Sovereign Debt and Structural Reforms*

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Abstract

We construct a dynamic theory of sovereign debt and structural reforms with limited enforcement and moral hazard. A sovereign country in recession would like to smooth consumption and can make costly reforms to speed up recovery. The sovereign can renge on contracts by suffering a stochastic cost. The Constrained Optimum Allocation (COA) prescribes non-monotonic dynamics for consumption and effort and imperfect risk sharing. The COA is decentralized by a competitive Markov equilibrium with markets for renegotiable GDP-linked one-period debt. The equilibrium features debt overhang: reform effort decreases in a high debt range. We also consider environments with less complete markets.

**JEL Codes**: E62, F33, F34, F53, H12, H63

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In this paper, we propose a normative and positive dynamic theory of sovereign debt in an environment characterized by informational frictions. The theory rests on two building blocks. First, sovereign debt is subject to limited enforcement, and countries can renege on their obligations subject to real costs as in, e.g., Aguiar and Gopinath (2006), Arellano (2008) and Yue (2010). Second, countries can undertake structural policy reforms to speed up recovery from an existing recession. The reform effort is assumed to be unobservable and subject to moral hazard.

The theory is motivated by the recent debt crisis in Europe, where sovereign debt and economic reforms emerged as salient and intertwined policy issues. Greece, for instance, saw its debt-GDP ratio soar from 103% in 2007 to 172% in 2011 despite a 53% haircut in 2011. While creditors and international organizations pushed the Greek government to introduce structural reforms that would help the economy recover and meet its international financial obligations, such reforms were forcefully resisted internally. Opposers maintained that the reforms would imply major sacrifice for domestic residents while a large share of the benefits would accrue to foreign lenders. Meanwhile, international organizations stepped in to provide financial assistance and access to new loans, asking in exchange fiscal restraint and a commitment to economic reforms. Our theory rationalizes these dynamics.

The model economy is a dynamic endowment economy subject to income shocks following a two-state Markov process. The economy (henceforth, the sovereign) starts in a recession with a stochastic duration. Costly structural reforms increase the probability that the recession ends. Consumers’ preferences induce a desire for consumption and effort smoothing. We first characterize the solution of two planning problems: the first best and the constrained optimum allocation (COA) subject to limited enforcement and moral hazard. In the first best, the planner provides the sovereign country with full insurance by transferring resources to it during recession and reversing the transfers once the recession ends. The sovereign exerts the efficient level of costly effort as long as the recession lasts.

The first best is not implementable in the presence of informational frictions for two reasons. First, the sovereign has access to a stochastic outside option whose realization is publicly observable. This creates scope for opportunistic deviations involving cashing in transfers for some time, and then unilaterally quit (i.e., default on) the insurance contract as soon as the realization of the outside option is sufficiently favorable. Second, the sovereign has an incentive to shirk and rely on the transfers rather than exerting the required reform effort to increase output.

The COA is characterized by means of a promised utility approach in the vein of Spear and Srivastava (1987), Thomas and Worrall (1988 and 1990), and Kocherlakota (1996). The optimal contract is subject to an incentive compatibility constraint (IC) that pins down the effort choice and a participation constraint (PC) that captures the limited commitment. The COA has the following features: throughout recession, within spells of a slack PC (i.e., when the realized cost of default is high), the planner front-loads the sovereign’s consumption and decreases it over time in order to provide dynamic incentives for reform effort (as in Hopenhayn and Nicolini 1997). In this case, the solution is dictated by the IC and is history-dependent: consumption and promised utility fall over time, while effort follows non-monotonic dynamics for reasons to which we return below. Whenever the PC binds (i.e., the sovereign faces an attractive outside option), the planner increases discretely consumption and promised utility in order to prevent the sovereign from leaving the contract.

Next, we study the decentralization of the COA, which is a key contribution of the paper. We show that the COA can be implemented by a competitive Markov equilibrium where the sovereign

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1 Examples of such reforms include labor and product market deregulation, and the establishment of fiscal capacity that allows the government to raise tax revenue efficiently (see, e.g., Ilzkovitz and Dierx 2011). While these reforms are beneficial in the long run, they entail short-run costs for citizens at large, governments or special-interest groups (see, e.g., Blanchard and Giavazzi 2003).
issues two one-period securities paying returns contingent on the aggregate state (GDP-linked bonds). The bonds are defaultable and renegotiable and are sold to profit-maximizing international creditors who hold well-diversified portfolios. When the sovereign faces a low realization of the default cost, she could, in principle, default, pay a cost, and restart afresh with zero debt. However, costly default can be averted by renegotiation: when a credible default threat is present, a syndicate of creditors offers a take-it-or-leave-it debt haircut. As in Bulow and Rogoff (1989), there is no outright default, but recurrent debt renegotiations.\(^2\) In order for this market arrangement to attain constrained efficiency, the creditors must, ex-post, have all the bargaining power. Importantly, this market arrangement is Markovian, and does not rely on complicated mechanisms to coordinate future punishment.

That a Markov equilibrium with only two assets decentralizes the COA may be surprising. In our environment, there is a continuum of states associated with the realizations of the stochastic outside option whereas only two securities are available. Moreover, there is moral hazard, and creditors cannot commit to punish opportunistic behavior, in contrast with the planner, who can design dynamic incentives under full commitment. Our decentralization result hinges on two features. First, the process of renegotiation (following a particular protocol) turns the two assets into state-contingent securities. Second, the equilibrium debt dynamics and its endogenously evolving price provide efficient dynamic incentives for the sovereign to exert the second-best reform effort. In fact, we prove that our environment with two renegotiable securities yields the same allocation as a full set (i.e., a continuum) of Arrow-Debreu securities with endogenous borrowing constraints in the spirit of Alvarez and Jermann (2000). Our decentralization is parsimonious and simple, in the sense that it requires only two assets and no need to solve for a set of endogenous borrowing constraints.

In the competitive equilibrium, debt accumulates and consumption falls over time as long as the recession lingers and debt is not renegotiated. Interestingly, the reform effort is a non-monotonic function of debt. This result stems from the interaction between limited enforcement and moral hazard. Under full enforcement, effort would increase monotonically over the recession as debt accumulates. Absent moral hazard, effort would be constant when the PC is slack and decrease every time debt is renegotiated. When both informational constraints are present, effort increases with debt at low levels. However, for sufficiently high debt levels the relationship is flipped: there, issuing more debt deters reforms because, due to the high probability of renegotiation, most of the gains from an economic recovery would accrue to foreign lenders in the form of capital gains on the outstanding debt. In this region, the reform effort falls over time as debt accumulates. This debt overhang curtails consumption smoothing: when sovereign debt is high, investors expect low reform effort, are pessimistic about the economic outlook, and request even higher risk premia. Interestingly, in our theory this form of debt overhang is constrained efficient under the postulated informational constraints.

After deriving the main decentralization result, we consider environments with more incomplete markets. In particular, we consider an economy in the spirit of Eaton and Gersovitz (1981) where the sovereign can issue only one asset – a non-contingent bond. This economy fails to attain the COA: the sovereign attains less consumption smoothing and provides an inefficient effort level.\(^3\) This extension is interesting because in reality markets for GDP-linked bonds are often missing. In this (arguably realistic) one-asset environment, there is scope for policy intervention. In particular, an international institution such as the IMF can improve welfare by means of an assistance program. During the recession, the optimal program entails a persistent budget support through extending loans on favorable terms. When the recession ends, the sovereign is settled with a (large) debt on market

\(^2\)Empirically, unordered defaults are indeed rare events. Tomz and Wright (2007) and Sturzenegger and Zettelmeyer (2008) documents a substantial heterogeneity in the terms at which debt is renegotiated.

\(^3\)We also study a case where renegotiation is ruled out. This further curtails consumption smoothing and welfare.
terms. We also discuss the possibility that the international institution takes control over the reform process, overcoming the friction associated with the non-contractible nature of the reform effort.

Our analysis is related to a large international and public finance literature. In a seminal contribution, Atkeson (1991) studies the optimal contract in an environment in which an infinitely-lived sovereign borrower faces a sequence of two-period lived lenders. There is moral hazard: the borrower can do (unobserved) investment in future productive capacity or consume the funds. Our paper differs from Atkeson’s in various aspects. First, the environments are different: in our theory, all agents have an infinite horizon and investments in structural reforms affect the future stochastic process of income, while in Atkeson’s model investments only affect next period’s income. Second, we provide a novel COA decentralization result through a Markov equilibrium with renegotiable one-period bonds. Third, Atkeson (1991) emphasizes the result that the optimal contract involves capital outflow from the borrower during the worst aggregate state. Our model predicts instead that in a recession the borrower keeps accumulating debt and renegotiates it periodically.

A number of recent papers deal with the dynamics of sovereign debt under a variety of informational and contractual frictions. Dovis (2017) studies the efficient risk-sharing arrangement between international lenders and a sovereign borrower with limited commitment and private information about domestic productivity. In his model the COA can be implemented as a competitive equilibrium with non-contingent defaultable bonds of short and long maturity. He does not consider the interaction between structural reforms and limited commitment. Aguiar et al. (2017) study a model à la Eaton and Gersovitz (1981) with limited commitment assuming, as we do, that the borrower has a stochastic default cost. Their research is complementary to ours insofar as it focuses on debt maturity in rollover crises, from which we abstract. Jeanne (2009) also studies a rollover crisis in an economy where the government takes a policy action that affects the return to foreign investors (e.g., the enforcement of creditor’s right) but this can be reversed within a time horizon that is shorter than that at which investors must commit their resources.

Our work is also related to the literature on debt overhang initiated by Krugman (1988). He constructs a static model with exogenous debt showing that a large debt can deter the borrower from undertaking productive investments. In this regime, it may be optimal for the creditor to forgive debt. This is never optimal in our model. Several papers consider distortions associated with high indebtedness in the presence of informational imperfections. Aguiar and Amador (2014) show that high debt increases the volatility of consumption by reducing risk sharing. Aguiar, Amador, and Gopinath (2009) consider the effect of debt on investment volatility. When an economy is indebted, productivity shocks give rise to larger dispersion in investment rates. Aguiar and Amador (2011) consider a politico-economic model where capital income can be expropriated ex-post and the government can default on external debt. A country with a large sovereign debt position has a greater temptation to default, and therefore investments are low. Conesa and Kehoe (2015) construct a theory where governments of highly indebted countries may choose to gamble for redemption.

Our research is related also to the literature on endogenous incomplete markets due to limited enforcement or limited commitment. This includes Alvarez and Jermann (2000) and Kehoe and Perri (2002). The analysis of constrained efficiency is related to the literature on competitive risk sharing contracts with limited commitment, including, among others, Thomas and Worrall (1988), Marcet and Marimon (1992), Phelan (1995), Kocherlakota (1996), and Krueger and Uhlig (2006).

Finally, our work is related, more generally, to recent quantitative models of sovereign default such as Aguiar and Gopinath (2006), Arellano (2008), and Chatterjee and Eyigungor (2012). Of particular

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4 Other papers studying restructuring of sovereign debt include Asomuma and Trebesch (2016), Bolton and Jeanne
interest is Abraham et al. (2017) who study an Arellano (2008) economy, extended to allow shocks to government expenditures, and compare quantitative outcomes of the market allocation to the optimal design of a Financial Stability Fund, interpreted as the solution to a planning problem. Broner et al. (2010) study the incentives to default when parts of the government debt is held by domestic residents. Song et al. (2012) and Müller et al. (2016a) study the politico-economic determination of debt in open economies where governments are committed to honor their debt.

The rest of the paper is organized as follows. Section 1 describes the model environment. Section 2 solves for the first best and the COA under limited commitment and moral hazard. Section 3 provides the main decentralization result. Section 4 considers one-asset economies and discusses policy interventions to restore efficiency. Section 5 concludes. Three online appendixes contain, respectively, the proofs of the main lemmas, propositions and corollaries (Appendix A) and additional technical material referred to in the text (Appendixes B and C).

1 The model environment

The model economy is a small open endowment economy populated by an infinitely-lived representative agent. A benevolent sovereign makes decisions on behalf of the representative agent. The stochastic endowment follows a two-state Markov switching process, with realizations \( \bar{w} \) and \( \bar{w} \), where \( 0 < \bar{w} < \bar{w} \). We label the two endowment states recession and normal time, respectively. An economy starting in recession remains in the recession with probability \( 1 - p \) and switches to normal time with probability \( p \). Normal time is assumed to be an absorbing state.\(^5\) This assumption aids tractability and enables us to obtain sharp analytical results. During recession, the sovereign can implement a costly reform policy to increase \( p \). In our notation \( p \) denotes both the reform effort and the probability that the recession ends. The sovereign can smooth consumption by contracting with a financial intermediary that has access to an international market offering a gross return \( R \).

The sovereign’s preferences are given by \( E_0 \sum \beta^t \left[ u(c_t) - \phi_{\text{I default in } t} - X(p_t) \right] \), where \( \beta = 1/R \).\(^6\) The function \( u \) is twice continuously differentiable and satisfies \( \lim_{c \rightarrow 0} u(c) = -\infty \), \( u'(c) > 0 \), and \( u''(c) < 0 \). \( I \in \{0,1\} \) is an indicator switching on when the economy is in a default state and \( \phi \) is an associated utility loss. In the planning allocation, the cost \( \phi \) accrues when the sovereign opts out of the contract offered by the planner. In the market allocation it accrues when sovereign unilaterally reneges on a debt contract with international lenders.\(^7\) In recession, \( \phi \) follows an i.i.d. process drawn from the p.d.f. \( f(\phi) \) with an associated c.d.f. \( F(\phi) \). We assume that \( F(\phi) \) is continuously differentiable everywhere, and denote its support by \( \phi \equiv [\phi_{\text{min}},\phi_{\text{max}}] \subseteq \mathbb{R}^+ \), where \( \phi_{\text{min}} < \phi_{\text{max}} < \infty \). The assumption that shocks are independent is for simplicity. In order to focus on debt dynamics in recessions we assume that there is full enforcement in normal time (i.e., in normal time \( \phi \) is arbitrarily large). This is again for simplicity. In an earlier version of the paper (Müller et al. 2016b), we show that our main results are robust to assuming the same distribution of \( \phi \) in normal time and in recession.

The function \( X(p) \) represents the reform cost, assumed to be increasing and convex in the probability of exiting recession, \( p \in [\hat{p}, \tilde{p}] \subseteq [0,1] \). \( X \) is assumed to be twice continuously differentiable, with

\(^{2007}\), Hatchondo et al. (2014), Mendoza and Yue (2012), and Yue (2010).

\(^6\)In a previous version of this paper, we considered the possibility that the economy could fall recurrently into recession.

\(^5\)Our insights carry over to the case in which \( \beta R < 1 \).

\(^7\)The cost \( \phi \) is exogenous and publicly observed, and captures in a reduced form a variety of shocks including both taste shocks (e.g., the sentiments of the public opinion about defaulting on foreign debt) and institutional shocks (e.g., the election of a new prime minister, a new central bank governor taking office, the attitude of foreign governments, etc.). Alternatively, \( \phi \) could be given a politico-economic interpretation, as reflecting special interests of lobbies.
the following properties: \( X(p) = 0 \), \( X'(p) = 0 \), \( X''(p) > 0 \) \( \forall p > \bar{p} \), \( X''(p) > 0 \), and \( \lim_{p \to \bar{p}} X' (p) = \infty \).

The time line of events is as follows: at the beginning of each period, the endowment state is observed; then, \( \phi \) is realized and publicly observed; finally, effort is exerted.

## 2 Planning allocation

We first characterize planning allocations. The planning problem is formulated as a one-sided commitment program following a promised-utility approach. The planner maximizes profits subject to a promise-keeping constraint (PK).

Let \( \nu \) denote the promised utility, i.e., the expected utility the sovereign is promised in the beginning of the period, before the realized \( \phi \) is observed. \( \nu \) is the key state variable of the problem. We denote by \( \omega_\phi \) the promised continuation utilities conditional on the realization \( \phi \) and on the economy staying in recession or switching to normal time, respectively. We denote by \( P(\nu) \) the expected present value of profits accruing to the planner conditional on delivering the promised utility \( \nu \) in the most cost-effective way. The optimal value \( P(\nu) \) satisfies the following functional equation:

\[
P(\nu) = \max_{\{c_\phi, p_\phi, \omega_\phi, \tilde{\omega}_\phi\} \in \mathcal{K}} \int_{\mathbb{R}} [w - c_\phi + \beta ((1 - p_\phi) P(\omega_\phi) + p_\phi P(\tilde{\omega}_\phi))] \, dF(\phi),
\]

where the maximization is subject to a promise-keeping constraint (PK)

\[
\int_{\mathbb{R}} [u(c_\phi) - X(p_\phi) + \beta ((1 - p_\phi) \omega_\phi + p_\phi \tilde{\omega}_\phi)] \, dF(\phi) = \nu, \tag{2}
\]

the planner’s profit function in normal time,

\[
\tilde{P}(\tilde{\omega}_\phi) = \max_{\tilde{\phi} \in [0, \bar{\phi}]} \bar{w} - \tilde{c} + \beta \tilde{P}(\beta^{-1} [\tilde{\omega}_\phi - u(\tilde{\phi})]),
\]

and the boundary conditions \( c_\phi \in [0, \bar{c}] \), \( p_\phi \in [p, \bar{p}] \), \( \nu, \omega_\phi \in [\omega, \tilde{\omega}] \), and \( \tilde{\omega}_\phi \in [\tilde{\omega}, \bar{\omega}] \), where \( \bar{c}, \bar{\omega}, \tilde{\omega} \) and \( \omega \) are generous bounds that will never bind in equilibrium.

We introduce two informational frictions. The first is limited enforcement: the sovereign can quit the contract when the realization of \( \phi \) makes such action attractive ex-post. This is captured by a PC. The second is moral hazard: the reform effort is chosen by the sovereign and is not observed by the planner. This is captured by an IC.

Under limited enforcement, the planner must provide a utility that exceeds the sovereign’s outside option if she quits the contract, \( \alpha - \phi \), where \( \phi \) is the shock discussed above and \( \alpha \) is the sovereign’s value of not being in the contract. Thus, the allocation is subject to the following set of PCs:

\[
u = \max_{\{c_\phi, p_\phi, \omega_\phi, \tilde{\omega}_\phi\} \in \mathcal{K}} \int_{\mathbb{R}} [w - c_\phi + \beta ((1 - p_\phi) \omega_\phi + p_\phi \tilde{\omega}_\phi)] \, dF(\phi), \tag{2}
\]

and to a lower bound on initial and future promised-utility, i.e., \( \nu \geq \alpha - E[\phi] \) and

\[
\omega_\phi \geq \alpha - E[\phi], \tag{5}
\]

\footnote{The problem would be identical under two-sided lack of commitment under some mild restrictions on the state space. In particular, one should impose an upper bound on the sovereign’s initial promised utility to ensure that the principal does not find it optimal to ever exit the contract. We return to this point below.}
where $\alpha - E[\phi]$ is the expected utility that the sovereign could guarantee by leaving the contract in the next period regardless of $\phi$. In addition, we assume that $\alpha$ is sufficiently low to ensure that it is never efficient to terminate the program (even under the realization $\phi = \phi_{\min}$). Note that here we simply treat $\alpha$ as an exogenous parameter. Later we endogenize $\alpha$.

When the planning problem is subject to moral hazard, the allocation is also subject to the following IC, reflecting the assumption that the sovereign chooses effort after the planner has set the future promised utilities:

$$p_\phi = \arg \max_{p \in [\bar{p}, \tilde{p}]} -X(p) + \beta((1 - p) \omega_\phi + p\bar{\omega}_\phi). \quad (6)$$

### 2.1 First best

We start by characterizing the first-best allocation given by the program (1)–(3). The proof, which follows standard methods, can be found in Appendix C.

**Proposition 1** Given a promised utility $\nu$, the first-best allocation satisfies the following properties.

The sequences for consumption and promised utilities are constant at the level $c_{FB}(\nu)$, $\omega_{FB} = \nu$, and $\bar{\omega}_{FB} = \nu + X(p_{FB}(\nu))/(1 - \beta(1 - p_{FB}(\nu)))$, implying full consumption insurance, $c_{FB}(\nu) = c_{FB}(\bar{\omega}_{FB})$. Moreover, effort $p_{FB}(\nu)$ is constant over time throughout recession. $c_{FB}(\nu)$ and $p_{FB}(\nu)$ are strictly increasing and strictly decreasing functions, respectively, satisfying:

$$\frac{\beta}{1 - \beta(1 - p_{FB}(\nu))}\left(\frac{\bar{\omega} - \omega}{u}(c_{FB}(\nu)) + \frac{X(p_{FB}(\nu))}{\text{output increase if recovery}} \right) = X'(p_{FB}(\nu)) \quad (7)$$

$$\frac{u(c_{FB}(\nu))}{1 - \beta} - \frac{X(p_{FB}(\nu))}{1 - \beta(1 - p_{FB}(\nu))} = \nu. \quad (8)$$

The solution to the functional equation (3) in normal time is given by

$$\hat{P}(\bar{\omega}_{\phi}) = \frac{\bar{\omega} - c_{FB}(\bar{\omega}_{\phi})}{1 - \beta}, \quad c_{FB}(\bar{\omega}_{\phi}) = u^{-1}(1 - \beta)\bar{\omega}_{\phi}. \quad (9)$$

The first-best allocation yields perfect insurance: the sovereign enjoys a constant consumption irrespective of the endowment state and exerts a constant reform effort during recession. Moreover, consumption $c_{FB}$ is strictly increasing in $\nu$ while effort $p_{FB}$ is strictly decreasing in $\nu$: a higher promised utility is associated with higher consumption and lower effort.

### 2.2 Constrained Optimum Allocation (COA)

Next, we characterize the COA. The planning problem (1)-(3) is subject to the PC (4), the lower bound on $\omega (5)$, and the IC (6). Note that the planning problem is evaluated after the uncertainty about the endowment state has been resolved, but before the realization of $\phi$. For didactic reasons, we first study limited enforcement separately, assuming that effort is controlled directly by the planner. Then,
we generalize the analysis to the more interesting case in which there is moral hazard. We start by establishing a property of the COA that holds true in both environments.\footnote{The functions $P$ and $\bar{P}$ are value functions of the planning problem. Proving that $\bar{P}$ is strictly concave is straightforward. The strict concavity of $P$ is more difficult to establish analytically. We prove below that $P$ is strictly concave in the case without moral hazard, while in the case with moral hazard we will guess and verify it numerically.}

**Lemma 1** Assume that the profit functions $P$ and $\bar{P}$ are strictly concave. Define the sovereign's discounted utility conditional on the promised utility $\nu$ and the realization $\phi$ as $\mu_\phi(\nu) \equiv u(c_\phi(\nu)) - X(p_\phi(\nu)) + \beta \left[ (1 - p_\phi(\nu)) \omega_\phi(\nu) + p_\phi(\nu) \bar{\omega}_\phi(\nu) \right]$. Then, the COA features a unique threshold function $\bar{\phi}(\nu)$ such that the PC binds if $\nu < \bar{\phi}(\nu)$ and is slack if $\nu \geq \bar{\phi}(\nu)$. Moreover,

$$
\mu_\phi(\nu) = \begin{cases} 
\alpha - \phi & \text{if } \phi \in [\phi_{\text{min}}, \bar{\phi}(\nu)], \\
\alpha - \bar{\phi}(\nu) & \text{if } \phi \in [\bar{\phi}(\nu), \phi_{\text{max}}].
\end{cases}
$$

The lemma formalizes the intuitive property that (i) if the planner promises the agent more than her reservation utility in a state $\phi_0$, then it is then optimal for her to do so for all $\phi > \phi_0$; moreover, promised utility is equalized across all such states; (ii) if the planner promises the agent the reservation utility in a state $\phi_1$, then it is then optimal for her to also do so for all $\phi < \phi_0$. Thus, $\mu_\phi(\nu)$ is linearly decreasing in $\phi$ for $\phi < \bar{\phi}(\nu)$, and constant thereafter.

### 2.2.1 Limited Enforcement without Moral Hazard

In this environment, there is no IC and the planner can choose the effort level. We prove in Appendix C (Proposition 9) that the planner’s profit function $P(\nu)$ is strictly decreasing, strictly concave and differentiable for all interior $\nu$, i.e., $\nu > \alpha - E[\phi]$. Moreover, the first-order conditions (FOCs) of the program are necessary and sufficient. The proof follows the strategy in Thomas and Worrall (1990).

