcal{D})\).

The dynamic programming equation associated to our optimization problem is the degenerate elliptic integro-differential variational inequality

\[ \max \left( G(D_v); F(x, v, D_v, D^2_v, \mathcal{B}^\pi v) \right) = 0 \quad \text{in } \mathcal{D}, \]

where the nonlinearity \(F\) and the gradient constraint \(G\) are given by

\[ F(x, v, D_v, D^2_v, \mathcal{B}^\pi v) = -\delta r + \max_{\pi \in [0, 1]} \left( (\mathcal{L}^\pi + \mathcal{B}^\pi) v \right) + U(x_2), \quad G(D_v) = \left\langle \beta, D_v \right\rangle. \]

The dynamic programming equation (24) is augmented by the Dirichlet boundary condition (see [7, 8] for a state constraint boundary condition)

\[ v(0, x_2) = \psi(x_2), \quad x_2 \geq 0, \]
where $\psi : [0, \infty) \to \mathbb{R}$ is defined by
\[
\psi(x_2) = \int_0^\infty e^{-\delta t} U(x_2 e^{-\delta t}) \, dt.
\]

Let
\[
\text{USC}(\mathcal{F}) := \{ \text{upper semicontinuous functions } v : \mathcal{F} \to \mathbb{R} \}, \quad \text{and}
\]
\[
\text{LSC}(\mathcal{F}) := \{ \text{lower semicontinuous functions } v : \mathcal{F} \to \mathbb{R} \}.
\]

For $v \in \text{USC}(\mathcal{F})$ (LSC(\mathcal{F})) and $\bar{\gamma} > 0$, we write $v \in \text{USC}_{\bar{\gamma}}(\mathcal{F})$ (LSC$_{\bar{\gamma}}(\mathcal{F})$) whenever $v$ satisfies the growth condition (23).

In Benth, Karlsen, and Reikvam [7, 8], the authors define a notion of viscosity solution to (24) (which we do not repeat here), and characterize the value function as the unique viscosity solution of (24). More precisely, the following theorem was proved:

**Theorem 1.** The value function $V$ defined in (20) is non-decreasing, concave and uniformly continuous in $\mathcal{F}$. Furthermore, $V$ is non-negative, and satisfies the growth condition $0 \leq V(x) \leq K (1 + x_1 + x_2)^\gamma$, for $x \in \mathcal{F}$. Moreover, if for some $\alpha \in (0,1]$, we have $\delta > k(\gamma)$ and $U \in C^{0,\alpha}((0,\infty))$, then $V \in C^{0,\alpha}(\mathcal{F})$. The value function $V$ in (20) is the unique viscosity solution of the dynamic programming equation (24)-(25).

The uniqueness assertion in Theorem 1 follows from the following strong comparison principle [7, 8]:

**Theorem 2.** Suppose $v \in \text{USC}_{\bar{\gamma}}(\mathcal{F})$ is a subsolution of (24)-(25) and $\bar{v} \in \text{LSC}_{\bar{\gamma}}(\mathcal{F})$ is a supersolution of (24)-(25). Then
\[
v \leq \bar{v} \quad \text{on } \mathcal{F}.
\]

We refer to [13], [17] for a general overview of viscosity solutions theory.

### 4. Approximating the Financial Market

Our aim is to approximate $L$ by a simple Lévy process. This simpler process will eventually lead to a simpler (and hence more numerically tractable) version of the dynamic programming equation (24). We shall call a Lévy process $(L_t)_{t \geq 0}$ simple if it is given by an independent sum:
\[
L_t = bt + \sigma B_t + \sum_{k=0}^d z_k N^k_t,
\]
of a drift $bt$, a Brownian motion $(B_t)_{t \geq 0}$, and a compound Poisson process $\sum_{k=0}^d z_k N^k_t$. The Poisson processes $(N^k)_{k=0,\ldots,d}$, with parameters $\lambda^0, \ldots, \lambda^d$ respectively, are independent. The jump locations $z_0, \ldots, z_k$ are different non-null numbers (deterministic). Thus, the generating triplet of $L$ is given by $T(L) = \left(h, \sigma, \sum_{k=0}^d \lambda^k \delta_{z_k}\right)$.

