Sovereign Debt and Structural Reforms*

Andreas Müller[†] Kjetil Storesletten[‡] Fabrizio Zilibotti[§]

September 23, 2016

Abstract

We construct a dynamic theory of sovereign debt and structural reforms with three interacting frictions: limited enforcement, limited commitment, and incomplete markets. A sovereign country in recession issues debt to smooth consumption and makes reforms to speed up recovery. The sovereign can renege on debt by suffering a stochastic cost, in which case debt is renegotiated. The competitive Markov equilibrium features large fluctuations in consumption and reform effort. We contrast the equilibrium with an optimal contract with one-sided commitment. A calibrated model can match several salient facts about debt crises. We quantify the welfare effect of relaxing different frictions.

JEL Codes: E62, F33, F34, F53, H12, H63

Keywords: Austerity, Commitment, Debt Overhang, Default, European Debt Crisis, Markov Equilibrium, Moral hazard, Renegotiation, Risk premia, Risk Sharing, Sovereign Debt, Structural Reforms.

*We would like to thank Manuel Amador, George-Marios Angeletos, Cristina Arellano, Marco Bassetto, Tobias Broer, Fernando Broner, Alessandro Dovis, Jonathan Eaton, Patrick Kehoe, Enrique Mendoza, Juan-Pablo Nicolini, Ugo Panizza, Aleh Tsyvinski, Jaume Ventura, Christopher Winter, Tim Worrall, and seminar participants at Annual Meeting of the Swiss Society of Economics and Statistics, Barcelona GSE Summer Forum, Brown University, CEMFI, Columbia University, CREi, EIEF Political Economy Workshop, ESSIM 2016, European University Institute, Goethe University Frankfurt, Graduate Institute of Geneva, Humboldt University, Istanbul School of Central Banking, NORMAC, Oxford, Royal Holloway, Swiss National Bank, Università Cà Foscari, Universitat Autonoma Barcelona, University College London, University of Cambridge, University of Konstanz, University of Oslo, University of Oxford, University of Pennsylvania, University of Padua, University of Toronto, University of Zurich, and Yale University. We acknowledge support from the European Research Council (ERC Advanced Grant IPCDP-324085).

[†]University of Oslo, Department of Economics, andreas.mueller@econ.uio.no.

[‡]University of Oslo, Department of Economics, kjetil.storesletten@econ.uio.no.

[§]University of Zurich, Department of Economics, fabrizio.zilibotti@econ.uzh.ch.

1 Introduction

Sovereign debt crises and economic reforms have been salient intertwined policy issues throughout the Great Recession, especially in Europe. Economic theory offers two simple policy prescriptions for countries suffering a temporary decline in output. First, they should borrow on international markets to smooth consumption. Second, they should undertake reforms – possibly painful ones in the short run – to speed up economic recovery. However, these prescriptions run into difficulties in the presence of limited enforcement issues. On the one hand, risk sharing is hampered by rising default premia. On the other hand, a large outstanding debt can reduce the borrower's incentive to undertake economic reforms to boost economic growth since some of the gains from growth would accrue to the lenders.

To cast light on these trade-offs and to derive positive and normative predictions, this paper proposes a dynamic theory of sovereign debt that rests on four building blocks. The first is that sovereign debt is subject to limited enforcement, and that countries can renege on their obligations subject to real costs as in, e.g., Aguiar and Gopinath (2006), Arellano (2008) and Yue (2010). The second building block is that whenever creditors face a credible default threat, they can make a renegotiation offer to the indebted country. This approach conforms with the empirical observations that unordered defaults are rare events, and that there is great heterogeneity in the terms at which debt is renegotiated, as documented by Tomz and Wright (2007) and Sturzenegger and Zettelmeyer (2008). The third building block is the possibility for the government of the indebted country to make *structural* policy reforms that speed up recovery from an existing recession.[1] The fourth building block is that reform effort is not contractible nor can markets commit to punish the past bad behavior of sovereign governments. This idea is captured by the notion of a Markov-perfect equilibrium, which excludes reputational mechanisms. The interaction between limited enforcement of sovereign debt and lack of commitment to discipline the structural reform effort is the focal point of our paper.

More formally, we construct a dynamic model of an endowment economy subject to income shocks following a two-state Markov process. A benevolent local government (henceforth, the *sovereign*) can issue debt to smooth consumption. The country starts in a recession of an unknown duration. The probability that the recession ends is endogenous, and hinges on its reform effort. Debt issuance is subject to a limited enforcement problem: the sovereign can, *ex-post,* repudiate its debt, based on the publicly observable realization of a stochastic default cost. When this realization is sufficiently low relative to the outstanding debt, the default threat is credible. In this case, a syndicate of creditors makes a take-it-or-leave-it debt haircut offer, as in Bulow and Rogoff (1989). In equilibrium, there is no outright default, but recurrent debt renegotiations. Haircuts are more frequent during recessions, and more frequent the larger is the outstanding sovereign debt. Consumption increases after a renegotiation, in line with the empirical evidence that economic conditions improve in the aftermath of debt relief, as documented in Reinhart and Trebesch (2016). Thereafter, debt growth resumes, as long as the recession continues.[2]

We first characterize the competitive (Markov) equilibrium. During recessions, the sovereign issues debt in order to smooth consumption. As debt accumulates, the probability of renegotiation increases,

[1] Examples of such reforms include labor and product market deregulation, and the establishment of fiscal capacity that allows the government to raise tax revenue efficiently (see, e.g., Ilzkovitz and Dierx 2011). While these reforms are beneficial in the long run, they entail short-run costs for citizens at large, governments or special-interest groups (see, e.g., Blanchard and Giavazzi 2003, and Boeri 2005).

[2] These debt dynamics are in line with the evidence for Greece, the hardest-hit country in the Europen debt crises. The Greek debt-GDP ratio soared from 107% in 2008 to 170% in 2011. At that point creditors had to agree to a debt haircut implying a 53% loss on its face value. Thereafter debt started increasing again until a new crisis erupted in the summer of 2015, culminating in the Greek government's request of a new renegotiation.

implying a rising risk premium and consumption volatility. The reform effort exhibits a non-monotonic pattern: it is increasing with debt at low levels of debt because of the disciplining effect of recession. However, for sufficiently high debt levels the relationship is flipped: higher debt levels deter reforms because most of the gains accrue to foreign lenders in the form of capital gains on the outstanding debt. The debt overhang exacerbates the country's inability to achieve consumption smoothing: at high debt levels, creditors expect little reform effort, are pessimistic about the economic outlook, and request an even higher risk premium. The main results carry over to an economy in which the sovereign can issue GDP-linked debt, i.e., securities whose payments are contingent on the stochastic realization of the endowment.

Next, we characterize the optimal dynamic contract when the planner, contrary to investors in the competitive equilibrium, can commit to punish the sovereign for deviations from the optimal contract. The extent of the punishment is limited: out of equilibrium, the sovereign suffers the default cost and is excluded from future contractual relations, but can resort to market financing at the competitive equilibrium terms. We consider two alternative cases. If the planner can observe (as do investors in the competitive equilibrium) the reform effort, the optimal contract with observable effort is qualitatively different from the Markov equilibrium: it features non-decreasing consumption and non-increasing reform effort during the recession, and overall less fluctuations. Consumption and effort remain constant whenever the country's participation and incentive constraints are not binding. When either constraint is binding the planner increases the country's promised utility and reduces the required reform effort. In contrast, if the planner cannot observe the reform effort, the optimal contract attains the same allocation as the market equilibrium with GDP-linked debt.

We interpret the optimal contract as the intervention of an external institution (e.g., the IMF) that provides assistance to the economy in recession, with the commitment to quit if the country does not implement the required reforms. During the recession, the optimal program entails a persistent budget support through extending loans on favorable terms. When the recession ends, the sovereign is settled with a (large) debt on market terms. A common objection to schemes implying deferred repayment is that the country may refuse to repay when the economy recovers. In our theory, this risks is factored in as part of the contract. Interestingly, whenever the country can credibly threaten to default, the international institution improves the terms of the agreement for the debtor by granting her higher consumption and a lower reform effort.

To evaluate the theory quantitatively, we extend the model to a world in which deep recessions are rare but recurrent events. In this case, for a range of low interest rates the competitive equilibrium and the planning solution feature a stationary long-run distribution of debt. We calibrate the model economy to match salient moments of observed debt-to-GDP ratios and default premia. The model can match realistic debt-to-GDP ratios, as well as default premia, renegotiation frequencies, and recovery rates. We regard this as a contribution in itself as the existing quantitative literature has difficulties to sustain high debt levels in equilibrium.[3]

We use the calibrated model to assess the quantitative welfare effects of policy interventions aimed at mitigating frictions. The effects are generally large: for instance, the assistance program outlined above is more valuable than the outright cancellation of a debt for an economy starting from a 100% debt-GDP ratio. On the contrary, the commitment to not renegotiate debt, with or without the imposition of fiscal austerity – an approach that is often portrayed in the policy debate as conducive to better incentives – is inefficient as it generates costly crises along the equilibrium path.

[3] For example, a recent study by Collard, Habib, and Rochet (2015) estimates that OECD countries can sustain debt-GDP ratios even in excess of 200%

1.1 Literature review

Our paper relates to several streams of the literature on sovereign debt. By focusing on Markov equilibria, we abstract from reputational mechanisms, being close in the spirit to the direct-punishment approach proposed by Bulow and Rogoff (1989).[4] Our work is related to the more recent quantitative models of sovereign default such as Aguiar and Gopinath (2006), Arellano (2008), and Chatterjee and Eyigungor (2012).[5] This literature does not consider the efficient allocation nor economic reforms. Moreover, we pursue an analytical characterization of the properties of the model.

In terms of the moral hazard in reform effort, our paper is related to Krugman (1988), Atkeson (1991) and Jeanne (2009). Krugman (1988) constructs a static model with exogenous debt showing that when a borrower has a large debt, productive investments might not be undertaken (the "debt overhang"). Atkeson (1991) studies the optimal contract in an environment in which an infinitely-lived borrower faces a sequence of two-period lived lenders. The borrower can use funds to invest in productive future capacity or to consume the funds. However, the lenders cannot observe the allocation to investment or consumption. Our paper differs from Atkeson's in various aspects. First, in our model we focus on Markov equilibria where the borrower cannot commit the reform effort, but the lender can observe it. This seems a plausible abstraction in the context of, for example, the European debt crisis. Second, in the constrained optimum the planner can observe the effort, but its power to punish deviations is limited by the ability of the sovereign to revert to the competitive (Markov) equilibrium. Third, in our theory structural reforms affect the future stochastic process of income, while his model investments only affect next period's income. Finally, in our model all agents have an infinite horizon. The results are different. Atkeson (1991) shows that the optimal contract involves capital outflow from the borrower during the worst aggregate state. Our model predicts instead that in a recession the borrower keeps accumulating debt and renegotiates it periodically. Moreover, in our model the constrained optimal allocation (though not necessarily the competitive equilibrium) has non-decreasing consumption when reform effort is observable. Jeanne (2009) studies an economy where the government takes a policy action that affects the return to foreign investors (e.g., the enforcement of creditor's right) but this can be reversed within a time horizon that is shorter than that at which investors must commit their resources.

Dovis (2016) studies the efficient risk-sharing arrangement between international lenders and a sovereign borrower with limited commitment and private information about domestic productivity. In his model the constrained efficient allocation can be implemented as a competitive equilibrium with non-contingent defaultable bonds of short and long maturity. Default episodes are *ex post* inefficient but occur nevertheless along the (*ex ante* efficient) equilibrium path. We focus instead on the interaction between structural reforms and limited commitment in a decentralized Markov equilibrium where international markets lack the commitment to coordinate on *ex post* inefficient punishments. Consequently, market outcomes are inefficient relative to the allocation of a planner who can observe (or has some information about) past reforms.

Hopenhayn and Werning (2008) study the optimal corporate debt contract between a bank and a risk-neutral borrowing firm. As we do, they assume that the borrower has a stochastic default cost.

[4] The distinction between the reputation approach and the punishment approach as the two main conceptual frameworks in the literature on sovereign debt crisis has been introduced recently by Bulow and Rogoff (2015).

Pioneer contributions to the analysis of debt repudiation based on reputational mechanisms such as the threat of future exclusion from credit markets include Eaton and Gersovitz (1981), Grossman and Van Huyck (1988), and Fernandez and Rosenthal (1989).

[5] Other papers studying restructuring of sovereign debt include Asonuma and Trebesch (2016), Benjamin and Wright (2009), Bolton and Jeanne (2007), Dovis (2016), Hatchondo et al. (2014), Mendoza and Yue (2012), and Yue (2010).

Different from us, they focus on the case when this outside option is not observable to the lender and show that this implies that default can occur in equilibrium. They do not study reform effort nor do they analyze the case of sovereign debt issued by a country in recession.

Conesa and Kehoe (2015) construct a theory predicting that the government of the borrowing country may opt to "gamble for redemption." Namely, it runs an irresponsible fiscal policy that sends the economy into the default zone if the recovery does not happen soon enough. The source and the mechanism of the crisis are different from ours. Their model is based on the framework of Cole and Kehoe (2000) featuring multiple equilibria and sunspots.

Our paper is related also to the literature on endogenous incomplete markets due to limited enforcement or limited commitment. This includes Alvarez and Jermann (2000) and Kehoe and Perri (2002). The analysis of constrained efficiency is related to the literature on competitive risk sharing contracts with limited commitment, including Thomas and Worrall (1988), Kocherlakota (1996), and Krueger and Uhlig (2006). An application of this methodology to the optimal design of a Financial Stability Fund is provided by Abraham, Carceles-Poveda, and Marimon (2014). In our model all debt is held by foreign lenders. Recent papers by Broner, Martin, and Ventura (2010), Broner and Ventura (2011), and Brutti and Sauré (2016) study the implications for the incentives to default of having part of the government debt held by domestic residents. Song *et al.* (2012) and Müller *et al.* (2016) focus, as we do, on Markov equilibria to study the politico-economic determination of debt in open economies where governments are committed to honor their debt. An excellent review of the sovereign debt literature is provided by Aguiar and Amador (2014).

From an empirical perspective, our paper is related to the findings of Tomz and Wright (2007). Using a dataset for the period 1820–2004, they find a negative but weak relationship between economic output in the borrowing country and default on loans from private foreign creditors. While countries default more often during recessions, there are many cases of default in good times and many instances in which countries have maintained debt service during times of very bad macroeconomic conditions. They argue that these findings are at odds with the existing theories of international debt. Our theory is consistent with the pattern they document. In our model, due to the stochastic default cost, countries may default during booms (though this is less likely, consistent with the data) and can conversely fail to renegotiate their debt during very bad times. Their findings are reinforced by Sturzenegger and Zettelmeyer (2008) who document that even within a relatively short period (1998-2005) there are very large differences between average investor losses across different episodes of debt restructuring. The observation of such a large variability in outcomes is in line with our theory, insofar as the bargaining outcome hinges on an outside option that is subject to stochastic shocks. In particular, our calibrated economy matches the cross-sectional variance of realized haircuts, as well as the frequency of debt restructuring. Borensztein and Panizza (2009) evaluate empirically the costs that may result from an international sovereign default, including reputation costs, international trade exclusion, costs to the domestic economy through the financial system, and political costs to the authorities. They find that the economic costs are generally short-lived. Finally, the relationship between consumption and renegotiations is in line with the evidence documented by Reinhart and Trebesch (2016), as discussed above. For a thorough review of the evidence, see also Panizza *et al.* (2009).