Combining the FOCs with respect to $c_\phi$, $\omega_\phi$, and $\bar{\omega}_\phi$ with the envelope condition (see proof of Proposition 2 in Appendix C) yields:\footnote{Note that since there is full enforcement during normal time, $\bar{c}(\bar{\omega}_\phi)$ and $\bar{P}(\bar{\omega}_\phi)$ are as in the first best (cf. Equation 9). Moreover, the first-best allocation implies that $\bar{P}'(\bar{\omega}_\phi) = -1/\bar{u}'(\bar{c}(\bar{\omega}_\phi))$. Thus, equations (10)-(11) imply that the marginal cost associated with providing promised utilities is equalized across the two endowment states, i.e., $P'(\bar{\omega}_\phi) = P'(\bar{\omega}_0)$.}

\begin{align}
\frac{1}{u'(\bar{c}(\bar{\omega}_\phi))} - \frac{1}{u'(c_\phi)} &= 0 \quad (10) \\
u'(c_\phi) &= -\frac{1}{P'(\omega_\phi)}, \quad \forall \omega_\phi > \alpha - E[\phi]. \quad (11)
\end{align}

Combining these with the FOC with respect to $p_\phi$ yields

$$
X'(p_\phi) = \beta \left( (\bar{\omega}_\phi - \omega_\phi) + u'(c_\phi) \times (\bar{P}(\bar{\omega}_\phi) - P(\omega_\phi)) \right). \quad (12)
$$

Equation (10) establishes that the planner provides the sovereign with full insurance against the endowment shock, i.e., she sets $\bar{c}(\bar{\omega}_\phi) = c_\phi$ and equates the marginal profit loss associated with promised utilities in the two states. Equation (12) establishes that effort is set at the constrained efficient level: The marginal cost of effort equals the sum of the marginal benefits accruing to the sovereign and to the planner, respectively. The next proposition provides a formal characterization of the COA.\footnote{The proof is in Appendix C. The proof of Proposition 3 below includes the proof of Proposition 2 as a special case.}
Proposition 2 The COA is characterized as follows. The threshold function $\bar{\phi}(\nu)$ is decreasing and implicitly defined by the condition

$$
\nu = \alpha - \left[ \int_{\phi_{\min}}^{\hat{\phi}(\nu)} \phi dF(\phi) + \hat{\phi}(\nu) \left[ 1 - F(\hat{\phi}(\nu)) \right] \right].
$$

Moreover:

1. If $\phi < \hat{\phi}(\nu)$, the PC is binding, and the allocation $(c_\phi, p_\phi, \omega_\phi, \bar{\omega}_\phi)$ is determined by (10), (11), (12), and by (4) holding with equality. Moreover, $\omega_{\hat{\phi}} = \nu' > \nu$.

2. If $\phi \geq \hat{\phi}(\nu)$, the PC is slack, and the allocation $(c_\phi, p_\phi, \omega_\phi, \bar{\omega}_\phi)$ is given by $\omega_{\hat{\phi}} = \nu' = \nu$, $c_\phi = c(\nu)$, $\bar{\omega}_\phi = \bar{\omega}(\nu)$, and $p_\phi = p(\nu)$, where the functions $c(\nu)$, $\bar{\omega}(\nu)$ and $p(\nu)$ are determined by

$$
u(c(\nu)) - X(p(\nu)) + \beta [p(\nu)\bar{\omega}(\nu) + (1 - p(\nu))\nu] = \alpha - \hat{\phi}(\nu),
$$

(10), and (12), respectively. The solution is history-dependent. The reform effort is strictly decreasing and consumption and future promised utility are strictly increasing in $\nu$.

The COA under limited enforcement has standard back-loading properties. Whenever the PC is slack, consumption, effort and promised utility remain constant over time. Consumption remains constant even as the recession ends. Thus, the COA yields full consumption insurance across all states in which the PC is slack. Whenever the PC binds, the planner increases the sovereign’s consumption and promised utilities while reducing her effort in order to meet her PC. In this case, $\nu' > \nu$.

The upper panels of Figure 1 describe the dynamics of consumption and effort (left panel), and promised utilities (right panel) under a particular sequence of realizations of $\phi$. In this numerical example, the recessions last for 13 periods. Thereafter, the economy attains full insurance. Consumption is back-loaded and effort is front-loaded. In periods 9 and 11, the PC binds and the planner must increase consumption and promised utility, and reduce effort. Consumption and effort remain constant when the PC is slack. Note also that consumption remains constant when the recession ends.

2.2.2 Limited Enforcement with Moral Hazard

Next, we consider the more interesting case in which the planner cannot observe effort and is subject to both a PC and an IC. The COA features an important qualitative difference from the case without moral hazard: within each spell in which the PC is slack, the planner front-loads consumption and promised utility to incentivize the sovereign to provide effort. Therefore, moral hazard prevents full insurance even across the states of nature in which the PC is slack.

Let us start the analysis from the IC (6). The FOC yields $X'(p_\phi) = \beta (\bar{\omega}_\phi - \omega_\phi)$, or equivalently

$$
p_\phi = \Upsilon (\bar{\omega}_\phi - \omega_\phi).
$$

where $\Upsilon(x) \equiv (X')^{-1}(\beta x)$. The properties of $X$ imply that $p_\phi$ is increasing in the promised utility gap $\bar{\omega}_\phi - \omega_\phi$. Equation (14) is the analogue of (12). Effort is distorted because the sovereign does not internalize the benefits accruing to the planner.

---

\[13\] Although we have assumed that the planner controls effort directly, the same allocation would obtain if the planner did not control effort ex ante but could observe it ex post and punish deviations. Details are available in the working paper version (Müller et al. 2016b).

\[14\] In all numerical examples, the parameters of the model are calibrated as described in Appendix C.
Figure 1: Simulation of consumption, effort, and promised utilities for a particular sequence of \( \phi \)'s. In this particular simulation the recession ends in period 13. The top panels show the planner allocation without moral hazard, the bottom panels with moral hazard.

The FOCs with respect to \( \omega_\phi \) and \( \bar{\omega}_\phi \), together with the envelope condition yield (see proof of Proposition 3 in Appendix A):

\[
\frac{1}{u'(c(\omega_\phi))} - \frac{1}{u'(c_\phi)} = \frac{\gamma'(\bar{\omega}_\phi - \omega_\phi)}{\bar{\gamma}(\bar{\omega}_\phi - \omega_\phi)} \left[ \bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) \right] \tag{15}
\]

\[
\frac{1}{u'(c_\phi)} = -P'(\omega_\phi) + \frac{\gamma'(-\bar{\omega}_\phi - \omega_\phi)}{1 - \bar{\gamma}(\bar{\omega}_\phi - \omega_\phi)} \left[ \bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) \right], \quad \forall \omega_\phi > \alpha - E[\phi] \tag{16}
\]

\[
0 = \theta_\phi \times [\omega_\phi - (\alpha - E[\phi])], \tag{17}
\]

where \( \theta_\phi \geq 0 \) is the Lagrange multiplier on the constraint \( \omega_\phi \geq \alpha - E[\phi] \). This constraint binds when \( \nu \) is sufficiently low, since the planner must take into account that she cannot promise a utility below the lower bound, and this constrains her ability to provide dynamic incentives.\(^{15}\) Equation (16) is then replaced by the conditions \( \omega_\phi = \alpha - E[\phi] \) and \( \theta_\phi > 0 \).

\(^{15}\)The issue did not arise in the case without moral hazard because the optimal \( \omega_\phi \) was non-decreasing, so the lower bound on \( \omega_\phi \) was never binding. Nor does the problem arise for the choice of \( \bar{\omega}_\phi \) since there is no PC in the normal state.
The FOC (15) is the analogue of (10). With moral hazard, the planner does no longer provide the sovereign with full insurance against the realization of the endowment shock: by promising a higher consumption if the economy recovers (thereby curtailing insurance), she incentivizes effort provision. Note that the effort wedge is proportional to the elasticity of effort $\Upsilon' / \Upsilon$.

The FOC (16) is the analogue of (11). There, the marginal utility of consumption simply equaled the profit loss associated with an increase in promised utility. Here, the planner finds it optimal to open a wedge that is proportional to the elasticity of effort: she front-loads consumption in order to make the sovereign more eager to leave a recession.

We can now proceed to a full characterization of the COA with moral hazard. As is common for problems with both limited enforcement and moral hazard, it is difficult to prove that the program is globally concave and to establish analytically the curvature of the profit function. We therefore assume that $P(\nu)$ is strictly concave in $\nu$, and verify this property numerically.\footnote{We prove that under the assumption that $P$ is strictly concave, $P(\nu)$ must be differentiable for all interior $\nu$ and that the the FOCs are necessary (see Lemma 3 in Appendix C). Although we cannot establish in general that they are also sufficient, this turns out to be the case in all parametric examples we considered.}

**Proposition 3** Assume that $P$ is strictly concave. The COA with unobservable effort is characterized as follows: (i) $P_\phi = \Upsilon (\tilde{\omega}_\phi - \omega_\phi)$ as in equation (14); (ii) the threshold function $\tilde{\phi}(\nu)$ is decreasing and implicitly defined by equation (13). Moreover:

1. If $\phi < \tilde{\phi}(\nu)$, the PC is binding and the allocation $(c_\phi, \omega_\phi, \tilde{\omega}_\phi, \theta_\phi)$ is determined by (15), (16), (17), and by (4) holding with equality.
2. If $\phi \geq \tilde{\phi}(\nu)$, the PC is slack and the allocation $(c_\phi, \omega_\phi, \tilde{\omega}_\phi, \theta_\phi)$ is determined by (15), (16), (17), and

   \[ u(c_\phi) - X(p_\phi) + \beta [p_\phi \tilde{\omega}_\phi + (1 - p_\phi) \omega_\phi] = \alpha - \tilde{\phi}(\nu). \]  

   The solution is history-dependent, i.e., $c_\phi = c(\nu)$, $\omega_\phi = \omega(\nu)$, and $\tilde{\omega}_\phi = \tilde{\omega}(\nu)$. Promised utility falls over time, i.e., $\omega(\nu) = \nu < \nu$ and $\omega(\nu) = \nu = 0$ when $\nu$ is sufficiently large. The function $c(\nu)$ is strictly increasing. The effort function $p(\nu) = \Upsilon (\tilde{\omega}(\nu) - \omega(\nu))$ is strictly increasing in a range of low $\nu$. In this range effort declines over time when the PC is slack.

Figure 2 summarizes the results of Proposition 3 based on a numerical example. All panels show policy functions for an economy in recession, conditional on a slack PC. Promised utilities $\omega_\phi(\nu)$ and $\tilde{\omega}_\phi(\nu)$ and consumption $c(\nu)$ are weakly increasing in $\nu$. The upper left panel shows the law of motion of $\nu$ when the recession lingers and the PC remains slack. The fact that $\omega_\phi(\nu)$ is below the 45-degree line implies that promised utility falls over time and converges to the lower bound $\nu \equiv \alpha - E[\phi]$.

In the range $[\nu, \nu^-]$ the planner is constrained by the inability to abase promised utility below $\nu$. The two left panels imply that, if the recession lingers and the PC is slack, consumption declines over time. Instead, the dynamics of effort are non-monotonic.

We now show some additional analytical properties of the COA. We start with consumption dynamics. Combining (16) with the envelope condition $P'(\omega_\phi) = -1/u(c(\omega_\phi))$ for $\omega_\phi > \nu$ and denoting $\nu' = \omega_\phi$ and $\nu' = \tilde{\omega}_\phi$, yields:

\[ \frac{1}{u'(c_\phi)} - \frac{1}{u'(c(\nu'))} = \frac{\Upsilon'(\nu' - \nu)}{1 - \Upsilon'(\nu' - \nu)} \left( \tilde{P}(\nu') - P(\nu') \right). \]  

\[ \text{(19)} \]
We label equation (19) a Conditional Euler Equation (CEE). The CEE describes the optimal consumption dynamics for states where the PC does not bind next period and $\nu > \nu^-$. Recall that the elasticity $\Upsilon'/(1-\Upsilon)$ captures moral hazard. In its absence, $\Upsilon' = 0$ and the planner delivers perfect consumption smoothing (as in Proposition 2). Under moral hazard, the right-hand side of (19) is positive. Thus, consumption decreases over time as long as the economy remains in recession and the PC is slack, echoing the optimal consumption dynamics in Hopenhayn and Nicolini (1997).\footnote{The right-hand side of (19) is positive since $\Upsilon' > 0$ and $\hat{P}(\hat{\nu}) > P(\nu')$ (see the proof of Proposition 3 in Appendix A). This implies that the marginal utility of consumption must be rising over time as $\nu$ falls.}

Combining the CEE (19) with the Euler equation describing the consumption dynamics upon recovery (15), yields a conditional version of the so-called Inverse Euler Equation (CIEE):

$$\frac{1}{u'(c_{\nu})} = (1 - \Upsilon(\hat{\nu}' - \nu')) \frac{1}{u'(c'(\nu'))} + \Upsilon(\hat{\nu}' - \nu') \frac{1}{u'(c'(\nu'))}. $$

(20)

The CIEE equates the inverse marginal utility in the current period with next period’s expected inverse marginal utility conditional on the PC being slack and $\nu > \nu^-$. The key difference relative to the standard inverse Euler equation in the dynamic contract literature (cf. Rogerson 1985) is that our CIEE holds only in states where the PC is slack. If there were no limited enforcement, then our CIEE would boil down to the standard inverse Euler equation.

Figure 2: Policy functions for state-contingent promised utility, consumption, and effort conditional on the maximum cost realization $\phi_{\max}$.
Next, we turn to the effort dynamics. In analogy with consumption, one might expect that, when the PC is slack, the planner would back-load effort to incentivize its provision. However, this conjecture is incorrect. Effort is indeed decreasing in promised utility when $\nu$ is high, inducing back-loading of effort. However, when $\nu$ is sufficiently low, effort is increasing in promised utility (cf. Proposition 3 and its proof), implying that, as $\nu$ falls, effort decreases over time. In this range, the planner front-loads effort even within spells when the PC is slack. The reason for this result is that the planner cannot reduce indefinitely the sovereign’s promised utility. When $\nu \leq \nu^-$, the constraint $\omega_\phi \geq \alpha - E[\phi]$ is binding, and the planner sets $\omega_\phi = \nu$, inducing $p_\phi(\nu) = \Upsilon(\tilde{\omega}_\phi(\nu) - \nu)$. Since both $\Upsilon$ and $\tilde{\omega}_\phi$ are increasing functions (the planner can decrease $\tilde{\omega}_\phi$ since promises in normal time are not subject to limited commitment), $p_\phi$ must be increasing in $\nu$. Hence, effort declines over time if the recession lingers for sufficiently long in recession and the PC is slack.

The lower panels of Figure 1 illustrate simulated dynamics of consumption, promised utilities, and effort in the case of moral hazard under the same sequence of realizations of $\phi$ as in the upper panels. Consumption and promised utilities fall over time when the PC is slack, while effort falls or rises over time depending on $\nu$. When $\nu$ becomes sufficiently low (i.e., in periods 7, 8, and 9), the reform effort starts falling. Note that, as the recession ends, consumption increases, implying that the endowment risk is not fully insured.

In conclusion, the combination of limited enforcement and moral hazard delivers effort dynamics qualitatively different from models with only one friction. In our model effort is hump-shaped over time, even when the PC remains slack. In contrast, effort is monotone increasing in many pure moral hazard models and weakly decreasing in pure limited enforcement models. The dynamics of consumption echo the typical properties of models with dynamic moral hazard as long as the PC is slack, namely the planner curtails consumption in order to extract higher effort over time. However, the planner periodically increases consumption and promised utility whenever the PC is binding. This averts the immiseration that would arise in the absence of limited enforcement.

### 2.3 Primal Formulation

We have characterized the COA by solving a dual planning problem, i.e., maximizing profits subject to a promised-utility constraint. We can alternatively characterize the COA by solving a primal problem where the planner maximizes discounted utility subject to a promised-expected-profit constraint. We study the primal program because its formulation is directly comparable with the market equilibrium. This is useful when deriving the main decentralization result.

Let $\mu^d(\pi)$ and $\mu^d_\phi(\pi)$ denote the sovereign’s discounted utility, respectively, before and after the realization of $\phi$. We can write the primal problem as:

$$
\mu^{pl}(\pi) = \max_{\{c_\phi, p_\phi, \pi_\phi, \pi_\phi^d\}} \int_{\mathcal{S}} \mu^d_\phi(\pi) \ dF(\phi) \\
= \max_{\{c_\phi, p_\phi, \pi_\phi, \pi_\phi^d\}} \int_{\mathcal{S}} \left[ u(c_\phi) - X(p_\phi) + \beta \left( (1 - p_\phi) \mu^d(\pi_\phi) + p_\phi \mu^d(\pi_\phi^d) \right) \right] \ dF(\phi),
$$

Lemma 2 in Appendix C shows that effort is decreasing in promised utility when $\nu$ is large. This result is subject to a sufficient condition, namely that $\lim_{p \to p^-} X''(p) > 0$. However, numerical analysis suggests that this is true more generally.

By continuity, this property extends to a contiguous range of $\nu$ above $\nu^-$. 

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where \( \tilde{\mu}^{pl} (\tilde{\pi}) = u (\tilde{w} - (1 - \beta) \tilde{\pi}) / (1 - \beta) \), subject to the promised-profit constraint

\[
\pi = \int_{\mathbb{R}} b_\phi dF (\phi),
\]

(22)

having defined \( b_\phi = w - c_\phi + \beta \left( (1 - p_\phi) \pi'_\phi + p_\phi \tilde{\pi}'_\phi \right) \) as the planner’s discounted profit after the realization of \( \phi \). The problem is subject to a set of PCs and ICs

\[
u (c_\phi) - X (p_\phi) + \beta \left( (1 - p_\phi) \mu^{pl} (\pi'_\phi) + p_\phi \tilde{\mu}^{pl} (\tilde{\pi}'_\phi) \right) \geq \alpha - \phi, \quad \phi \in \mathbb{R},
\]

(23)

\[
p_\phi = \arg \max_{p \in [\mathbb{P}, \mathbb{P}^\mu]} -X (p) + \beta \left( (1 - p) \mu^{pl} (\pi'_\phi) + p \tilde{\mu}^{pl} (\tilde{\pi}'_\phi) \right),
\]

(24)

and to the boundary conditions \( c_\phi \in [0, \bar{c}], \tilde{\pi}'_\phi \in [\bar{\pi}, \tilde{\pi}], \) and \( \pi, \pi'_\phi \in \left[ \bar{\pi}, \right. \left. (\Phi^{pl})^{-1} (\phi_{max}) \right] \), where \( \Phi^{pl} \) is defined below and \( \bar{\pi}, \tilde{\pi} \) are generous bounds that will never bind in equilibrium.

The primal allocation is identical to the dual one as long as \( \pi = P (\nu) \). Conversely, the dual allocation is identical to the primal one as long as \( \nu = \mu^{pl} (\pi) \). Thus, \( \mu^{pl} = P^{-1} \). The analysis of the dual problem established that the COA features threshold properties, with a threshold function \( \tilde{\phi} (\nu) \) decreasing in \( \nu \). The same property is inherited by the primal problem, where the threshold \( \Phi^{pl} (\pi) = \phi (P^{-1} (\pi)) \) is an increasing function of \( \pi \).

It is useful to precede the characterization result by some definitions.

**Definition 1**

1. \( \Psi^{pl} (\pi', \tilde{\pi}') = Y (\tilde{\mu}^{pl} (\tilde{\pi}') - \mu^{pl} (\pi')) ; \)

2. \( \left( \Pi^{pl} (\pi), \Pi^{pl} (\pi), C^{pl} (\pi) \right) = \arg \max_{c, \pi, \tilde{\pi}} \left\{ -u (c) - X (\Psi^{pl} (\pi', \tilde{\pi}')) + \beta \left( \Psi^{pl} (\pi', \tilde{\pi}') \times \tilde{\mu}^{pl} (\tilde{\pi}') + (1 - \Psi^{pl} (\pi', \tilde{\pi}')) \times \mu^{pl} (\pi') \right) \right\} \), subject to \( c = w - b^{pl} (\pi) + \beta \left( (1 - \Psi^{pl} (\pi', \tilde{\pi}')) \pi' + \Psi^{pl} (\pi', \tilde{\pi}') \tilde{\pi}' \right) \) and \( c \in [0, \bar{c}], \tilde{\pi}' \in [\bar{\pi}, \tilde{\pi}], \pi \leq \pi' \leq (\Phi^{pl})^{-1} (\phi_{max}) \equiv \pi_{max} \), where the function \( b^{pl} : \left[ \bar{\pi}, \pi_{max} \right] \rightarrow \mathbb{R} \) is recursively defined as follows:

\[
b^{pl} (\pi) = \pi - \int_{\phi_{min}}^{\Phi^{pl} (\pi)} \phi \left( \Psi^{pl} (\pi', \tilde{\pi}) \right) dF (\phi) \left( \Phi^{pl} (\pi) \right)^{-1} (\phi) \]  \(dF (\phi) \) \]

(25)

3. \( W^{pl} (\pi) = u (C^{pl} (\pi)) - X (\Psi^{pl} (\Pi^{pl} (\pi), \Pi^{pl} (\pi))) + \beta \left( \frac{1 - \Psi^{pl} (\Pi^{pl} (\pi), \Pi^{pl} (\pi))) \mu^{pl} (\Pi^{pl} (\pi)) + \psi^{pl} (\Pi^{pl} (\pi), \Pi^{pl} (\pi))) \mu^{pl} (\Pi^{pl} (\pi)) \right) \).

\( \Psi^{pl} \) is the optimal incentive-compatible effort function. The functions in parts 2 and 3 of Definition 1 are all conditional on realizations of \( \phi \) such that the PC is slack: \( \Pi^{pl} \) and \( \Pi^{pl} \) are the discounted profits under recession and normal state; \( b^{pl} \) denotes the transfer from the sovereign to the principal; and, \( W^{pl} \) is the discounted utility conditional on realizations of \( \phi \) such that the PC does not bind. Finally, note that \( W^{pl} (\pi) = \alpha - \Phi^{pl} (\pi) \). We can now move to the primal characterization of the COA.

**Proposition 4**

The planning problem has the following primal characterization; \( c_\phi = C^{pl} (\mathbb{B}^{pl} (\pi, \phi)) \), \( \pi'_\phi = \Pi^{pl} (\mathbb{B}^{pl} (\pi, \phi)) \), \( \tilde{\pi}'_\phi = \Pi^{pl} (\mathbb{B}^{pl} (\pi, \phi)) \), and \( p_\phi = \Psi^{pl} (\Pi^{pl} (\mathbb{B}^{pl} (\pi, \phi)) \), \( \Pi^{pl} (\mathbb{B}^{pl} (\pi, \phi)) \), where

\[
\mathbb{B}^{pl} (\pi, \phi) = \begin{cases} \tilde{\pi}'_\phi & \text{if } \phi < \Phi^{pl} (\pi) \\ \pi & \text{if } \phi \geq \Phi^{pl} (\pi) \end{cases}
\]

13
\[ \Phi^{pl}(\pi) \equiv \tilde{\phi}(P^{-1}(\pi)), \text{ and } \tilde{\pi}_\phi^{pl} = (\Phi^{pl})^{-1}(\phi). \] The sovereign attains the utility \( W^{pl}(\tilde{\pi}_\phi^{pl}) = \alpha - \phi \) if \( \phi < \Phi^{pl}(\pi) \) (i.e., if the PC binds), and \( W^{pl}(\pi) = \alpha - \Phi^{pl}(\pi) \) if \( \phi \geq \Phi^{pl}(\pi) \) (i.e., if the PC is slack). The value function satisfies:

\[
\mu^{pl}(\pi) = \alpha - \left(1 - F\left(\Phi^{pl}(\pi)\right)\right) \Phi^{pl}(\pi) - \int_{\phi_{\min}}^{\Phi^{pl}(\pi)} \phi dF(\phi).
\]

When the PC binds, consumption and promised expected profits are pinned down by \( \phi \) and are history-independent. The sovereign’s discounted utility equals \( \alpha - \phi \). To keep notation compact we define the pseudo-expected profit \( \tilde{\pi}_\phi^{pl} \) as the lowest initial expected profit such that, conditional on the realized \( \phi \), the PC would have been slack. This change of variable allows us to pin down the optimal choices by simply applying the functions \( \Pi^{pl}, \tilde{\Pi}^{pl}, \text{ and } C^{pl} \) to \( \tilde{\pi}_\phi^{pl} \) instead of \( \pi \). When the PC is slack, consumption and promised future expected profits are history-dependent, and the sovereign receives the utility \( \alpha - \Phi^{pl}(\pi) \), irrespective of \( \phi \). This implies a simple value function formulation.