Under (H1) we have $\int_{\mathbb{R}} |z|^2 \nu(dz) < \infty$, and it is therefore not necessary to use the truncation function $\mathbf{1}_{\{ |z| > 1\}}$. More precisely, we set
\[
\begin{align*}
\mathbf{b}' &= b + \int_{\{ |z| \geq 1\}} z \nu(dz), \\
(s')^2 &= \sigma^2 + \int_{\mathbb{R}} z^2 \nu(dz),
\end{align*}
\]
and of course we have
\[
\Psi_{b,\sigma,\nu}(\theta) = \Psi_{b', s', \nu}(\theta) = i \theta b' + \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (e^{i \theta z} - 1 - i \theta z) \nu(dz).
\]
Note that \( b' = \mathbb{E}[L_t] \). In terms of the Lévy-Khintchine decomposition, equation (28) translates to

\[
L_t = b't + \sigma B_t + \int_0^t \int_{\mathbb{R}} z \tilde{N}(ds, dz).
\]

Let \( \kappa \in (0,1) \). Then we may write

\[
L_t = L^{(\kappa)} + J^{(\kappa)}, \quad \text{where}
\]

\[
J^{(\kappa)} : = \int_0^t \int_{0 < |z| < \kappa} z \tilde{N}(ds, dz), \quad \text{and}
\]

\[
L^{(\kappa)} : = b't + \sigma B_t + \int_0^t \int_{|z| \geq \kappa} z \tilde{N}(ds, dz).
\]

4.1. **Approximating** \( L^{(\kappa)} \). The process \( L^{(\kappa)} \) is written as an independent sum of drift \( b't \), a Brownian part \( \sigma B \), and a compound Poisson process \( \tilde{T}^{\kappa} : = \int_0^t \int_{|z| \geq \kappa} \tilde{N}(ds, dz) \) with

\[
\begin{cases}
\lambda^{(\kappa)}(\nu) = \nu \left( \{ |z| \geq \kappa \} \right), \\
\nu^{(\kappa)} = 1_{\{ |z| \geq \kappa \}} \nu(dz) \text{ denotes its Lévy measure.}
\end{cases}
\]

To approximate the process \( L^{(\kappa)} \) by a simple Lévy process, we consider a sequence of non-null real-numbers (jump heights)

\[
\bar{z}_0 < \bar{z}_1 < \cdots < \bar{z}_k = -\kappa, \quad \kappa = \bar{z}_{k+1} < \cdots < \bar{z}_d,
\]

and let

\[
\begin{cases}
\Lambda_k = (\bar{z}_{k-1}, \bar{z}_k), \quad \text{for } 1 \leq k \leq \bar{k}, \\
\Lambda_k = (\bar{z}_k, \bar{z}_{k+1}), \quad \text{for } \bar{k} + 1 \leq k \leq d.
\end{cases}
\]

Here we have assumed that \( \bar{z}_{-1} = -\infty, \bar{z}_{d+1} = +\infty \). Since \( b \) (the first characteristic of \( L \)) depends on the choice of the truncation function (in our case \( h(z) = \mathbb{1}_{\{ |z| < 1 \}} \)), we need to assume that there exists \( k, k' \in \{0, \ldots, d\} \) such that \( \bar{z}_k = -1, \bar{z}_{k'} = 1 \).

Let \( \ell \) be a parameter that determines the size of the partition \( \Lambda_k \).

For each \( k \), the process \( N^k = (N^k_t)_{t \geq 0} \) given by

\[
N^k_t = \sum_{0 < s \leq t} \mathbb{1}_{\Lambda_k}(\Delta L_s),
\]

is a Poisson process with jump arrival rate

\[
\lambda^k = \nu(\Lambda_k).
\]

For \( k \neq l \), the processes \( N^k, N^l \) never jump simultaneously, and are therefore independent. Moreover, the sum process \( \sum_{k=0}^d N^k_t \) is a Poisson process with jump arrival rate \( \sum_{k=0}^d \lambda^k = \lambda^{(\kappa)} \), where the aggregated jump intensity is given in (32).

As the representation (31) suggests, we approximate the process \( L^{(\kappa)} \) by the process \( L^{(\kappa,\ell)} = (L^{(\kappa,\ell)}_t)_{t \geq 0} \), defined by

\[
L^{(\kappa,\ell)}_t = b'() + \sigma B_t + \sum_{k=0}^d \bar{z}_k \left( N^k_t - t\lambda^k \right),
\]