The rest of the paper is organized as follows. Section 2 describes the model environment. Section 3 characterizes the competitive Markov equilibrium. Section 4 solves for the optimal dynamic contract under the assumption that the principal (e.g., a syndicate of creditors) has full commitment, whereas the agent (i.e., the sovereign) is subject to limited commitment. A decentralized interpretation of the optimal contract is provided. Section 5 presents quantitative positive and normative implications of the theory with the aid of a calibrated economy. Section 6 concludes. Two online appendixes contain, respectively, the proofs of the main propositions and lemmas (Appendix A) and additional technical material referred in the text (Appendix B).

2 The model environment

The model economy is a small open endowment economy populated by an infinitely-lived representative agent. The endowments follow a two-state Markov switching process, with realizations $w \in \{\underline{w}, \bar{w}\}$, where $0 < \underline{w} < \bar{w}$. We label the two endowment states, respectively, *recession* and *normal times.* Normal times is assumed to be an absorbing state. If the economy starts in a recession, it switches to normal times with probability p and remains in the recession with probability $1-p$. The sovereign can implement a costly reform policy to increase the probability of a recovery. In our notation, p is both the *reform effort* and the probability that the recession ends. The assumption that normal times is an absorbing state aids tractability and enables us to obtain sharp analytical results. In Section 5, we generalize the model to the case of recurrent recessions.

The preferences of the representative agent are described by the following expected utility function:

$$E_0 \sum \beta^t \left[u\left(c_t\right) - \phi_t I_{\{\text{default in } t\}} - X\left(p_t\right) \right].$$

The utility function u is twice continuously differentiable and satisfies $\lim_{c\to 0} u(c) = -\infty$, $u'(c) > 0$, and $u''(c) < 0$. $I \in \{0,1\}$ is an indicator switching on when the economy is in a default state and ϕ is a stochastic default cost assumed to be *i.i.d.* over time and to be drawn from the p.d.f. $f(\phi)$ with an associated c.d.f. $F(\phi)$. We assume that $F(\phi)$ is continuously differentiable everywhere, and denote its support by $\aleph \equiv [0, \phi_{\max}] \subseteq \mathbb{R}^+$, where $\phi_{\max} < \infty$. The assumption that shocks are independent is inessential, but aids tractability. X is the cost of reform, assumed to be an increasing convex function of the probability of exiting recession, $p \in [\underline{p}, \bar{p}] \subseteq [0,1]$. X is assumed to be twice continuously differentiable, with the properties that $X\left(\underline{p}\right) = 0$, $X'(p) > 0$ and $X''(p) > 0$. In normal times, $X = 0$.

To establish a benchmark, we characterize the optimal allocation under full insurance and full enforcement (labelled the *first-best* allocation). The economy is assumed to start in a recession with an outstanding obligation b given an implicit gross rate of return of $R = \beta^{-1}$. The first-best allocation entails perfect insurance: the country enjoys a constant stream of consumption and exerts a constant reform effort during recession (during normal times, there is no effort). The level of b lowers consumption and increases reform effort in recession.

Proposition 1 *Let $W^{FB}(b,w)$, $c^{FB}(b,w)$ and $p^{FB}(b)$ denote, respectively, the discounted utility, consumption and effort as a function of the outstanding obligation b, with $w \in \{\underline{w}, \bar{w}\}$ denoting the initial state of productivity. Then, for an economy starting in recession:*

$$\begin{aligned} c^{FB}(b,\underline{w}) &= \frac{(1-\beta)\underline{w} + \beta p^{FB}(b)\bar{w}}{1-\beta\left(1-p^{FB}(b)\right)} - (1-\beta) b, \\ W^{FB}(b,\underline{w}) &= \frac{u\left(c^{FB}(b,\underline{w})\right)}{1-\beta} - \frac{X\left(p^{FB}(b)\right)}{1-\beta\left(1-p^{FB}(b)\right)} \end{aligned}$$

where $p^{FB}(b)$ is the reform effort exerted for as long as the economy stays in recession. $p^{FB}(b)$ is the *unique solution for p^{FB} satisfying the following condition:*

$$\frac{\beta}{1-\beta\left(1-p^{FB}\right)}\left(\underbrace{(\bar{w}-\underline{w})\times u'\left(c^{FB}\left(b,\underline{w}\right)\right)}_{\textit{increase in output if econ. recovers}}+\underbrace{X\left(p^{FB}\right)}_{\textit{saved effort cost if econ. recovers}}\right)=X'\left(p^{FB}\right). \qquad (1)$$

Moreover, when effort is interior, $c^{FB}\left(b,\underline{w}\right)$ and $p^{FB}\left(b\right)$ are, respectively, decreasing and increasing functions of b.

3 Competitive equilibrium

In the competitive equilibrium, the sovereign can issue a one-period discount bond to smooth consumption. The bond, b, is a claim to one unit of the next-period consumption good, which sells today at the price $Q\left(b,w\right)$. Bonds are purchased by a representative risk-neutral foreign creditor who has access to an international risk-free portfolio paying the world interest rate R. For simplicity, we focus on the case in which $\beta R=1$, although our main insights carry over to the case in which $\beta R<1$ (see Section 5). After issuing debt, the country decides its reform effort.

The key assumptions are that (i) the country cannot commit to repay its sovereign debt, and (ii) the reform effort is not contractible. At the beginning of each period, the sovereign observes the realization of the default cost ϕ, and decides whether to repay the debt that reaches maturity or to announce default on all its debt. The cost ϕ is publicly observed, and captures in a reduced form a variety of shocks including both taste shocks (e.g., the sentiments of the public opinion about defaulting on foreign debt) and institutional shocks (e.g., the election of a new prime minister, a new central bank governor taking office, the attitude of foreign governments, etc.).[6] If a country defaults, no debt is reimbursed.[7]

When the sovereign announces its intention to default, a syndicate of creditors can make a take-it-or-leave-it renegotiation offer that we assume to be binding for all creditors. By accepting the renegotiation offer, the sovereign averts the default cost. In equilibrium, a haircut is offered only if the default threat is credible, i.e., if the realization of ϕ is sufficiently low to make the country prefer default to full repayment. When they offer renegotiation, creditors make the debtor indifferent between an outright default and the proposed haircut.

In summary, the timing is as follows: The sovereign enters the period with the pledged debt b, observes the realization of w and ϕ, and then decides whether to announce default. If the threat is credible, the creditors offer a haircut. Next, the country decides whether to accept or decline the offer. Then, the sovereign issues new debt subject to the period budget constraint $Q\times b'=\mathbb{B}\left(b,\phi,w\right)+c-w$, where $\mathbb{B}\left(b,\phi,w\right)\leq b$ denotes the debt level after the renegotiation stage. For technical reasons we also impose that debt is bounded, $b\in[\underline{b},\tilde{b}]$ where $\underline{b}\in\left(-\infty,0\right]$ and $\tilde{b}=\bar{w}/\left(R-1\right)$ is the natural borrowing constraint in normal times. In equilibrium, these bounds will never be binding. If the country could

[6] Alternatively, ϕ could be given a politico-economic interpretation, as reflecting special interests of the groups in power. For instance, the government may care about the cost of default to its constituency rather than to the population at large. In the welfare analysis, we stick to the interpretation of a benevolent government and abstract from politico-economic factors, although the model could be extended in this direction.

[7] For simplicity, we assume that ϕ captures all costs associated with default. In an earlier version of this paper, we assumed that the government could not issue new debt in the default period, but were allowed to start issuing bonds already in the following period. The results are unchanged. One could even consider richer post-default dynamics, such as prolonged or stochastic exclusion from debt markets. Since outright default does not occur in equilibrium, the details of the post-default dynamics are immaterial.

commit to honor its debt, it would sell bonds at the price $Q = 1/R$. However, due to the risk of default or renegotiation, it sells at a discount, $Q \leq 1/R$. Next, consumption is realized, and finally the sovereign decides its reform effort.

3.1 Definition of Markov equilibrium

In the characterization of the competitive equilibrium, we restrict attention to Markov-perfect equilibria where the set of equilibrium functions only depend on the pay-off relevant state variables, b, ϕ, and w. This rules out that the sovereign's decisions can be affected by the desire to establish or maintain a reputation.

Definition 1 *A Markov-perfect equilibrium is a set of value functions $\{V, W\}$, a threshold renegotiation function Φ, an equilibrium debt price function Q, a set of optimal decision rules $\{\mathbb{B}, B, C, \Psi\}$, such that, conditional on the state vector $(b, \phi, w) \in \left([\underline{b}, \tilde{b}] \times [0, \phi_{\max}] \times \{\underline{w}, \bar{w}\}\right)$, the sovereign and the international creditors maximize utility, and markets clear. More formally:*

- *The value function V satisfies*

$$V(b, \phi, w) = \max\{W(b, w), W(0, w) - \phi\}, \tag{2}$$

where $W(b, w)$ is the value function conditional on the debt level b being honored,

$$W(b, w) = \max_{b' \in [\underline{b}, \tilde{b}]} u\left(Q\left(b', w\right) \times b' + w - b\right) + Z\left(b', w\right),$$

and where Z is defined as

$$\begin{aligned} Z\left(b', \underline{w}\right) &= \max_{p \in [\underline{p}, \bar{p}]} \left\{-X(p) + \beta\left(p \times E\left[V\left(b', \phi', \bar{w}\right)\right] + (1-p) \times E\left[V\left(b', \phi', \underline{w}\right)\right]\right)\right\}, & (3) \\ Z\left(b', \bar{w}\right) &= \beta E\left[V\left(b', \phi', \bar{w}\right)\right], & (4) \end{aligned}$$

and $E\left[V\left(x, \phi', w\right)\right] = \int_{\aleph} V(x, \phi, w)\, dF(\phi)$.

- *The threshold renegotiation function Φ satisfies*

$$\Phi(b, w) = W(0, w) - W(b, w). \tag{5}$$

- *The debt price function satisfies the following arbitrage conditions:*

$$Q(b, \bar{w}) = \hat{Q}(b, \bar{w}) \tag{6}$$

$$Q(b, \underline{w}) = \Psi(b) \times \hat{Q}(b, \bar{w}) + [1 - \Psi(b)] \times \hat{Q}(b, \underline{w}) \tag{7}$$

where $\hat{Q}(b, w)$ is the bond price conditional on next period being in state w,

$$\hat{Q}(b, w) \equiv \frac{1}{R}(1 - F(\Phi(b, w))) + \frac{1}{R}\frac{1}{b}\int_0^{\Phi(b,w)} \hat{b}(\phi, w) \times f(\phi)\, d\phi, \tag{8}$$

and where $\hat{b}(\phi, w)$ is the new debt after a renegotiation given a realization ϕ. $\hat{b}$ is implicitly defined by the condition $W\left(\hat{b}(\phi, w), w\right) = W(0, w) - \phi$.

- *The set of optimal decision rules comprises:*

  1. *A take-it-or-leave-it debt renegotiation offer:*

$$\mathbb{B}(b,\phi,w) = \begin{cases} \hat{b}(\phi,w) & \text{if} \quad \phi \le \Phi(b,w), \\ b & \text{if} \quad \phi > \Phi(b,w). \end{cases} \tag{9}$$

  2. *An optimal debt accumulation and an associated consumption decision rule:*

$$B(\mathbb{B}(b,\phi,w),w) = \arg\max_{b' \in [\underline{b},\tilde{b}]} \left\{ u\left(Q\left(b',w\right) \times b' + w - \mathbb{B}(b,\phi,w)\right) + Z\left(b',w\right) \right\}, \tag{10}$$

$$C(\mathbb{B}(b,\phi,w),w) = Q(B(\mathbb{B}(b,\phi,w),w),w) \times B(\mathbb{B}(b,\phi,w),w) + w - \mathbb{B}(b,\phi,w). \tag{11}$$

  3. *An optimal effort decision rule:*

$$\Psi\left(b'\right) = \arg\max_{p \in [\underline{p},\bar{p}]} \left\{ -X(p) + \beta\left(p \times E\left[V\left(b',\phi',\bar{w}\right)\right] + (1-p) \times E\left[V\left(b',\phi',\underline{w}\right)\right]\right) \right\}. \tag{12}$$

- *The equilibrium law of motion of debt is* $b' = B(\mathbb{B}(b,\phi,w),w)$.

- *The probability that the recession ends is* $p = \Psi(b')$.

V and W denote the sovereign's value functions. Equation (2) implies that there is renegotiation if and only if $\phi < \Phi(b,w)$. Since, *ex-post*, creditors have all the bargaining power, the discounted utility accruing to the sovereign equals the value that she would get under outright default. Thus,

$$V(b,\phi,w) = \begin{cases} W(b,w) & \text{if} \quad b \le \hat{b}(\phi,w), \\ W(0,w) - \phi & \text{if} \quad b > \hat{b}(\phi,w). \end{cases}$$

Consider, next, the equilibrium debt price function. Since creditors are risk neutral, the expected rate of return on the sovereign debt must equal the risk-free rate of return. Then, the arbitrage conditions (6)–(7) ensure market clearing in the bond market and pin down the equilibrium bond price in normal times and recession, respectively. The function $\hat{Q}$ defined in Equation (8) yields the bond price after the state w has realized but before knowing ϕ. With probability $1 - F(\Phi(b,w))$ debt is honored, where $\Phi(b,w)$ denotes the threshold default shock realization such that, conditional on the debt b, the sovereign cannot credibly threaten to default for all $\phi \ge \Phi(b,w)$. With probability $F(\Phi(b,w))$, debt is renegotiated to a level that depends on the realization of ϕ. This level is given by $\hat{b}(\phi,w)$ which, recall, denotes the renegotiated debt level that keeps the sovereign indifferent between accepting the creditors' offer and defaulting. In the rest of the paper, we use the more compact notation $EV(b,w) \equiv E[V(b,\phi,w)]$ and $EV(b',w) \equiv E\left[V\left(b',\phi',w\right)\right]$.

Consider, finally, the set of decision rules. (9) stipulates that creditors always extract the entire surplus at the renegotiation stage. Equations (10)-(11) yield the optimal consumption-saving decisions subject to a resource constraint. Equation (12) yields the optimal effort decision. Note that the effort exerted depends on b', since effort is chosen after the new debt is issued.

3.2 Existence of a Markov equilibrium

We start by establishing an intuitive property linking $\hat{b}$ and Φ:

Lemma 1 *Suppose a value function* $W(b, w)$ *exists and is strictly decreasing in* b*. Then,* $\hat{b}(\Phi(b, w), w) = b$*. Moreover,* $\Phi(b, w)$ *is strictly increasing in* b*, hence,* $\hat{b}(\phi, \bar{w}) = \bar{\Phi}^{-1}(\phi)$ *and* $\hat{b}(\phi, \underline{w}) = \underline{\Phi}^{-1}(\phi)$*, where* $\bar{\Phi}(b) \equiv \Phi(b, \bar{w})$*, and* $\underline{\Phi}(b) \equiv \Phi(b, \underline{w})$*.*

The lemma follows from the definitions of $\hat{b}$ and Φ. On the one hand, $\hat{b}(\phi, w)$ is the debt level that, conditional on ϕ, makes the debtor indifferent between honoring and defaulting. On the other hand, $\Phi(b, w)$ is the realization of ϕ that, conditional on b, makes the debtor indifferent between honoring and defaulting.