As a first step towards a market decentralization of the COA, consider the following interpretation of the primal problem. In the initial period, the risk-neutral principal is endowed with a claim whose expected value is \( \pi \). This claim is backed by a (possibly, negative) output transfer from the sovereign in the current period and by rolling over the claim to the next period. The transfer resembles the returns of a state-contingent bond. The principal receives the agreed return \( b^{pl}(\pi) \) in all states in which the PC is slack. Otherwise, she takes a haircut \( b^{pl}(\tilde{\pi}_\phi^{pl}) < b^{pl}(\pi) \). In addition, the planner adjusts optimally the future claims (contingent on the endowment state), as if the contract had undergone a renegotiation. The case without moral hazard is especially intuitive. If the PC is slack, the planner keeps the obligation constant at its initial level (\( \pi' = \pi \)). If the PC binds, she reduces the future obligation so as to keep the sovereign in the contract (\( \pi' < \pi \)). Under moral hazard, \( \pi \) changes over time even when the PC is slack in order to provide the optimal dynamic incentives for effort provision.

### 3 Decentralization

In this section, we show that the COA can be decentralized by a market allocation where the sovereign issues one-period defaultable bonds, held by risk-neutral international creditors. In the market economy, the planner is replaced by a syndicate of international investors (the creditors) who can borrow and lend at the gross interest rate \( R \). We assume that the sovereign can only issue one-period securities with return contingent on the endowment state (GDP-linked bonds). The first (\( \hat{b} \)) pays one unit of good if the economy switches to a normal state. The second (\( \bar{b} \)) pays one unit if the economy remains in recession. Effort is not verifiable and there is no market for securities with return contingent on the effort level. Moreover, there is no market to insure against the realization of \( \phi \). Interestingly, despite the stark market structure restrictions, the Markov equilibrium decentralizes the COA.

We restrict attention to Markov-perfect equilibria where agents condition their strategies on pay-off relevant state variables, ruling out reputational mechanisms. We view this assumption as realistic in the context of sovereign debt since it is generally difficult for creditors to commit to punishment strategies, especially when new lenders can enter and make separate deals with the sovereign.

We label the two securities \( \text{recession-contingent debt} \) and \( \text{recovery-contingent debt} \) and denote their prices by \( Q(b, \hat{b}) \) and \( Q(b, \bar{b}) \). At the beginning of each recession period, the sovereign observes the realization of the default cost \( \phi \) and decides whether to honor the recession-contingent debt that reaches maturity or to announce default. When default is announced, a renegotiation protocol is triggered, described below. Since debt is honored in normal time, no arbitrage implies that \( Q = pR^{-1} \). If the
country could commit to pay its debt also in recession, we would have $Q = (1 - p)R^{-1}$. However, due to the risk of renegotiation, recession-contingent debt sells at a discount, $Q \leq (1 - p)R^{-1}$.

We now describe the renegotiation protocol. If the sovereign announces default, the syndicate of creditors can offer a take-it-or-leave-it haircut that we assume to be binding for all creditors. By accepting this offer, the sovereign averts the default cost. In equilibrium, a haircut is offered only if the default threat is credible, i.e., if the realization of $\phi$ is sufficiently low to make the sovereign prefer default to full repayment. Note that the creditors have, ex-post, all the bargaining power, and their offer makes the sovereign indifferent between an outright default and the proposed haircut.

The timing of a debt crisis can be summarized as follows: The sovereign enters the period with the pledged debt $b$, observes the realization $\phi$, and then decides whether to announce default on all its debt. If the default threat is credible, the creditors offer a haircut $\bar{b} \leq b$. Next, the country decides whether to accept or decline this offer. Then, the sovereign issues new debt subject to the period budget constraint $c = Q (b', \bar{b}') \times \bar{b}' + Q (b', \bar{b}') \times \bar{b}' + w - b$. To facilitate comparison with the COA we start by assuming that the outside option following outright default is $\alpha$. Thus, in the out-of-equilibrium event that the sovereign declines the offered haircut, the default cost $\phi$ is triggered, the debt is canceled, and realized utility is $\alpha - \phi$. We will later endogenize $\alpha$ by assuming that the sovereign can subsequently resume access to financial markets.

### 3.1 Markov Equilibrium

In Markov-perfect equilibria, the equilibrium functions depend only on the pay-off relevant state variables, $b$ and $\phi$. For technical reasons, we impose that debt is bounded, $b \in [\underline{b}, \bar{b}]$ where $\underline{b} > -\infty$ and $\bar{b} = \bar{w}/(1 - R^{-1})$ is the natural borrowing constraint in normal time. In equilibrium, these bounds never bind. We define the equilibrium for an economy starting in a recession with debt $b$. We omit the formal definition for normal time, since then the first-best allocation obtains.

**Definition 2** A Markov-perfect equilibrium is a set of value functions $\{V, W\}$, a threshold renegotiation function $\Phi$, a set of equilibrium debt price functions $\{Q, Q\}$, and a set of optimal decision rules $\{B, \bar{B}, C, \Psi\}$ such that, conditional on the state vector $(b, \phi) \in [\underline{b}, \bar{b}] \times [\phi_{\min}, \phi_{\max}]$, the sovereign maximizes utility, the creditors maximize profits, and markets clear. More formally:

- The value function $V$ satisfies
  $$V (b, \phi) = \max \{W (b), \alpha - \phi\},$$
  where $W (b)$ is the value function conditional on the debt level $b$ being honored,
  $$W (b) = \max_{(b', \bar{b}') \in [\underline{b}, \bar{b}]^2} u (Q (b', \bar{b}') \times \bar{b}' + \bar{Q} (b', \bar{b}') \times \bar{b}' + w - b) + Z (b', \bar{b}'),$$

---

This is a strong assumption. Note that in our environment there would be no reason for a subset of creditors to deviate and seek a better deal. In reality, this issue may arise if deviants may hope that some courts would rule more favorably for them, as in the dispute involving the Argentinian government vs. a group of vulture funds led by Elliott Management. In Section 4, below, we show that ruling out renegotiation altogether reduces welfare, ex-ante. Therefore, our theory emphasizes the value of making haircut agreements binding for all creditors.

By assumption, the sovereign has always the option to simply honor the debt contract. Thus, the creditors’ take-it-or-leave-it offer cannot demand a repayment larger than the face value of outstanding debt.

Our focus on renegotiable debt with a stochastic outside option is in line with the evidence of Sturzenegger and Zettelmeyer (2008) who document that even within a relatively short period (1998-2005) there are large differences in investor losses across various debt restructurings (see also Panizza et al. 2009 and Reinhart and Trebesch 2016).
continuation utility $Z$ is defined as
\[ Z (b', \bar{b}') = \max_{p \in [\underline{p}, \overline{p}]} \left\{ -X (p) + \beta (p \times \bar{\mu} (b') + (1 - p) \times \mu (b')) \right\}, \tag{27} \]
and the value of starting in recession with debt $b$ and in normal time with debt $\bar{b}$ are $\mu (b) = \int_{\mathbb{R}} V (b, \phi) \, dF (\phi)$ and $\bar{\mu} (b) = u (\bar{w} - (1 - R^{-1}) \bar{b}) / (1 - \beta)$, respectively.

- The threshold renegotiation function $\Phi$ satisfies
  \[ \Phi (b) = \alpha - W (b). \tag{28} \]
- The recovery and recession-contingent debt price functions satisfy the arbitrage conditions:
  \[ \tilde{Q} (b', \bar{b}') \times \bar{b}' = \Psi (b', \bar{b}') \, R^{-1} \times \bar{b}' \tag{29} \]
  \[ Q (b', \bar{b}') \times b' = [1 - \Psi (b', \bar{b}')] \, R^{-1} \times \Pi (b') \tag{30} \]
where $\Pi (b')$ is the expected repayment of the recession-contingent bonds conditional on next period being a recession,
\[ \Pi (b) = (1 - F (\Phi (b))) \, b + \int_{\phi_{\min}}^{\Phi (b)} \tilde{b} (\phi) \times dF (\phi), \tag{31} \]
and where $\tilde{b} (\phi) = \Phi^{-1} (\phi)$ is the new post-renegotiation debt after a realization $\phi$.

- The set of optimal decision rules comprises:
  1. A take-it-or-leave-it debt renegotiation offer:
     \[ \mathbb{B} (b, \phi) = \begin{cases} \tilde{b} (\phi) & \text{if } \phi \leq \Phi (b), \\ b & \text{if } \phi > \Phi (b). \end{cases} \tag{32} \]
  2. An optimal debt accumulation and an associated consumption decision rule:
     \[ \langle B (\mathbb{B} (b, \phi)), \bar{B} (\mathbb{B} (b, \phi)) \rangle = \arg \max_{(b', \bar{b}') \in [\underline{b}, \overline{b}]} \left\{ u (Q (b', \bar{b}') \times b' + \bar{Q} (b', \bar{b}') \times \bar{b}' + \bar{w} - \mathbb{B} (b, \phi)) + Z (b', \bar{b}') \right\}, \tag{33} \]
     \[ C (\mathbb{B} (b, \phi)) = Q (B (\mathbb{B} (b, \phi)), \bar{B} (\mathbb{B} (b, \phi))) \times B (\mathbb{B} (b, \phi)) + \bar{Q} (B (\mathbb{B} (b, \phi)), \bar{B} (\mathbb{B} (b, \phi))) \times \bar{B} (\mathbb{B} (b, \phi)) + \bar{w} - \mathbb{B} (b, \phi). \tag{34} \]
  3. An optimal effort decision rule:
     \[ \Psi (b', \bar{b}') = \arg \max_{p \in [\underline{p}, \overline{p}]} \left\{ -X (p) + \beta (p \times \bar{\mu} (b') + (1 - p) \times \mu (b')) \right\}. \tag{35} \]

- The equilibrium law of motion of debt is $(b', \bar{b}') = \langle B (\mathbb{B} (b, \phi)), \bar{B} (\mathbb{B} (b, \phi)) \rangle$.
- The probability that the recession ends is $p = \Psi (b', \bar{b}')$. 

16
Equation (26) implies that there is renegotiation if and only if $\phi < \Phi(b)$. Since, ex-post, creditors have all the bargaining power, the discounted utility accruing to the sovereign equals the value that she would get under outright default. Thus

$$V(b, \phi) = W(\mathbb{E}(b, \phi)) = \begin{cases} W(b) & \text{if } b \leq \hat{b}(\phi), \\ \alpha - \phi & \text{if } b > \hat{b}(\phi). \end{cases}$$

Consider, next, the equilibrium price functions. Since creditors are risk neutral, the expected rate of return on each security must equal the risk-free rate of return. Then, the arbitrage conditions (29) and (30) ensure market clearing in the security markets and pin down the equilibrium price of securities in recession. The function $\Pi(b)$ defined in equation (31) yields the market value of the outstanding debt $b$ conditional on the endowment state being a recession but before the realization of $\phi$. The obligation $b$ is honored with probability $1 - F(\Phi(b))$, where $\Phi(b)$ denotes the largest realization of $\phi$ such that the sovereign can credibly threaten to default. With probability $F(\Phi(b))$, debt is renegotiated to a level that depends on $\phi$ denoted by $\hat{b}(\phi)$. The haircut $\hat{b}(\phi)$ keeps the sovereign indifferent between accepting the creditors’ offer and defaulting. Consequently, $W(\hat{b}(\phi)) = \alpha - \phi$.

Consider, finally, the set of decision rules. Equations (33)-(34) yield the optimal consumption-saving decisions while equation (35) yields the optimal reform effort. The effort depends on $b'$ and $\bar{B}$, since it is chosen after the new securities are issued. Note also that since $F(\Phi(b)) = 0$ for $b' \geq (\Phi)^{-1}(\phi_{\max})$, the bond price function (31) implies that debt exceeding this level will not raise any debt revenue. Thus, it is optimal to choose $b' \leq (\Phi)^{-1}(\phi_{\max})$.

### 3.2 Decentralization Through Renegotiable Debt

We now establish a key result of the paper, namely, that the Markov-perfect equilibrium with one-period renegotiable bonds decentralizes the COA with unobservable effort. This result embeds an existence proof for the Markov-perfect equilibrium.

In the equilibrium allocation, the outstanding debt level $b$ replaces the expected profit $\pi$ in the primal planning problem as the endogenous state variable. The equilibrium debt price function identifies a one-to-one relationship between $b$ and $\pi$ through the function $\Pi(b)$, see (31) – recall that $\Pi(b)$ is the expected repayment of the recession-contingent bond before $\phi$ is realized. Since $\Pi$ is an increasing function, we can invert it and write $b(\pi) = \Pi^{-1}(\pi)$, where $b(\pi)$ satisfies

$$b(\pi) = \frac{\pi - \int_{\phi_{\min}}^{\Phi(b(\pi))} \hat{b}(\phi) dF(\phi)}{1 - F(\Phi(b(\pi)))}, \quad \hat{b}(\phi) = W^{-1}(\alpha - \phi).$$

Note that $b(\pi)$ has the same form as $b^{pl}(\pi)$ in Definition 1. The counterpart in normal time is $\hat{b}(\pi) = \bar{b}$, due to full commitment. The decentralization result will be stated under the condition that $\pi = \Pi(b)$ (or, identically, that $b = b(\pi)$), namely, the sovereign’s initial obligation is the same in the COA and in the market equilibrium.

**Proposition 5** Suppose the sovereign’s outside option is $\alpha$. Then, there exists a Markov equilibrium that decentralizes the COA of Section 2.3. Namely, given equilibrium price functions $\{Q, \bar{Q}\}$ consistent with (29)–(30) (i) the equilibrium policy functions for consumption and effort are the same as in the COA: $C(\mathbb{E}(b(\pi), \phi)) = C^{pl}(\mathbb{E}(\pi, \phi)), \Psi(b(\pi), \bar{b}(\pi)) = \Psi^{pl}(\pi, \bar{b})$; (ii) the threshold functions for debt renegotiation is the same as the threshold for which the PC binds in the planning problem: $\Phi(b(\pi)) = \Phi^{pl}(\pi)$; (iii) the equilibrium law of motion of debt $(b' = B(\mathbb{E}(b(\pi), \phi))$ and
\( \bar{Y} = \hat{B} (B (b (\pi), \phi)) \) is consistent with the law of motion of promised profits in the COA \( (\pi' = \Pi^d (\pi) \) and \( \bar{\pi}' = \bar{\Pi}^d (\pi) \)) ; (iv) the sovereign has the same discounted utility, \( \mu (b (\pi)) = \bar{\mu} (b (\bar{\pi})) = \bar{\mu}^d (\bar{\pi}) \).

The proof establishes that the program solved by the sovereign and by the creditors in the competitive equilibrium (including market clearing) is the same as the primal planning problem of Section 2.3. The strategy of establishing equivalence between the two programs is similar to Aguiar et al. 2017. The equilibrium is therefore characterized equivalently to the COA of Section 2.2.2 as long as \( \pi = \Pi (b) = P (\nu) \). Since both \( \Pi \) and \( P \) are monotonic functions, one can invert them and write \( \nu = P^{-1} (\Pi (b)) \).

The decentralization hinges on the equilibrium price functions \( Q \) and \( \bar{Q} \). These in turn require that (i) there exists two GDP-linked bonds; (ii) the bonds are renegotiable; (iii) renegotiation entails no cost; and (iv) the renegotiation protocol involves full ex-post bargaining power for the creditors.\(^2\)

When selling one-period bonds at the prices \( Q \) and \( \bar{Q} \), the sovereign implicitly offers creditors an expected future profit that takes into account the probability of renegotiation. This is equivalent to the profit promised by the social planner in the primal problem. There are two noteworthy features. First, although there is a continuum of states of nature, two securities are sufficient to decentralize the COA. This is due to the state-contingency embedded in the renegotiable bonds. Second, although the Markov equilibrium rules out reputational mechanisms (while the planner has no such constraint), it nevertheless provides efficient dynamic incentives.

The equivalence result of Proposition 5 holds for any exogenous outside option \( \alpha \). Next, we endogenize \( \alpha \). In a sovereign debt setting it is natural to focus on a case where the sovereign can resume participation in financial markets after defaulting. For simplicity, we assume that access to new borrowing is immediate, in which case \( \alpha = W (0) \) in the Markov equilibrium. This value for \( \alpha \) offers also a natural interpretation of the planning problem, namely, that the sovereign can leave the optimal contract and revert to the market with zero debt after suffering the punishment \( \phi \).

With some abuse of notation let \( W (b; \alpha) \) denote the value function conditional on honoring debt \( b \) in an economy with outside option \( \alpha \). Then, we look for an allocation satisfying the fixed point condition \( \alpha = W (0; \alpha) \). The next corollary shows that there exists one and only one such fixed point.

**Corollary 1** There exists a unique COA such that the outside option is equal to the value of starting with zero debt in the market equilibrium, \( \alpha = W (0; \alpha) \).

It is useful to note here that if the planner had access to a better technology to punish deviations such as forcing stronger types of market exclusions (of which autarky would be an extreme example), then she could attain higher utility. We return to this point in Section 4 below.

Finally, we briefly return to the discussion about one- vs. two-sided commitment in the planning problem of Section 2.2.1. In footnote 8 we argued that commitment on behalf of the principal is not an issue as long as \( \nu \) is sufficiently low. In the equilibrium, this amounts to assuming that \( b \geq 0 \). In this case, recession debt will always remain positive along the equilibrium path. Since this claim has a non-negative value, the creditors would never unilaterally terminate existing contracts, nor would the principal have any commitment problem if she promised the corresponding utility in the planning program.

\(^2\)If any of these assumptions fail, the market equilibrium will in general be suboptimal. The assumption of full commitment under normal time is instead inessential, as we show in an earlier version of this paper (Müller et al. 2016b).
3.2.1 Debt Overhang and Debt Dynamics

The equivalence result in Proposition 5 implies that the Markov equilibrium inherits the same properties as the COA. A binding PC in the planning problem corresponds to an episode of sovereign debt renegotiation in the equilibrium. Thus, as long as the recession continues and debt is not renegotiated, consumption falls. When debt is renegotiated down, consumption may increase. Debt dynamics mirror the promised utility dynamics in the COA. A fall in promised utility corresponds to an increase of sovereign debt. In the COA of Section 2.2.2, $\nu$ decreases over time during a recession unless the PC binds and the promised utilities $\omega(\nu)$ and $\bar{\omega}(\nu)$ are increasing functions (where, recall, $\nu' = \omega(\nu)$ if the recession continues and the PC is slack). Correspondingly, as long as debt is honored and the recession continues, both recession- and recovery-contingent debt are increasing over time.$^{25}$

The left panel of Figure 3 shows the equilibrium policy function for recession- and recovery-contingent debt (solid lines) conditional on no renegotiation and on the recession lingering. Recession-contingent debt converges to $b^{\max}$, which corresponds to the lower bound on promised utility $\nu$ discussed in Section 2.2.2 and displayed in Figure 2. At this level, debt is renegotiated with certainty if the recession continues, and issuing more debt would not increase the expected repayment in recession. The figure also shows the level of debt $b^+$ corresponding to $\nu^-$ in Figure 2. In the range $b > b^+$ the sovereign would like to issue recession-debt above $b^{\max}$ but is constrained to issue $b' = b^{\max}$. However, she is not subject to any constraints when issuing recovery-contingent debt, so $b'$ must be increasing in $b$ in this range.

The right panel of Figure 3 shows the equilibrium effort as a function of $b$ (solid line). This is the mirror image of the lower right panel in Figure 2: it is increasing at low debt levels and falling high debt levels. Intuitively, when debt is low, a higher debt increases the desire for the sovereign to escape recession (this force is also present in the first best of Proposition 1 where effort is decreasing in promised utility). However, as debt increases, the probability that debt is fully honored in recession falls, increasing the share of benefits from recovery accruing to the creditors and making moral hazard more salient. In this region, there is a debt overhang problem reminiscent of Krugman (1988). In our model debt overhang is an equilibrium outcome, namely, a long recession may lead the sovereign to rational choose to push indebtedness into this region and creditors to rationally buy it. Creditors are willing to buy recession-contingent debt from a highly indebted countries in the hope of obtaining favorable terms in the renegotiation process.

3.3 Alternative Decentralization

An alternative decentralization of the COA follows Alvarez and Jermann (2000), henceforth AJ, who show that the COA of a dynamic principal-agent model with enforcement constraints can be attained through a full set of Arrow-Debreu securities subject to appropriate borrowing constraints. In our economy, this requires markets for a continuum of securities paying off in recession – one asset for each $\phi \in \Phi$ – plus one recovery-contingent bond. In this section, we show that the AJ decentralization attains the same allocation as our decentralization through two renegotiable securities.

Consider, first, the case without moral hazard, which is more directly comparable to the original AJ model. Suppose that the sovereign can issue securities $b'_{AJ,\phi}$ that are claims to output in the following period if the recession lingers and state $\phi$ is realized. These securities are non-renegotiable, and ex-post the sovereign can either deliver the payment $b'_{AJ,\phi}$ or default and pay the penalty $\phi$. Under perfect

$^{25}$In the absence of moral hazard (e.g., if effort were contractible), consumption and both types of debt would remain constant over time unless there is renegotiation.
enforcement, the security \(b'_{AJ,\phi} \) sells at a price \(Q_{AJ,\phi} = (1-p)f(\phi)R^{-1} \), where \(p\) is the probability that the recession ends. However, to ensure that the sovereign has an incentive to repay in all states, one must impose some borrowing constraints. Our decentralization yields an informed guess for these constraints: one must impose that \(b'_{AJ,\phi} \leq \hat{b}(\phi) \) where \(\hat{b}(\phi) = \Phi^{-1}(\phi) \) is derived from our Markov equilibrium above. No borrowing constraint has to be imposed for the recovery-contingent bond. Let \(b\) denote the payment due in the current period (after the realization of \(\phi\) is known). In Appendix A we prove that the AJ equilibrium yields an allocation that is equivalent to our decentralized equilibrium under the assumption that \(b'_{AJ,\phi} = \hat{b}(\phi) \) for all \(\phi \leq \Phi(B(b))\) and \(b'_{AJ,\phi} = B(b)\) for all \(\phi > \Phi(B(b))\), where, recall, \(B\) is the equilibrium debt function in our Markov equilibrium with two renegotiable assets. The crux of the proof is to establish that the revenue obtained from issuing recession-contingent debt in the Markov equilibrium is identical to that raised in the AJ economy. In other words, in our two-asset economy with renegotiable securities, the planner is de facto issuing a full set of state-contingent promises equivalent to an AJ portfolio subject to the endogenous borrowing constraints.

Proposition 6 in Appendix A establishes that an AJ equilibrium with borrowing constraints \(b'_{AJ,\phi} \leq \hat{b}(\phi)\) sustains the COA even when the effort choice is subject to an IC (although the original AJ framework does not entail any moral hazard). Intuitively, in the AJ environment the borrowing constraints \(\hat{b}(\phi)\) take into account the effect of issuing debt on the incentives to do effort. We believe this equivalence applies to a larger class of models, although we do not pursue such generalization in this paper.

In summary, this section establishes that the COA can also be decentralized by a full set of non-renegotiable AJ securities with appropriate borrowing constraints. Our decentralization with two renegotiable securities is parsimonious relative to the AJ equilibrium that requires as many securities as there are states (in our environment, this means a continuum of securities). Parsimony is important in realistic extensions in which it is costly for creditors to verify the overall exposure of the sovereign. In our model, creditors must only verify the issued quantity for two assets, while in the AJ framework they must verify that a large number of borrowing constraints are not violated.

### 3.4 Contractible effort

To conclude the analysis of decentralization, we sketch a market arrangement that decentralizes the COA with observable effort of Section 2. This decentralization requires that reform effort be observable and verifiable, and that there exist a market for effort-deviation securities whose return is contingent on the reform effort. In particular, assume that there exists a (defaultable) security \(b_e\) that pays one unit of good in the next period if the reform effort is lower than the constrained efficient effort level denoted by \(p^*(b)\).²⁶ Let \(Q_e(b',\hat{b}',b'_e)\) denote the price of this debt. If the sovereign fails to deliver \(p^*(b)\) at time \(t\), then \(b_e\) comes due at time \(t+1\). After observing \(\phi\) at \(t+1\) the sovereign can either honor the debt \(b_e\) or announce default and trigger the usual renegotiation protocol.