where

\[
b'(\kappa) = b + \sum_{|z_k| \geq 1} \lambda^k z_k.
\]
The jump size \( z_k \) in each interval \((\Lambda_k)_{k=0,...,d}\), is chosen to satisfy
\[
\int_{\Lambda_k} \left( e^z - 1 - z1_{\{|z|<1\}} \right) \nu^{(k)}(dz) = \lambda^k \left( e^{z_k} - 1 - z_k1_{\{|z|<1\}} \right),
\]
so that we get \( \mathbb{E}[\exp(L_\kappa^{(k)})] = \mathbb{E}[\exp(L_\kappa^{(k)})] \). One can prove that equation (37) admits a unique solution, see [15] for details. Note that the approximating process \( L^{(\kappa,\ell)} \) corresponds to approximating
\[
\begin{align*}
\{ \nu^{(k)}(dz) \text{ by the Lévy measure } & \nu^{(k,\ell)}(dz) = \sum_{k=0}^d \lambda^k \delta_{z_k}(dz), \text{ which} \\
\{ Q^{(\kappa,\ell)}(dz) & = \sum_{k=0}^d q_k \delta_{z_k}(dz). \}
\end{align*}
\]

The probability weights \( (q_k)_{k=0,...,d} \) are given by
\[
q_k = \frac{\lambda^k}{\lambda^{(k)}}.
\]

For a sequence of Lévy processes \( (L^{(n)})_{n \geq 0} \) and \( L \) with characteristics triplet \( \mathbb{T}(L^{(n)}) = (b, \sigma^2, \nu^{(n)}) \) and \( \mathbb{T}(L) = (b, \sigma^2, \nu) \), respectively, we shall write
\[
\mathbb{T}(L^{(n)}) \to \mathbb{T}(L),
\]
whenever the following conditions hold:

(C1) \( b_n' \to b' \),
(C2) \( \sigma_n' \to \sigma' \),
(C3) \( \int \mathbb{R} g(z) \nu^{(n)}(dz) \to \int \mathbb{R} g(z) \nu(dz) \) for all \( g \in C_2(\mathbb{R}) \).

Here \( C_2(\mathbb{R}) : = \text{The set of all continuous bounded functions: } \mathbb{R} \to \mathbb{R} \text{ which are 0 around 0 and have a limit at infinity.} \)

Assume \( z^2 \) is uniformly integrable w.r.t. the sequence of Lévy measures \( \nu^{(n)} \). Then using [20, Theorem VII.2.14 and Corollary VII.3.6] and the above notation we have
\[
\begin{align*}
\mathbb{T}(L^{(n)}) \to \mathbb{T}(L), \text{ implies} \\
L^{(n)} \overset{D}{\to} L, \text{ or equivalently, } L_1^{(n)} \overset{D}{\to} L_1,
\end{align*}
\]
where \( \overset{D}{\to} \) denotes convergence in law.

Clearly we get \( \mathbb{T}(L^{(\kappa,\ell)}) \to \mathbb{T}(L) \), by sending first \( \ell \uparrow \infty \) and then \( \kappa \downarrow 0 \), and therefore have
\( L^{(\kappa,\ell)} \overset{D}{\to} L \) as \( \kappa \downarrow 0, \ell \uparrow \infty \).

Moreover, by the choice of the sequence of jump locations \( (z_k)_{k=0,...,d} \) as in (37), we have
(P1) \( \nu^{(k,\ell)} \to \nu^{(k)} \) weakly,
(P2) \( \int e^z \nu^{(k,\ell)}(dz) < \infty, \int e^z \nu^{(k)}(dz) < \infty, \)
(P3) \( \int e^z \nu^{(k,\ell)}(dz) = \int e^z \nu^{(k)}(dz) \).

We can conclude, using Chow and Teicher [12, Thm 8.1.2], that \( \exp(z) \) is uniformly integrable relative to \( \nu^{(k,\ell)} \).

Define the class of functions
\[
\mathcal{G} : = \left\{ g \in C^2(\mathbb{R}) \bigg| g(z) = O(z^2) \text{ as } z \to 0 \text{ and } g(z) = O(e^z) \text{ as } z \to \infty \right\}.
\]
For any \( g \in \mathcal{G} \), the boundedness condition \( 0 \leq g(z) \leq e^z \) implies the uniform integrability of the function \( g(z) \) w.r.t. the same sequence. Another application of the Chow and Teicher [12, Thm. 8.1.2] yields the following
\[
\lim_{\kappa \downarrow 0} \lim_{\ell \uparrow \infty} \int \mathbb{R} g(z) \nu^{(k,\ell)}(dz) = \int \mathbb{R} g(z) \nu(dz)
\]
for all \( g \in \mathcal{G} \). We refer to [15] for more details.
4.2. Approximating $J^\kappa$. The idea is to throw the small jumps $J^\kappa$ of $L$ away and approximate $(L_t)_{t \geq 0}$ by the process $L^{(\kappa)}$ given by (31). Neglecting $J^\kappa$ for small $\kappa$ is not appropriate, however, since small jumps are dominating the behavior of the Lévy process in most Lévy models found in the literature (see Section 2.2). Analyzing the behavior of the densities of both the NIG and HYP Lévy motions around zero, reveals this fact, since we have:

$$d\nu^{(\text{NIG})}/dz = O(z^{-2}) \quad \text{and} \quad d\nu^{(\text{HYP})}/dz = O(z^{-2}) \quad \text{as} \quad z \to 0.$$ 