The next proposition establishes the existence of a Markov equilibrium. The crux of the proof lies in establishing the existence of the value function W. This is done by showing that the value function W is a fixed-point of a monotone mapping following Theorem 17.7 in Stokey and Lucas (1989). Once the existence of W is established, all the equilibrium functions $(V, \Phi, \hat{b}, Q, Z, \mathbb{B}, B, C, \Psi)$ can be derived from the set of definitions above.

Proposition 2 *A Markov equilibrium exists, i.e., there exists a set of equilibrium functions* $(V, W, \Phi, \hat{b}, Q, Z, \mathbb{B}, B, C, \Psi)$ *satisfying Definition 1. The value functions* V *and* W *are continuous and non-increasing in* b*. The equilibrium functions* Φ*,* Q *and* Ψ *are also continuous in* b*. The bond revenue,* $Q(b, w)b$*, is non-decreasing in* b*. The policy function* $B(b, w)$ *is non-decreasing in* b*.*

Proposition 2 establishes the existence but not the uniqueness of the Markov equilibrium. The corollary of Theorem 17.7 in Stokey and Lucas (1989) provides a strategy to verify numerically whether an equilibrium is unique. We state this as Corollary 1 in Appendix A.

The next proposition establishes local differentiability properties of the value function W, and that the first-order conditions are necessary. Due to the possibility that debt is renegotiated, the value functions are not necessarily concave. In spite of this, we can establish that the equilibrium functions are differentiable at all debt levels that can be the result of an optimal choice given some initial debt level. We define formally the set of such debt levels as $\mathcal{B}(w) = \{x \in [\underline{b}, \tilde{b}] \mid B(\mathbb{B}(b, \phi, w), w) = x$, for $b \in [\underline{b}, \tilde{b}]\}$.[8]

Proposition 3 *The equilibrium functions* $W(b, w)$*,* $Z(b, w)$*,* $\Phi(b, w)$*,* $Q(b, w)$*,* $\Psi(b)$ *are differentiable for all* $b \in \mathcal{B}(w)$*. Moreover, for any* $b' \in \mathcal{B}(w)$*, the first-order condition* $(\partial/\partial b')\, u(Q(b', w)b' + w - b) + (\partial/\partial b')\, Z(b', w) = 0$ *and the envelope condition* $\partial W(b', w)/\partial b' = -u'(C(b', w))$ *holds true.*

The proof follows from the envelope theorem of Clausen and Strub (2013) that applies to problems including endogenous functions such as default probabilities and interest rates (see also Arellano *et al.* 2014).

Hereafter, for simplicity, we refer to the competitive Markov equilibrium as the *competitive equilibrium.*

[8] We prove later that during normal times the equilibrium functions are differentiable everywhere. However, this is not true in recession. In this case Proposition 3 shows that differentiability still holds for all b that can be attained as an optimal choice.

3.3 Competitive equilibrium in normal times

In this section, we characterize the equilibrium when the economy is in normal times. The next lemma establishes properties of the debt revenue function at the optimal interior debt choice.

Lemma 2 *The debt revenue function* $Q(b', \bar{w})b'$ *is concave in* $b' \in [\underline{b}, \tilde{b}]$ *and differentiable for all* $b' \in \mathcal{B}(w)$ *where* $\partial\left(Q\left(b', \bar{w}\right) b'\right) / \partial b' = R^{-1}\left(1 - F\left(\Phi\left(b', \bar{w}\right)\right)\right)$.

An immediate implication of the lemma is that if we define $\tilde{b}$ to be the lowest debt inducing renegotiation almost surely (i.e., such that $\lim_{b' \to \tilde{b}} F\left(\Phi(b', \bar{w})\right) = 1$), then, $\bar{b}$ is also the top of the Laffer curve, i.e., the endogenous debt limit. More formally, $\bar{b} \equiv \min\left\{\arg\max_{b \in [\underline{b}, \tilde{b}]} \left\{Q\left(b, \bar{w}\right) b\right\}\right\} < \tilde{b}$. Although the borrower could issue debt exceeding $\bar{b}$, the marginal debt revenue would be zero for $b' > \bar{b}$ since this debt would never be honored.

We now characterize the consumption and debt dynamics. We introduce a definition that will be useful throughout the paper.

Definition 2 *A Conditional Euler Equation (CEE) describes the (expected) marginal rate of substitution between current and next-period consumption in all states of nature* ϕ' *that induce the sovereign to honor its debt next period.*

Next, we characterize formally the CEE. The sovereign solves the consumption-saving problem given by (10). The first-order condition and the envelope theorem yield the following result.

Proposition 4 *If the realization of* ϕ' *induces no renegotiation, then the following CEE holds true:*

$$\beta R \frac{u'\left(c'|_{H,\bar{w}}\right)}{u'\left(c\right)} = 1, \tag{13}$$

where $c = C\left(\mathbb{B}\left(b, \phi, \bar{w}\right), \bar{w}\right)$ *is current consumption and* $c'|_{H,\bar{w}} = C\left(b', \bar{w}\right) = C\left(B\left(\mathbb{B}\left(b, \phi, \bar{w}\right), \bar{w}\right), \bar{w}\right)$ *is next-period consumption conditional on no renegotiation. Since* $\beta R = 1$, *then* $b' = B\left(b, \bar{w}\right) = b$, *and consumption remains constant. Moreover, for all* $b < \bar{b}$, *the value function* $W\left(b, \bar{w}\right)$ *is strictly decreasing, strictly concave and twice continuously differentiable in* b, *and consumption* $C(b, \bar{w})$ *is strictly falling in* b.

Although the CEE (13) resembles a standard Euler Equation under full commitment, the similarity is deceiving: R is *not* the realized interest rate when debt is fully honored; this in fact higher due to the default premium.

When debt is renegotiated, consumption increases discretely, hence $u'\left(c_t\right) / u'\left(c_{t+1}\right) > \beta R$. This is not surprising, since the country benefits from a reduction in debt repayment.[9] Thus, consumption and debt are, respectively, increasing and decreasing step functions over time: they remain constant

[9] The prediction that whenever debt is renegotiated consumption increases permanently is extreme, and hinges on the assumptions that $\beta R = 1$ and that ϕ is i.i.d. with a known distribution. In Section B.3 of Appendix B we extend the model to a setting where there is uncertainty about the true distribution of ϕ and the market learns about this distribution by observing the sequence of ϕ's. In this case, a low realization of ϕ has two opposing effects on consumption: on the one hand, a low ϕ triggers debt renegotiation which on its own would increase consumption; on the other hand, a low ϕ affects the beliefs about the distribution of ϕ, inducing the market to regard the country as less creditworthy (namely, the country draws from a distribution where low ϕ is more likely). This tends to increase the default premium on bonds and to lower consumption.

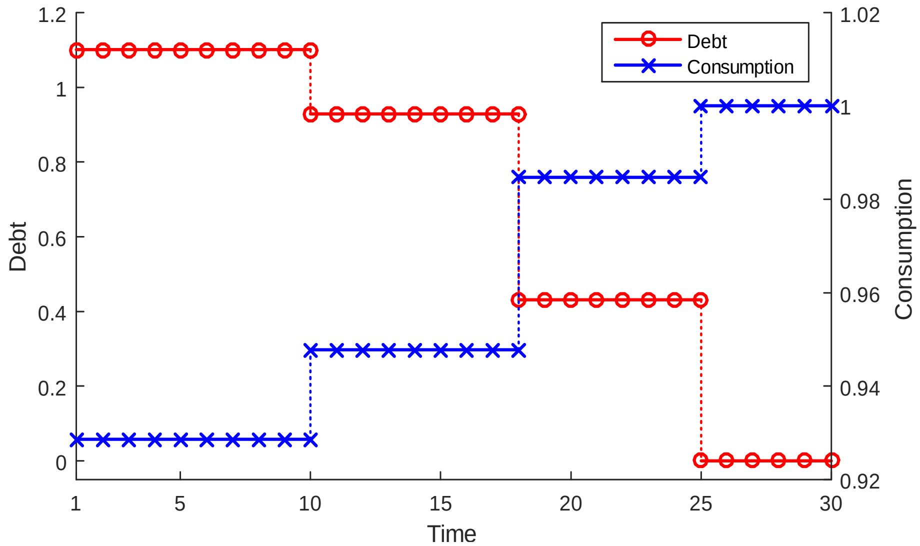

Figure 1: Simulation of debt and consumption for a particular sequence of ϕ's during normal times.

in every period in which the country honors its debt, while changing discretely upon every episode of renegotiation. Figure 1 illustrates a simulation of the consumption and debt dynamics. Note that the sequence of renegotiations eventually brings the debt to a sufficiently low level where the risk of renegotiation vanishes. This consumption path is different from the first-best allocation where consumption and debt are constant for ever. Interestingly, in the long run, consumption is higher in the competitive equilibrium with the risk of repudiation than in the first best allocation.

It is straightforward to generalize the results to the case of $\beta R < 1$ under the assumption that utility features constant relative risk aversion. In this case, when the debt is honored debt would increase and consumption would fall. After each episode of renegotiation the economy would start again accumulating debt. In a world comprising economies with different β, e.g., some with $\beta R = 1$ and some with $\beta R < 1$, economies with low β would experience recurrent debt crises.

3.4 Equilibrium under recession

When the economy is in recession the sovereign chooses, sequentially, whether to honor the current debt, how much new debt to issue, and how much reform effort to exert. In this section, we assume that the sovereign cannot issue GDP-linked debt, i.e., securities whose payment is contingent on the stochastic realization of the endowment. In Section 3.5 below we relax this restriction.

A natural property of the competitive equilibrium is that $C(b, \underline{w}) < C(b, \bar{w})$ for all $b \leq \bar{b}$: conditional on honoring a giving debt level, consumption is higher in normal times than in a recession. Although we could find no numerical counterexample to this property, it is difficult to prove it in general because the equilibrium functions for consumption, effort and debt price are determined simultaneously. However, we can provide a sufficient condition.

Proposition 5 *The following conditions are sufficient to ensure that $C(b, \underline{w}) < C(b, \bar{w})$ for all $b \in [0, \bar{b}]$: (i) $\bar{w} - \underline{w} > \frac{\beta}{1-\beta}\bar{w}$, and (ii) $F[(u(\bar{w}) - u((1-\beta)(\bar{w} - \underline{w}))) / (1-\beta)] = 0$.*

When $C(b, \underline{w}) < C(b, \bar{w})$, it is straightforward to show, using Definition 1, that $\Phi(b, \underline{w}) > \Phi(b, \bar{w})$,

$\hat{b}(\phi, \underline{w}) < \hat{b}(\phi, \bar{w})$, $Q(b, \underline{w}) < Q(b, \bar{w})$ and $W(b, \underline{w}) < W(b, \bar{w})$. Note, in particular, that the price of the bond increases if the recession ends because of the associated reduction in the probability of renegotiation. The property that $\Phi(b, \underline{w}) > \Phi(b, \bar{w})$ implies that one can partition the state space into three regions:

- if $b < b^-$, the country honors the debt with a positive probability, irrespective of the aggregate state (the probability of renegotiation being higher if the recession continues than if it ends);[10]
- if $b \in [b^-, \bar{b}]$, the country renegotiates with probability one if the recession continues, while it honors the debt with a positive probability if the recession ends;
- if $b > \bar{b}$, the country renegotiates its debt with probability one, irrespective of the aggregate state.

Note that the risk of repudiation introduces some state contingency, since debt is repaid with different probabilities under recession and normal times.

3.4.1 Reform effort in equilibrium

We denote by $\Psi(b')$ the equilibrium policy function for effort, i.e., the probability that the recession ends next period, as a function of the newly-issued debt. More formally, the first-order condition from (12) yields:[11]

$$X'(\Psi(b')) = \beta \left[\int_0^\infty V(b', \phi', \bar{w})\, dF(\phi) - \int_0^\infty V(b', \phi', \underline{w})\, dF(\phi) \right]. \tag{14}$$

The sovereign's incentive to exert reform effort hinges on the increase in expected utility associated with the end of the recession. However, effort is not provided efficiently. To see why, recall that the bond price increases upon economic recovery. Thus, the creditors reap part of the welfare gain from economic recovery, whereas the country bears the full burden of the effort cost.

We can prove that effort is inefficiently provided with the aid of a simple one-period deviation argument. Consider an equilibrium effort choice path consistent with (14) – corresponding to the case of non-contractible effort. Next, suppose that, only in the initial period, the country could contract effort before issuing new debt. The following lemma shows that, in this case, the country would choose a higher reform effort than in the competitive equilibrium.

Lemma 3 *Suppose that $b' > 0$ and that the borrower can, in the initial period, commit to an effort level upon issuing new debt. Then, the reform effort would be strictly larger than in the case in which effort is never contractible.*

If the sovereign could commit to reform, its effort would be monotonically increasing in the debt level, since a high debt increases the hardship of a recession. However, in equilibrium reform effort exhibits a non-monotonic behavior. More precisely, $\Psi(b)$ is increasing at low levels of debt, and decreasing in a range of high debt levels, including the entire region $[b^-, \bar{b}]$. Proposition 6 establishes this result more formally.

[10] b^- is implicitly determined by the equation $W(b^-, \underline{w}) = W(0, \underline{w}) - \phi_{\max}$.

[11] Nota that the continuity and monotonicity of X, together with the continuity of the value functions ensure that Ψ is a continuous function.

Proposition 6 *There exist three ranges,* $[0, b_1] \subseteq [0, b^-]$, $[b_2, \bar{b}] \supseteq [b^-, \bar{b}]$, *and* $\left[\bar{b}, \infty\right)$ *such that:*

1. *If* $b \in [0, b_1)$, $\Psi'(b) > 0$;
2. *If* $b \in \left(b_2, \bar{b}\right)$, $\Psi'(b) < 0$;
3. *If* $b \in \left[\bar{b}, \infty\right)$, $\Psi'(b) = 0$.

The following argument establishes the result. Consider a low (possibly negative) debt range where the probability of renegotiation is zero. In this range, there is no moral hazard. Thus, a higher debt level has a disciplining effect, i.e., it strengthens the incentive for economic reforms: due to the concavity of the utility function, the discounted gain of leaving the recession is an increasing function of debt.

As one moves to a larger initial debt, however, moral hazard becomes more severe, since the reform effort decreases the probability of default, and shifts some of the gains to the creditors. The effect of *debt overhang* (cf. Krugman 1988) dominates over the disciplining effect in the region $[b^-, \bar{b}]$. In this range, debt has a stark state contingency. If the economy remains in recession, it is renegotiated for sure, rendering the continuation utility independent of b. If the recession ends, the continuation utility is decreasing in b. Therefore, in this region the value of reform effort necessarily decreases in b. By continuity, the same argument applies to a range of debt below b^-.[12]

The debt-overhang effect hinges on the presence of some renegotiation risk and an associated premium on debt. If the borrower instead could commit to repay the debt, the price of debt would be $1/R$, so an economic recovery would not yield any benefits to the lenders and the effort function would be monotone increasing in debt.