Since along the equilibrium path the sovereign exerts the effort level \(p^*\), the effort-deviation debt will be priced at \(Q_e = 0\) in equilibrium. In the proposed equilibrium, the sovereign issues the maximum feasible effort-deviation debt \(b_e = \hat{b}\) and raises no additional revenue. After issuing \(b_e\), the sovereign will not find it profitable to deviate and set \(p < p^*\). To see why, consider one such deviation. Since, with a positive probability the economy would recover (even if effort were set to \(p > 0\), then the effort-deviation debt would come due yielding an arbitrarily low consumption and expected utility.²⁷

²⁶Here, we define \(p^*(b) = p\left(P^{-1}(\Pi(b))\right)\). Recall that \(p(\nu)\) denotes the constrained efficient effort level when effort is observable and that \(\nu = P^{-1}(\Pi(b))\).

²⁷The assumption that there is full enforcement in normal time simplifies the argument but is not essential. The decentralization could alternatively be attained if the sovereign could issue two GDP-linked effort-deviation debt instruments.
Therefore, deviations are never profitable, and the allocation is equivalent to the COA of Proposition 2 in which the planner controls the effort.

The assumption that reform effort is verifiable is very strong: it requires that international courts can accept and verify evidence about insufficient reform effort when ruling about the breach of contractual agreements. In reality, we do not see such contracts, arguably because the extent to which a country passes and, especially, enforces reforms is opaque.

4 Less Complete Markets

In this section, we consider a competitive (Markov) equilibrium subject to more severe market frictions: in the spirit of Eaton and Gersovitz (1981), the sovereign can issue only a one-period non-state-contingent bond. It is fruitful to analyze this market environment because of its empirical appeal: in the real world, government bonds typically promise repayments that are independent of the endowment state. As is common in the sovereign debt literature, we do not investigate the microfoundations of this market incompleteness. Rather, we take it as exogenous and study its effect on the allocation. We first maintain the same renegotiation protocol as in the earlier sections; then, as an extension, we rule out renegotiation as in the original Eaton-Gersovitz model.

The one-asset economy does not attain the COA. In the COA, the planner trades off the gains from risk sharing against the cost of moral hazard in an optimal way. We proved that two instruments (two defaultable securities) are sufficient to replicate the COA. However, in the one-asset economy, the shortage of securities forces a particular correlation structure between future consumption in recovery and recession that is generally suboptimal, resulting in less risk sharing in equilibrium.

A formal definition, proof of equilibrium existence, and characterization of equilibrium is deferred to Appendix B (Definition 3, and Propositions 7 and 8.) Existence is established by proving that the program is a contraction mapping. Here, we emphasize the salient features of the equilibrium.

Consider, first, the equilibrium policy function for effort $\Psi(b)$, where $b$ now denotes a claim to one unit of output next period, irrespective of the endowment state. The FOC for the effort choice yields $X'(\Psi(b')) = \beta [\bar{\mu}(b') - \mu(b')]$. The equilibrium features debt overhang, namely, the effort function $\Psi$ is falling in $b$ for $b$ sufficiently large.\(^{28}\) Conversely, $\Psi(b)$ is increasing for $b$ sufficiently low. Thus, effort is non-monotone in debt and shares the qualitative properties of the benchmark allocation with two securities.

Consider, next, consumption dynamics. Even in the one-asset economy the risk of repudiation introduces some state contingency, since debt is repaid with different probabilities under recession and normal time. This provides a partial substitute for state-contingent contracts, although not sufficient to decentralize the COA. Recall that in the benchmark economy consumption was determined by two CEEs, (15) and (19). Issuing optimally two types of debt allowed the sovereign to mimic the planner’s ability control the promised utility in each of the two states. This is not feasible in the one-asset economy: there is only one CEE, which pins down the expected marginal rate of substitution Then, the sovereign would issue the maximum sustainable recession-linked effort-deviation debt. The expected value of a deviation would in this case be $\alpha - E[\phi]$ which is a lower bound to the continuation value under no deviation (cf. Equation 5 in the COA). Therefore, the sovereign would prefer to exert the effort level $p^*$.

\(^{28}\)The reason for debt overhang is similar to that in benchmark economy with GDP-linked bonds. There exists a threshold debt $b^{EG} = \Phi^{-1}(\hat{\phi}_{\text{max}})$ such that debt is renegotiated with probability one if $b' \geq b^{EG}$ and the recession continues, while the debt is honored with positive probability if $b' < b^{EG}$. Recall that if the recession ends debt is repaid even for $b' \geq b^{EG}$. Thus, for $b' \geq b^{EG}$ the difference $\bar{\mu}(b') - \mu(b')$ is decreasing in $b'$ implying a decreasing effort, i.e., debt overhang. This feature extends to a contiguous range below $b^{EG}$.

21
in consumption conditional on debt being honored. In Appendix B (Proposition 8), we show that the CEE with non-state-contingent debt takes the form

\[
E \left\{ \frac{u'(c')}{u'(c)} \mid \text{debt is honored at } t+1 \right\} = 1 + \frac{\Psi'(b') \times [b' - \Pi(b')]}{\Pr(\text{debt is honored at } t+1)}
\]  

(36)

where \( b' - \Pi(b') \) is the difference between the expected debt repayment if the economy recovers or remains in recession, and \( c' \) denotes future consumption.

Consider, first, the case with no moral hazard, i.e., \( \Psi' = 0 \). In the equilibrium with GDP-linked bonds, the sovereign could obtain full insurance against the realization of the endowment shock in this case. In the one-asset economy, this is no longer true. To see why, note that the CEE (36) requires that the expected marginal utility in the CEE be equal to the current marginal utility. For this to be true, consumption growth must be positive if the recession ends and negative if it continues.

In the general case where \( \Psi' \neq 0 \), the market incompleteness interacts with the moral hazard friction introducing a novel strategic motive for debt. When the outstanding debt is low, then \( \Psi' > 0 \) and the right-hand side of (36) is larger than unity. In this case, issuing more debt strengthens the ex-post incentive to exert reform effort. The CEE implies then higher debt accumulation and lower future consumption growth than in the absence of moral hazard. In contrast, in the region of debt overhang (\( \Psi' < 0 \)) more debt weakens the ex-post incentive to exert reform effort. To remedy this, the sovereign issues less debt than in the absence of moral hazard. Thus, when debt is large the moral hazard friction magnifies the reduction in consumption insurance.

Figure 3 shows the policy functions for debt and effort in the one-asset economy (dashed lines).\(^\text{29}\) Debt is always higher (lower) than recession-debt (recovery-debt) in the two-security economy. This shows that there is less consumption smoothing in the one-asset economy. Effort is hump-shaped in both the one-asset economy and in the COA. Conditional on debt, effort is higher in the one-asset

\(^{29}\)Note that the policy rules for consumption and debt feature discontinuities. In Appendix B (Proposition 8), we prove that such discontinuities only arise in correspondence of debt levels that are never chosen in equilibrium (unless there is renegotiation). Moreover, the generalized Envelope Theorem in Clausen and Strub (2016) ensures that the FOCs are necessary for interior optima despite the fact that the decision rules are discontinuous.
economy than in the COA reflecting the fact that there is less insurance against the continuation of a recession, making the sovereign more eager to leave the recession itself.

4.1 No Renegotiation

Next, we consider the effect of shutting down renegotiation. Namely, we assume that the sovereign can either default or fully honor the outstanding debt. This alternative environment has three implications. First and foremost, there is costly default in equilibrium. The real costs suffered by the sovereign yields no benefit to creditors. This is in contrast to the equilibrium with renegotiation, where no real costs accrue and creditors recover a share of the face value of debt. Second, renegotiation affects the price function of debt, and thus the incentive for the sovereign to accumulate debt. In particular, the bond price now becomes

\[ Q^{NR}(b') = R^{-1}(\psi^{NR}(b') + [1 - \Psi^{NR}(b')] (1 - F(\Phi^{NR}(b')))). \]

When renegotiation is ruled out the lender expects a lower repayment. Thus, the risk premium associated with each debt level is higher, and it becomes more costly for the sovereign to roll over debt. Therefore, the sovereign will be more wary of debt accumulation. This limits consumption smoothing and lowers welfare. Third, conditional on the debt level, the range of \( \phi \) for which debt is honored is different across the two economies.\textsuperscript{30} While this is in general ambiguous, our numerical analysis, discussed below, suggests that ruling out renegotiation reduces the likelihood that a given level of debt is honored.

Figure 4 in Appendix B compares the policy functions in an Eaton-Gersovitz economy without renegotiation (solid lines) with a one-asset economy with renegotiation (dashed lines). Ruling out renegotiation implies lower consumption for each debt level than in the economy where debt can be renegotiated. Effort is larger when renegotiation is ruled out, reflecting the fact that the debt price is lower in recession and that there is less insurance. Finally, the probability of full debt repayment is lower without renegotiation than when debt can be renegotiated (note that the outside option \( \alpha \) is different across the two cases, since \( W^{NR}(0) < W(0) \)).

4.2 Assistance Program

The possibility of market failures discussed in the previous section provides scope for policy intervention. Consider an assistance program conducted by an international institution, e.g., the IMF. The assistance program mimics the COA through a sequence of transfers to the sovereign during recession in exchange of the promise of a repayment once the recession is over. The IMF can commit but has – like the planner – limited enforcement: it can inflict the same stochastic punishment (\( \phi \)) as can markets.

The program has two key features. First, the terms of the program are renegotiable: whenever the country credibly threatens to abandon it, the IMF sweetens the deal by increasing the transfers and reducing the payment the country owes when the recession ends. When there is no credible default threat, the transfer falls over time, implying the constrained optimum sequence of declining consumption and time-varying reform effort prescribed by the COA. Second, when the recession ends, the IMF receives a payment from the sovereign, financed by issuing debt in the market. This payment depends on the length of the recession and on the history of renegotiations.

More formally, let \( \nu^\phi \) denote the discounted utility guaranteed to the sovereign when the program is first agreed upon. At the beginning of that period, the IMF buys the debt \( b_0 \) with an expenditure

\textsuperscript{30}More formally, in the equilibrium with renegotiation the sovereign renegotiates if \( \phi < W(0) - W(b) \) whereas in the no-renegotiation equilibrium she defaults if \( \phi < W^{NR}(0) - W^{NR}(b) \), where \( W^{NR} \) is the value function under no renegotiation and recession. As long as \( W^{NR} \) is falling more steeply in \( b \) than \( W \), then, conditional on the debt level, the sovereign is more likely to honor the debt in the benchmark equilibrium than in the no-renegotiation equilibrium.
Then, the IMF transfers to the country $T(\nu^\phi) = c(\nu^\phi) - w$ where $c(\nu^\phi)$ is as in the COA of Proposition 3 unless the realization of $\phi$ makes the sovereign want to terminate the program, in which case the country receives $T_\phi = c_\phi - w$, where, again, $c_\phi$ is as in Proposition 3. In the subsequent periods, consumption and promised utility evolve in accordance with the law of motion of the COA. In other words, the IMF commits to a sequence of state-contingent transfers that mimics the COA in the planning allocation. Note that this implies, by construction, that the sovereign exerts the incentive-compatible reform effort level. As soon as the recession ends, the country owes a debt $b_N$ to the international institution, determined by the equation $R^{-1} \times b_N = \tilde{c}(\hat{\omega}_N) - \hat{w} + b_N$, where the discounted utility $\hat{\omega}_N$ depends on the duration of the recession and the history of realizations of $\phi$. Note that $\tilde{c}$ is first-best consumption, exactly like in the COA, and that in normal time the country resumes market access to refinance its debt at the constant level $b_N$.

How large $\nu^\phi$ can be depends on how many resources the IMF is willing to commit to sustain the assistance program. A natural benchmark is to pin down $\nu^\phi$ at the level that allows the IMF to break even in expectation. Whether, ex-post, the IMF makes net gains or losses hinges on the duration of the recession and on the realized sequence of $\phi$'s.

Can such a program improve upon the market allocations? The answer hinges on the extent of market and contract incompleteness. If there exists a market for GDP-linked bonds and if the renegotiation process is frictionless and efficient, the assistance program cannot improve upon the market allocation. More formally, $\nu^\phi = P^{-1}(\Pi(b_0))$. This follows immediately from our decentralization result in Section 3. However, in the one asset economies (with or without renegotiation), the assistance program yields higher efficiency than the competitive equilibrium. Interestingly, the assistance program would in this case yield higher utility than the equilibrium with GDP-linked debt. Since market incompleteness makes deviations less attractive, the IMF has a more powerful threat to discipline the sovereign’s behavior. The outside option $\alpha = W(0)$ is lower the more incomplete are markets. Thus, with more pervasive market incompleteness, the assistance program is closer to first best.

The assistance program would be even more powerful if the IMF could observe and verify the reform effort (e.g., by taking temporarily control over the reform process). In this case, the policy intervention could implement the COA without moral hazard of Section 2.2.1, acting as a stand-in for the missing market of effort-deviation debt discussed in Section 3.4. Clearly, policies infringing on a country’s sovereignty run into severe politico-economic implementability constraints whose discussion goes beyond the scope of our paper.

4.3 Welfare Effects

In a previous version of the paper (Müller et al. 2016b), we quantified the effects of the different informational frictions and of assistance programs when markets are incomplete. The quantitative analysis required some generalization of the stylized model in order to align it with the data. In particular, we relaxed the assumption of full commitment in normal time and assumed, instead of an absorbing state, that during normal time the economy can transit to recession with an exogenous probability. We also relaxed the assumption that $\beta R = 1$ and endogenized the interest rate so that the stationary debt distribution and associated bond prices match key moments of the observed debt distributions and default premia for Southern European countries. The analysis showed that the calibrated model can successfully replicate salient empirical moments not targeted in the calibration, including the bond spread, the frequency of renegotiations, the average haircut of the debt’s face value

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31$\Pi$ depends on the market structure. If there are two securities, then $\Pi$ is given by equation (31). If there is only one asset, the corresponding definition given in Appendix B applies.
and its variance across renegotiation episodes. We found that an assistance program can deliver sizeable welfare gains, especially if it can alleviate the moral hazard problem. The welfare effects of completing markets in a one-asset economy are significant when renegotiation is ruled out in the market economy. The details are available upon request.

5 Concluding Remarks

This paper proposes a novel theory of sovereign debt dynamics under limited enforcement and moral hazard. A sovereign country issues debt to smooth consumption during a recession whose duration is uncertain and endogenous. The expected duration of the recession depends on the intensity of (costly) structural reforms. Both elements – the risk of repudiation and the need for structural reforms – are salient features of the European debt crisis during the last decade.

A key result is that a Markovian competitive equilibrium with renegotiable one-period GDP-linked securities implements the COA. The crux of this result is that, under the assumption that creditors have all the ex-post bargaining power, the renegotiable securities are a stand-in for missing markets for state-contingent debt. In addition, these markets provide the optimal trade-off between insurance and incentives that a fully committed planner subject only to informational constraints can achieve. A surprising element is that the market needs no reputational mechanism to discipline the sovereign’s effort provision over time. In fact, it attains the COA with very "simple" instruments, i.e., two one-period securities.

We also study the effect of additional exogenous restrictions on market arrangements, including assuming that the sovereign can only issue non-contingent debt and, in the spirit of Eaton-Gersovitz (1981), ruling out renegotiation. In this case, the market equilibrium is not constrained efficient. We discuss an assistant program that can restore efficiency and the associated welfare gains.

To retain tractability, we make important assumptions that we plan to relax in future research. First, in our theory the default cost follows an exogenous stochastic process. In a richer model, this would be part of the equilibrium dynamics. Strategic delegation is a potentially important extension. Voters may have an incentive to elect a government that undervalues the cost of default. In our current model, however, the stochastic process governing the creditor’s outside option is exogenous, and is outside of the control of the sovereign and creditors.

Second, again for simplicity, we assume that renegotiation is costless, that creditors can perfectly coordinate and that they have full bargaining power in the renegotiation game. Each of these assumptions could be relaxed. For instance, in reality the process of negotiation may entail costs. Moreover, as in the recent contention between Argentina and the so-called vulture funds, some creditors may hold out and refuse to accept a restructuring plan signed by a syndicate of lenders. Finally, the country may retain some bargaining power in the renegotiation. All these extensions would introduce interesting additional dimensions, and invalidate some of the strong efficiency results. However, we are confident that the gist of the results is robust to these extensions.

Third, we make the convenient assumption that reform effort and consumption are separable in the utility function. If reforms are especially costly during recession, our results would be weakened. However, if one interprets the effort cost as political resistance rather than a burden on the population at large, a long recession might actually strengthen the viability of structural reforms. Finally, by focusing on a representative agent, we abstract from conflicts of interest between different groups of agents within the country. Studying the political economy of sovereign debt would be an interesting extension. We leave the exploration of these avenues to future work.
References


A Appendix A: Proofs of main lemmas, propositions and corollaries

Proof of Lemma 1. To prove this result, we ignore technicalities related to the continuum of states. Namely, we assume that there exist \( N \) states with associated positive probabilities, and view the continuum as an approximation of the discrete state space for \( N \rightarrow \infty \). This is without loss of generality (one can replace statements about single states \( \phi \) by statements about small intervals of positive measure). The proof involves two steps.

1. We start by proving that, if \( \phi_2 > \phi_1 \), then, \( \mu_{\phi_1} > \alpha - \phi_1 \Rightarrow \mu_{\phi_2} = \mu_{\phi_1} > \alpha - \phi_2 \). To see why, recall that the planner’s objective is to deliver the promised utility \( \nu \) given by (2) in a profit-maximizing way. Since \( u, P, \) and \( \bar{P} \) are strictly concave, and \( X \) is strictly convex, then, profit maximization is attained by setting \( c_{\phi_2} = c_{\phi_1}, p_{\phi_2} = p_{\phi_1}, \omega_{\phi_2} = \omega_{\phi_1}, \) and \( \bar{\omega}_{\phi_2} = \bar{\omega}_{\phi_1} \). This planning choice implies that \( \mu_{\phi_2} (\nu) = \mu_{\phi_1} (\nu) \). This is feasible since the assumption that the PC is slack in state \( \phi_1 \) implies that \textit{a fortiori} the PC is slack in state \( \phi_2 \). Two cases are then possible: either \( \exists \tilde{\phi}(\nu) \) such that \( \mu_{\phi} (\nu) = \alpha - \tilde{\phi} (\nu) \) for all \( \phi \geq \tilde{\phi} (\nu) \); or \( \mu_{\phi} (\nu) = \mu (\nu) > \alpha - \phi_{\min} \forall \phi \in [\phi_{\min}, \phi_{\max}] \). The latter case would imply that no PC ever binds and can be ignored.

2. Next, we prove that, if \( \phi_2 > \phi_1 \), then, \( \mu_{\phi_2} = \alpha - \phi_2 \) implies that \( \mu_{\phi_1} = \alpha - \phi_1 \). To derive a contradiction, suppose that this is not the case, and that \( \mu_{\phi_1} > \alpha - \phi_1 > \mu_{\phi_2} \) (the opposite inequality would violate the PC and is not feasible). Then, the planner could deliver to the agent the promised utility \( \nu \) by uniformly increasing \( \omega_2 \), and \( \bar{\omega}_2 \) and reducing \( \omega_1 \), and \( \bar{\omega}_1 \) also uniformly, so as to keep \( \nu \) unchanged, while leaving consumption and effort constant (it is easy to check that this is feasible). The strict concavity of \( P \) and \( \bar{P} \) guarantees that this change increases profits. Let \( \tilde{\phi} (\nu) \) denote the largest \( \phi \) such that \( \mu_{\phi (\nu)} (\nu) = \alpha - \phi (\nu) \). Then, \( \mu_{\phi} (\nu) = \alpha - \phi \) for all \( \phi \leq \phi (\nu) \).

Parts 1. and 2. above jointly establish that the threshold \( \tilde{\phi} (\nu) \) is unique. ■

Proof of Proposition 3. Preliminary: Lemma 3 in Appendix C establishes that \( P \) is strictly differentiable at the interior of its support (under the assumption that \( P \) is concave).

For the characterization result, the planner solves (1) subject to (2)-(5), and (14). The Lagrangian
yields

\[
\mathcal{L} = \int_\mathcal{N} \left[ w - c_\phi + \beta \left( (1 - p_\phi) P(\omega_\phi) + p_\phi P(\tilde{\omega}_\phi) \right) \right] f(\phi) d\phi \\
+ \vartheta \left( \int_\mathcal{N} \left( u(c_\phi) - X(p_\phi) + \beta \left( (1 - p_\phi) \omega_\phi + p_\phi \tilde{\omega}_\phi \right) \right) f(\phi) d\phi - \nu \right) \\
+ \int_\mathcal{N} \lambda_\phi \left( u(c_\phi) - X(p_\phi) + \beta \left( (1 - p_\phi) \omega_\phi + p_\phi \tilde{\omega}_\phi \right) - [\alpha - \phi] \right) d\phi \\
+ \int_\mathcal{N} \chi_\phi \left( -X'(p_\phi) + \beta(\tilde{\omega}_\phi - \omega_\phi) \right) d\phi + \int_\mathcal{N} \theta_\phi \left( \omega_\phi - (\alpha - E[\phi]) \right) d\phi,
\]

(37)

where \( \lambda_\phi \geq 0, \theta_\phi \geq 0, \vartheta, \) and \( \chi_\phi \) denote the multipliers. The first-order conditions (FOCs) with respect to \( c_\phi, \omega_\phi, \tilde{\omega}_\phi, p_\phi \) and \( \chi_\phi \) yield

\[
0 = -f(\phi) + [\vartheta f(\phi) + \lambda_\phi] u'(c_\phi), \\
0 = \beta p_\phi P'(\tilde{\omega}_\phi) f(\phi) + [\vartheta f(\phi) + \lambda_\phi] \beta p_\phi + \chi_\phi \beta \\
0 = \beta(1 - p_\phi)P'(\omega_\phi)f(\phi) + [\vartheta f(\phi) + \lambda_\phi] \beta(1 - p_\phi) - \chi_\phi \beta, \quad \forall \omega_\phi > \alpha - E[\phi] \\
0 = \beta \left( \tilde{P}(\tilde{\omega}_\phi) - P(\omega_\phi) \right) f(\phi) + [\vartheta f(\phi) + \lambda_\phi] \left( -X'(p_\phi) + \beta(\tilde{\omega}_\phi - \omega_\phi) \right) - \chi_\phi X''(p_\phi) \\
0 = -X'(p_\phi) + \beta(\tilde{\omega}_\phi - \omega_\phi)
\]

Lemma 4 in Appendix C rules out the possibility of corner solutions establishing that the FOCs are necessary optimality conditions. The envelope condition yields \(-P'(\nu) = \vartheta\) for all \( \nu > \alpha - E[\phi]\) (note that \( P \) is only differentiable at the interior support of \( \nu \)), while the slackness condition for \( \theta_\phi \) implies \( 0 = \theta_\phi \left( \omega_\phi - (\alpha - E[\phi]) \right) \). Combining the FOCs and the envelope condition, and noting that \(-P'(\tilde{\omega}_\phi) = 1/u'(\tilde{\omega}_\phi))\), yields

\[
\frac{1}{u'(c_\phi)} = -P'(\nu) + \frac{\lambda_\phi}{f(\tilde{\phi}(\nu))}, \quad \forall \nu > \alpha - E[\phi],
\]

(38)

and equations (15) and (16) in the text. Note that we have also used the facts that \( p_\phi = Y(\tilde{\omega}_\phi - \omega_\phi) \) (established in the text, see equation (14)) and \( Y'(\tilde{\omega}_\phi - \omega_\phi) = \beta/X''(\tilde{\omega}_\phi - \omega_\phi) \). When \( \lambda_\phi = 0 \), then (38) implies \(-P'(\nu) = 1/u'(c(\nu)) = \vartheta > 0\) for all \( \nu > \alpha - E[\phi]\).