The same asymptotic behavior of the density of the Lévy measure around 0 holds for the general class of GH. One approach, as has been suggested in some cases (Rydberg [26]), is to assume that

$$J^\kappa \overset{D}{\to} \sigma(\kappa) \tilde{B}, \quad \text{as} \quad \kappa \downarrow 0,$$

and replace $J^\kappa$ by the limit process $\sigma(\kappa) B_t$. Here

$$(43) \sigma^2(\kappa) := \text{Var} (J^\kappa) = \int_{\{ |z| < \kappa \}} z^2 \nu(dz),$$

and $\tilde{B}$ is a Brownian motion independent of $B$ and $N$.

When the following condition

$$\frac{\sigma(\kappa)}{\kappa} \to 0 \quad \text{as} \quad \kappa \downarrow 0,$$

is satisfied, one can show that indeed a Brownian limit holds, see [15]. Using this condition, together with the asymptotic behavior of the density of the Lévy measure, one can show that a Brownian limit holds for the financial Lévy models: GH, NIG, HYP, and also for the CGMY (for $Y \in [1, 2]$) model.

Remark 1. Recall that the class of Lévy processes is closed under convergence in law ([see 20, Chap. VII.2.6]). Since the sequence $\{ \mathcal{L}(\sigma^2(\kappa)^{-1} J^\kappa) \}, 0 < |\kappa| < 1$ is tight. One can always find a sequence of truncations $\kappa_n \downarrow 0$ and a Lévy process $K$ such that $\sigma^2(\kappa_n)^{-1} J^\kappa_n \overset{D}{\to} K$.

We assume that the error in the approximation is approximately normal and consider the following approximating process $L^{(\kappa, \ell)}$ to $L$ given by

$$(44) L_t^{(\kappa, \ell)} = b'(\kappa)t + \tilde{\sigma}(\kappa)B_t + \sum_{k=0}^d z_k \left( N_t^k - t\lambda^k \right),$$

where

$$(45) \tilde{\sigma}_k = \left( \sigma^2 + \sigma^2(\kappa) \right)^{1/2}.$$ 

Observe that, $L^{(\kappa, \ell)}$ is an independent sum of a drift, a Brownian component and $(d+1)$ compound Poisson processes.

4.3. Approximating the investment-consumption problem. Consider the financial Lévy market with stock price driven by $L^{(\kappa, \ell)}$

$$(46) S^{(\kappa, \ell)} := \exp \left( L^{(\kappa, \ell)} \right),$$

where $L^{(\kappa, \ell)}$ is given by (44). Since the mapping $\alpha \mapsto \alpha \circ \exp$ is continuous from $D[0, \infty)$ to $D[0, \infty)$, we conclude that the sequence of stock price models $(S^{(\kappa, \ell)})$ converges in law to the original exponential Lévy model $S$ given by (1).

Denote by $X^{(\kappa, \ell)} = (W^{(\kappa, \ell)}, Y^{(\kappa, \ell)})$ the corresponding level of wealth and satisfaction process. Define the value function corresponding to the portfolio problem in this approximating market:

$$(47) V^{(\kappa, \ell)}(x) := \sup_{(\pi, C) \in \mathscr{A}_x} \mathbb{E}_x \left( \int_0^\infty e^{-\delta t} U(Y^{\kappa, \ell}_t) \, dt \right).$$

Relying on the stability properties of viscosity solutions under the passage to weak (or half-relaxed) limits ([1, 13]) as well as the strong comparison principle for the viscosity solutions found in Theorem 2 we are able to show the following
Theorem 3. Let $V^{(\kappa, \ell)}$ be the value function (47) of the approximating investment-consumption problem and let $V$ be the value function (20) of the original investment-consumption problem. Then $V^{(\kappa, \ell)} \to V$ as $\kappa \downarrow 0, \ell \uparrow \infty$. The convergence is uniform on any compact subset of $\mathcal{D}$.