3.4.2 Debt issuance and consumption dynamics

In this section, we characterize the equilibrium dynamics of consumption and debt. We proceed in two steps. First, we derive the properties of the CEE. Then, we summarize its characterization in a formal proposition.

The first-order condition of (10) together with the envelope theorem yields the following CEE:

$$\begin{aligned} & E\left\{\frac{MU_{t+1}}{MU_t}|\text{debt is honored at } t+1\right\} \\ = \quad & 1 + \frac{\Psi'(b_{t+1})}{\Pr(\text{debt is honored at } t+1)} R\left[Q(b_{t+1}, \bar{w}) - \hat{Q}(b_{t+1}, \underline{w})\right] b_{t+1}. \end{aligned} \tag{15}$$

Equation (15) differs from (13) in two terms. First, the left-hand side has the *expected* ratio between the marginal utilities, due to the uncertainty about the future aggregate state. Second, there is a new term on the right-hand side capturing the effect of debt on reform effort.

For expositional purposes, consider first the case in which the probability that the recession ends is exogenous, i.e., $\Psi' = 0$. In this case, the CEE requires that the expected marginal utility be constant. For this to be true, consumption growth must be positive if the recession ends and negative if it continues. The lack of consumption insurance stems from the incompleteness of financial markets,

[12] In a variety of numerical simulations, we have always found Ψ to be hump-shaped with a unique peak (see Figure 3), although in general this depends on the distribution $F(\phi)$.

and would disappear if the sovereign could issue GDP-linked debt. In Section 3.5 below, we show that this conclusion does not carry over to the economy with moral hazard.

Consider, next, the general case. Moral hazard introduces a new strategic motive since the level of newly-issued debt affects the sovereign's *ex-post* incentive to make reforms. The sign of this strategic effect hinges on the sign of Ψ' (see Proposition 6). When the outstanding debt is low, $\Psi' > 0$. Then, more debt strengthens the *ex-post* incentive to reform, thereby increasing the price of the newly-issued debt. The right-hand side of (15) is in this case larger than unity, and the CEE implies a lower consumption fall (hence, higher debt accumulation) than in the absence of moral hazard. In contrast, in the region of high initial debt, $\Psi' < 0$. In this case, the sovereign issues less debt than in the absence of moral hazard in order to mitigate the fall in debt price associated with moral hazard. Thus, when the recession continues, a highly indebted country will obtain less consumption insurance when the reform is endogenous than when p is exogenous.

We summarize the results in a formal proposition.

Proposition 7 *If the economy starts in a recession and the realization of ϕ' induces no renegotiation, the optimal debt level, $b' = B(\mathbb{B}(b,\phi,\underline{w}),\underline{w})$, induces a consumption sequence that satisfies the following CEE:*

$$\beta R \left(\begin{array}{c} \dfrac{\underbrace{\left[1-\Psi\left(b'\right)\right]\times\left(1-F\left(\underline{\Phi}\left(b'\right)\right)\right)}_{\textit{prob. of repayment and continuing recession}}}{\underbrace{\Pr\left(H|b'\right)}_{\textit{unconditional prob. of repayment}}} \times \underbrace{\dfrac{u'\left(c'|_{H,\underline{w}}\right)}{u\left(c\right)}}_{\textit{MRS if rec. cont.}} \\ + \dfrac{\underbrace{\Psi\left(b'\right)\times\left(1-F\left(\bar{\Phi}\left(b'\right)\right)\right)}_{\textit{prob. of repayment and end of recession}}}{\underbrace{\Pr\left(H|b'\right)}_{\textit{unconditional prob. of repayment}}} \times \underbrace{\dfrac{u'\left(c'|_{H,\bar{w}}\right)}{u\left(c\right)}}_{\textit{MRS if rec. ends}} \end{array} \right) = 1 + \frac{\Psi'\left(b'\right)}{\Pr\left(H|b'\right)} \times \underbrace{R\left(Q\left(b',\bar{w}\right) - \hat{Q}\left(b',\underline{w}\right)\right)b'}_{\textit{gain to lenders if recession ends}} \tag{16}$$

where $c = C(\mathbb{B}(b,\phi,\underline{w}),\underline{w})$ is current consumption, $c'|_{H,w} = C(b',w) = C(B(\mathbb{B}(b,\phi,w),w),w)$ is next-period consumption conditional on w and no renegotiation, and $\Pr(H|b')$ is the unconditional probability that the debt b' be honored, i.e., $\Pr(H|b') = [1-\Psi(b')]\times\left(1-F\left(\bar{\Phi}(b')\right)\right) + \Psi(b')\times(1-F(\underline{\Phi}(b')))$.

We end this section by noting that the top of the Laffer curve of debt corresponds to a lower debt level in recession than during normal times.

Lemma 4 *Let $\bar{b} = \min\left\{\arg\max_{b\in[\underline{b},\tilde{b}]}\left\{Q(b,\bar{w})\,b\right\}\right\}$ and $\bar{b}^R = \min\left\{\arg\max_{b\in[\underline{b},\tilde{b}]}\left\{Q(b,\underline{w})\,b\right\}\right\}$. Then, $\bar{b}^R \leq \bar{b}$, with equality holding only if $\Psi(b) = p$ (i.e., if the probability of staying in a recession is exogenous).*

The reason why the top of the Laffer curve under recession is located strictly to the left of $\bar{b}$ is that the reform effort is decreasing in debt (i.e., $\Psi' < 0$) for b close to $\bar{b}$, as established in Proposition 6. This implies that for b close to but smaller than $\bar{b}$, bond revenue is strictly decreasing in b. By reducing the newly-issued debt, the borrower increases the subsequent reform effort, which in turn increases the current bond price and debt revenue.

3.4.3 Contracting on effort

In equilibrium, there is no contracting on effort, even though this is not ruled out at the outset. The reason is that in a Markov equilibrium the market has no commitment power to dispense retrospective punishment. To see why, consider the possibility for a syndicate of creditors to write a contract specifying a reform effort. In the spirit of the limited commitment approach of our paper, assume that the maximum feasible punishment is to treat any deviation from the agreed effort level as equivalent to a default in the bond market. Namely, a sovereign who deviates from the agreed reform effort would be forced to default on the outstanding debt and pay the stochastic default cost ϕ. For simplicity, we assume that a deviation at t triggers punishment in period $t+1$ when debt is defaulted (although this timing assumption is not essential).

This threat could discipline, ex-ante, the sovereign's effort. However, it would not be optimal for the syndicate of creditors to carry out the punishment ex-post, since this would induce a loss for creditors (the country would not repay its debt). More generally, once the sovereign has failed to deliver the efficient effort level, it is never time-consistent to punish her. Therefore, the lack of commitment embedded in the Markov equilibrium implies that effort is not contractible.

3.4.4 Taking stock

The previous sections have established the main properties of the competitive equilibrium. The first property is that moral hazard induces an inefficient provision of reform effort in equilibrium, especially for high debt levels. Figure 2 shows the effort function $\Psi(b)$ in a calibrated economy. Note that the reform effort plunges for high debt levels.[13] The hump-shaped effort function contrasts sharply with the optimum effort in Proposition 1. In the first best, reform effort is monotone increasing in the initial debt level, and remains constant over time. The second property is that the possibility of renegotiating debt may improve risk sharing. This is *per se* welfare-enhancing but it exacerbates the moral hazard in reform effort.

The third property is that in periods when debt is fully honored, the equilibrium features positive debt accumulation if the economy remains in recession, and constant debt when the economy returns to normal times. An implication of the first and third property is that, as the recession persists, the reform effort initially increases, but then, for high debt levels, it declines over time. Figure 3 illustrates a time path for debt and consumption (left panel) and of the corresponding reform effort (right panel) for a particular simulated sequence of ϕ's. The volatility in consumption and effort contrast sharply with the optimal allocation of Proposition 1 where consumption and reform effort are constant over time.

The fourth property concerns post-renegotiation debt dynamics. Debt accumulation resumes immediately after the haircut, while consumption increases upon debt relief and starts falling again thereafter. This prediction is broadly consistent with the empirical evidence that economic conditions of debtors improve following a debt relief, as documented in Reinhart and Trebesch (2016). It is also consistent with the recent debt dynamics of Greece – after the 2011 debt relief, the debt-GDP ratio fell from 171% to 157%, but subsequently it increased back to 177%. Interestingly, the theory predicts that for highly indebted countries a large haircut may enhance the reform effort, contrary to the common view that pardoning debt would have perverse effects on incentives.

[13] This prediction is consistent with the casual observation that in the recent European debt crisis structural reforms have met stronger opposition in highly indebted countries. Countries with moderate initial debt levels, such as for instance Spain, have arguably been more prone to enact structural reforms than has Greece.

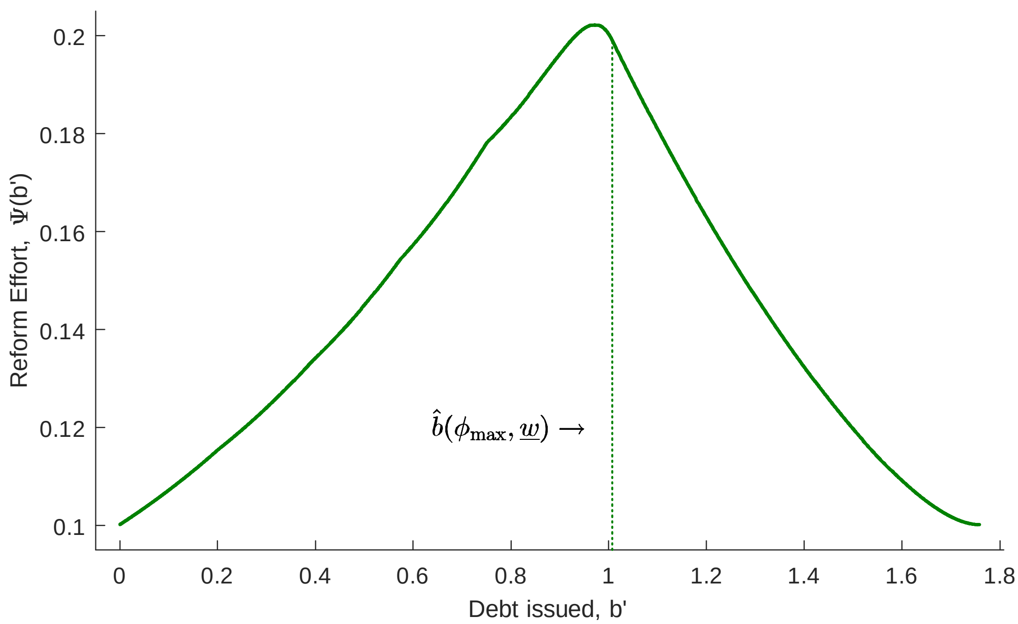

Figure 2: Reform effort function $\Psi(b)$ resulting from the benchmark calibration in Section 5.2.

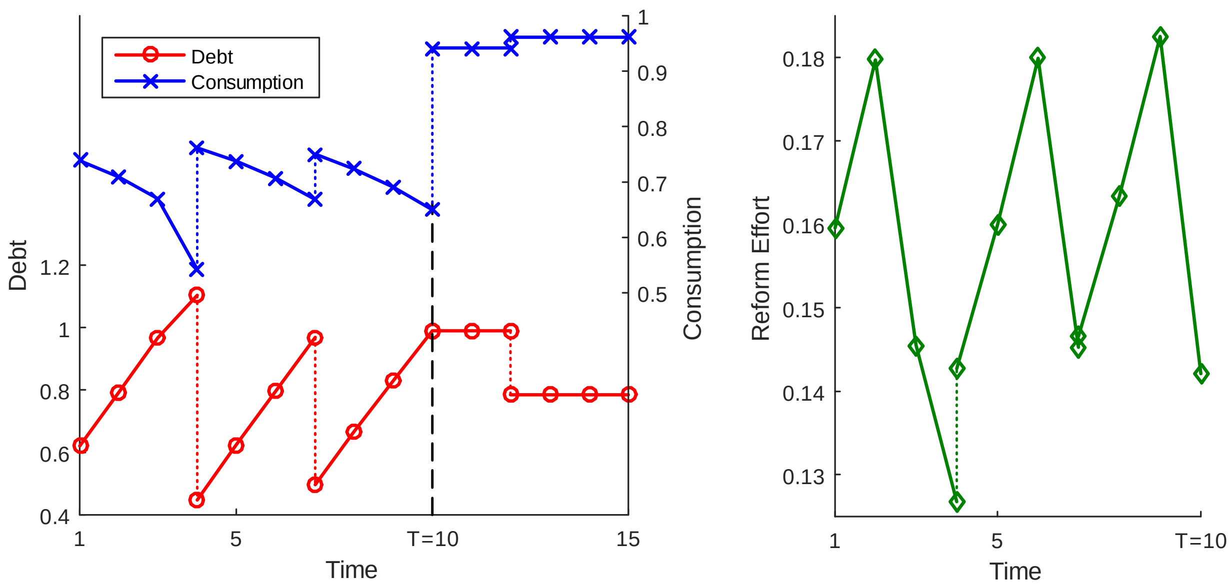

Figure 3: Simulation of debt, consumption and effort for a particular sequence of ϕ's in the competitive equilibrium. In this particular simulation the recession ends at time $T = 10$.

For simplicity we have assumed that the sovereign can only issue one-period debt. Issuing debt at multiple maturities could in principle allow the borrower to obtain some additional insurance. In a world without moral hazard, this could complete the markets (cf. Angeletos 2002, Dovis 2016). However, as we show in Section 3.5 below, in our model even an economy with GDP-linked debt would fail to overcome the moral hazard problem associated with structural reforms. This mitigates the concern about the loss of generality associated with the assumption that there is only one-period debt. Moreover, we conjecture that if the borrower could issue debt at multiple maturities, it would only issue one-period debt in steady state in order to limit the moral hazard problem.[14]

Finally, our focus on Markov equilibrium yields the extreme implication that renegotiations do not affect the terms at which the country can borrow in future. In particular, conditional on the debt level, the risk premium is independent of the country's credit history. This implies that renegotiations entail no cost for the sovereign. In Appendix B (Section B.3), we present a simple extension where sovereigns can be of different types, and the frequency of renegotiations induces learning thereby affecting bond prices. In this extension, renegotiations are less benign as they ruin the borrower's reputation.

3.5 Competitive equilibrium with GDP-linked debt

The analysis of the competitive equilibrium was carried out thus far under the assumption that the sovereign can issue only a non-contingent asset. In this section, we extend the analysis and allow for GDP-linked debt. We continue to focus on Markov equilibria.