In order to prove \( \tilde{\phi}(\nu) \) is strictly decreasing, note that Lemma 1 establishes that all PCs associated with \( \phi < \tilde{\phi}(\nu) \) are binding, while all those with \( \phi \geq \tilde{\phi}(\nu) \) are slack. Hence, the PK can be written as \( \nu = \int_{\tilde{\phi}(\nu)}^{\alpha - \phi} dF(\phi) + \left( 1 - F\left( \tilde{\phi}(\nu) \right) \right) \left( \alpha - \tilde{\phi}(\nu) \right), \) which can be rearranged to yield equation (13). Standard arguments establish then that \( \tilde{\phi}(\nu) \) is strictly decreasing.

Consider part 1. in the proposition. For all \( \phi < \tilde{\phi}(\nu) \), the PC is binding and holds with equality. In these cases, the optimal choices are independent of \( \nu \) and \( c_\phi, \omega_\phi, \tilde{\omega}_\phi, \) and \( \theta_\phi \) are determined by the PC (4) holding with equality, and by the FOCs (15)--(17). Consider, next, part 2. For all \( \phi \geq \tilde{\phi}(\nu) \), the PC is slack, and Lemma 1 implies equation (18). The solution is history dependent: \( c_\phi = c(\nu), \omega_\phi = \omega(\nu), \tilde{\omega}_\phi = \tilde{\omega}(\nu), \theta_\phi = \theta(\nu), \) and \( \omega(\nu), \tilde{\omega}(\nu), \) and \( \theta(\nu) \) are determined by equations (15)--(18). To prove that \( c(\nu) \) is an increasing function, note that, since \( P \) and \( u \) are strictly concave, then (38) implies that \( c(\nu) \) is strictly increasing for \( \nu > \alpha - E[\phi] \). Moreover, consumption is also falling at the lower bound, \( c(\nu) < c(\nu) \forall \nu > \nu \). This can be proved by a contradiction argument: Suppose
that \( \exists \nu > \nu \) such that \( c(\nu) = c(\nu) \). The optimal allocation associated with \( c(\nu) \) yields utility \( \nu > \nu \). Thus, \( c(\nu) = c(\nu) \) would violate the PK, which is not feasible.

Next, we prove that promised utility falls over time, i.e., \( \omega(\nu) = \omega(\nu) \leq \nu \), with \( \omega(\nu) < \nu \) for \( \theta = 0 \) (i.e., if \( \nu \) is sufficiently large). To this aim, we establish the following properties in three steps: (i) \( \bar{P}(\omega(\nu)) = P(\omega(\nu)) > 0 \); (ii) if \( \nu > \nu \), then \( c(\omega(\nu)) > c(\nu) > c(\omega(\nu)) \) and \( \omega(\nu) < \nu \) in all states \( \phi \) in which the PC is slack; (iii) \( \omega(\nu) \) is strictly increasing and \( \omega(\nu) \) constant on the interval \( [\nu, \nu^-] \).

We first prove that \( \bar{P}(\omega(\nu)) - P(\omega(\nu)) > 0 \) for \( \nu > \nu^- \) (i.e., when \( \omega(\nu) > \alpha - E[\phi] \) and \( \theta = 0 \)). Suppose, to derive a contradiction, that \( \bar{P}(\omega(\nu)) - P(\omega(\nu)) \leq 0 \) for \( \theta = 0 \). Then (15), (16), and (17) imply the following consumption ordering

\[
\bar{c}(\omega(\nu)) \leq c(\nu) \leq c(\omega(\nu)). \tag{39}
\]

To arrive at a contradiction suppose that \( \omega(\nu) \geq \nu \). Once the economy recovers, promised-utility and profits remain constant such that consumption can be written as

\[
\bar{c}(\omega(\nu)) = \bar{w} - \bar{P}(\omega(\nu)) + \beta \bar{P}(\omega(\nu)) = \bar{w} - \bar{P}(\omega(\nu)) + \frac{p}{R} \bar{P}(\omega(\nu)) = \frac{1 - p}{R} P(\omega(\nu))
\]

We used \( \bar{w} > w \) to derive the first inequality, the fact that \([1 - p]/R - 1 \bar{P}(\omega(\nu)) > [(1 - p)/R - 1] P(\omega(\nu)) \)

since \( [(1 - p)/R - 1] < 0 \) and \( \bar{P}(\omega(\nu)) \leq P(\omega(\nu)) \) to derive the second inequality, and \( \omega(\nu) \geq \nu \) along with \( P'(\nu) < 0 \) to derive the last inequality. \( \bar{c}(\omega(\nu)) > c(\nu) \) contradicts the ordering in (39). Thus, we have proven that \( \bar{P}(\omega(\nu)) - P(\omega(\nu)) \leq 0 \Rightarrow \omega(\nu) < \nu \) for \( \nu > \nu^- \). Next, recall that consumption is strictly increasing in the promise (since \( P \) is strictly concave) such that \( \omega(\nu) < \nu \Rightarrow c(\omega(\nu)) < c(\nu) \leq c(\nu) \). This again contradicts the ordering in (39). Thus, we have shown that \( \bar{P}(\omega(\nu)) - P(\omega(\nu)) > 0 \) for \( \nu > \nu^- \).

Combining, \( \bar{P}(\omega(\nu)) - P(\omega(\nu)) > 0 \), \( \theta = 0 \), (15), (16), and (17) implies that

\[
\bar{c}(\omega(\nu)) > c(\nu) \geq c(\nu) > c(\omega(\nu)).
\]

Since \( c(\nu) \) is strictly falling in \( \nu \), the last inequality also implies that \( \omega(\nu) < \nu \), as long as the recession lasts, the PC remains slack, and the lower bound \( \nu \) has not been reached. Once \( \omega(\nu) = \nu \), promised utility and consumption remains constant as long as the recession lasts and the PC remains slack. The first inequality also implies that consumption increases when the recession ends \( \bar{c}(\omega(\nu)) > c(\nu) \).

Next, we use Topkis’s theorem to show that \( \omega(\nu) \) must be strictly increasing in \( \nu \) when \( \omega(\nu) = \nu \) and that \( \bar{P}(\omega(\nu)) - P(\omega(\nu)) > 0 \) also for \( \nu < \nu^- \). Given the threshold property, the planner solves the following maximization problem for all states \( \omega \) with a slack PC

\[
\langle \omega(\nu), \omega(\nu) \rangle = \max_{\omega \in [\bar{\omega}, \omega; \omega \in [\alpha - E[\phi]; \omega]} u - u^{-1}(x(\nu, \omega, \omega)) + \beta \left[ (1 - Y(\omega - \omega)) P(\omega) + Y(\omega - \omega) P(\omega) \right],
\]

where \( x(\nu, \omega, \omega) = \alpha - \phi(\nu) + X(\gamma(\omega - \omega) - \beta \left[ (1 - Y(\omega - \omega) + Y(\omega - \omega) \right] \). The strictly positive cross-derivative of the objective function

\[
\frac{u''(x(\nu, \omega, \omega)\phi(\nu)}{u'(x(\nu, \omega, \omega))^2} \times \beta Y(\omega - \omega) > 0,
\]

III
shows that the objective is supermodular in \((\nu, \tilde{\omega})\) for a given promised-utility \(\omega\). Thus, by Topkis’s theorem, \(\tilde{\omega}(\nu)\) is strictly increasing in \(\nu\) when \(\omega(\nu) = \nu\). This also implies that \(\tilde{P}(\tilde{\omega}(\nu)) - P(\nu) \geq \tilde{P}(\tilde{\omega}(\nu^-)) - P(\nu) > 0\), for \(\nu \leq \nu^-\). Thus, we have shown that, \(\tilde{P}(\tilde{\omega}(\nu)) - P(\omega(\nu)) > 0\) for all \(\nu\).

Finally, we establish that effort is increasing in \(\nu\) in a low range of \(\nu\). We established above that \(\tilde{\omega}(\nu)\) is strictly increasing in \(\nu\) and \(\omega(\nu)\) is constant for \(\nu \leq \nu^-\). This implies that \(\tilde{\omega}(\nu) - \omega(\nu)\), and hence effort, is strictly increasing in \(\nu \in [\nu^-, \nu]\). By continuity, the result extends to a contiguous range of \(\nu > \nu^-\). This concludes the proof of the proposition. ■

**Proof of Proposition 4.** Consider first the set of realizations for which the PC is slack, \(\phi \geq \Phi^{pl}(\pi)\). In this case, it follows immediately from points (1)–(3) of Definition 1 that the planner’s optimal choice is \(c_\phi = C^{pl}(\pi), \pi'_\phi = \Pi^{pl}(\pi), \pi''_\phi = \Pi^{pl}(\pi)\), and \(p_\phi = \Psi^{pl}(\Pi^{pl}(\pi), \Pi^{pl}(\pi))\). In this case, the sovereign attains the utility \(W^{pl}(\pi) = \alpha - \Phi^{pl}(\pi)\), namely, the outside option utility associated with the largest realization of \(\phi\) for which the PC binds. When the PC binds, the solutions are dictated from the PC. In particular, the sovereign receives a utility \(\alpha - \phi\). We can then define \(\pi^{pl}_\phi\) as a pseudo-profit such that, conditional on the realization \(\phi\) and on \(\pi = \pi^{pl}_\phi\), the PC is just binding, i.e., \(\pi^{pl}_\phi = (\Phi^{pl})^{-1}(\phi)\) and \(W^{pl}(\pi^{pl}_\phi) = \alpha - \phi\). Given this definition of \(\pi^{pl}_\phi\) and Definition 1, we can write the planner’s optimal choice when \(\phi \geq \Phi^{pl}(\pi)\) as \(c_\phi = C^{pl}(\pi^{pl}_\phi), \pi'_\phi = \Pi^{pl}(\pi^{pl}_\phi), \pi''_\phi = \Pi^{pl}(\pi^{pl}_\phi)\), and \(p_\phi = \Psi^{pl}(\Pi^{pl}(\pi^{pl}_\phi), \Pi^{pl}(\pi^{pl}_\phi))\). The definition of \(\mathbb{E}^{pl}(\pi, \phi)\) allows us to write the solution in a more compact form. The expression for the value function \(\mu^{pl}(\pi)\) follows from the fact that the sovereign receives the utility \(\alpha - \Phi^{pl}(\pi)\) if the PC is slack and \(\alpha - \phi\) if the PC binds. ■

**Proof of Proposition 5.** We start by guessing that \(\Phi(b(\pi)) = \Phi^{pl}(\pi)\). We show that under this guess the equilibrium is characterized by the same program as the COA. We then verify the guess, establishing that there exists a Markov equilibrium that decentralizes the COA.

We start by proving that \(\mathbb{B}(b(\pi), \phi) = b^{pl}(\mathbb{B}^{pl}(\pi, \phi))\), i.e.,

\[
\mathbb{B}(b(\pi), \phi) = \begin{cases} 
\hat{b}(\phi) = b^{pl}(\pi^{pl}_\phi) & \text{if } \phi < \Phi(b(\pi)) = \Phi^{pl}(\pi), \\
\hat{b}(\pi) = b^{pl}(\pi) & \text{if } \phi \geq \Phi(b(\pi)) = \Phi^{pl}(\pi).
\end{cases}
\]

Consider, first, the case in which the PC binds. We must show that, then, \(\hat{b}(\phi) = b^{pl}(\pi^{pl}_\phi)\). Setting \(b = \hat{b}(\phi)\) in the bond revenue equation (31) for the competitive equilibrium yields the recursive equation

\[
\hat{b}(\phi) = \Pi(\hat{b}(\phi)) - \int_{\phi_{\min}}^{\phi} \hat{b}(\phi') dF(\phi') - 1 - F(\phi).
\]

This has the same form as the recursion (25) in the primal planning problem, where, if we set \(\pi = \pi^{pl}_\phi\),

\[
b^{pl}(\pi^{pl}_\phi) = \frac{\hat{\pi}^{pl}_\phi - \int_{\phi_{\min}}^{\phi} b^{pl}(\pi^{pl}_\phi) dF(\phi')}{1 - F(\phi)}.
\]

Both equations only depend on \(\phi\). In particular, the two equations are identical for \(\hat{b}(\phi) = b^{pl}(\pi^{pl}_\phi)\) and \(\hat{\pi}^{pl}_\phi = \Pi(\hat{b}(\phi))\).

IV
Consider, next, the case in which the PC is slack. In this case, one obtains

\[ b^{pl} (\pi) = \left( \pi - \int_{\phi_{\min}}^{\Phi^{pl}(\pi)} b^{pl} \left( \frac{\pi}{\phi} \right) dF (\phi) \right) = \left( \Pi (b(\pi)) - \int_{\phi_{\min}}^{\Phi(b(\pi))} \bar{b}(\phi) dF (\phi) \right) = b(\pi). \]

Moreover, conditional on the guess of a common threshold, it is immediate to verify that \( \mu (b (\pi)) = \mu^{pl} (\pi) \) and \( \bar{b} (\bar{\pi}) = \bar{\mu}^{pl} (\bar{\pi}) \). Then, since the effort function stems from the same IC in the equilibrium and in the planning problem, it follows immediately that \( \Psi (b (\pi), \bar{b} (\bar{\pi})) = \Psi^{pl} (\pi, \bar{\pi}) \).

Next, we show that \( C (b (\pi)) = C^{pl} (\pi) \). The gist of the argument is that we can rewrite the program (33)–(34) as the optimal choice of consumption and future expected debt repayment instead of consumption and future debt. More formally, the equilibrium functions solve

\[
\begin{align*}
\langle B (B (b, \phi)), \bar{B} (B (b, \phi)) \rangle &= \arg \max_{(b', \bar{b}') \in [b, \bar{b}]^2} \left\{ u \left( Q (b', \bar{b}') \times b' + \bar{Q} (b', \bar{b}') \times \bar{b}' + w - B (b, \phi) \right) + \right. \\
- &\left. X \left( \Psi (b', \bar{b}') \right) + \beta \left( \Psi (b', \bar{b}') \times \bar{b}' (1 - \Psi (b', \bar{b}') ) \right) \right\},
\end{align*}
\]

where

\[
C (B (b, \phi)) = Q (B (B (b, \phi)), \bar{B} (B (b, \phi))) \times B (B (b, \phi)) + \bar{Q} (B (B (b, \phi)), \bar{B} (B (b, \phi))) \times \bar{B} (B (b, \phi)) + w - B (b, \phi).
\]

Evaluate optimal consumption at \( b = b (\pi) \) and use the fact that \( Q (b', \bar{b}') \times b' + \bar{Q} (b', \bar{b}') \times \bar{b}' = \beta \left( \Psi (b', \bar{b}') \times b' + (1 - \Psi (b', \bar{b}') ) \right) \Pi (b') \), to yield

\[
C (B (b, \pi)) = \beta \left( 1 - \Psi (b (\Pi^c (B (b (\pi), \phi))), \bar{b} (\Pi^c (B (b (\pi), \phi)))) \right) \Pi^c (B (b (\pi), \phi)) + \bar{b} (\Pi^c (B (b (\pi), \phi))) \Pi^c (B (b (\pi), \phi)) + w - B (b, \phi).
\]

subject to \( b \leq b (\pi') \leq (\Phi)^{-1} (\phi_{\max}), \leq \bar{b} \leq \bar{b} (\pi') \leq \bar{b}, \) and \( c = w - B (b, \phi) + \beta (1 - \Psi (b (\pi'), \bar{b} (\pi'))) \pi' + \Psi (b (\pi'), \bar{b} (\pi'))) \pi'. \) Since (40), the competitive equilibrium solves the same program as the planning problem, and \( \langle \Pi^c (b (\pi)), \Pi^c (b (\pi)), C (b (\pi)) \rangle = \langle \Pi^{pl} (\pi), \Pi^{pl} (\pi), C^{pl} (\pi) \rangle \). The fact that in equilibrium \( \pi' = \Pi^c (b (\pi)) = \Pi^{pl} (\pi) \) and \( \bar{\pi}' = \Pi^c (b (\pi)) = \Pi^{pl} (\pi) \) implies that the equilibrium law of motion for discounted profits is the same as in the COA. This set of equivalences also establishes that \( W (b (\pi)) = W^{pl} (\pi) \).

To conclude the proof, we must verify the guess that \( \Phi (b (\pi)) = \Phi^{pl} (\pi) \). To this aim, note that, in equilibrium \( W (b (\pi)) = \alpha - \Phi (b (\pi)) \), whereas in the COA \( W^{pl} (\pi) = \alpha - \Phi^{pl} (\pi) \). Both are fixed-point conditions, since \( W \) depends on \( \Phi \) and \( W^{pl} \) depends on \( \Phi^{pl} \). Moreover, we have established that \( \Phi (b (\pi)) = \Phi^{pl} (\pi) \Rightarrow W (b (\pi)) = W^{pl} (\pi). \) Therefore, the fixed point condition is the same in the Markov equilibrium and in the COA. Hence, the COA can be decentralized by the Markov equilibrium. Note that we cannot rule out the existence of multiple fixed points.

**Proof of Corollary 1.** With slight abuse of notation we write \( x(0; \alpha) = x(0) \) to make the dependence of a variable \( x \) on \( \alpha \) explicit. We show first that \( \Phi (0; \alpha) \) is strictly increasing in \( \alpha \). Suppose to
the opposite that $\partial \Phi(0; \alpha)/\partial \alpha \leq 0$. Then, the indifference condition implies that $\partial W(0; \alpha)/\partial \alpha > 0$ since $\partial \Phi(0; \alpha)/\partial \alpha = 1 - \partial W(0; \alpha)/\partial \alpha$. This also implies that, $\partial EV(0; \alpha)/\partial \alpha > 0$ since $EV(0; \alpha) = \int_{\mathbb{R}} \max\{W(0; \alpha), \alpha - \phi\} dF(\phi)$. At the same time, $\Pi(0; \alpha) = 0 \forall \alpha$. However, we know from Proposition 5 that $P(\nu; \alpha) = \Pi(0; \alpha)$ implies that $\nu = EV(0; \alpha)$. Thus, for any $\alpha' > \alpha$ there exists an allocation that yields the same profits $\Pi(0; \alpha') = \Pi(0; \alpha')$ but higher promised-utility $EV(0; \alpha') > EV(0; \alpha)$ than the COA for $\alpha$. This must be a contradiction, since - for a given $\alpha$ - it is always feasible for the planner to replicate the allocation with outside option $\alpha'$ by relaxing all the PCs by $\alpha' - \alpha$. However, the planner chose not to do so because its not optimal. Thus, $\partial \Phi(0; \alpha)/\partial \alpha > 0$.

Next, by the Theorem of the Maximum $W(0; \alpha)$ (and therefore $\Phi(0; \alpha)$) is continuous in $\alpha$. Moreover, at the fixed point $W(0) = W(0; W(0)) = W(0) - \Phi(0; W(0))$ the threshold must be zero. Then, since $\Phi(0; \alpha)$ is continuous and strictly increasing in $\alpha$ on the interval $[u(w)/(1 - \beta), \bar{\mu}(0)]$ and $u(w)/(1 - \beta) < W(0) < \bar{\mu}(0)$, it is sufficient to state that $\Phi(0; u(w)/(1 - \beta)) < \Phi(0, W(0)) = 0$ and $\Phi(0; \bar{\mu}(0)) > \Phi(0, W(0)) = 0$ to establish that the fixed-point $W(0)$ exists in the interior of this interval and is unique. 

**Proposition 6** Assume that the economy has an AJ market structure, where the sovereign can issue non-renegotiable debt contingent on the realization of GDP and on $\phi$, subject to borrowing constraints. Then, the COA can be decentralized by an AJ equilibrium whose borrowing constraints are set as follows: $b'_{AJ, \phi} \leq \tilde{b}(\phi)$. The borrowing constraint is binding for $\phi \leq \Phi(B(b))$ and slack otherwise, where $b$ denotes the debt repayment in the current period-state and $B$ is the debt equilibrium function in Definition 2.

**Proof of Proposition 6.** Let $\phi$-specific recession debt be denoted $b'_{AJ, \phi} = b'_\phi$, and let the recession and normal time value functions be given by $W_{AJ}$ and $\tilde{W}_{AJ}$, respectively. Since debt is non-renegotiable and subject to non-default borrowing constraints, the prices of recession debt and normal time debt are $(1 - p_{AJ})f(\phi)R^{-1}$ and $p_{AJ}R^{-1}$, respectively, where $p_{AJ}$ is expected effort. The problem for a sovereign who owes $b$ in recession is then

$$W_{AJ}(b) = \max \left\{ b'_\phi \in \mathbb{R} \mid b'_{AJ} = \tilde{b}(\phi) \right\} \left\{ u \left( (1 - p_{AJ}) R^{-1} \times \int_{\mathbb{R}} b'_\phi f(\phi) d\phi + p_{AJ} R^{-1} \times \tilde{b}_{AJ} + w - b \right) - X \left( p_{AJ} + \beta (1 - p_{AJ}) \int_{\mathbb{R}} W_{AJ}(b'_\phi) dF(\phi) + \beta p_{AJ} \tilde{W}_{AJ}(\tilde{b}_{AJ}) \right) \right\},$$

where $p_{AJ} = \arg \max_{p \in [0, 1]} \left\{ -X(p) + \beta (1 - p) \int_{\mathbb{R}} W_{AJ}(b'_\phi) F(\phi) + \beta p \tilde{W}_{AJ}(\tilde{b}_{AJ}) \right\}$, subject to a set of no-default borrowing constraints

$$b'_\phi \leq J(\phi), \; \forall \phi \in \mathbb{R},$$

and $b'_\phi \geq b, \; \tilde{b}_{AJ} \in [b, \tilde{b}]$.

The proof strategy is to show that the market allocation of Definition 2 is feasible, rules out default, is consistent with the FOCs, and yields the same expected utility in the AJ economy. Let the policy functions $\tilde{b}(\phi), \Phi(B(b)), \Pi(b), \bar{B}(\bar{b}), B(b), \mathbb{E}(b, \phi), C(b), \tilde{C}(\tilde{b}), W(b), \Psi(b, \tilde{b}),$ and $\bar{\mu}(\tilde{b})$ be given by the market allocation.

We guess that (i) the borrowing constraints are $J(\phi) = \tilde{b}(\phi)$, (ii) optimal debt issuance is given by:

$$b'_\phi = \begin{cases} B(b) & \text{for } \phi \geq \Phi(B(b)), \\ \tilde{b}(\phi) & \text{for } \phi < \Phi(B(b)) \end{cases},$$

VI
and \( \bar{b}_{AJ} = \bar{B}(b) \), and (iii) optimal consumption is \( C_{AJ}(b) = C(b) \) and \( \bar{C}_{AJ}(b) = \bar{C}(b) \). Since \( b'_\phi = \mathbb{B}(b, \phi) \), the realized debt payments are equivalent to the market allocation. Given that repayments and consumption are as in the market economy in all states, the equilibrium effort \( p_{AJ} \) must also be the same as in the market economy. It follows that the revenue from issuing recession debt is identical in the two economies, since

\[
(1 - p_{AJ}) R^{-1} \times \int_{\mathbb{R}} b'_\phi f(\phi) d\phi = (1 - \Psi(B(b), \bar{B}(b))) R^{-1} \times \left( (1 - F(\Phi(B(b)))) \cdot B(b) + \int_{\phi_{\text{min}}}^{\Phi(B(b))} \hat{b}(\phi) f(\phi) d\phi \right)
\]

\[
= (1 - \Psi(B(b), \bar{B}(b))) R^{-1} \times \Pi(B(b)).
\]

Hence, the proposed allocation \( \{b'_\phi, \bar{b}_{AJ}, C_{AJ}, \bar{C}_{AJ}, p_{AJ}\} \) satisfies the budget constraint and is therefore feasible.

Since the allocation \( \{C_{AJ}, \bar{C}_{AJ}, p_{AJ}\} \) is the same as in the market economy, the discounted value must be the same as well; \( \bar{W}_{AJ}(b) = \bar{\mu}(b) \) and \( W_{AJ}(b) = W(b) \), where \( b \) is the realized debt payment after realization of \( \phi \). Since \( W_{AJ}(J(\phi)) = W(\bar{b}(\phi)) = \alpha - \phi \forall \phi \in \mathbb{R} \), it follows that \( J(\phi) \) is not too tight, in the sense that the PC holds with equality at the borrowing constraint. Hence, \( J \) rules out default.