We refer to [15] for the proof.

5. The Markov Chain scheme and its Transition Probabilities

Following [16, 19, 21], we present and analyze in this section a Markov chain approximation scheme for the investment-consumption problem described in the previous section. Our starting point is the approximating investment-consumption problem described in Subsection 4.3. Again, we refer to [15] for a more detailed presentation.

Given a discretization parameter $h > 0$, we let $h\mathbb{Z}^2$ denote the lattice $\{(ih, jh) \mid i, j \in \mathbb{Z}\}$ and consider the discretization of the state-space $\mathcal{D} = \mathcal{Y} \cap h\mathbb{Z}^2$ of the process $X^{\kappa, \ell} = (X_{t}^{\kappa, \ell})_{t \geq 0}$. We divide the set $\mathcal{D}$ into disjoint subsets $\mathcal{D}_h = \mathcal{Y} \cap h\mathbb{Z}^2$ and $\partial \mathcal{D}_h$, where the latter represent a discretization of the boundary of $\mathcal{D}$. For a fixed $h > 0$, we introduce the space $\mathcal{A}^h$ of discrete strategies consisting of a discrete investment strategy $\pi = (\pi_n)_{n \geq 0}$ and a discrete consumption strategy $C = (C_n)_{n \geq 0}$:

$$\mathcal{A}^h = \left\{ \left( \pi, C \right) \mid \pi_n \in [0, 1], C_n = C_{n-1} + \Delta C, \Delta C = 0 \text{ or } \Delta C = \frac{h}{\max(1, \beta)}, n = 0, 1, \ldots \right\}.$$ 

Our first chief goal is to approximate the continuous-time controlled jump-diffusion process $X^{\kappa, \ell}$ by a discrete-time discrete-state controlled Markov chain

$$\xi^h = (\xi^h_n)_{n \geq 0} = \left( W^{\kappa, \ell, h}_n, Y^{\kappa, \ell, h}_n \right)_{n \geq 0},$$

whose state-space is $\mathcal{D}_h$ and $n$ denotes the discrete time level of the chain. The chain $\xi^h$ is parameterized by the parameter $h$ such that, as $h \downarrow 0$, the local properties of the chain resemble closely those of the original process $X^{\kappa, \ell}$, see Subsection 5.1 for details. To simplify the notation, we suppress the dependency on $\kappa, \ell$ in the Markov chain $\xi^h$. Before we continue, let us note here that our wealth equation $W^{\kappa, \ell}$ has control dependent volatility and “jump-volatility”, as opposed to most of the problems treated in [21].

The approximating Markov chain is characterized by its transition probabilities:

$$p^h(x, x' \mid (\pi, C)) : = \mathbb{P} \left[ \xi^h_{n+1} = x' \mid \xi^h_n = x, \pi_n = \pi, C_n = C \right],$$

for $\pi \in [0, 1], C \in \mathbb{R}$ (here we abuse the notation and denote the control variable by the same letter as the control processes), and an appropriate interpolation interval $\Delta t^h$ which in our case will depend on the state variable $x \in \mathcal{D}$. We shall use the following notation:

- $\Delta t^h_n = \Delta t^h(\xi^h_n)$: the $n$th transition time interval.
- $t^h_n = \sum_{m=1}^{n-1} \Delta t^h_m$ : the time of occurrence of the $n$th transition.
- $\Delta \xi^h_n = \xi^h_{n+1} - \xi^h_n$ : the incremental difference.

The chain $\xi^h$ has the property that, at each time level $n$, there is a choice between investment and consumption, that is, the controlled transition probabilities of the chain verify

$$p^h(x, x' \mid (\pi, C)) = \begin{cases} p^h(x, x' \mid \pi), & \Delta C = 0, \\ p^h(x, x' \mid C), & \Delta C = h/\max(1, \beta), \end{cases}$$

where $p^h(x, x' \mid \pi)$ denotes the transition probabilities corresponding to investment and $p^h(x, x' \mid C)$ denotes the transition probabilities corresponding to consumption. Since $X^{\kappa, \ell}$ is a jump-diffusion process, we shall need to further distinguish between a diffusion-step and a jump-step:

$$p^h(x, x' \mid \pi) = \begin{cases} p^D_p(x, x' \mid \pi), & \text{diffusion-step}, \\ p^J_p(x, x' \mid \pi), & \text{jump-step}. \end{cases}$$
We will specify the transition probabilities \( p^h_D(x, x' \mid \pi) \), \( p^h_J(x, x' \mid \pi) \), \( p^h(x, x' \mid C) \) below, while for the actual derivation of them we refer to [15], see also [16, 19, 21]. We mention that the transition probabilities of \( \xi^h \) are derived from a monotone, consistent and stable finite difference scheme.