Let $b_{\underline{w}}$ and $b_{\bar{w}}$ denote two securities paying one unit of output if the economy is in a recession or in normal times, respectively. We label these securities *recession-contingent debt* and *recovery-contingent debt*, respectively, and denote by $Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ and $Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ their corresponding prices. The budget constraint in a recession is given by:

$$Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} = \mathbb{B}\left(b, \phi, \underline{w}\right) + c - \underline{w}. \tag{17}$$

Under limited commitment, the price of each security depends on the two outstanding debt levels, as both affect the reform effort and the probability of renegotiation.[15] The sovereign's value function can be written as:

$$\begin{aligned} V\left(b, \phi, \underline{w}\right) \quad = \quad & \max_{\left\{b'_{\underline{w}}, b'_{\bar{w}}\right\} \in \left([\underline{b}, \tilde{b}] \times [\underline{b}, \tilde{b}]\right)} \left\{ u\left[Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \mathbb{B}\left(b, \phi, \underline{w}\right)\right] \right. \qquad (18) \\ & \left. - X\left(\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) + \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right] EV\left(b'_{\underline{w}}, \underline{w}\right) + \beta \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) EV\left(b'_{\bar{w}}, \bar{w}\right) \right\}. \end{aligned}$$

Mirroring the analysis in the case of non-state-contingent debt, we proceed in two steps. First, we characterize the optimal reform effort. This is determined by the difference between the discounted utility conditional on the recession ending and continuing, respectively (cf. Equation (14)):

$$X'\left(\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) = \beta\left[\int_0^\infty V\left(b'_{\bar{w}}, \phi', \bar{w}\right) dF\left(\phi\right) - \int_0^\infty V\left(b'_{\underline{w}}, \phi', \underline{w}\right) \, dF\left(\phi\right)\right]. \tag{19}$$

Note that the incentive to reform would vanish under full insurance.

[14] Aguiar and Amador (2013) reach a similar conclusion in a different model. From an empirical standpoint, Broner, Lorenzoni, and Schmukler (2013) document that in emerging markets governments issue mostly short term debt.

[15] Note that these assets are not Arrow-Debreu assets since their payoffs are not conditional on the realization of ϕ. An alternative approach would have been to follow Alvarez and Jermann (2000) and issue an Arrow-Debreu asset for each state (w, ϕ) and let the default-driven participation constraint serve as an endogenous borrowing constraint.

Next, we characterize consumption and debt issuance. To this aim, consider first the equilibrium asset prices. The prices of the recession- and recovery-contingent debt are given by, respectively:

$$Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \frac{1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)+\frac{1}{b'_{\underline{w}}}\int_0^{\underline{\Phi}\left(b'_{\underline{w}}\right)}\left(\underline{\Phi}^{-1}\left(\phi\right)\, dF\left(\phi\right)\right)\right), \tag{20}$$

$$Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \frac{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)+\frac{1}{b'_{\bar{w}}}\int_0^{\bar{\Phi}\left(b'_{\bar{w}}\right)}\left(\bar{\Phi}^{-1}\left(\phi\right)\, dF\left(\phi\right)\right)\right). \tag{21}$$

The next proposition characterizes the CEE with GDP-linked debt.

Proposition 8 *Assume that there exist markets for two securities delivering one unit of output if the economy is in recession and in normal times, respectively, and subject to the risk of renegotiation. Suppose that the economy is initially in recession. The following CEEs are satisfied in the competitive equilibrium:*
(I) If the recession continues,

$$\underbrace{\frac{u'\left(c'|_{H,\underline{w}}\right)}{u'\left(c\right)}}_{MRS\ if\ rec.\ continues} = 1 + \underbrace{\frac{\partial}{\partial b'_{\underline{w}}}\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}_{>0} \times \underbrace{\frac{R \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)\left(1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)}}_{>0}. \tag{22}$$

(II) If the recession ends,

$$\underbrace{\frac{u'\left(c'|_{H,\bar{w}}\right)}{u'\left(c\right)}}_{MRS\ if\ rec.\ ends} = 1 + \underbrace{\frac{\partial}{\partial b'_{\bar{w}}}\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}_{<0} \times \underbrace{\frac{R \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}}_{>0}, \tag{23}$$

where

$$\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \equiv \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} - \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\times b'_{\underline{w}}}{1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} \geq 0. \tag{24}$$

Moreover,

$$\begin{aligned}
c &= Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\times b'_{\bar{w}} + \underline{w} - \mathbb{B}\left(b, \phi, \underline{w}\right), \\
c'|_{H,\underline{w}} &= Q_{\underline{w}}\left(B_{\bar{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right)\times B_{\underline{w}}\left(b'_{\underline{w}}\right) + Q_{\bar{w}}\left(B_{\underline{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right)\times B_{\bar{w}}\left(b'_{\underline{w}}\right) + \underline{w} - b'_{\underline{w}}, \\
c'|_{H,\bar{w}} &= Q\left(B\left(b'_{\underline{w}}, \bar{w}\right), \bar{w}\right)\times B\left(b'_{\underline{w}}, \bar{w}\right) + \bar{w} - b'_{\bar{w}},
\end{aligned}$$

where $B_{\underline{w}}\left(b'_{\underline{w}}\right)$ *and* $B_{\bar{w}}\left(b'_{\underline{w}}\right)$ *denote the optimal level of newly-issued recession- and recovery-contingent debt when the recession continues, and debt is honored.*

If the probability that the recession ends were exogenous, $\Psi' = 0$, consumption would be independent of the realization of the aggregate state. In this case, the CEEs imply constant consumption $c'|_{H,\underline{w}} = c'|_{H,\bar{w}} = c$ where, recall, $c'|_{H,w}$ is consumption conditional on debt being honored in the next period. However, in the general case with moral hazard, consumption falls (and recession-contingent debt increases) whenever the economy remains in recession and debt is honored, as shown by Equation (22). When the recession ends, consumption increases as shown by Equation (23). Therefore, the competitive equilibrium features imperfect insurance, even conditional on honoring the debt.

The intuition is as follows. By issuing more recession-contingent debt, the country strengthens its incentive to make reforms, since $\partial\Psi/\partial b'_{\underline{w}} > 0$. This induces the sovereign to issue more recession-contingent debt than in the absence of moral hazard. This effect is stronger the larger is $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ which can be interpreted as the net expected gain accruing to the lenders from a marginal increase in the probability that the recession ends. On the contrary, issuing more recovery-contingent debt weakens the incentives to do reform. As a result, consumption increases if the recession ends and falls if the recession continues (and debt is honored). This result highlights the trade-off between insurance and incentives: the country must give up insurance in order to gain credibility about its willingness to do reforms. In addition, debt influences the reform effort: this is increasing in the newly-issued recession-contingent debt and decreasing in the newly-issued recovery-contingent debt. In summary, the moral hazard problem in reform limits the possibility for the equilibrium with GDP-linked debt to smooth consumption and reform effort.

4 Optimal contract with one-sided commitment

In the competitive equilibrium of the previous section, the sovereign cannot commit to the efficient reform effort and creditors cannot commit to punishments that are ex-post suboptimal. In this section, we characterize the allocation chosen by a benevolent social planner who can commit to enforce a contract even by dispensing punishments that are not ex-post optimal. However, the borrower continues to be subject to limited commitment. This limits the planner's ability to punish deviations from the optimal contract. In particular, the maximum punishment the planner can impose is to terminate the contract and let the sovereign resort to the competitive equilibrium. In the next section, we interpret this allocation as the result of an assistance program managed by an international agency (e.g., the IMF) that can commit to terminate its program in case of non-compliance. We consider two scenarios. In the first, the reform effort is observable, while in the second it is not. We continue to assume, as in the competitive equilibrium, that the realization of ϕ is publicly observable.

The problem is formulated as a one-sided commitment program, following Ljungqvist and Sargent (2012) and based on a promised-utility approach in the vein of Spear and Srivastava (1987), Thomas and Worrall (1988 and 1990) and Kocherlakota (1996).

We denote by ν the utility promised to the risk-averse agent in the beginning of the period, before the realization of ϕ. ν is the key state variable of the problem. We denote by $\bar{\omega}_{\phi}$ and $\underline{\omega}_{\phi}$ the promised continuation utilities conditional on the realization ϕ and on the aggregate state $\bar{w}$ and $\underline{w}$, respectively. $\underline{P}(\nu)$ and $\bar{P}(\nu)$ denote the expected present value of profits accruing to the principal conditional on delivering the promised utility ν in the most cost-effective way in recession and in normal times, respectively. The planning problem is evaluated after the uncertainty about the aggregate state has been resolved (i.e., the economy is either in recession or in normal times in the current period), but before the realization of ϕ is known. In Appendix B (Proposition 16), we prove, following the strategy in Thomas and Worrall (1990), that the functional equations defined in Equations (25) and (30) below are contraction mappings, that the profit functions $\underline{P}(\nu)$ and $\bar{P}(\nu)$ are decreasing, strictly concave and continuously differentiable, and that the associated maximands are unique.

4.1 Normal times

In normal times, the optimal value $\bar{P}(\nu)$ satisfies the following functional equation:

$$\bar{P}(\nu) = \max_{\{c_\phi, \bar{\omega}_\phi\}_{\phi \in \aleph}} \int_{\aleph} \left[\bar{w} - \bar{c}_\phi + \frac{1}{R} \bar{P}(\bar{\omega}_\phi) \right] dF(\phi), \tag{25}$$

where the maximization is subject to the constraints

$$\begin{aligned} \int_{\aleph} [u(\bar{c}_\phi) + \beta \bar{\omega}_\phi] \, dF(\phi) &\geq \nu, & (26) \\ u(\bar{c}_\phi) + \beta \bar{\omega}_\phi &\geq W(0, \bar{w}) - \phi, \quad \phi \in \aleph, & (27) \\ \bar{c}_\phi \in [0, \bar{w}], \ \nu, \bar{\omega}_\phi &\in [W(0, \bar{w}) - E[\phi], W(0, \bar{w})]. \end{aligned}$$

The inequality (26) is a promise-keeping constraint, whereas (27) is a participation constraint (PC). Note that the outside option for the agent is equivalent to the value of default in the competitive equilibrium.

The application of recursive methods allows us to establish the following proposition.

Proposition 9 *Assume the economy is in normal times. (I) For all realizations ϕ such that the PC of the agent, (27), is binding, $\bar{\omega}_\phi > \nu$ and the solution for $(\bar{c}_\phi, \bar{\omega}_\phi)$ is determined by the following conditions:*

$$u'(\bar{c}_\phi) = -\frac{1}{\bar{P}'(\bar{\omega}_\phi)}, \tag{28}$$

$$u(\bar{c}_\phi) + \beta \bar{\omega}_\phi = W(0, \bar{w}) - \phi. \tag{29}$$

The solution is not history-dependent, i.e., the initial promise, ν, does not matter. (II) For all realizations ϕ such that the PC of the agent, (27), is not binding, $\bar{\omega}_\phi = \nu$ and $\bar{c}_\phi = \bar{c}(\nu)$, where $\bar{c}(\nu)$ is determined by (28). The solution is history-dependent.

The constrained optimal allocation (COA) has standard properties. Whenever the agent's PC is not binding, consumption and promised utility remain constant over time. Whenever the PC binds, the planner increases the agent's consumption and promised utility in order to meet her PC.

4.2 Recession

When the economy is in recession, the contract specifies also an effort level. We consider first the case in which the reform effort is observable. In this case, the planner terminates the contract whenever the agent deviates from the efficient effort level. We assume that when the agent deviates at time t, she is settled with the default cost at $t+1$. Note that this allocation is equivalent to a decentralized equilibrium in which the sovereign can issue debt contingent both on the aggregate level and on the borrower's effort.

The reform effort associated with a deviation is given by

$$\begin{aligned} p_{dev} &= \arg \max_{p \in [\underline{p}, \bar{p}]} -X(p) + \beta \left((1-p) W(0, \underline{w}) + pW(0, \bar{w}) \right) \\ &\rightarrow \quad X'(p_{dev}) = \beta \left(W(0, \bar{w}) - W(0, \underline{w}) \right) \end{aligned}$$

Then, the continuation utility from a deviation is given by

$$\zeta_{dev} \equiv -X(p_{dev}) + \beta\left((1-p_{dev})W(0,\underline{w}) + p_{dev}W(0,\bar{w}) - E\left[\phi'\right]\right),$$

where $E\left[\phi'\right]$ denotes the expected value of ϕ'.

We can now characterize the optimal contract under recession

$$\underline{P}(\nu) = \max_{\left\{\underline{c}_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\right\}_{\phi\in\aleph}} \int_\aleph \left[\underline{w} - \underline{c}_\phi + \frac{1}{R}\left((1-p_\phi)\underline{P}\left(\underline{\omega}_\phi\right) + p_\phi\bar{P}(\bar{\omega}_\phi)\right)\right] dF(\phi), \tag{30}$$

where the maximization is subject to the constraints

$$\begin{aligned}
\int_\aleph \left(u\left(\underline{c}_\phi\right) - X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right)\right) dF(\phi) &\geq \nu, &(31)\\
u\left(\underline{c}_\phi\right) - X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right) &\geq W(0,\underline{w}) - \phi, \quad \phi\in\aleph, &(32)\\
-X(p_\phi) + \beta\left((1-p_\phi)\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right) &\geq \zeta_{dev} &(33)
\end{aligned}$$

$\underline{c}_\phi \in [0,\bar{w}]$, $p_\phi \in [\underline{p},\bar{p}]$, $\nu, \underline{\omega}_\phi \in [W(0,\underline{w}) - E[\phi], W(0,\bar{w})]$, $\bar{\omega}_\phi \in [W(0,\bar{w}) - E[\phi], W(0,\bar{w})]$.

We prove in Appendix B (Proposition 16) that the program is concave, and that the FOCs are necessary and sufficient. The FOCs with respect to $\bar{\omega}_\phi$, $\underline{\omega}_\phi$, and p_ϕ (see Equations (63)–(66) in Appendix A) yield:[16]

$$\begin{aligned}
\bar{P}'(\bar{\omega}_\phi) &= \underline{P}'\left(\underline{\omega}_\phi\right) &(34)\\
X'(p_\phi) &= \beta\left(\bar{\omega}_\phi - \underline{\omega}_\phi\right) - \frac{R^{-1}}{\underline{P}'\left(\underline{\omega}_\phi\right)}\left(\bar{P}(\bar{\omega}_\phi) - \underline{P}\left(\underline{\omega}_\phi\right)\right). &(35)
\end{aligned}$$

Equation (34) establishes that the planner equates the marginal profit loss associated with promised utilities in the two aggregate states. (35) establishes that effort is set at the constrained efficient level. The two terms on the right hand-side are the benefits accruing to the agent and to the principal, respectively. Note that in the competitive equilibrium the sovereign only takes into consideration the private gain of exerting effort, so the second term is missing.

The IC constraint may or may not be binding. When it is binding, (34), (35) and the IC constraint pin down a unique (constant) level of promised utilities and effort. We state this formally in the following Lemma.