Finally, it is straightforward to verify that the proposed allocation satisfies the FOCs in the AJ economy. Since the proposed allocation is feasible, satisfies the AJ optimality conditions, and at the same time attains the same utility as in the COA, it must represent an equilibrium allocation in the AJ economy.
B Appendix B: Analysis in Section 4

In this section, we provide the technical analysis of the results summarized in Section 4 and one related figure. We consider the competitive (Markov) equilibrium for an economy where the sovereign can issue only a one-period renegotiable non-state-contingent bond. The extension in which we rule out renegotiation is qualitatively similar. The only difference is that the bond price, which is

\[ Q(b) = R^{-1} \left[ 1 - \Psi(b) \right] \times \left( 1 - F(\Phi(b)) \right) + \frac{1}{b} \int_{\phi_{\min}}^{\Phi(b)} \tilde{b}(\phi) \times dF(\phi) + \Psi(b') \]

when renegotiation is allowed, becomes

\[ Q^{NR}(b) = R^{-1} \left[ 1 - \Psi^{NR}(b) \right] \times \left[ 1 - F(\Phi^{NR}(b)) \right] + \Psi^{NR}(b) \]

when renegotiation is ruled out. In both cases we assume that the outside option is the market equilibrium with a zero debt position, i.e., \( \alpha = W(0) \) in the case with renegotiation, and \( \alpha = W^{NR}(0) \), if we rule out renegotiation.

We provide (i) a definition of Markov equilibrium for the one-asset economy; (ii) a proof of existence and uniqueness of \( W \) and the associated equilibrium functions; (iii) a derivation of the CEE in equation (36). All proofs are in a separate section of this appendix below.

**Definition 3** A Markov-perfect equilibrium with non-state contingent renegotiable debt is a set of value functions \( \{V, W\} \), a threshold renegotiation function \( \Phi \), an equilibrium debt price function \( Q \), and a set of optimal decision rules \( \{B, C, \Psi\} \) such that, conditional on the state vector \( (\phi, \phi') \in \left( [b, \bar{b}] \times [\phi_{\min}, \phi_{\max}] \right) \), the sovereign maximizes utility, the creditors maximize profits, and markets clear. More formally:

- **The value function** \( V \) satisfies
  \[ V(b, \phi) = \max \{W(b), W(0) - \phi\}, \]
  where \( W(b) \) is the value function conditional on the debt level \( b \) being honored,
  \[ W(b) = \max_{b' \in [b, \bar{b}]} u \left( Q(b') \times b' + \underline{w} - b \right) + Z(b'), \]
  continuation utility \( Z \) is defined as
  \[ Z(b') = \max_{p \in [\underline{p}, \bar{p}]} \left\{ -X(p) + \beta \left( p \times \bar{\mu}(b') + (1 - p) \times \mu(b') \right) \right\}, \]
  \[ \text{the value of starting in recession with debt } b \text{ and in normal time with debt } \bar{b} \text{ are } \mu(b) = \int_{\phi} V(b, \phi) \, dF(\phi) \]
  \[ \text{and } \bar{\mu}(b) = u \left( \bar{w} - \left( 1 - R^{-1} \right) \bar{b} \right) / (1 - \beta), \]
  \[ \text{respectively.} \]

- **The threshold renegotiation function** \( \Phi \) satisfies
  \[ \Phi(b) = W(0) - W(b). \]
The debt price function satisfies the following arbitrage condition:

\[
Q(b') \times b' = R^{-1}\left(1 - \Psi(b') + \Pi(b') + \Psi(b')\right) \times b'
\]  

(42)

where \(\Pi(b')\) is the expected repayment of the non-contingent bond conditional on next period being a recession,

\[
\Pi(b) = (1 - F(\Phi(b))) + \int_{\phi_{\min}}^{\Phi(b)} \hat{b}(\phi) \times dF(\phi),
\]

(43)

and where \(\hat{b}(\phi) = \Phi^{-1}(\phi)\) is the new post-renegotiation debt after a realization \(\phi\).

The set of optimal decision rules comprises:

1. A take-it-or-leave-it debt renegotiation offer:

\[
B(b, \phi) = \begin{cases}
\hat{b}(\phi) & \text{if } \phi \leq \Phi(b), \\
b & \text{if } \phi > \Phi(b).
\end{cases}
\]

2. An optimal debt accumulation and an associated consumption decision rule:

\[
B(B(b, \phi)) = \arg \max_{b' \in [b, b]} \left\{ u(Q(b') \times b' + w - \mathbb{B}(b, \phi)) + Z(b') \right\}
\]

\[
C(B(b, \phi)) = Q(B(B(b, \phi))) + B(B(b, \phi)) + w - \mathbb{B}(b, \phi).
\]

3. An optimal effort decision rule:

\[
\Psi(b') = \arg \max_{p \in [p, \bar{p}]} \left\{ -X(p) + \beta(p \times \hat{b}(b') + (1 - p) \times \mu(b')) \right\}.
\]

The equilibrium law of motion of debt is \(b' = B(B(b, \phi))\).

The probability that the recession ends is \(p = \Psi(b')\).

We prove, next, that a Markov Equilibrium with non-contingent debt exists and that the set of equilibrium functions \(\{V, W, \Phi, Q, B, \Psi\}\) is unique.

Proposition 7 The Markov equilibrium with non-contingent renegotiable debt exists and the set of equilibrium functions \(\{V, W, \Phi, Q, B, \Psi\}\) satisfying Definition 3 is unique. The value functions \(\{V, W\}\) are continuous, \(W\) is strictly decreasing in \(b\), and \(V\) is non-increasing in \(b\).

Finally, we derive formally the CEE of equation (36). Since the CEE is derived from FOCs, the proposition must first establish appropriate differentiability properties that ensures that the FOCs are necessary conditions for an equilibrium. It turns out that in the one-asset economy the equilibrium functions are not globally continuous and differentiable. However, we can prove that they are differentiable at all interior level of debt that can be the result of an optimal choice. Moreover, the discontinuities in the policy functions do not invalidate the fact that the FOCs are necessary conditions for an equilibrium. It is then useful to define \(\bar{B}\) as the set of debt levels \(b'\) that can be the result of an optimal interior choice given debt \(b\).
Definition 4 \( \hat{B} = \{ b' \in (\bar{b}, \tilde{b}) | B(b, \phi) = b', \text{ for } b \in [\bar{b}, \tilde{b}] \} \).

Proposition 8 Let \( \bar{C}(b) \) denote the consumption function in normal time. The equilibrium functions \( W(b'), \Phi(b'), Q(b'), \) and \( \Psi(b') \) are differentiable for all \( b' \in \hat{B} \). Moreover, for any \( b' \in \hat{B} \), the FOC \( (\partial/\partial b')u(Q(b')b' + \bar{w} - b) + (\partial/\partial b')Z(b') = 0 \) and the envelope condition \( \partial W(b')/\partial b' = -u'(C(b')) \) holds true, such that the conditional Euler equation (CEE)

\[
\frac{(1 - \Psi(b')) [1 - F(\Phi(b'))]}{(1 - \Psi(b')) [1 - F(\Phi(b'))] + \Psi(b') u'(C(b'))} + \frac{\Psi(b')}{(1 - \Psi(b')) [1 - F(\Phi(b'))] + \Psi(b') u'(C(b'))} = 1 + \frac{u'(\hat{C}(b'))}{(1 - \Psi(b')) [b' - \Pi(b')]}
\]

is a necessary condition for an interior optimum.

Note that the left-hand side of (44) is the expected ratio between next-period and current-period marginal utility conditional on debt being honored in the next period. More precisely, the term \( u'(C(b'))/u'(C(b)) \) is the ratio between the marginal utilities if the recession continues, whereas the term \( u'(\hat{C}(b'))/u'(C(b)) \) is the ratio between the marginal utilities if the recession ends. Therefore, equation (44) is identical to equation (36).

B.1 Figure 4

In this section we show Figure 4 comparing the properties of the one-asset economy with and without renegotiation. This figure is discussed in Section 4.1 in the text.
Figure 4: Policy functions of the one-asset economy, conditional on the maximum cost realization $\phi_{max}$. The dashed lines show the Markov equilibrium with renegotiation, the solid lines the equilibrium where renegotiation is ruled out.
B.2 Proofs

Proof of Proposition 7. We prove the existence and uniqueness of the equilibrium functions by showing that the value function $W$ is a fixed-point of a contraction mapping following Stokey and Lucas (1989, Theorem 3.3).

Let $\Gamma$ be the space of bounded, continuous, and decreasing functions defined over $[b, \bar{b}]$. Moreover, let $d_\infty$ denote the supremum norm such that $(\Gamma, d_\infty)$ is a complete metric space. Let $z \in \Gamma$ and $\alpha$ be a real constant representing the outside option under outright default. The operator $T(z; \alpha)$ is similar to the Bellman equation of the Markov equilibrium in Definition 3, but differs in that the value of outright default in the recession state is exogenously given by $\alpha - \phi$ in place of $W(0) - \phi$. We first establish the uniqueness of the equilibrium functions for an exogenous $\alpha$, and then extend the argument to the endogenous outside option $W(0)$ as in the Markov equilibrium.

Define the following mapping:

$$T(z; \alpha)(b) = \max_{\nu' \in [b, \bar{b}]} u(Q(b', z(b'); \alpha)b' - b + \nu) + Z(z(b'); \alpha),$$

$$\equiv \max_{\nu' \in [b, \bar{b}]} O(b, b'; z(b'); \alpha),$$

where $\Phi(z(b'); \alpha) = \alpha - z(b')$, and $z \left( \bar{b}(\phi; z, \alpha) \right) = \alpha - \phi$. In addition, let $T^0(z; \alpha) = z$, $T^n(z; \alpha) = T(T^{n-1}(z; \alpha); \alpha)$, $n = 1, 2, \ldots$, and $z^n \equiv T^n(z; \alpha)$. The elements that enter the above objective function $O(b, b', z(b'); \alpha)$ are given by

$$Z(z(b'); \alpha) = \max_{p \in [p, \bar{p}]} -X(p) + \beta \left[ p\mu(b') + (1-p)Ez(b') \right]$$

$$\Psi(z(b'); \alpha) = \arg \max_{p \in [p, \bar{p}]} -X(p) + \beta \left[ p\mu(b') + (1-p)Ez(b') \right]$$

$$Ez(b') = \left[ 1 - F \left( \Phi(z(b'); \alpha) \right) \right] z(b') + \int_{\phi_{\min}}^{\Phi(z(b'); \alpha)} [\alpha - \phi'] f(\phi')d\phi'$$

$$Q(b', z(b'); \alpha)b' = R^{-1} \left[ \left( 1 - \Psi(z(b'); \alpha) \right) \Pi(b', z(b'); \alpha) + \Psi(z(b'); \alpha) \right] b'$$

$$\Pi(b', z(b'); \alpha) \equiv \left[ 1 - F \left( \Phi(z(b'); \alpha) \right) \right] b' + \int_{\phi_{\min}}^{\Phi(z(b'); \alpha)} \bar{b}(\phi; z, \alpha) f(\phi')d\phi'. $$

Moreover, we define $W_{MIN} \equiv u(\bar{w})/(1 - \beta) - \phi_{\max}$, and $W_{MAX} \equiv u(\bar{w} - \bar{b}(1 - R^{-1}))/ (1 - \beta)$. It is straightforward to see that $W_{MIN}$ and $W_{MAX}$ are, respectively, lower and upper bounds to the value the country can attain in equilibrium.

We establish first that $T(z; \alpha)(b)$ is a uniformly continuous, bounded and strictly decreasing (in $b$) mapping of the function space $\Gamma$ into itself. Continuity follows by the Theorem of the Maximum, and since $[b, \bar{b}]$ is compact the mapping must be uniformly continuous in $b$. Boundedness follows from the fact that utility is bounded because consumption, reform effort, the support of the default cost, and the elements of $\Gamma$ are bounded. Finally, to establish that the mapping $T$ is strictly decreasing in $b$ note
that, for any $\delta > 0$, $T(z; \alpha)(b + \delta) < T(z; \alpha)(b)$ since

$$T(z; \alpha)(b + \delta) = \max_{b' \in [b, b]} O(b + \delta, b', z(b')); \alpha,$$

where $B(b; z, \alpha) = \arg \max_{b' \in [b, b]} O(b, b', z(b'); \alpha)$. Thus, the second inequality follows from the fact that issuing debt as if the current debt level was $b + \delta$ must be weakly worse than issuing the optimal amount given the debt level $b$.

Next we show that the operator $T(z; \alpha)(b)$ discounts. Following the same proof strategy as in Stokey and Lucas (1989, Lemma 17.1), we proceed by showing that the objective function $O(b, b', y; \alpha)$ is differentiable in its last argument $y = z(b')$ for any given tuple $(b, b')$. Then we can apply the Mean Value Theorem to show that bound the derivative by $\beta \in (0, 1)$. The derivative of $O(b, b', y; \alpha)$ with respect to $y$ is given by

$$O_3(b, b', y; \alpha) = u'(Q(b', y; \alpha)b' - b + w) \times \frac{\partial Q(b', y; \alpha)b'}{\partial y} + \frac{\partial Z(y; \alpha)}{\partial y},$$

where the partial derivatives of $Q(b', y; \alpha)b'$, $\Psi(y; \alpha)$ and $Z(y; \alpha)$ with respect to $y$ exist and are given by

$$\frac{\partial Q(b', y; \alpha)b'}{\partial y} = \frac{\partial \Psi(y; \alpha)}{\partial y} R^{-1} [b' - \Pi(b', y; \alpha)],$$

$$\frac{\partial \Psi(y; \alpha)}{\partial y} = -\frac{\beta [1 - F(\Phi(y; \alpha))]}{X''(\Psi(y; \alpha))},$$

$$\frac{\partial Z(y; \alpha)}{\partial y} = \beta (1 - \Psi(y; \alpha)) [1 - F(\Phi(y; \alpha))],$$

such that the derivative can be bounded for any given $(b, b', y)$ by

$$O_3(b, b', y; \alpha) = -u'() \frac{\beta [1 - F(\Phi(y; \alpha))]}{X''(\Psi(y; \alpha))} R^{-1} [b' - \Pi(b', y; \alpha)]$$

$$+ \beta (1 - \Psi(y; \alpha)) [1 - F(\Phi(y; \alpha))],$$

$$\leq \beta - u'() \frac{\beta [1 - F(\Phi(y; \alpha))]}{X''(\Psi(y; \alpha))} R^{-1} [b' - \Pi(b', y; \alpha)]$$

$$\leq \beta \in (0, 1). \quad (45)$$

Note that $b' - \Pi(b', y; \alpha) \geq 0$, thus for any triple $(b, b', y)$ the derivative is bounded by $\beta \in (0, 1)$. We now invoke the Mean Value Theorem to show that $T$ discounts. Since $O(b, b', y; \alpha)$ is continuous in $y$ on the closed interval $[z(b'), z(b') + a]$ and differentiable in $y$ on the open interval $(z(b'), z(b') + a)$ there exists an interior point $x(b, b', z(b')) \in (z(b'), z(b') + a)$ such that

$$O(b, b', z(b') + a; \alpha) - O(b, b', z(b'); \alpha)$$

$$= O_3(b, b', x(b, b', z(b')); \alpha) \times [z(b') + a - z(b')], \quad (46)$$
where \( a \geq 0 \) is a real constant. Finally, we can derive the inequality

\[
T(z + a; \alpha)(b) = \max_{b' \in [\hat{b}, \bar{b}]} O(b, b', z(b') + a; \alpha) \\
= \max_{b' \in [\hat{b}, \bar{b}]} O(b, b', z(b'); \alpha) + O_3(b, b', x(b, b', z(b')); \alpha) a \\
\leq \max_{b' \in [\hat{b}, \bar{b}]} O(b, b', z(b'); \alpha) + \beta a = T(z; \alpha)(b) + \beta a,
\]

where we have used (46) to derive the second equality, and (45) to derive the first inequality.

Next, we establish that \( T(z; \alpha) \) is monotone in \( z \), and that the mapping’s fixed-point \( z^*(z; \alpha)(b) = \lim_{n \to +\infty} T^n(z; \alpha)(b) \) exists and is unique. To this aim, consider \( z, z^+ \in \Gamma \) with \( z(b) \leq z^+(b), \forall b \in [\hat{b}, \bar{b}] \). Since \( z \) and \( z^+ \) are decreasing in \( b \), implying that \( z^+ (\hat{b}(\phi; z^+, \alpha)) = \alpha - \phi \geq z (\hat{b}(\phi; z^+, \alpha)) \), it follows immediately that \( \hat{b}(\phi; z, \alpha) \leq \hat{b}(\phi; z^+, \alpha) \). Consequently, \( \Phi(z(b); \alpha) \geq \Phi(z^+(b); \alpha) \) for all \( b \in [\hat{b}, \bar{b}] \), which in turn implies that \( \Pi(b', z(b'); \alpha) \leq \Pi(b', z^+(b'); \alpha) \). We establish now that \( z^+ \geq z \Rightarrow T(z^+; \alpha)(b) \geq T(z; \alpha)(b) \), since

\[
T(z^+; \alpha)(b) = \max_{b' \in [\hat{b}, \bar{b}]} O(b, b', z^+(b'); \alpha) \\
\geq u \left( (1 - \Psi(z(B(b; z, \alpha)); \alpha)) \times R^{-1} \Pi(B(b; z, \alpha), z^+(B(b; z, \alpha)); \alpha) \right) \\
\geq + \Psi(z(B(b; z, \alpha)); \alpha) R^{-1} B(b; z, \alpha) - b + w \\
- X(\Psi(z(B(b; z, \alpha)); \alpha)) \\
+ \beta \left[ \Psi(z(B(b; z, \alpha)); \alpha) \mu(B(b; z, \alpha)) + (1 - \Psi(z(B(b; z, \alpha)); \alpha)) E z^+(B(b; z, \alpha)) \right] \\
\geq \max_{b' \in [\hat{b}, \bar{b}]} O(b, b', z(b'); \alpha) = T(z; \alpha)(b)
\]

where the first inequality follows from the fact that the optimal debt issuance \( B(b; z, \alpha) \) and effort \( \Psi(z(B(b; z, \alpha)); \alpha) \) given \( z \) is feasible but yields a weakly lower utility relative to the optimal \( B(b; z^+, \alpha) \) and \( \Psi(z^+(B(b; z^+, \alpha)); \alpha) \) given \( z^+ \). The second inequality follows from the fact that for a given effort and debt level expected debt repayment \( \Pi(b', z^+(b'); \alpha) \geq \Pi(b', z; \alpha) \) and the continuation value is weakly higher for \( z^+ \) than for \( z \).

We have established that for all \( b \in [\hat{b}, \bar{b}] \) and \( z \in \Gamma \) (i) \( T(z; \alpha)(b) \) is a monotone operator with the sup norm \( d_{\infty} \), and (ii) that \( T(z + a; \alpha)(b) \leq T(z; \alpha)(b) + \beta a \). Thus, Stokey and Lucas (1989, Theorem 3.3) applies and the fixed point \( z^*(z; \alpha)(b) = \lim_{n \to +\infty} T^n(z; \alpha)(b) \) exists and is unique since \( T \) is a contraction mapping.

Thus far, we have proven the uniqueness of the fixed point of the mapping \( T \) for any exogenous outside option, \( \alpha \in [W_{MIN}, W_{MAX}] \). We now use a different fixed point argument to show that, conditional on an initial \( z \), there exists a unique fixed point such that the outside option in recession is \( z^* \) with the following properties: \( z^*(z; \alpha_z)(0) = \alpha_z^* \in (W_{MIN}, W_{MAX}) \). To see why, note that, by the Theorem of the Maximum, \( z^*(z; \alpha)(b) = \lim_{n \to +\infty} T^n(z; \alpha)(b) \) is continuous in \( \alpha \). Moreover, \( z^* \) is bounded since \( z^*(z; \alpha) \in [W_{MIN}, W_{MAX}] \). Thus, the Brouwer fixed-point theorem ensures that there exists a \( \alpha_z \in [W_{MIN}, W_{MAX}] \) such that \( z^*(z; \alpha_z)(0) = \alpha_z \). Since \( z^*(z; \alpha_z)(0) = \alpha_z - \Phi(0; z^*, \alpha_z) \), this is equivalent to say that, at each fixed point, \( \Phi(0; z^*, \alpha_z) = 0. \) To prove uniqueness, we note then
that \( \Phi(0; z^*_u, \alpha) \) is monotone increasing for all \( \alpha \in [W_{MIN}, W_{MAX}] \), as the set of (potential) states of nature in which the outside option is attractive expands when \( \alpha \) increases. Therefore, there exists a unique fixed point \( \alpha^*_z \) such that \( \Phi(0; z^*_u, \alpha^*_z) = 0 \). In particular, \( \alpha^*_z = W(0) \in (W_{MIN}, W_{MAX}) \).

The results proven thus far allow us to claim the existence and uniqueness of an equilibrium value function \( W \) such that \( W(b) = T(W; W(0))(b) \). The existence of all remaining equilibrium functions then follows from Definition 3. The uniqueness of \( V, \Phi, Q, \) and \( B \) follows from Definition 3 and the uniqueness of \( \Psi \) from the strict convexity of \( X \). Note that we do not claim uniqueness of \( B \) and \( C \) (we cannot rule out that more than one debt level \( b' \) achieves the value \( W(b) \) given \( b \)). The continuity of the value function \( W(b) \) in \( b \) follows from the Theorem of the Maximum, and implies that also the equilibrium functions \( V, \Phi, Q, \) and \( \Psi \) are continuous in \( b \). Since \( T(z; W(0)) \) maps decreasing functions into strictly decreasing functions, it follows that the fixed-point \( W \) is strictly decreasing in \( b \) and, hence, that \( V(b, \phi) = \max \{ W(b), W(0) - \phi \} \) is non-increasing in \( b \). Note that the This concludes the proof of the proposition.

**Proof of Proposition 8.** The proof is an application of the generalized envelope theorem in Clausen and Strub (2016) which allows for discrete choices (i.e., repayment or renegotiation) and non-concave value functions. Consider the program \( W(b) = \max_{b' \in [b, u]} O(b'), O(b') \equiv u(Q(b')b' - b + w) + \beta Z(b') \). Theorem 1 in Clausen and Strub (2016) ensures that if we can find a differentiable lower support function (DLSF) for \( O \), then \( O \) is differentiable at all interior optimal debt choices \( b' \in B \) where \( B \) was defined in Definition 4 above.

To construct a DLSF for \( O \), we follow the strategy of Benveniste and Scheinkman (1979), and consider the value function of a pseudo borrower with post-renegotiation debt \( b \) that chooses debt issuance \( b' = B(x) \) instead of the optimal \( b' = B(b) \),

\[
\tilde{W}(b, x) = u (Q(B(x)) B(x) - b + w) + Z(B(x)).
\]

Note that \( \tilde{W} \) is differentiable and strictly decreasing in \( b \). Since debt issuance is chosen suboptimally, it must be that \( \tilde{W}(b, x) \leq W(b) \) with equality holding at \( x = b \). Furthermore, let the pseudo borrower set the default threshold at the level \( \tilde{W}(b, x) = W(0) - \tilde{W}(b, x) \), where \( \tilde{\Phi}(b, x) = \Phi(b) \). Thus, the pseudo borrower renegotiates even for some \( \phi \) larger than \( \Phi(b) \). Note that \( \tilde{\Phi}(b, x) \) is differentiable and strictly increasing in \( b \). Thus, the inverse function exists and is such that \( \tilde{\Phi}^{-1}(\phi) \leq b(\phi) \) (where we define \( \tilde{\Phi}(b) \equiv \Phi(b, x) \)).

Let

\[
\tilde{O}(b', x) = u (\tilde{Q}(b', x)b' - b + w) + \tilde{Z}(b', x),
\]

where \( \tilde{Q}(b', x)b' \) and \( \tilde{Z}(b', x) \) are given by

\[
\tilde{Q}(b', x)b' = \tilde{R}^{-1} \left[ (1 - \tilde{\Psi}(b', x)) \left( 1 - F(\tilde{\Phi}(b', x)) \right) b' + \int_{\phi_{\min}}^{\tilde{\Phi}(b', x)} \tilde{\Phi}^{-1}(\phi) dF(\phi) + \tilde{\Psi}(b', x)b' \right],
\]

\[
\tilde{Z}(b', x) = -X(\tilde{\Psi}(b', x)) + \beta \times \left[ \left( 1 - F(\tilde{\Phi}(b', x)) \right) \tilde{W}(b', x) + \int_{\phi_{\min}}^{\tilde{\Phi}(b', x)} [W(0) - \phi] dF(\phi) \right] \times \left( 1 - F(\tilde{\Phi}(b', x)) \right) \tilde{W}(b', x) + \int_{\phi_{\min}}^{\tilde{\Phi}(b', x)} [W(0) - \phi] dF(\phi) \right],
\]

having defined \( \tilde{\Psi}(b', x) \) as

\[
\tilde{\Psi}(b', x) = (X')^{-1} \left[ \beta \left[ \tilde{\mu}(b') - \left( 1 - F(\tilde{\Phi}(b', x)) \right) \tilde{W}(b', x) + \int_{\phi_{\min}}^{\tilde{\Phi}(b', x)} [W(0) - \phi] dF(\phi) \right] \right].
\]
Note that $\tilde{Q}$, $\tilde{Z}$ and $\tilde{\Psi}$ are differentiable in $b'$ since we established above that $\tilde{W}$ and $\tilde{\Phi}$ are differentiable.