Fix a chain \( \xi^h = (\xi^h_n)_{n \geq 0} \) and its corresponding discretization parameter \( h \). The transition probabilities for this Markov chain are specified as follows:

(a) In the case of no consumption (\( \Delta C = 0 \)), the chain advances from its current state \( \xi^h_n = x \) to \( \xi^h_{n+1} \) by either a diffusion-step or a jump-step. In the first case, the chain can remain at its current state \( x \) or move to one of the three neighboring points \( x \mp he_1 \) and \( x - he_2 \). In the second case, the chain jumps to one of the points in \( x + j(x, \pi, z_k) \). The transition probabilities are given in the form

\[
p^h(x, x' \mid \pi) = \left( 1 - \lambda^{(\pi)} \Delta t^h(x) \right) p^h_D(x, x' \mid \pi) + \lambda^{(\pi)} \Delta t^h(x) p^h_J(x, x' \mid \pi),
\]

where one can think of \( 1 - \lambda^{(\pi)} \Delta t^h(x) \) as the probability of a diffusion-step and \( \lambda^{(\pi)} \Delta t^h(x) \) as the probability of a jump-step. The diffusion-step transition probabilities \( p^h_D(x, x' \mid \pi) \) are given by

\[
\begin{align*}
p^h_D(x, x - he_1 \mid \pi) &= \left( h x_1 \pi r + h \lambda^{(\pi)} \sum_{k=k+1}^d q_k x_1 \pi (e^{\varepsilon_k} - 1) + x_1^2 \pi^2 \sigma^2 / 2 \right) / Q^h(x), \\
p^h_D(x, x + he_1 \mid \pi) &= \left( h x_1 (r + \pi \hat{b}_{n,t}) - h \lambda^{(\pi)} \sum_{k=0}^d q_k x_1 \pi (e^{\varepsilon_k} - 1) + x_1^2 \pi^2 \sigma^2 / 2 \right) / Q^h(x), \\
p^h_D(x, x - he_2 \mid \pi) &= \frac{h \beta x_2}{Q^h(x)}, \quad \text{and} \\
p^h_D(x, x \mid \pi) &= 1 - \sum_{x'} p^h_D(x, x' \mid \pi), \quad x' = x \mp he_1, x - he_2.
\end{align*}
\]

The constant \( \hat{b}_{n,t} \) is given by

\[
\hat{b}_{n,t} = b + \frac{1}{2} \sigma^2 + \lambda^{(\pi)} \sum_{k=0}^d q_k \left( e^{\varepsilon_k} - 1 - 1 \{ |z_k| < 1 \} \right).
\]

The normalization factor \( Q^h : \mathcal{D} \rightarrow \mathbb{R} \) is defined as

\[
Q^h(x) = \max_{\pi \in [0,1]} \left( h x_1 \pi r + h x_1 \left( r + \pi \hat{b}_{n,t} \right) + h \beta x_2 + x_1^2 \pi^2 \sigma^2 \right. \\
+ \left. h \lambda^{(\pi)} \sum_{k=k+1}^d q_k x_1 \pi (e^{\varepsilon_k} - 1) - h \lambda^{(\pi)} \sum_{k=0}^d q_k x_1 \pi (e^{\varepsilon_k} - 1) \right).
\]

Note that the normalization factor \( Q^h(x) \) is independent of the control \( \pi \), and \( Q^h(x) > 0 \) for any \( x \in \mathcal{D}_h \). The interpolation time is given by

\[
\Delta t^h : \mathcal{D} \rightarrow \mathbb{R}, \quad \Delta t^h(x) = \frac{h^2}{Q^h(x)}.
\]

The jump-step transition probabilities \( p^h_J(x, x' \mid \pi) \) are given by

\[
p^h_J(x, x + j(x, \pi, z_k) \mid \pi) = q_k, \quad \text{for all } k = 0, \ldots, d.
\]