Lemma 5 *When the IC is binding, effort and promised utilities are constant at the levels $\bar{\omega}_\phi = \bar{\omega}^*$, $\underline{\omega}_\phi = \underline{\omega}^*$ and $p_\phi = p^*$, where the triplet $(p^*, \underline{\omega}^*, \bar{\omega}^*)$ is uniquely determined by the IC constraint (33) holding with equality, (34), and (35).*

When the IC constraint is not binding, consumption is pinned down by the following standard FOC (see Equations (63) and (64) in Appendix A):

$$u'\left(\underline{c}_\phi\right) = -\frac{1}{\underline{P}'\left(\underline{\omega}_\phi\right)}. \tag{36}$$

[16] To see why the solution to (34)–(35) is unique, note that the strict concavity and monotonicity of $\underline{P}$ and $\bar{P}$ imply that Equation (34) determines a strictly positive relationship between $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$. Thus, Equation (35) yields an implicit strictly decreasing relationship between p_ϕ and $\underline{\omega}_\phi$ and an implicit strictly increasing relationship between p_ϕ and $\bar{\omega}_\phi$.

This condition does not hold when the IC constraint is binding, since the planner must in this case distort the consumption margin.

We characterize the optimal contract by distinguishing two cases. Proposition 10 covers the case in which the initial promised utility is high and the IC constraint is not binding irrespective of the realization of ϕ. Proposition 11 covers the case in which the initial promised utility is low and the IC constraint is binding for a non-empty range of realizations of ϕ.

It is useful to define $\tilde{\phi}(\nu)$ as the threshold realization of ϕ such that the participation constraint is binding for a given ν. In particular, $\tilde{\phi}(\nu)$ is implicitly defined by the promise-keeping constraint (31),

$$\nu = W\left(0, \underline{w}\right) - \left[\int_0^{\tilde{\phi}(v)} \phi dF(\phi) + \tilde{\phi}(\nu)\left[1 - F(\tilde{\phi}(v))\right]\right], \tag{37}$$

where $\tilde{\phi}(\nu)$ is decreasing in ν.

Proposition 10 *Suppose that the economy starts in a recession with promised utility $\nu \geq \underline{\omega}^*$. Then, the IC constraint (33) is never binding (irrespective of ϕ), and the optimal contract is characterized as follows:*

1. *If $\phi < \tilde{\phi}(\nu)$, the PC is binding, and the solution for $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ is determined by (34), (35), (36) and by (32) holding with equality. Moreover, $\underline{\omega}_\phi > \nu$.*

2. *If $\phi \geq \tilde{\phi}(\nu)$, the PC is slack, and the solution for $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ is given by $\underline{\omega}_\phi = \nu$, $c_\phi = \underline{c}\left(\nu\right)$, $\bar{\omega}_\phi = \bar{\omega}\left(\nu\right)$, and $p_\phi = p\left(\nu\right)$, where the functions $\underline{c}\left(\nu\right)$, $\bar{\omega}\left(\nu\right)$ and $p\left(\nu\right)$ are determined by (36), (34), (35), respectively. The solution is history-dependent. The reform effort is decreasing and consumption and future promised utility are increasing in ν.*

When $\nu < \underline{\omega}^*$, the solution of Proposition 10 would violate the IC constraint in some states. Thus, the planner must either reduce the demands on effort or increase the promise utilities so that the IC constraint holds. The following proposition characterizes the optimal contract in this case.

Proposition 11 *Suppose that the economy starts in a recession with promised utility $\nu < \underline{\omega}^*$. Then, the IC constraint (33) is binding in some states, and the optimal contract is characterized as follows:*

1. *If $\phi < \tilde{\phi}(\underline{\omega}^*)$, the PC is binding while the IC is not binding. The solution is not history-dependent and is determined as in Proposition 10, part 1 (in particular, $\underline{\omega}_\phi > \underline{\omega}^*$ and $p_\phi < p^*$).*

2. *If $\phi \in [\tilde{\phi}(\underline{\omega}^*), \tilde{\phi}(\nu)]$, both the PC and the IC are binding. Effort and promised utilities are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$ as given by Lemma 5. Consumption is determined by (32) and (33) jointly, which yield:*

$$c_\phi^* = u^{-1}\left(W\left(0, \underline{w}\right) - \phi - \zeta_{dev}\right). \tag{38}$$

Consumption and effort are lower and promised utilities are higher than in the absence of an IC constraint.

3. *If $\phi > \tilde{\phi}(\nu)$, the IC is binding, while the PC is not binding. Effort and promised utilities are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$. Consumption is constant across ϕ and is determined by (31) and (33), which yield:*

$$c_\phi^* = u^{-1}\left(W\left(0, \underline{w}\right) - \tilde{\phi}(\nu) - \zeta_{dev}\right). \tag{39}$$

Consumption and effort are lower and promised utilities are higher than in the absence of an IC constraint.

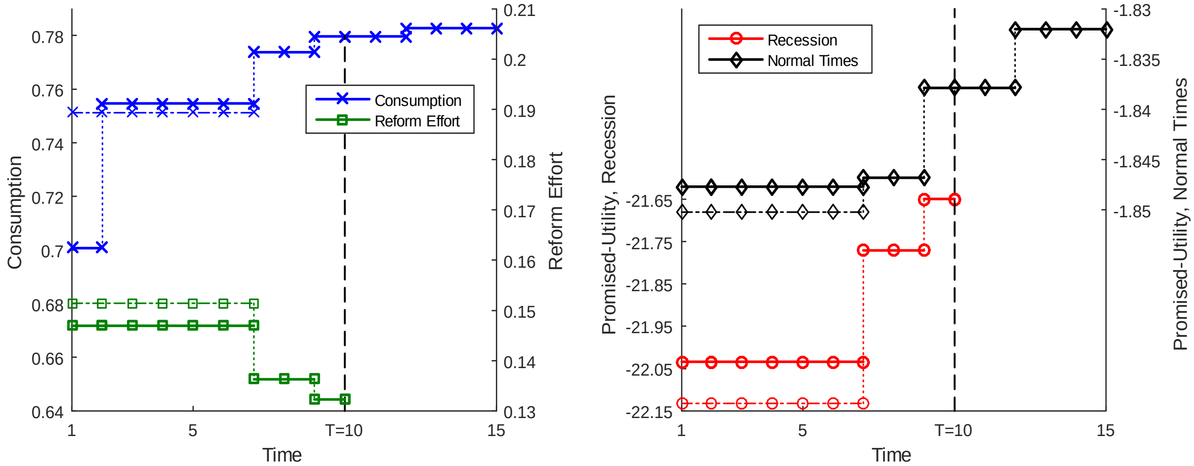

Figure 4: Simulation of consumption, effort, and promised utilities for a particular sequence of ϕ's where the IC is initially binding. Solid lines refer to the planner solution with the IC constraint. Dashed lines refer to the planner solution without the IC constraint. In this particular simulation the recession ends at time $T = 10$.

Consider an economy where, initially, $\nu < \underline{\omega}^*$. If $\phi < \tilde{\phi}(\underline{\omega}^*)$ (case 1), the binding PC induces the planner to set an effort level so low that the IC is not binding. The allocation is not history-dependent, and the characterization of Proposition 10 applies. For all levels of ϕ larger than $\tilde{\phi}(\underline{\omega}^*)$, the IC is binding, and Lemma 5 implies that effort and promised utility are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$, i.e. the maximum effort and the minimum promised future utilities consistent with the IC. In particular, if $\phi \in [\tilde{\phi}(\underline{\omega}^*), \tilde{\phi}(\nu)]$ (case 2), consumption is pinned down jointly by the PC and IC. In this case, consumption is decreasing in ϕ. Finally, if $\phi > \tilde{\phi}(\nu)$ (case 3) the PC is slack, and consumption is constant across ϕ and determined by the promise-keeping constraint. Note that whenever the IC constraint is binding (cases 2 and 3), both consumption and effort are lower than in Proposition 10. Intuitively, the planner satisfies the IC and promise-keeping constraints by reducing current consumption and effort, and by increasing promised utilities relative to the case in which the IC constraint is not binding. Thus, the contract provides less consumption insurance but more effort smoothing.

Note that the IC constraint binds for at most one period. After that either the recession ends, or the planner sets the promised utility to the level $\underline{\omega}^*$. Either way, the IC constraint becomes irrelevant, and the equilibrium is characterized as in Proposition 10.

Figure 4 represents an economy in which the IC is binding in the initial period, i.e., $\nu < \underline{\omega}^*$. It shows simulated paths of consumption, effort and promised utility. For comparison, the figure also displays (dashed lines) the allocation in an otherwise identical economy where the planner can control the effort without the IC constraint. In the first period, consumption and effort are lower in the economy with an IC constraint. In contrast, promised utility is higher. In other words, the IC constraint forces the planner to provide less insurance by making consumption and effort initially lower, but growing at a higher speed. As of the second period, the dynamics of both economies are the same.

Note the sharp contrast of these dynamics relative to the competitive equilibrium of Section 3.4. There, consumption is falling (and debt accumulates) when the country honors its debt. In contrast, in the COA the planner insures the agent's consumption by keeping it constant whenever the PC is not binding. Therefore, the competitive equilibrium underprovides insurance. The dynamics of the reform effort also are sharply different. In the COA, effort is a monotone decreasing function of promised utility which is in turn step-wise increasing over time (cf. Figure 4). In contrast, in the competitive equilibrium the reform effort is hump-shaped in debt. Since debt increases over time (unless it is renegotiated), effort is also hump-shaped over time conditional on no renegotiation.

4.3 Unobservable reform effort

When the reform effort is not observable, deviations in effort cannot be sanctioned. Thus, ζ_{dev} is replaced by $\tilde{\zeta}_\phi \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right) = -X\left(p_\phi\right) + \beta\left(p_\phi \bar{\omega}_\phi + (1-p_\phi)\underline{\omega}_\phi\right)$, where

$$\begin{aligned} p_\phi &= \arg\max_p -X(p) + \beta\left(p\bar{\omega}_\phi + (1-p)\underline{\omega}_\phi\right) \\ &\rightarrow X'(p_\phi) = \beta\left(\bar{\omega}_\phi - \underline{\omega}_\phi\right) \\ &\rightarrow p_\phi = \Upsilon\left(\bar{\omega}_\phi - \underline{\omega}_\phi\right) \end{aligned} \tag{40}$$

Moreover, the IC constraint always holds. Note that effort is now inefficiently provided, since the agent does not internalize the benefit of effort provision that accrues to the planner.

The FOCs with respect to $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$, together with the envelope condition, yield (see proof of Proposition 12 in Appendix A):

$$\underline{P}'\left(\underline{\omega}_\phi\right) - \bar{P}'\left(\bar{\omega}_\phi\right) = \left(\frac{\Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)}{\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)} + \frac{\Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)}{1-\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)}\right)\left[\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right] \tag{41}$$

$$\frac{1}{u'(\underline{c}_\phi)} = -\left[\underline{P}'\left(\underline{\omega}_\phi\right) - \frac{\Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)}{1-\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)}\left[\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right]\right]. \tag{42}$$

The FOC (41) is the analogue of (34). Note that the planner does no longer equalize the marginal cost of promised utility in the two states. The reason is that increasing the difference in promised utility is the only way for the planner to increase effort provision. Thus, the unobservability of effort reduces insurance.

We can now establish the following proposition.

Proposition 12 *Suppose that the economy starts in a recession and effort is not observable. Then, the optimal contract is characterized as follows: (i) $p_\phi = \Upsilon\left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)$ as in (40), and (ii):*

1. *If $\phi < \tilde{\phi}(\nu)$, the PC is binding, and the solution for $\left(c_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ is determined by (41), (42), and by (32) holding with equality.*
2. *If $\phi \geq \tilde{\phi}(\nu)$, the PC is slack, and the solution for $\left(c_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ is determined by (41), (42), and by the FOC for consumption*
$$u'\left(\underline{c}_\phi\right) = -\frac{1}{\underline{P}'(\nu)}.$$
The solution is history-dependent (i.e., $\underline{c}_\phi = \underline{c}(\nu)$, $\underline{\omega}_\phi = \underline{\omega}(\nu) < \nu$, and $\bar{\omega}_\phi = \bar{\omega}(\nu)$). As long as the recession persists and the PC remains slack, consumption and promised utilities are falling and effort is increasing over time.

The results when effort is not unobservable differ sharply from the case in which effort is observable and the planner has commitment. In this case, the dynamics are more similar to those of the competitive equilibrium without commitment. Consumption falls over time whenever ϕ is sufficiently high. This is the way the planner gives dynamic incentives: she curtails insurance in order to extract higher effort over time.

4.4 Comparison between the COA and the competitive equilibrium

The first result is that in normal times the planning allocation in Proposition 9 is identical to the competitive equilibrium. To establish the equivalence result we return, first, to the competitive equilibrium. Let

$$\bar{\Pi}(b) = \left(1 - F\left(\bar{\Phi}(b)\right)\right) b + \int_0^{\bar{\Phi}(b)} \hat{b}(\phi, \bar{w})\, dF(\phi) \tag{43}$$

denote the expected value for the creditors of an outstanding debt b before the current-period uncertainty is resolved. Note that $\bar{\Pi}(b)$ yields the expected debt repayment, which is lower than the face value of debt, since in some states of nature debt is renegotiated. To prove the equivalence, we postulate that $\bar{\Pi}(b) = \bar{P}(\nu)$, and show that in this case $\nu = EV(b, \bar{w})$.[17] If the planning allocation were more efficient than the equilibrium, then we would find that $\nu > EV(b, \bar{w})$.

Proposition 13 *Assume that the economy is in normal times. The competitive equilibrium is equivalent to the planning allocation in Proposition 9, namely,* $\bar{\Pi}(b) = \bar{P}(\nu) \Leftrightarrow \nu = EV(b, \bar{w})$.

Intuitively, renegotiation provides the market economy with sufficiently many state contingencies to attain second-best efficiency. This result hinges on two features of the renegotiation protocol. First, renegotiation averts any real loss associated with unordered default. Second, creditors have all the bargaining power in the renegotiation game.[18] Moreover, note that in normal times there is no issue of commitment since effort is only exerted in recession.

The equivalence result of Proposition 13 hinges on the assumption that normal times is an absorbing state that will be relaxed in Section 5 below. Moreover, even in the current environment, it does not carry over to recessions. We will show that in recession the result hinges on two critical assumptions about the planning problem: whether effort is observable and whether the sovereign can issue GDP-linked debt.

It is instructive to start by analyzing with a case in which there is no moral hazard problem, i.e., the probability that the recession ends is independent of the reform effort (i.e., $\Psi = p$).

Proposition 14 *If the probability that the recession ends is independent of the reform effort (i.e.,* $\Psi = p$*), then the competitive equilibrium with GDP-linked debt is constrained efficient conditional on* p*. Namely, if effort is set at the constrained optimum level the equilibrium allocation is identical to the planning allocation of Proposition 12 where the outside option* $W(0, w)$ *in Equation (32) is the value function associated with a competitive equilibrium with GDP-linked debt.*

[17] Recall that $EV(b, \bar{w}) = \int_{\aleph} V(b, \phi, \bar{w})\, dF(\phi)$ denotes the discounted utility accruing to a country with the debt level b in the competitive Markov equilibrium.

[18] We view this as a useful benchmark. In reality, renegotiations may entail costs associated with legal proceeds and lawsuits, trade retaliation, temporary market exclusion, etc. Also, creditors may not have the full ex-post bargaining power at the renegotiation stage as in Yue (2010). This would reduce the amount of loans creditors can recover. In all these cases, the competitive equilibrium would fail to implement the COA.