Then, $\tilde{O}$ is a DLSF for $O$ such that $\tilde{O}(b', x) \leq O(b')$ with equality (only) at $b' = x$. Thus, Theorem 1 in Clausen and Strub (2016) ensures that $O(b')$ is differentiable at all optimal interior choices $b' \in \tilde{B}$ and that $\partial O(B(b)) / \partial B(b) = \partial \tilde{O}(B(b), B(b)) / \partial B(b) = 0$. In this case, a standard FOC holds

$$\frac{\partial u (Q(B(b))B(b) - w)}{\partial B(b)} + \frac{\partial Z(B(b))}{\partial B(b)} = 0.$$ 

Moreover, Lemma 3 in Clausen and Strub (2016) ensures that also the functions $W(b')$, $Z(b')$, $\Phi(b')$, $Q(b')$, and $\Psi(b')$ are differentiable in $b' \in \tilde{B}$ and that a standard envelope condition applies, namely,

$$\frac{\partial Z(B(b))}{\partial B(b)} = \beta \left[ (1 - \Psi(B(b))) \left[ 1 - F(\Phi(B(b))) \right] \frac{\partial W(B(b))}{\partial B(b)} + \Psi(B(b)) \frac{\partial \mu(B(b))}{\partial B(b)} \right],$$

$$\frac{\partial W(B(b))}{\partial B(b)} = -u' (C(B(b))) < 0.$$ 

This shows that the FOC stated in Proposition 8 is indeed necessary for an interior optimum. ■
C Appendix C: Additional Technical Analysis

This appendix contains additional technical details. In particular, it provides: (i) proofs of Proposition 1 (first best) and Proposition 2 (COA without moral hazard); (ii) Lemma 2, and Lemmas 3 and 4 used to prove Proposition 3 in Appendix A; (iii) details of the numerical example used to generate the figures; and (iv) technical details of the analysis in Sections 2.2.1.

C.1 First best

In this section, we provide the proof of Proposition 1.

Proof of Proposition 1. In the first part of the proof we take as given that the profit function $P$ has the solution (9) and verify this below. Moreover, we also take as given that $P$ is strictly decreasing, strictly concave, and differentiable in $\nu$. We delegate the formal proof of these properties to Proposition 9 further below in this appendix.

The Lagrangian of the planner’s problem in recession reads as

$$
L = \int_R \left[ w - c_\phi + \beta ((1 - p_\phi) P(\omega_\phi) + p_\phi \tilde{P}(\bar{\omega}_\phi)) \right] f(\phi) d\phi 
+ \vartheta \left( \int_R \left[ u(c_\phi) - X(p_\phi) + \beta ((1 - p_\phi) \omega_\phi + p_\phi \tilde{\omega}_\phi) \right] f(\phi) d\phi - \nu \right)
$$

where the Lagrange multiplier on the PK is given by $\vartheta$. The FOCs with respect to the controls $c_\phi$, $\omega_\phi$, $\bar{\omega}_\phi$, and $p_\phi$ yield:

$$
f(\phi) = u'(c_\phi) \vartheta f(\phi), \tag{47}
$$

$$
\vartheta f(\phi) = -P'(\omega_\phi) f(\phi), \tag{48}
$$

$$
\vartheta f(\phi) = -\tilde{P}'(\bar{\omega}_\phi) f(\phi), \tag{49}
$$

$$
\beta \left( \tilde{P}(\bar{\omega}_\phi) - P(\omega_\phi) \right) f(\phi) = \vartheta f(\phi) \left( X'(p_\phi) - \beta (\bar{\omega}_\phi - \omega_\phi) \right), \tag{50}
$$

while the envelope condition is given by

$$
-P'(\nu) = \vartheta. \tag{51}
$$

First, since $f(\phi) > 0$ over the relevant support of $\phi$ the optimal allocation is independent of the default cost realization. Thus, the planner fully insures the agent against the risk in $\phi$. The optimality condition in (49) implies that $\vartheta > 0$, since $-\tilde{P}'(\bar{\omega}_\phi) > 0$. The optimality conditions (48) and (51) imply $\omega^{FB}(\nu) = \nu$ such that promised utility, consumption, and reform effort stay constant during recessions. Equations (47)-(49) together with (9) imply that the planner provides the agent with full consumption insurance across the income states, $u'(c^{FB}(\nu)) = u'(\bar{c}(\bar{\omega}_h^{FB}(\nu))) \Leftrightarrow c^{FB}(\nu) = \bar{c}(\bar{\omega}^{FB}(\nu)) = u^{-1} [(1 - \beta)\bar{\omega}^{FB}(\nu)]$. Given the constant allocation, equation (8) in Proposition 1 follows immediately from the PK (2). Moreover, since in normal time the agent gets the same consumption as in recession (but reform effort is absent), equation (8) implies that promised utility in normal time can be expressed as $\bar{\omega}_h^{FB}(\nu) = \nu + X(p^{FB}(\nu)) \left( 1 - \beta (1 - p^{FB}(\nu)) \right) = u(c^{FB}(\nu))/(1 - \beta)$. The FOC with respect to effort (50) can then be expressed as

$$
\beta \left( \tilde{P}(\bar{\omega}_h^{FB}(\nu)) - P(\nu) \right) = u'(c^{FB}(\nu))^{-1} \left( X'(p^{FB}(\nu)) - \beta (\bar{\omega}_h^{FB}(\nu) - \nu) \right), \tag{52}
$$
where
\[
\tilde{P} \left( \tilde{\omega}^{FB}(\nu) \right) - P(\nu) = \bar{w} - w + \beta \left( 1 - p^{FB}(\nu) \right) \left( \tilde{P} \left( \tilde{\omega}^{FB}(\nu) \right) - P(\nu) \right)
\]
\[
= \frac{1}{1 - \beta (1 - p^{FB}(\nu))} (\bar{w} - w)
\]
\[
\tilde{\omega}^{FB}(\nu) - \nu = \frac{X \left( p^{FB}(\nu) \right)}{1 - \beta (1 - p^{FB}(\nu))}.
\]
Substituting for \( \tilde{P} \left( \tilde{\omega}^{FB}(\nu) \right) - P(\nu) \) and \( \tilde{\omega}^{FB}(\nu) - \nu \) in (52) yields equation (7) in Proposition 1. Note that the profit function in recession
\[
P(\nu) = \frac{w - c^{FB}(\nu)}{1 - \beta (1 - p^{FB}(\nu))} + \frac{\beta p^{FB}(\nu)}{1 - \beta} \frac{\bar{w} - c^{FB}(\nu)}{1 - \beta (1 - p^{FB}(\nu))}
\]
\[
= \frac{\bar{w} - c^{FB}(\nu)}{1 - \beta} - \frac{w - \bar{w}}{1 - \beta (1 - p^{FB}(\nu))},
\]
defines a positively sloped locus in the plane \((p, c)\), while equation (7) defines a negatively sloped locus in the same plane. The two equations pin down a unique interior solution for \( p^{FB}(\nu) \) and \( c^{FB}(\nu) \). Now, consider the comparative statics with respect to \( \nu \). An increase in \( \nu \) yields a strict increase in consumption \( c^{FB}(\nu) \) according to (47) and (51) since \( P \) is strictly concave, while (7) is independent of \( \nu \) such that the increase in \( c^{FB}(\nu) \) must come with a strict decrease in \( p^{FB}(\nu) \). Finally, set \( \bar{w} = \tilde{w} \) in (53) to see that the profit function in normal time is indeed given by (9). This concludes the proof of the proposition. 

C.2 Constrained Optimum: Limited Commitment without Moral Hazard

In this section, we provide the proof of Proposition 2.

Proof of Proposition 2. In this proof we take as given that \( P \) is strictly decreasing, strictly concave, and differentiable in \( \nu \) at its interior support, and that the FOC of the planner problem are necessary and sufficient to characterize the COA. We delegate the formal proof of these properties to Proposition 9 further below in this appendix. Moreover, we limit the proof to the arguments that do not overlap with those already made in the proof of Proposition 3 in Appendix A.

The Lagrangian of the planner’s problem is the same as in (37), except that we can drop all terms that involve the incentive constraint \((\chi_\phi = 0)\). The FOCs yield, then:
\[
f(\phi) = u'(c_\phi) (\partial f(\phi) + \lambda_\phi),
\]
\[
\theta f(\phi) + \lambda_\phi = -P'(\omega_\phi)f(\phi), \quad \forall \omega_\phi > \alpha - E[\phi],
\]
\[
\theta f(\phi) + \lambda_\phi = -\tilde{P}'(\tilde{\omega}_\phi)f(\phi),
\]
\[
\beta \left( \tilde{P}(\tilde{\omega}_\phi) - P(\omega_\phi) \right) f(\phi) = (\theta f(\phi) + \lambda_\phi)(X'(p_\phi) - \beta (\tilde{\omega}_\phi - \omega_\phi)).
\]
The envelope condition yields \(-P'(\nu) = \theta > 0, \forall \nu > \alpha - E[\phi]\), and the slackness condition for \( \theta_\phi \) reads \( 0 = \theta_\phi (\omega_\phi - (\alpha - E[\phi])) \).

Note that \( P \) is strictly concave, thus Lemma 1 implies that the solution remains characterized by the unique threshold \( \tilde{\phi}(\nu) \) stated in (13) where the PC binds only for \( \phi < \tilde{\phi}(\nu) \).

The FOCs (54)–(57) imply equations (10)-(12) in the text. Combine equation (55) and the envelope condition to yield \( P'(\nu) = P'(\omega_\phi) + \lambda_\phi/f(\phi) \geq P'(\omega_\phi) \) for \( \nu, \omega_\phi > \alpha - E[\phi] \). Since \( P \) is strictly
concave this implies that promised utility is weakly increasing conditional on staying in recession, \( \omega_\phi \geq \nu > \alpha - E[\phi] \). Note that this property extends to the lower bound where \( \nu = \alpha - E[\phi] = \omega(\nu) \) if the PC is slack. This can be proved by a contradiction argument: Suppose that \( \exists \nu > \nu \) such that \( \omega(\nu) = \omega(\nu) \). The optimal allocation associated with \( \omega(\nu) \) yields utility \( \nu > \nu \). Thus, \( \omega(\nu) = \omega(\nu) \) would violate the PK, which is not feasible. In summary, promised utility is weakly increasing \( \omega_\phi \geq \nu \), such that its lower bound is never relevant and we can drop the multiplier \( \theta_\phi \) from the further analysis.

Since \( \lambda_\phi \) enters the optimality conditions, the solution will depend on whether the PC is slack or binding:

1. When the PC is binding and the recession continues, \( \phi < \phi(\nu), \lambda_\phi > 0, \omega_\phi > \nu \), and

\[
 u(c_\phi) - X(p_\phi) + \beta((1 - p_\phi)\omega_\phi + p_\phi\omega_\phi) = \alpha - \phi. \tag{58}
\]

Then, (10), (12), (11) and (58) determine jointly the solution for \((c_\phi, p_\phi, \omega_\phi, \omega_\phi)\). In this case, there is no history dependence, i.e., \( \nu \) does not matter.

2. When the PC is not binding, \( \phi \geq \phi(\nu) \) and \( \lambda_\phi = 0 \). Then, \( \omega_\phi = \nu \), and \( c_\phi = c(\nu), p_\phi = p(\nu) \), and \( \omega_\phi = \omega_\phi(\nu) \) are determined by (18), (10), and (12), respectively. The solution is history dependent.

By the same argument made in the proof of Proposition 3, \( c(\nu) \) must be strictly increasing in \( \nu \). In turn \( 1/u'(c(\nu)) = 1/u'(\bar{c}(\omega(\nu))) \) implies that also \( \bar{\omega}(\nu) \) is strictly increasing in \( \nu \). Finally, equation (12) implies that

\[
u'(c(\nu))\left[\bar{P}(\bar{\omega}(\nu)) - P(\nu)\right] + [\bar{\omega}(\nu) - \nu] = \beta^{-1}X'(p(\nu)).
\]

For \( \nu > \nu \), differentiating the left-hand side yields

\[
u''(c(\nu))c'(\nu) \times \left[\bar{P}(\bar{\omega}(\nu)) - P(\nu)\right] + \left[u'(c(\nu)) P'(\nu) + 1\right] (\bar{\omega}'(\nu) - 1)
\]

\[
= u''(c(\nu))c'(\nu) \times (\bar{P}(\bar{\omega}(\nu)) - P(\nu)) < 0
\]

since, recall, (11) implies that \( P'(\nu) = -1/u'(c(\nu)) \) and Lemma 9 stated below establishes that \( \bar{P}(\bar{\omega}(\nu)) - P(\omega(\nu)) = \bar{P}(\bar{\omega}(\nu)) - P(\nu) > 0 \). This implies that the right-hand side must also be strictly decreasing in \( \nu \). Since \( X \) is convex and increasing, this implies in turn that \( p(\nu) \) must be strictly decreasing in \( \nu > \nu \). Note that this property extends to the lower bound, \( p(\nu) > p(\nu) \forall \nu > \nu \). Suppose not, \( p(\nu) \leq p(\nu) \), then \( \nu = \omega(\nu) > \omega(\nu) = \nu \) which contradicts the fact \( \omega(\nu) \) and \( p(\nu) \) are optimal given \( \nu \) and the same \( \phi \). Thus, effort \( p(\nu) \) must be strictly decreasing for all \( \nu \).

This concludes the proof of Proposition 2. ■

C.3 Lemmas 2, 3, and 4.

This section contains three lemmas that apply to the planner problem with moral hazard. Lemmas 3 and 4 are needed to prove Proposition 3 in Appendix A. Lemma 2 provides a sufficient condition for the effort function to be falling in promised utility when \( \nu \) is sufficiently large.

**Lemma 2** Suppose \( \lim_{\nu \to \infty} u'(c) = 0 \) and that \( \lim_{\nu \to \nu} X''(p) > 0 \). Then \( \lim_{\nu \to \infty} p(\nu) = p \).
Proof. We conjecture that in the limit the COA is given by $\lim_{\nu \to \infty} \{ \tilde{\omega}(\nu) - \omega(\nu) \} = 0$, $\lim_{\nu \to \infty} c(\nu) = \lim_{\nu \to \infty} c(\omega(\nu)) = \lim_{\nu \to \infty} \hat{c}(\tilde{\omega}(\nu)) = \infty$, and $\lim_{\nu \to \infty} P(\nu) = p$, where $p(\nu) \equiv Y(\tilde{\omega} - \omega)$. We verify that this allocation satisfies the necessary FOCs of the COA. First, eq. (2) implies $\lim_{\nu \to \infty} \omega(\nu) = \lim_{\nu \to \infty} \tilde{\omega}(\nu) = \infty$. Second, $\lim_{\nu \to \infty} X'(p(\nu)) = 0$ satisfies eq. (14). Third, the lower bound on $\omega$ and the PC (4) become irrelevant when $\nu$ is sufficiently large. Fourth, note that $Y'(\tilde{\omega} - \omega) = \beta/X''(Y(\tilde{\omega} - \omega))$. Equations (15)-(16) can then be rewritten as

$$1 - p(\nu) = (1 - p(\nu)) \frac{u'(c(\nu))}{u'(c(\omega(\nu)))} + u'(c(\nu)) \frac{\beta}{X''(p(\nu))} [\tilde{P}(\tilde{\omega}(\nu)) - P(\omega(\nu))]$$  \hspace{1cm} (59)$$

$$p(\nu) = p(\nu) \frac{u'(c(\nu))}{u'(c(\omega(\nu)))} - u'(c(\nu)) \frac{\beta}{X''(p(\nu))} [P(\tilde{\omega}(\nu)) - P(\omega(\nu))].$$  \hspace{1cm} (60)

Consider the limit when $\nu \to \infty$. Equations (59)-(60) hold as $\nu \to \infty$ since $\lim_{\nu \to \infty} p(\nu) = p$, $\lim_{\nu \to \infty} \{ \tilde{P}(\tilde{\omega}(\nu)) - P(\omega(\nu)) \} = (\bar{w} - \bar{w}) / [1 - \beta(1 - p)]$, $\lim_{\nu \to \infty} u'(c(\nu)) = 0$, $\lim_{\nu \to \infty} \{ u'[c(\nu)] / u'[c(\omega(\nu))]\} = \lim_{\nu \to \infty} \{ u'[c(\nu)] / u'[c(\bar{w}(\nu))]\} = 1$, and $X''(p)$ is bounded away from zero by assumption. Finally, note that the conjectured COA coincides with the FB in the limit, since $\lim_{\nu \to \infty} u'(c^{FB}(\nu)) (\bar{w} - \bar{w}) = 0$ in (7). Thus, it must yield the maximal profits given $\nu$. This implies that the conjectured limiting allocation is indeed the COA. 

**Lemma 3** Assume $P$ is strictly concave. Then, $P$ is differentiable at the interior of its support with $P'(\nu) = -1/u'(c(\nu)) < 0$.

**Proof.** The proof is an application of Benveniste and Scheinkman (1979, Lemma 1). Consider the profit of a pseudo planner that is committed to deliver the initial promise $\tilde{\nu}$, but suboptimally chooses effort and future promised-utility like in the optimal contract given an initial promise $\nu$

$$\tilde{P}(\tilde{\nu}, \nu) = \int_{\phi_{max}}^{\hat{\phi}(\tilde{\nu})} \left[ \bar{w} - x(\phi, p_\phi(\nu), \bar{\omega}_\phi(\nu), \omega_\phi(\nu)) + \beta \left( (1 - p_\phi(\nu))P(\omega_\phi(\nu)) \right. \right] dF(\phi)$$

$$+ \int_{\hat{\phi}(\tilde{\nu})}^{\infty} \left[ \bar{w} - x(\phi, p_\phi(\nu), \bar{\omega}_\phi(\nu), \omega_\phi(\nu)) + \beta \left( (1 - p_\phi(\nu))P(\omega_\phi(\nu)) \right. \right] dF(\phi),$$

where consumption provided by the pseudo planner is determined by

$$x(\phi, p, \bar{\omega}, \omega) = u^{-1} \left( \alpha - \phi + X(p) - \beta \left[ (1 - p)\omega + p\bar{\omega} \right] \right)$$

and $p_\phi(\nu) = Y(\bar{\omega}_\phi(\nu) - \omega_\phi(\nu))$. Note that for $\tilde{\nu} = \nu$, the pseudo planner achieves the same profit as in the optimal contract, $\tilde{P}(\tilde{\nu}, \tilde{\nu}) = P(\tilde{\nu})$, but profits must be weakly lower otherwise, $\tilde{P}(\tilde{\nu}, \nu) \leq P(\tilde{\nu})$. Furthermore, $\tilde{P}(\tilde{\nu}, \nu)$ is twice differentiable in $\tilde{\nu}$ and strictly concave. Then, Lemma 1 in Benveniste and Scheinkman (1979) applies and the profit function $P(\nu)$ is differentiable at the interior of its support $\nu$ with derivative

$$P'(\nu) = \tilde{P}_1(\nu, \nu) = -1/u'(c(\nu)) < 0.$$ 

This concludes the proof of the Lemma 3. 

**Lemma 4** The FOCs of the planning problem are necessary for optimality.
This also implies that realization (we have Greece in mind). Finally, we parameterize country starting with a 100% debt-output ratio in recession recovers in expectation after one decade \( R \) fall of real GDP per capita for Greece between 2007 and 2016. Assume that the recession causes a drop in income of 25%, i.e., an average default premium of \( \exp \) exponential with rate parameter \( X \). Where the years 2007 and 2016 correspond to the peak and the trough, respectively, of real GDP per capita (GIIPS) during the Great Recession.

Parameters so as to match salient moments observed for Greece, Ireland, Italy, Portugal, and Spain (GIIPS) of Section 4 since this is a more realistic positive representation of the world. We choose in the ...gures of the paper. We focus on the quantitative properties of the one-asset economy (with renegotiation) of Section 4 since this is a more realistic positive representation of the world. We choose

\[
\text{Proof.} \quad \text{That the optimal e¤ort is interior follows from the assumed properties of the } X \text{ function } (X'(p) = 0, X'(p) > 0 \text{ for } p > p, \text{ and } \lim_{p \to p} X'(p) = +\infty). \text{ The optimality condition for effort } X'(p^*(\nu)) = \beta(\omega(\nu) - \omega(\nu)) \text{ implies then } \omega(\nu) > \omega(\nu), \text{ and that there exists an interior maximum e¤ort level } p^* = \max \{p^*(\nu)\} \text{ if the } \omega(\nu) \geq \alpha - E[\phi], \text{ then } \omega(\nu) > \alpha - E[\phi]. \text{ In conclusion, the optimal choice of } p^* \text{ is interior and } \omega(\nu) \text{ will never be at the lower bound } \omega. \text{ Note that the possibility } \omega(\nu) = \alpha - E[\phi] \text{ is taken into account by the stated FOCs.}

Next, consider the upper bound \( \bar{\omega} \) for \( \omega(\nu) \) which is sufficiently high that none of the PCs will bind if the economy starts at \( \bar{\omega} \), i.e., \( \omega(\bar{\omega}) = \omega(\bar{\omega}) > \alpha - \phi_{\min}. \text{ Then, the FOC with respect to } \omega(\nu) \text{ implies that - if the planner was not constrained by } \omega(\nu) \text{ - profits are maximized when } \omega(\nu) < \bar{\omega}. \text{ This allocation is feasible in the constrained problem thus it must also be the optimal choice when } \omega(\nu) \text{ is bounded by } \bar{\omega}. \text{ The same applies to any level of promised-utility below } \bar{\omega} \text{ when the PC is slack. Finally, in states where the PC binds, } \omega(\nu) \text{ always remains below } \alpha - \phi_{\min} < \bar{\omega}. \text{ Thus, } \omega(\nu) \text{ always remains strictly below } \bar{\omega}. \text{ In turn, the optimality condition for e¤ort then implies that the } \omega(\nu) \text{ can never be higher than } X'(p^+)/\beta + \bar{\omega} < \bar{\omega}, \text{ where } X'(p^+) < +\infty. \text{ In summary, the optimal choices of } \omega(\nu) \text{ and } \omega(\nu) \text{ are also interior (apart from the corner solution, } \omega(\nu) = \alpha - E[\phi]). \text{ Finally, consumption must always be positive since } \lim_{c \to 0} u(c) = -\infty \text{ and it is interior because promised-utility and e¤ort is interior. Thus, the solution to the planner problem must be interior and the stated FOCs are necessary.}

\[ \text{C.4 Parameterization} \]

In this section, we provide details of the parameterization underlying the numerical examples shown in the figures of the paper. We focus on the quantitative properties of the one-asset economy (with renegotiation) of Section 4 since this is a more realistic positive representation of the world. We choose parameters so as to match salient moments observed for Greece, Ireland, Italy, Portugal, and Spain (GIIPS) during the Great Recession.

A model period corresponds to one year. We normalize the GDP during normal time to \( \bar{w} = 1 \) and assume that the recession causes a drop in income of 25%, i.e., \( w = 0.75 \times \bar{w} \). This corresponds to the fall of real GDP per capita for Greece between 2007 and 2016.\(^{32}\) The annual real gross interest rate is set to \( R = 1.02 \). The utility function is assumed to be CRRA with a relative risk aversion of 2. We assume an isoelastic effort cost function, \( X(p) = \frac{k}{1+1/\eta}(p)^{1+1/\eta} \), and calibrate \( \xi = 14.371 \) so that a country starting with a 100% debt-output ratio in recession recovers in expectation after one decade (we have Greece in mind). Finally, we parameterize \( f(\phi) \) and its support. The maximum default cost realization \( \phi_{\max} = 2.275 \) is calibrated to target a debt limit during recession of \( \frac{p_{\max}}{\bar{w}} = 178\% \) in line with Collard et al. (2015, Table 3, Column 1).\(^{33}\) Finally, we assume that \( \phi_{\max} - \phi \) is distributed exponential with rate parameter \( \eta = 1.625 \) and truncation point \( \phi_{\max}. \)\(^{34}\) The model then generates an average default premium of 4.04% for a country with a debt-output ratio of 100% in recession.