The points \( x + j(x, \pi, z_k) \) are not necessarily grid points. For each \( k \), we can represent the point \( x + j(x, \pi, z_k) \) as a convex combination of the four points of the rectangle \( \mathcal{R}^h (x + j(x, \pi, z_k)) \) in which it falls. Let \( w^h (x, x'; \pi) \) be the weights used for the convexification. To produce transition probabilities to grid points we let the transition probabilities from \( x \) to each point \( x' \) in \( \mathcal{R}^h (x + j(x, \pi, z_k)) \) be

\[
p^h_J(x, x' \mid \pi) = w^h (x, x'; \pi) p^h_J(x, x + j(x, \pi, z_k) \mid \pi).
\]
(b) When consumption occurs \((\Delta C = h/\max(1, \beta))\), the chain jumps immediately along the direction \(\tilde{\beta}\) from its current state \(\xi_n^h = x\) to the state \(\xi_{n+1}^h = x' = x + h\tilde{\beta}\) with probability 1. Then the one-step transition probability is given by

\[
p^h(x, x + h\tilde{\beta} \mid C) = 1, \quad \text{and} \quad \Delta t^h(x) = 0.
\]

Since the point \(x + h\tilde{\beta}\) does not in general coincide with a grid point on \(\mathcal{D}_h\), to produce grid transition probabilities to grid points we proceed as follows:

(a) When \(\beta < 1\), then \(\Delta C = h\), and the point \((x_1 - h, x_2 + h\beta)\) belongs to the segment spanned by the grid points \(x - he_1\) and \(x - he_1 + he_2\). We randomize between these two points such that the expected random increments will be along the direction \(\tilde{\beta}\), and therefore define the transition probabilities as

\[
p^h(x, x - he_1 \mid C) = 1 - \beta, \quad p^h(x, x - he_1 + he_2 \mid C) = \beta.
\]

(b) When \(\beta > 1\), then \(\Delta C = h/\beta\), and the point \((x_1 - h, x_2 + h\beta)\) belongs to the segment spanned by the grid points \(x - he_1 + he_2\) and \(x + he_2\). We define the transition probabilities as

\[
p^h(x, x + he_2 \mid C) = 1/\beta, \quad p^h(x, x - he_1 + he_2 \mid C) = 1 - 1/\beta.
\]

5.1. Consistency of the Markov chain. We may write the jump part of the process \(X^{\kappa,\ell}\)

\[
J^{\kappa,\ell}_t := \int_0^t \int_{\mathbb{R}} j (X^{\kappa,\ell}_s, \pi_s, z) N^{\kappa,\ell}(ds, dz) \quad \text{as} \quad J^{\kappa,\ell}_t = \sum_{n:0<\tau_n\leq t} J (X^{\kappa,\ell}_{\tau_n}, \pi_{\tau_n}, \tilde{z}_n),
\]

with \((\tau_{n+1} - \tau_n)\) and \((\tilde{z}_n)_{n \geq 1}\) being mutually independent random variables; \((\tau_{n+1} - \tau_n)\) is exponentially distributed with mean value \(1/\lambda(\kappa)\), and \((\tilde{z}_n)_{n \geq 1}\) have the common distribution \(Q^{(\kappa,\ell)}(dz)\).

Because \((\tau_{n+1} - \tau_n)\) is exponentially distributed, we have

\[
P\left(\left\{X^{\kappa,\ell}_t \text{ jumps on } [t, t + \Delta] \mid X^{\kappa,\ell}_s, \pi_s, B_s, N^{\kappa,\ell}_s, s \leq t\right\}\right) = \lambda(\kappa)\Delta + o(\Delta),
\]

for any small \(\Delta > 0\). By the independence properties and the definition of \((\tilde{z}_n)_{n \geq 1}\), we have

\[
P\left(\left\{X^{\kappa,\ell}_t - X^{\kappa,\ell}_{t^-} \in H \mid X^{\kappa,\ell}_{t^-} = x, \pi_{t^-} = \pi, X^{\kappa,\ell}_s, \pi_s, B_s, N^{\kappa,\ell}_s, s < t\right\}\right)
\]

\[
= Q^{(\kappa,\ell)}\left(\left\{z : j(x^{\kappa,\ell}_t, \pi_{t^-}, z) \in H\right\}\right), \quad H \in \mathcal{B}(\mathbb{R}).
\]

(A) In the case of no-consumption, the process \(X^{\kappa,\ell}\) can thus be viewed as a process which evolves as a diffusion, with jumps occurring at random according to the rate defined by (51). Given that the \(n\)th jump occurs at time \(\tau_n\), its values are given according to the probability law (52). The approximating Markov chain as specified above, is therefore derived by piecing together an approximation of the continuous-part of \(X^{\kappa,\ell}\) with an approximation of the jump-part of \(X^{\kappa,\ell}\). More precisely, we proceed as follows: We define “step” to be random variable, taking either the value “jump-step” with probability \(\lambda(\kappa)\Delta t^h(\xi_n^h)\), or “diffusion-step” with probability \((1 - \lambda(\kappa)\Delta t^h(\xi_n^h))\), given that the time level is \(n\), \(\xi_n^h = x\), and \(\pi_n = \pi\). If “step” is a “jump-step”, the next state \(\xi_{n+1}^h = x'\) is determined via the transition probability \(p_D^h(x, x' \mid \pi)\). If “step” is a “diffusion-step”, then the next state \(\xi_{n+1}^h = x'\) is determined via the transition probability \(p_D^h(x, x' \mid \pi)\).