The equivalence of Proposition 14 breaks down if there is moral hazard, and the market cannot commit to punish deviations in reform effort. The qualitative dynamics are also different. In the equilibrium with GDP-linked debt of Section 3.5, consumption falls (and recession-contingent debt increases) whenever the economy remains in recession and debt is honored, as shown by Equation (22). On the contrary, consumption increases whenever the recession ends, as shown by Equation (23).

For the equilibrium with GDP-linked debt to decentralize the planning allocation of Proposition 10, the sovereign should be able to commit to the efficient reform level. In particular, the planner should issue securities that are conditioned not only on GDP but also on the exerted effort level. If a market for such securities existed, the sovereign would promise a repayment that is equivalent to the optimal contract at the optimal effort level. For any other effort level, the payment would be that associated with the maximum debt level $\bar{b}$. This implies that if there is a deviation from the equilibrium effort debt would be renegotiated with certainty in the following period. This ensures that the IC constraint holds in the competitive equilibrium. Clearly, this type of state-contingent debt is a stand-in for commitment. Their existence requires reform effort to be *both* observable and verifiable in courts, which we view as a strong assumption.

Finally, the competitive equilibrium with GDP-linked debt can sustain a COA where the planner cannot observe effort. The proof of this equivalence is harder, as it is difficult to prove that the planning problem is concave in general when effort is not observable. This is a common problem in the literature (see Renner and Schmedders 2015). Therefore, the equivalence is stated under the assumption that the first-order conditions are sufficient for the planning problem. This assumption can be verified numerically, as we do in the numerical analysis below.

Proposition 15 *Assume that the economy is in recession. Consider the planning allocation with unobservable effort of Proposition 12 where the outside option $W(0, w)$ in Equation (32) is the value function associated with a competitive equilibrium with GDP-linked debt. Assume that the first-order conditions are necessary and sufficient. This planning allocation can be sustained as a competitive equilibrium with state contingent debt (cf. Proposition 3.5), namely,* $\underline{\Pi}(b) = \underline{P}(\nu) \Leftrightarrow \nu = EV(b, \underline{w})$.

The proposition establishes that if effort is not observable, then, commitment is of no value to the planner. Then, the market decentralizes the planner allocation in the vein of Prescott and Townsend (1984).[19] An immediate corollary of Proposition 15 is that the planning allocation of Proposition 12 is more efficient than the equilibrium without GDP-linked debt.

4.5 Interpreting the COA as an austerity program

In this section, we discuss a policy-relevant institutional interpretation of the COA. Consider a stand-by program run by an international institution, e.g., the IMF. Like the planner, and unlike the market, the IMF can punish deviations, but cannot get around the limited commitment problem, i.e., the indebted country can pay the default cost and walk away unilaterally. We show that the planner allocation can be interpreted as a combination of transfers (or loans), repayment schedules, reform program and renegotiation strategy. This program has two key features. First, the country cannot run an independent fiscal policy, i.e., it is not allowed to issue additional debt in the market. Second, the program is subject to renegotiation. More precisely, whenever the country credibly threatens to

[19] See also Dovis (2016) for a related result in a setting where the sovereign has private information about productivity shocks and a market with short and long-lived bonds can implement the constrained optimum.

abandon the program, the international institution should sweeten the deal by increasing the transfers, reducing the required effort, and reducing the debt the country owes when the recession ends. When no credible threat of default is on the table, consumption and reform effort should be held constant as long as the recession lasts. When the recession ends, the international institution receives a payment from the country, financed by issuing debt in the market.

Let ν denote the present discounted utility guaranteed to the country when the program is first agreed upon. Let $c^o(\nu)$ and $p^o(\nu)$ be the consumption and reform effort associated with the promised utility in the planning problem. Upon entering the program, the country receives a transfer equal to $T(\nu) + b_0$, where $T(\nu) = c^o(\nu) - \underline{w}$ (note that $T(\nu)$ could be negative). In the subsequent periods, the country is guaranteed the transfer flow $T(\nu)$ so long as the recession lasts and there is no credible request of renegotiating the terms of the austerity program. In other words, the international institution first bails out the country from its obligations to creditors, and then becomes the sole residual claimant of the country's sovereign debt. The country is also asked to exert a reform effort $p^o(\nu)$. If the country faces a low realization of ϕ and threatens to leave the program, the institution improves the terms of the program so as to match the country's outside option. Thereafter, consumption and effort are held constant at new higher and lower levels, respectively, as in the planner's allocation. And so on, for as long as the recession continues.

As soon as the recession ends, the country owes a debt b_N to the international institution, determined by the equation

$$Q(b_N, \bar{w}) \times b_N = c^o(\nu_N) - \bar{w} + b_N.$$

Here ν_N is the expected utility granted to the country after the most recent round of renegotiation. After receiving this payment, the international institution terminates the program and lets the country finance its debt in the market.

This program resembles an *austerity* program, in the sense that the country is prevented from running an independent fiscal policy and reform program. In particular, the country would like to issue extra debt after entering the stand-by agreement, so austerity is a binding constraint. In addition, the country would like to shirk on the reform effort prescribed by the agreement. Thus, the sovereign would like to (temporarily) deviate from the optimal plan, and promises about future transfers is an essential feature of the program.

A distinctive feature of the assistance program is that the international institution sets "harsh" entry conditions in anticipation of future renegotiations. How harsh these conditions are depends on ν. In turn, ν may reflect a political decision about how many (if any) own resources the international institution wishes to commit to rescuing the indebted country. A natural benchmark is to set ν such that the international institution makes zero profits (and zero losses) in expectation. Whether, ex-post, the international institution makes net gains or losses hinges on the duration of the recession and on the realized sequence of ϕ's.

Another important policy implication of our analysis is that it would be suboptimal for the international institution to commit never to accept any renegotiation. On the contrary, such a policy would lead to welfare losses because, on the one hand, there would be inefficient default in equilibrium; on the other hand, the country could not expect future improvements, and therefore would not accept a very low initial consumption, or a very high reform effort. If the international institution's expected profit were zero in both programs, the country would receive a lower expected utility from the alternative (no renegotiation) program.

In summary, our theory prescribes a pragmatic approach to debt renegotiation. Credible threats of default should be appeased by reducing the debt and softening the austerity program. Such approach is often criticized for creating bad incentives. In our model, such appeasement is precisely the optimal policy under the reasonable assumption that penalties on sovereign countries for breaking an agreement are limited.

5 Recurrent recessions and quantitative analysis

In this section, we generalize the model and study its quantitative properties from a positive and normative standpoint.

5.1 Recurrent recessions

In order to align the model with the data, we relax the assumption that there exists an absorbing state and assume, instead, that in normal times the economy falls into a deep recession with an exogenous probability $\hat{p}$. Additionally, we relax the assumption that $\beta R = 1$. In particular, we emphasize the case in which $\beta R < 1$ since this ensures that the competitive equilibrium has a non-degenerate stationary distribution (cf. Aiyagari 1994).

While most properties discussed in the previous section carry over to this generalization, the economy will feature some qualitative differences relative to the analysis in Sections 3-4 above. First, in normal times, the sovereign engages in precautionary savings to accumulate a buffer in expectation of future recessions. Therefore, consumption and wealth are not constant during normal times even when debt is honored.[20] The qualitative debt dynamics in the stationary equilibrium have the following features. In normal times, debt (when honored) tends to a target level. During recession, when honored, debt increases unambiguously with dynamics qualitatively similar to those of Section 3.4.

Assuming $\beta R < 1$ also affects the planning allocation. The first-best now features ever-decreasing consumption and increasing effort when the economy is in recession. The planning allocation with one-sided commitment and observable effort of Section 4.2 is affected in a similar fashion: when neither the participation nor the IC constraint are binding, the allocation yields rising effort and declining consumption and promised utility. Consequently, the IC constraint may bind recurrently: the IC constraint sets a floor to the continuation utility below which the agent would choose to exit the contract and resort to market financing.

5.2 Calibration

We calibrate the model economy to match salient moments of observed debt-to-GDP ratios and default premia for Greece, Ireland, Italy, Portugal, and Spain (GIIPS). A model period corresponds to one year. We set $\hat{p} = 0.01$. This low probability is intended to capture rare and severe downturns ignoring standard fluctuations on a business cycle frequency. We normalize the GDP during normal times to $\bar{w} = 1$ and assume that the recession causes a drop in income of 40%, i.e., $\underline{w} = 0.6 \times \bar{w}$. This corresponds to the fall of GDP per capita for Greece between 2007 and 2013, relative to trend.[21] Since we focus on the return on sovereign debt, the annual real gross interest rate is set to $R = 1.02$. The utility function is assumed to be CRRA with a relative risk aversion of 2.

[20] Moreover, as anticipated above, Proposition 13 is no longer true.

[21] GDP per capita of Greece fell from 22'700 to 16'800 Euro between 2007 and 2013 (Eurostat, nama_10_pc series). The annualized growth rate between 1997 and 2007 was 3.6%. The fall in output between 2007 and 2013 relative to trend was therefore 40%.

We calibrate the discount factor β to target a stationary average debt 54.9% of GDP in line with the evidence for the GIIPS over the period 1950-2015.[22] We assume an isoelastic effort cost function, $X(p) = \frac{\xi}{1+1/\varphi}(p)^{1+1/\varphi}$, where ξ regulates the average level of effort and φ regulates the elasticity of reform effort to changes in the return to reforms. We calibrate the two parameters, φ and ξ, so as to match two points on the equilibrium effort function $\Psi(b)$. In particular, we target the effort at the debt limit, $\Psi(\bar{b}) = 10\%$, so that a country at the debt limit would choose an effort inducing an expected duration of the recession of one decade (we have Greece in mind). Moreover, we target a maximum effort, $\max_b \Psi(b) = 20\%$, inducing an expected recession duration of five years (we have Iceland and Ireland in mind).

Finally, we calibrate the support and the distribution of the default cost ϕ so that the model matches key moments of the quantity and price of sovereign debt. One common problem in the quantitative literature on sovereign debt is that those models fail to match observed values of debt-to-GDP ratios under standard parameterization (Arellano 2008; Yue 2010). This is not a problem in our model. In fact, the maximum default cost realization $\bar{\phi}$ is calibrated to target a debt limit during normal times of $\bar{b}/\bar{w} = 178\%$ which corresponds to the maximum sustainable debt reported in Collard et al. (2015, Table 3, Column 1).[23] Moreover, the distribution $f(\phi)$ is parametrized to target an average default premium of 4.04% for a country which has a debt-output ratio of 100% in recession. This was the average debt and average default premium for the GIIPS during 2008-2012 (Eurostat). In particular, we assume that $\bar{\phi} - \phi$ is distributed exponential with rate parameter η and truncation point $\bar{\phi}$.[24] We then calibrate η to target the above default premium. Table 1 summarizes the targeted empirical moments and the resulting calibration of the parameters. The five parameters β,η, $\bar{\phi}$, φ, and ξ are calibrated simultaneously to minimize the squared distance (in percentage and with equal weights) between the empirical and the model generated moments.

5.3 Quantitative predictions

The model is solved by discretizing the state space and iterating on the value functions and the default threshold functions. The benchmark calibration uses 5000 grid points for debt and 600 for the default cost ϕ. Measured by the Euler Equation errors, the numerical approximation of the equilibrium is very accurate. See Appendix B for details on the algorithm.

Figure 5 illustrates the properties of the calibrated economy by showing a simulated path for an economy that starts in a recession with an initial debt-GDP ratio of 100% ($b = 0.6$). The dotted lines indicate the renegotiation episodes and the grey shades indicate recessions. Panel (a) shows the path for consumption and effort in the competitive equilibrium. The economy starts in a recession, then recovers at $T = 11$, then falls again into a recession at $T = 31$, and finally recovers at $T = 40$. Consumption is lower during recession, and it falls throughout both recession periods except after renegotiation. Effort follows a non-monotonic dynamic being increasing at moderate debt levels and falling in the debt overhang region. Panel (b) shows the associated debt dynamics. Note that during recessions debt accumulates rapidly and renegotiations are more likely (on average, the calibrated

[22] We use the debt-to-GDP ratios reported by Eurostat for the period 1995-2015. For earlier periods, we chain the debt levels back to 1950 with the series reported in the Reinhart and Rogoff (2010) dataset.

[23] We ignore the value of 282% for Korea which is a clear outlier.

[24] More formally, ϕ has the p.d.f.

$$f(\phi) = \frac{\eta e^{-\eta(\bar{\phi}-\phi)}}{1 - e^{-\eta\bar{\phi}}},\ \phi \in [0, \bar{\phi}].$$

Here, $f(\phi)$ is strictly increasing in ϕ, and higher values of η are associated with a larger probability mass in the upper tail of the distribution.

Target	Data	Model	Par.	Value
Average debt: (% GDP, GIIPS, 1950-2015)	54.9%	53.7%	β	0.972
Bond spread: (GIIPS, at 100% debt-output ratio, 2008-2012)	4.04%	3.99%	η	1.804
Maximum debt level: (% of normal output, Collard et al. 2015)	178%	176%	$\bar{\phi}$	2.134
Expected recession duration: (at max. reform effort, years)	5	4.95	φ	14.24
Expected recession duration: (at the debt limit $\bar{b}$, years)	10	9.99	ξ	14.55

Table 1: Model calibration

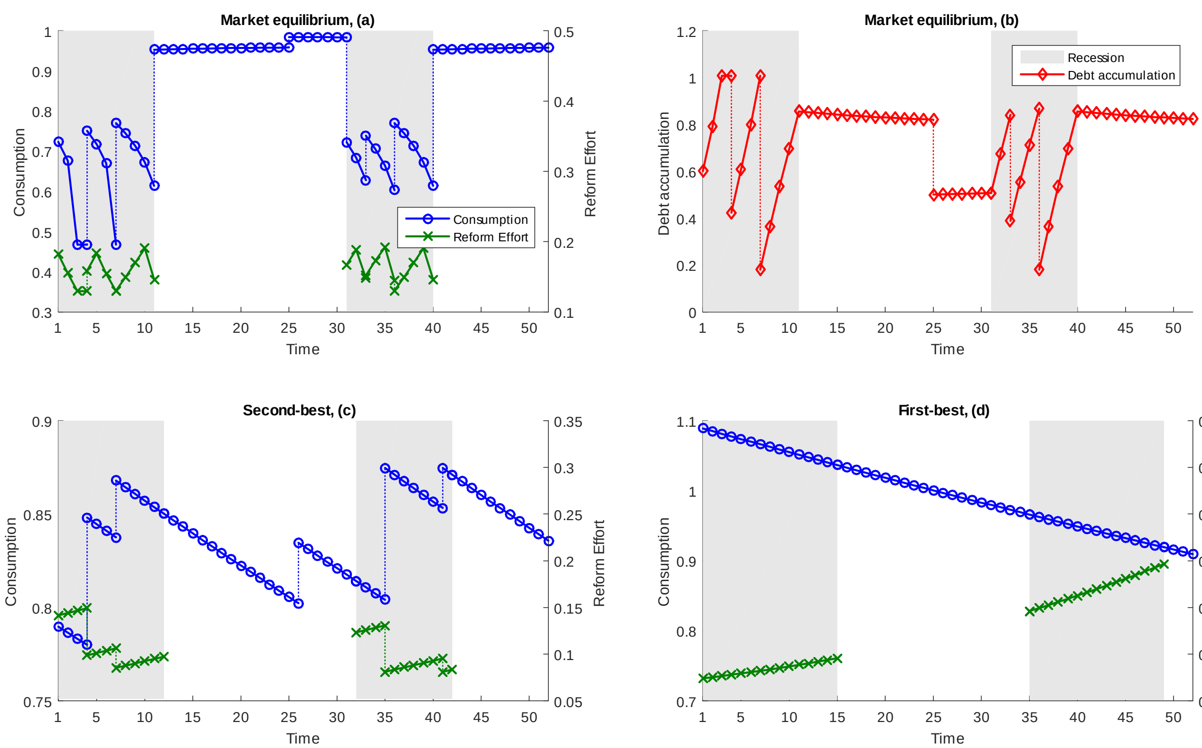

Figure 5: Simulation of competitive equilibrium, second-best, and first-best in the calibrated economy with recurrent recessions and $\beta R < 1$.

economy yields renegotiation 39% of the time in recession and 4% of the time in normal times). In normal times debt decreases or increases depending on whether the current debt level is above or below the target level.