\(^{32}\) Real GDP per capita of Greece fell from 22'700 to 17'100 Euro between 2007 and 2016 (Eurostat, nama_10_pc series). Where the years 2007 and 2016 correspond to the peak and the trough, respectively, of real GDP per capita relative to a 2% growth trend with base year 1995.

\(^{33}\) We ignore the value of 282% for Korea which is a clear outlier.

\(^{34}\) More formally, \( \phi \) has the p.d.f.

\[
\frac{\eta e^{-\eta(\phi_{\max}-\phi)}}{1-e^{-\eta \phi_{\max}}}, \phi \in [0, \phi_{\max}].
\]

This also implies that \( \phi_{\min} = 0. \)
This overlaps with the average debt and average default premium for the GIIPS during 2008-2012 (Eurostat).

### C.5 Technical Details of the Analysis of Section 2.2.1

**Proposition 9** The profit functions $P$ and $\tilde{P}$ that solve the Bellman equation (1) subject to (2)-(5), or, subject to (2)-(3), are strictly decreasing, strictly concave, and differentiable at the interior of their support. The FOCs of the planning problem are necessary and sufficient to characterize the COA.

The proof strategy follows Thomas and Worrall (1990, Proof of Proposition 1), i.e., we show first that the planner’s problem is a contraction mapping with a strictly concave fixed-point $P$ or $\tilde{P}$, respectively. The differentiability of $P$ and $\tilde{P}$ then follows from Benveniste and Scheinkman (1979, Lemma 1). Finally, we prove that $P$ and $\tilde{P}$ pin down uniquely interior promised utilities, effort and consumption.

The arguments used to prove Proposition 9 for $P$ and $\tilde{P}$ are mirror image of each other, except that the recession case is complicated by the presence of an effort choice and the PCs. For this reason, we prove the results for $P$ (assuming the properties of $\tilde{P}$ follow the proposition), omitting the simpler proof for $\tilde{P}$ (more precisely, the arguments are extended by setting $\bar{w} = \tilde{w}$, $X(p_\phi) = 0$, $p_\phi = 1$, and dropping the PCs).

We prove the results in the form of four lemmas and one corollary (Lemmas 5 to 8) and Corollary 2. We demonstrate the proof for the COA where the planner is subject to the PC in (4) and the lower bound (5). The properties of the first-best planner problem then follow immediately by simply dropping the PC and replacing the lower bound $\alpha - E[\phi]$ with $\tilde{\omega}$ in the constrained set $\Lambda(\nu)$ defined below.

Define, first, the mapping $T(z)(\nu)$ as the right-hand side of the planner’s functional equation

$$T(z)(\nu) = \max_{(\{c_\phi, p_\phi, \omega_\phi, \omega_\phi\}_{\phi \in \mathbb{R}}) \in \Lambda(\nu)} \int_{\mathbb{R}} [w - c_\phi + \beta \left[p_\phi \tilde{P}(\tilde{\omega}_\phi) + (1 - p_\phi)z(\omega_\phi)\right]] \, dF(\phi)$$

where maximization is constrained by the set $\Lambda(\nu)$ defined by

$$\int_{\mathbb{R}} [u(c_\phi) - X(p_\phi) + \beta [p_\phi \tilde{\omega}_\phi + (1 - p_\phi)\omega_\phi]] \, dF(\phi) = \nu$$

$$u(c_\phi) - X(p_\phi) + \beta [p_\phi \tilde{\omega}_\phi + (1 - p_\phi)\omega_\phi] \geq \alpha - \phi, \quad \forall \phi \in \mathbb{R},$$

$$c_\phi \in [0, \tilde{c}], \quad p_\phi \in [\tilde{p}, \bar{p}], \quad \nu, \omega_\phi \in [\alpha - E[\phi], \tilde{\omega}], \quad \tilde{\omega}_\phi \in [\omega, \tilde{\omega}].$$

We take as given that $\tilde{P}$ is strictly concave and bounded between $P_{MIN}$ and $P_{MAX}$.

**Lemma 5** $T(z)$ maps concave functions into strictly concave functions.

**Proof.** Let $\nu' \neq \nu'' \in [\alpha - E[\phi], \tilde{\omega}], \delta \in (0,1)$, $\nu = \delta \nu' + (1 - \delta)\nu''$, $P_k(\nu) = T(P_{k-1})(\nu)$, and $P_{k-1}$ be concave. Then,

$$P_{k-1}(\delta \nu' + (1 - \delta)\nu'') \geq \delta P_{k-1}(\nu') + (1 - \delta)P_{k-1}(\nu'').$$

We follow the strategy of Thomas and Worrall (1990, Proof of Proposition 1), i.e., we construct a feasible but (weakly) suboptimal contract, $\left\{c_\phi^o(\nu), p_\phi^o(\nu), \omega_\phi^o(\nu), \omega_\phi^o(\nu)\right\}_{\phi \in \mathbb{R}}$, such that even the profit generated by the suboptimal contract $P_k^o(\delta \nu' + (1 - \delta)\nu'') \leq P_k(\delta \nu' + (1 - \delta)\nu'')$ is higher than the
linear combination of maximal profits $\delta P_k(\nu') + (1 - \delta) P_k(\nu'')$. Define the weights $\delta, \tilde{\delta} \in (0, 1)$ and the 4-tuple $(c^o_\phi(\nu), p^o_\phi(\nu), \omega^o_\phi(\nu), \tilde{\omega}^o_\phi(\nu))$ such that

$$
\delta = \frac{\delta [1 - p_\phi(\nu')] \delta (1 - p_\phi(\nu')) + (1 - \delta)(1 - p_\phi(\nu''))}{1 - p^o_\phi(\nu)} = \frac{\delta [1 - p_\phi(\nu')] \delta (1 - p_\phi(\nu')) + (1 - \delta)(1 - p_\phi(\nu''))}{1 - p^o_\phi(\nu)}
$$

$$
\tilde{\delta} = \frac{\delta p_\phi(\nu') \delta p_\phi(\nu') + (1 - \delta)p_\phi(\nu'')(1 - \delta)p_\phi(\nu'')}{p^o_\phi(\nu)} = \frac{\delta p_\phi(\nu') \delta p_\phi(\nu') + (1 - \delta)p_\phi(\nu'')(1 - \delta)p_\phi(\nu'')}{p^o_\phi(\nu)}
$$

$$
\omega^o_\phi(\nu) = \frac{\delta \omega_\phi(\nu') + (1 - \delta)\omega_\phi(\nu'')}{\delta \omega_\phi(\nu') + (1 - \delta)\omega_\phi(\nu'')}
$$

$$
\tilde{\omega}^o_\phi(\nu) = \frac{\delta \tilde{\omega}_\phi(\nu') + (1 - \delta)\tilde{\omega}_\phi(\nu'')}{\delta \tilde{\omega}_\phi(\nu') + (1 - \delta)\tilde{\omega}_\phi(\nu'')}
$$

$$
c^o_\phi(\nu) = u^{-1}\left[ -\left[ \delta X(p_\phi(\nu')) + (1 - \delta)X(p_\phi(\nu'')) \right] + X(p_\phi(\nu')) + (1 - \delta)X(p_\phi(\nu'')) \right].
$$

Hence,

$$
(1 - p^o_\phi(\nu))\omega^o_\phi(\nu) = \delta (1 - p^o_\phi(\nu))\omega^o_\phi(\nu') + (1 - \delta)(1 - p^o_\phi(\nu''))\omega^o_\phi(\nu'')
$$

$$
p^o_\phi(\nu)\omega^o_\phi(\nu) = \delta p^o_\phi(\nu')\omega^o_\phi(\nu') + (1 - \delta)p^o_\phi(\nu'')\omega^o_\phi(\nu'')
$$

$$
u \left( c^o_\phi(\nu) \right) - X(p^o_\phi(\nu)) = \delta u(c^o_\phi(\nu')) + (1 - \delta)u(c^o_\phi(\nu'')) - \left[ \delta X(p^o_\phi(\nu')) + (1 - \delta)X(p^o_\phi(\nu'')) \right].
$$

By construction the suboptimal allocation satisfies

$$
c^o_\phi(\nu) \in [0, \bar{c}], \ p^o_\phi(\nu) \in [\underline{p}, \bar{p}], \ \omega^o_\phi(\nu) \in [\alpha - E[\phi], \bar{\omega}], \ \tilde{\omega}^o_\phi(\nu) \in [\omega, \bar{\omega}]
$$

and, given the promised-utility $\nu$, is also consistent with the PK

$$
\int_{\nu} \left[ u \left( c^o_\phi(\nu) - X(p^o_\phi(\nu)) + \beta \left[ (1 - p^o_\phi(\nu))\omega^o_\phi(\nu) + p^o_\phi(\nu)\tilde{\omega}^o_\phi(\nu) \right] \right) dF(\phi)
$$

$$
= \int_{\nu} \left[ \delta u \left( c^o_\phi(\nu') \right) + (1 - \delta)u \left( c^o_\phi(\nu'') \right) - \left[ \delta X(p^o_\phi(\nu')) + (1 - \delta)X(p^o_\phi(\nu'')) \right] 
+ \beta \left[ \delta (1 - p^o_\phi(\nu'))\omega^o_\phi(\nu') + (1 - \delta)(1 - p^o_\phi(\nu''))\omega^o_\phi(\nu'') \right] 
+ \beta \left[ \delta p^o_\phi(\nu')\omega^o_\phi(\nu') + (1 - \delta)p^o_\phi(\nu'')\omega^o_\phi(\nu'') \right] \right] dF(\phi)
$$

$$
= \delta \nu' + (1 - \delta)\nu'' = \nu.
$$

Moreover, the PC for any $\phi$ yields

$$
u \left( c^o_\phi(\nu) \right) - X(p^o_\phi(\nu)) + \beta \left[ (1 - p^o_\phi(\nu))\omega^o_\phi(\nu) + p^o_\phi(\nu)\tilde{\omega}^o_\phi(\nu) \right]
$$

$$
\geq \delta \left[ u \left( c_\phi(\nu') \right) - X(p_\phi(\nu') + \beta (1 - p_\phi(\nu'))\omega_\phi(\nu') + \beta p_\phi(\nu')\tilde{\omega}_\phi(\nu') \right] 
+ (1 - \delta) \left[ u \left( c_\phi(\nu'') \right) - X(p_\phi(\nu'') + \beta (1 - p_\phi(\nu''))\omega_\phi(\nu'') + \beta p_\phi(\nu'')\tilde{\omega}_\phi(\nu'') \right] \right]
$$

Thus, we have proven that the suboptimal allocation $\left\{ c^o_\phi(\nu), p^o_\phi(\nu), \omega^o_\phi(\nu), \tilde{\omega}^o_\phi(\nu) \right\}_{\phi \in \mathbb{R}}$ is feasible. Namely, it satisfies the PCs and delivers promised utility $\nu$. The profit function evaluated at the optimal contract
\{c_\phi(\nu), p_\phi(\nu), \omega_\phi(\nu), \bar{\omega}_\phi(\nu)\}_{\phi \in \mathbb{R}} \text{ then implies the following inequality,}

\[
\begin{align*}
\delta P_k(\nu') + (1 - \delta) P_k(\nu'') &= \delta T(P_{k-1})(\nu') + (1 - \delta) T(P_{k-1})(\nu'') \\
&= \int_{x} \begin{bmatrix}
\beta \left[ \delta (1 - p_\phi(\nu')) P_{k-1}(\omega_\phi(\nu')) + (1 - \delta)(1 - p_\phi(\nu'')) P_{k-1}(\omega_\phi(\nu'')) \right] \\
\beta \left[ \delta p_\phi(\nu') \bar{P}(\bar{\omega}_\phi(\nu')) + (1 - \delta) p_\phi(\nu'') \bar{P}(\bar{\omega}_\phi(\nu'')) \right]
\end{bmatrix} dF(\phi) \\
&= \int_{x} \begin{bmatrix}
w - [\delta c_\phi(\nu') + (1 - \delta)c_\phi(\nu'')] \\
\beta(1 - p_\phi(\nu')) [\delta P_{k-1}(\omega_\phi(\nu')) + (1 - \delta) P_{k-1}(\omega_\phi(\nu''))]
\end{bmatrix} dF(\phi) \\
&< \int_{x} \begin{bmatrix}
w - u^{-1} (\delta u(c_\phi(\nu')) + (1 - \delta) u(c_\phi(\nu''))) \\
\beta(1 - p_\phi^0(\nu')) P_{k-1}(\delta \omega_\phi(\nu') + (1 - \delta) \omega_\phi(\nu'')) \\
\beta p_\phi^0(\nu') P(\delta \bar{\omega}_\phi(\nu') + (1 - \delta) \bar{\omega}_\phi(\nu''))
\end{bmatrix} dF(\phi) \\
&= \int_{x} \begin{bmatrix}
w - c_\phi(\nu') + \beta \left[ p_\phi^0(\nu') \bar{P}(\bar{\omega}_\phi^0(\nu')) + (1 - p_\phi^0(\nu')) P_{k-1}(\omega_\phi^0(\nu')) \right]
\end{bmatrix} dF(\phi)
\end{align*}
\]

\[\equiv P_k(\nu) \leq P_k(\nu) = P_k(\delta \nu' + (1 - \delta) \nu'').\]

The first inequality follows from the strict concavity of \(u\) and \(\bar{P}\), along with the concavity of \(P_{k-1}\). The second inequality follows from \(0 \leq X(\delta p_\phi(\nu') + (1 - \delta) p_\phi(\nu'')) < \delta X(p_\phi(\nu') + (1 - \delta) X(p_\phi(\nu''))\) since \(X\) is strictly convex. The third inequality, \(P_k(\nu) \leq P_k(\nu)\) follows from the fact that the optimal allocation delivers (weakly) larger profits than the suboptimal one. We conclude that \(P_k(\delta \nu' + (1 - \delta) \nu'') > \delta P_k(\nu'') + (1 - \delta) P_k(\nu''), \text{ i.e., } P_k\) is strictly concave. This concludes the proof of the lemma. \(\blacksquare\)

Let \(\Omega\) denote the space of continuous functions defined over the interval \([\alpha - E[\phi], \bar{\omega}\] and bounded between \(P_{\text{MIN}} = (w - \bar{c} + \beta p_{\text{MIN}})/(1 - \beta(1 - p))\) and \(P_{\text{MAX}} = w/(1 - \beta)\). Moreover, let \(d_\infty\) denote the supremum norm, such that \((\Omega, d_\infty)\) is a complete metric space.

**Lemma 6** The mapping \(T(\nu)\) is an operator on the complete metric space \((\Omega, d_\infty)\), \(T(\nu)\) is a contraction mapping with a unique fixed-point \(P \in \Omega\).

**Proof.** By the Theorem of the Maximum \(T(\nu)(\nu)\) is continuous in \(\nu\). Moreover, \(T(\nu)(\nu)\) is bounded between \(P_{\text{MIN}}\) and \(P_{\text{MAX}}\) since even choosing zero consumption for any realization of \(\phi\) would induce profits not exceeding \(P_{\text{MAX}}\)

\[
w + \beta \int_{x} \left[ p_\phi \bar{P}(\bar{\omega}_\phi) + (1 - p_\phi) z(\omega_\phi) \right] dF(\phi) < w + \beta(1 - \beta) \bar{w}
\]

\[
= \frac{w}{1 - \beta} = P_{\text{MAX}}.
\]

and choosing the maximal consumption \(\bar{c}\) and promised utility \(\bar{\omega}\) and \(\tilde{\omega}\) for any \(\phi\) would induce profits no lower than \(P_{\text{MIN}}\). Thus, \(T(\nu)(\nu)\) is indeed an operator on \((\Omega, d_\infty)\).

According to Blackwell’s sufficient conditions \(T(\nu)\) is a contraction mapping (see Lucas and Stokey (1989, Theorem 3.3) if: (i) \(T\) is monotone, (ii) \(T\) discounts.
1. Monotonicity: Let \( z, y \in \Omega \) with \( z(\nu) \geq y(\nu), \forall \nu \in [\alpha - E[\phi], \tilde{\omega}] \). Then

\[
T(z)(\nu) = \max_{\{c_\phi, p_\phi, \omega_\phi, \tilde{\omega}_\phi\} \in \Lambda(\nu)} \int_{\mathbb{R}} \left[ w - c_\phi + \beta \left[ p_\phi \tilde{P}(\omega_\phi) + (1 - p_\phi) z(\phi) \right] \right] dF(\phi) \\
\geq \max_{\{c_\phi, p_\phi, \omega_\phi, \tilde{\omega}_\phi\} \in \Lambda(\nu)} \int_{\mathbb{R}} \left[ w - c_\phi + \beta \left[ p_\phi \tilde{P}(\omega_\phi) + (1 - p_\phi) y(\phi) \right] \right] dF(\phi) \\
= T(y)(\nu).
\]

2. Discounting: Let \( z \in \Omega \) and \( a \geq 0 \) be a real constant. Then

\[
T(z + a)(\nu) = \max_{\{c_\phi, p_\phi, \omega_\phi, \tilde{\omega}_\phi\} \in \Lambda(\nu)} \int_{\mathbb{R}} \left[ w - c_\phi + \beta \left[ p_\phi \tilde{P}(\omega_\phi) + (1 - p_\phi) (z(\phi) + a) \right] \right] dF(\phi) \\
= T(z)(\nu) + \beta a \int_{\mathbb{R}} (1 - p_\phi) dF(\phi) \\
\leq T(z)(\nu) + \beta a
\]

and \( \beta \in (0, 1) \).

Thus, \( T(z) \) is indeed a contraction mapping and according to Banach’s fixed-point theorem (see Lucas and Stokey (1989, Theorem 3.2)) there exists a unique fixed-point \( P \in \Omega \) satisfying the stationary functional equation,

\( P(\nu) = T(P)(\nu) \).

\[\Box\]

**Corollary 2** The profit function \( P \) is strictly concave.

This follows immediately from Lucas and Stokey (1989, Corollary 1). Since the unique fixed-point of \( T(z) \) is the limit of applying the operator \( n \) times \( T^n(z)(\nu) \) starting from any (and, in particular the concave ones) element \( z \) in \( \Omega \), and the operator \( T(z) \) maps concave into strictly concave functions the fixed-point \( P \) must be strictly concave.

**Lemma 7** The profit function \( P \) differentiable at its interior support with \( P'(\nu) = 1/u'(c(\nu)) < 0 \).

**Proof.** Given the strict concavity of the profit function, the proof is the same as for Lemma 3. The only difference is that \( p_\phi(\nu) \) denotes the optimal effort stated in Proposition 2 instead of Proposition 3. \[\Box\]

We can now establish that the FOCs of the COA are necessary and sufficient.

**Lemma 8** The FOCs of the planner problem without moral hazard are necessary and sufficient for optimality.

**Proof.** Lemma 5 implies that there cannot be two optimal contracts with distinct \( \omega_\phi \) and \( \tilde{\omega}_\phi \). Suppose not, so that there exists a 4-tuple of promised utilities \( \{\omega'_\phi, \omega''_\phi, \tilde{\omega}'_\phi, \tilde{\omega}''_\phi\} \) such that either \( \omega'_\phi(\nu) = \omega''_\phi(\nu) \) or \( \tilde{\omega}'_\phi(\nu) \neq \tilde{\omega}''_\phi(\nu) \) (or both). Then, from the strict concavity of \( P \) and \( \tilde{P} \), it would be possible to construct a feasible allocation that dominates the continuation profit implied by the proposed optimal
Thus, we have proven that \( \delta P(\tilde{\omega}_\phi' + (1 - \delta)\omega''_\phi) > \delta P(\omega'_\phi) + (1 - \delta)P(\omega''_\phi) \), or \( \bar{P}(\tilde{\omega}_\phi' + (1 - \delta)\bar{\omega}''_\phi) > \bar{\delta}P(\bar{\omega}_\phi') + (1 - \delta)\bar{\bar{P}}(\bar{\omega}''_\phi) \) (or both). This contradicts the assumption that the proposed allocations are optimal, establishing that the optimal contract pins down a unique pair of promised utilities, \( \{\omega_\phi, \bar{\omega}_\phi\} \).

Finally, we show that a unique pair of promised utilities pins down uniquely effort and consumption. The assumptions on \( X \) rule out corner solutions for effort, the assumptions on \( \nu \) and the fact that promised utility is interior (\( \omega_\phi \) and \( \bar{\omega}_\phi \) remain constant for \( \nu \geq \alpha - \phi_{\text{min}} \)) implies that also consumption is interior. Then the FOCs in (10) and (12) imply that

\[
-P'(\bar{\omega}_\phi(\nu))^{-1} = u'(c_\phi(\nu)),
\]

\[
X'(p_\phi(\nu)) = \beta \left(-P'(\bar{\omega}_\phi(\nu))^{-1} \left(\bar{P}(\bar{\omega}_\phi(\nu)) - P(\omega_\phi(\nu))\right) + (\bar{\omega}_\phi(\nu) - \omega_\phi(\nu))\right),
\]

which shows that, given \( \nu \) and \( \phi \), effort and consumption are uniquely determined as well. ■

This concludes the proof of Proposition 9.

**Lemma 9** In the planner problem without moral hazard future profits are strictly higher when the economy recovers, \( \bar{P}(\bar{\omega}_\phi(\nu)) - P(\omega_\phi(\nu)) > 0 \), \( \forall \omega_\phi(\nu) > \alpha - E[\phi] \).

**Proof.** We show that \( \bar{P}(\bar{\omega}_\phi(\nu)) - P(\omega_\phi(\nu)) \leq 0 \) runs into a contradiction. To simplify the notation, we abstract from the dependence of current optimal choices on the promise \( \nu \), i.e., \( \omega_\phi \) denotes \( \omega_\phi(\nu) \). The FOCs of the planner problem in equations (10) and (11) imply

\[
c(\bar{\omega}_\phi) = c_\phi = c(\omega_\phi).
\]

To arrive at a contradiction recall that \( \omega_\phi \geq \nu \). Then, once the economy recovers, promised-utility and profits remains constant such that consumption can be written as

\[
c(\bar{\omega}_\phi) = \bar{w} - \bar{P}(\bar{\omega}_\phi) + \beta \bar{P}(\bar{\omega}_\phi) = \bar{w} - \bar{P}(\bar{\omega}_\phi) + \frac{p_\phi}{R} \bar{P}(\bar{\omega}_\phi) + \left[1 - \frac{p_\phi}{R}\right] \bar{P}(\bar{\omega}_\phi)
\]

\[
> \bar{w} - \bar{P}(\bar{\omega}_\phi) + \frac{p_\phi}{R} \bar{P}(\bar{\omega}_\phi) + \frac{1 - p_\phi}{R} \bar{P}(\bar{\omega}_\phi)
\]

\[
\geq \bar{w} - P(\omega_\phi) + \frac{p_\phi}{R} \bar{P}(\bar{\omega}_\phi) + \frac{1 - p_\phi}{R} P(\omega_\phi)
\]

\[
\geq \bar{w} - P(\nu) + \frac{p_\phi}{R} \bar{P}(\bar{\omega}_\phi) + \frac{1 - p_\phi}{R} P(\omega_\phi) = c_\phi.
\]

We have used \( \bar{w} > w \) to derive the first inequality, \( [(1 - p_\phi)/R - 1] \bar{P}(\bar{\omega}_\phi) \geq [(1 - p_\phi)/R - 1] P(\omega_\phi) \) since \( [(1 - p_\phi)/R - 1] < 0 \) and \( \bar{P}(\bar{\omega}_\phi) \leq P(\omega_\phi) \) to derive the second inequality, and \( \omega_\phi \geq \nu \) along with \( P'(\nu) < 0 \) to derive the last inequality. \( c(\bar{\omega}_\phi) > c_\phi \) contradicts the equality \( c(\bar{\omega}_\phi) = c_\phi \) derived above. Thus, we have proven that \( \bar{P}(\bar{\omega}_\phi) - P(\omega_\phi) > 0 \). ■

**References of Online Appendix**