(i) The Markov chain with transition probabilities \(p_D^h(x, x' \mid \pi)\) is consistent with the continuous-part of the jump-diffusion process \(X^{\kappa,\ell}\), in the sense that it approximates the mean and the quadratic mean of its increments to the order \(\Delta t^h\) (see, e.g., [21]).

(ii) The transition states of the Markov chain with transition probabilities \(p_D^h(x, x' \mid \pi)\) have the appropriate distribution \(Q^{(\kappa,\ell)}\) (see (52)). Moreover, since “step” is a “jump-step” with probability \(\lambda(\kappa)\Delta t^h(\xi_n^h)\), the chain has the appropriate rate of jumps between transition times (see (51)). Therefore the chain is consistent with the local properties of the jump-part of \(X^{\kappa,\ell}\).
In the case of consumption, we have the local consistency relation

\[ v'_{n+1} = x' \left[ \psi_{n_m}, \pi_{n_m}, C_m, m \leq n \right] = \begin{cases} p^h\left( x_n', x' \mid \pi_n \right), & \Delta C_n = 0, \\ p^h\left( x_n', x' \mid C_n \right), & \Delta C_n = h / \max(1, \beta). \end{cases} \]

We let the class of admissible control laws \( \mathcal{A}^h \) be

\[ \mathcal{A}^h_{adm} = \{ (\pi, C) \in \mathcal{A}^h \mid (\pi, C) \text{ is Markov} \}. \]

The control problem for the discrete Markov chain \( \xi^h \) is then to solve the program:

\[ V^{(\kappa, \ell, h)}(x) = \max_{(\pi, C) \in \mathcal{A}^h_{adm}} \mathbb{E}^x \left[ \sum_{n=0}^{\infty} e^{-\delta t_n^h} U \left( V^{(\kappa, \ell, h)}_n \right) \Delta t_n^h \right], \quad x \in \mathcal{D}^h. \]

The discrete dynamic programming equation associated to our control problem (54) reads for each grid point \( x \in \mathcal{D}^h \),

\[ v(x) = \max \left( \sum_{x'} p^h(x, x' \mid C) v(x') ; \max_{\pi \in [0, 1]} \left[ e^{-\delta \Delta t^h(x)} \sum_{x'} p^h(x, x' \mid \pi) v(x') + U(x_2) \Delta t^h(x) \right] \right), \]

where we sum over all grid points \( x' \in \mathcal{D}^h \). To have a well-posed discrete problem, we must impose the boundary condition

\[ v(0, x_2) = \psi(x_2), \quad x_2 \geq 0, \]

where \( \psi \) is defined in (26).

To compute \( V^{(\kappa, \ell, h)} \) we need to solve the discrete problem (55)-(56). Methods for solving this problem will be discussed in our forthcoming work [15], which also must take into account a truncation of the computational domain. Roughly speaking, the problem (55)-(56) can be solved by any one of the standard policy or value iteration methods [21].

Our main result is the convergence of the sequence of discrete value functions \( (V^{(\kappa, \ell, h)})_{\kappa, \ell, h} \) to the value function of the original continuous-time investment-consumption problem.

**Theorem 4** (Convergence). Let \( V^{(\kappa, \ell, h)} \) be the value function (54) of the discrete Markov chain decision problem, and let \( V \) be the value function (20) of the continuous-time investment-consumption problem. By sending first \( h \downarrow 0 \), then \( \ell \uparrow \infty \), and finally \( \kappa \downarrow 0 \) (in this order), \( V^{(\kappa, \ell, h)} \to V \). The convergence is uniform on any compact subset of \( \mathcal{D} \).

The convergence proof relies on the stability properties of viscosity solutions under the passage to weak limits ([1, 13]), the convergence result in (42), as well as the strong comparison principle for the viscosity solutions found in Theorem 2. We refer [15] for the proof.
REFERENCES


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