Panel (c) shows consumption and effort when the latter is observable and there is a market for GDP-linked debt (i.e., the planning allocation of Section 4.2). Note that consumption falls by the annual factor $(\beta R)^{\frac{1}{\gamma}} \approx 0.992$ in non-renegotiation periods irrespective of the aggregate state. During recession, consumption falls less steeply than in the competitive equilibrium. Reform effort increases during recessions in non-renegotiation periods. Finally, panel (d) shows the reform effort and consumption in the first best. Here consumption is initially high and falls at the rate $(\beta R)^{\frac{1}{\gamma}}$ throughout. Effort increases over time, accordingly.

Table 2 shows the quantitative predictions of the competitive equilibrium for moments that we have not targeted, and compares them to their empirical counterparts. Our calibration yields a stationary bond spread with an average of 3.0% and a standard deviation of 8.0%. The average is close to the 2.5% bond spread reported for the GIIPS relative to Germany over the period 1992-2015, while the model yields too much variation in the spread compared to the data. The renegotiation probability in the stationary equilibrium is predicted to be 6.5%, which lies in the middle of the range of estimates reported in Tomz and Wright (2013, Section 4.2).[25] During renegotiation periods, the model generates recovery values and investor losses that are remarkably close to the ones reported in the literature. The simulations yield an average haircut 41% of the debt's face value, which is just above the interval of empirical estimates reported in Tomz and Wright (2013, Section 4.4). This is remarkable, given that this moment was not targeted in the calibration. The model also produces a high variation in haircuts which is just 2 percentage points below the one documented in Cruces and Trebesch (2013). Moreover, Reinhart and Trebesch (2016) document the average debt relief (in terms of market value) to have been 21% of GDP for advanced economies in the 1930s and 16% of GDP for emerging market economies in the 1980s/1990s. On average, our model yields a 21% debt relief in terms of GDP which is in line with their estimates. Our simulation results are also in line with Asonuma and Trebesch (2016, Table 2 and 3) who show that debt-GDP ratio's are higher in renegotiation periods (89.7%) compared to the average debt-GDP ratio (53.7%). Finally, a great recession in the model lasts on average 6.4 years and the unconditional probability of being in recession is 6.0%.

[25] Sovereign default and renegotiation are rare events. For haircuts and renegotiation probabilities we therefore use data for a longer time period and for a broader set of countries than the GIIPS during 1992-2015.

[26] The empirical moments of the bond spread are calculated from the EMU convergence criterion bond yields which are reported by Eurostat for the GIIPS over the period 1992-2015.

[27] Tomz and Wright (2013, Section 4.2) suggests this range of estimates considering several countries. Interestingly, they also report four default waves, where at least 30% of the worlds debtors where in default. This is close to the renegotation probability of 39% that we report for recession periods. For Argentina, Arellano (2008) targets a 3% default probability.

[28] In historical data on sovereign debt restructurings, Benjamin and Wright (2009, Table 1) report an average haircut 38% in terms of market value. Cruces and Trebesch (2013) report a 40% market value haircut, and a 37% haircut according to the Sturzenegger and Zettelmeyer (2008) methodology. Tomz and Wright (2013, Section 4.4) provide a more detailed overview of estimates. Since we only consider one-period discount bonds in the model, face value and market value haircuts mostly overlap according to the above methodologies.

[29] Reinhart and Trebesch (2016).

[30] Cruces and Trebesch (2013, Table 1).

	Data	Model	Recession	Normal
Bond spread, avg. (GIIPS)[26]	2.54%	3.0%	20.5%	1.6%
Bond spread, std. (GIIPS)	2.54%	8.0%	22.8%	1.3%
Renegotiation, prob.[27]	[1.7%,13%]	6.5%	39.0%	4.0%
	Renegotiation periods			
Haircut, avg.[28]	[37%,40%]	41.4%	36.8%	42.0%
Haircut, std.[29]	27%	24.7%	18.2%	29.0%
Investor loss (% GDP)[30]	[16%,21%]	21.1%	12.3%	23.7%
Debt (% GDP)	-	89.7%	185.5%	66.3%
	Recession periods			
Exp. duration recession	-	-	6.4 yrs	-
Prob. being in recession	-	-	6.0%	-

Table 2: Non-targeted moments

	Stationary Distribution		Recession ($b_0/y_0 = 1$)	
	Cons. Equiv.(%)	Debt Equiv.(%)	Cons.Equiv.(%)	Debt Equiv.(%)
First Best	6.1	241	13.2	580
GDP-linked debt	1.0	37	0.9	34
One-sided commitment	1.7	60	3.0	113
Full Commit. & Inc. Mkts	4.7	183	11.0	475

Table 3: Welfare gains of allocations with less frictions

5.4 Welfare comparison

We use the calibrated economy to evaluate the welfare gains of different policy scenarios relative to the competitive equilibrium. All thought experiments are performed according to the following principles. We start from a competitive equilibrium without GDP-linked debt, with a given inherited debt level and realized state of productivity at time t. Before ϕ_t is realized, the outstanding debt is bought back by the planner (or creditors in row 2 of Table 3) at the going market price so as to guarantee that investors who bought the debt at time $t-1$ receive the expected repayment in period t (so in expectation neither gains nor losses are accrued). Then, the planner calculates the expected utility she can provide to the sovereign under the constraint that the expected profit for the planner is equal to the cost of buying back the debt. We refer to this intervention as *cost neutral*.

The welfare gains are measured as the equivalent variation in terms of consumption. We also report the equivalent variation in terms of debt, namely, the market value of a reduction in the initial debt that keeps the borrower indifferent between staying in the competitive equilibrium (with the adjusted debt) and moving to an alternative allocation.

In Table 3, we report the welfare gains of moving from the competitive (Markov) equilibrium to counterfactual economies, starting from the stationary distribution of the competitive equilibrium (columns 1-2) and from a recession with an initial debt-output ratio of $b_0/y_0 = 100\%$ (columns 3-4). Note that, since $\beta R < 1$, all economies except the first best are stationary.

The welfare gains are generally large, especially when the economy is initially in a recession with a large debt. Naturally, going to the first best yields the largest gains (first row). The gains are also sizable in the planning economy with limited enforcement of Section 4 (third row): they amount to

1.7% when evaluated (in expected value) at the stationary distribution, and to 3% in a recession with a large debt. The equivalent debt reduction is also large. For instance, access to a contract with one-sided commitment when effort is observable delivers larger welfare gains than the outright cancellation of the outstanding debt. Welfare gains are, as expected, increasing in risk aversion (details available upon request).

The second row shows the value of access to GDP-linked debt. When evaluated at the stationary distribution, the consumption-equivalent welfare effect is 1%, or about 60% of the gains from the planning allocation with one-sided commitment. However, the gains are smaller when the economy starts in a recession with high debt, being less than a third of the welfare gains the planner can deliver (third row). As discussed above, this illustrates that the trade-off between moral hazard and insurance limits the gains associated with the possibility to issue GDP-linked debt in a recession. The planner can resolve this problem owing to her commitment to punish past deviations, hence the much larger welfare gain.

The last row shows the value of moving to an Aiyagari economy with full commitment to honor debt but no GDP-linked debt. The gains are three times larger than the planning allocation with limited commitment. This confirms the importance of limited enforcement.

5.4.1 Decomposition exercise

To understand better the result, it is useful to decompose the welfare gains. We focus for simplicity on the comparison between the first best and the competitive equilibrium. The welfare gains can be decomposed in three components:

1. Discounting: In the first best, the planner can frontload consumption and backload effort to satisfy the representative agent's impatience (recall that $\beta R < 1$). In particular, consumption falls to zero in the long run and effort tends to the maximum level. This cannot happen in the competitive equilibrium because the outside option shock (or, in the Aiyagari economy, the precautionary motive) bounds consumption away from zero.

2. Volatility: In the first best, there is no volatility of consumption or effort around the trend. In particular, shocks do not influence consumption.

3. Level: The present value of consumption and effort is different in the two economies.

The decomposition proceeds through the following steps. First, we construct a first best benchmark using as initial condition the stationary distribution of wealth. More precisely, we take the debt distribution associated with the competitive equilibrium of the calibrated economy, and calculate for each debt level the corresponding cost-equivalent first-best allocation. Then, we calculate a *pseudo-planner* allocation with constant consumption and effort. Namely, we calculate the present value of consumption in the first best and generate a constant consumption sequence with the same present value. Moreover, we calculate the constant effort level needed to sustain this allocation. The *discounting effect* is the welfare cost of going from the first-best to the pseudo-planner allocation. Next, we calculate level and volatility effects, building on the decomposition proposed by Atkinson (1970). This amounts to calculating the average consumption in the pseudo-planner allocation and in each of the alternatives. In all cases, we calculate a constant effort sequence that sustains the associated consumption. The welfare cost (possibly negative) of going from the pseudo-planner to the constant-consumption alternative is the *level effect*. Finally, we calculate the welfare of having the

Total	Volatility	Level	Discounting
-5.5	-1.1	-0.4	-4.1

Table 4: Welfare decomposition: from first best to competitive equilibrium

fluctuations in consumption and reform effort associated with each allocation, relative to the constant sequences. We label this *volatility effect*. In Appendix B we show that this decomposition is exact when consumption is log normally distributed and in the absence of reform effort.

The results of the decomposition are shown in Table 4. The discounting effect is large and accounts for 74% of the losses of going from the first best to the competitive equilibrium. This result illustrates that the ability to frontload consumption by overcoming the limited enforcement friction is very important. The reduction in volatility also yields significant welfare gains (20% of the total effect), while the level effect is small. A similar comparison can be made between the first best and the other economies considered in Table 3 (omitted here).

5.5 Ruling out renegotiation

In this section, we consider an environment in which there is no possibility to renegotiate debt: the sovereign can decide to either honor the debt or outright default. The purpose of this exercise is to investigate how ruling out renegotiations will influence welfare. In the economy with state-contingent debt and no moral hazard we can provide a sharp result: ruling out renegotiations will always be welfare reducing.[31]

In the general case, shutting down renegotiation has a number of negative implications. First and foremost, there will be costly default in equilibrium. The real costs suffered by the sovereign yields no benefit to creditors, in contrast with the renegotiation scenario, where real costs are averted and creditors recover a share of the face value of debt. Second, conditional on the debt level, the range ϕ for which the sovereign defaults is different across the two economies. More formally, in the benchmark equilibrium of Section 3 the sovereign renegotiates if $\phi < W(0, w) - W(b, w)$ whereas in the no-renegotiation equilibrium she defaults if $\phi < W_{NR}(0, w) - W_{NR}(b, w)$, where W_{NR} is the value function under no renegotiation. As long as W_{NR} is falling more steeply in b than W, then, conditional on the debt level, the sovereign is more likely to honor the debt in the benchmark equilibrium than in the no-renegotiation equilibrium.

This is the case in our calibrated economy, as illustrated by Figure 7 in Appendix B. The figure displays the renegotiation threshold functions $\Phi(b, w)$ of the calibrated competitive equilibrium and corresponding no-renegotiation scenario. The former are uniformly below the latter. Thus, for a given debt level, debt is honored with a higher probability in the benchmark economy where renegotiation is allowed. As foreign investors anticipate the larger risk of default and the larger haircuts (100%), the price of debt is lower under no renegotiation, and this curtails the sovereign's ability to smooth consumption. Moreover, the maximum debt level is lower in the no-renegotiation economy than in the benchmark equilibrium (125% of normal-time GDP rather than 176%).

Panel a of Figure 6 plots the welfare losses associated with ruling out renegotiation as a function of the initial debt level, starting from the benchmark economy. In particular, b_0 is the initial face value of debt in the benchmark economy. As in our earlier experiments, cost neutrality for the

[31] This result follows directly from Proposition 14 which shows that the competitive equilibrium allocation (in the economy with renegotiation) is equivalent to the planner allocation.

lenders is preserved by compensating initial debt holders for the change in the market value of the outstanding debt. This is attained by increasing the face value of debt at time t before the shock ϕ_t is realized.[32] For instance, for an economy in a recession with a 40% debt-to-GDP ratio the consumption equivalent welfare loss of ruling out renegotiation amounts to 1.67% of permanent consumption. The welfare losses are increasing in the initial debt level. The reason is that the set of states for which renegotiation prevents costly default is larger when debt is large. Therefore, ruling out renegotiation is especially costly when an economy is in a recession with high debt.

These results differ from the existing literature. For example, Yue (2010) and Hatchondo *et al.* (2014) find that ruling out renegotiation can be welfare improving in models without moral hazard. Their models differ in a number of respects from ours. In particular, in both papers renegotiation is costly, and the sovereign has bargaining power in the renegotiation game. Moreover, Hatchondo *et al.* (2014) assume that the sovereign issues long-term debt.

5.6 Austerity cum Grexit

In this section, we evaluate the welfare consequences of an austerity program, where any violation of the program's conditions triggers an immediate and permanent termination of the arrangement. This scenario is reminiscent of the so-called Grexit threat that was supported by some Eurozone leaders, most notably the German Finance Minister Wolfgang Schäuble, before the third bailout plan for Greece was finally settled in July 2015.[33]

Consider a once-and-for-all intervention of an external institution (the *Trojka*) that provides a guarantee on the sovereign's obligations, so that the market price of debt will be $1/R$ in all states. The Trojka requires in exchange fiscal austerity, i.e., the sovereign can roll over the outstanding debt, but cannot borrow additional resources on the market. Effort is observable, and the Trojka is committed to terminate the assistance program (*Grexit*) as soon as the sovereign either attempts to renegotiate the outstanding debt, or violates the fiscal austerity requirement. The abrupt termination of the contract triggers default: the sovereign pays the cost ϕ and renegotiates its outstanding debt. Even in this case, the Trojka reimburses the investors for losses on the debt issued before default. In case of no termination, the program continues until the recession ends. At that time the sovereign repays its debt and start borrowing at market terms.

This program has some attractive features: (i) the international guarantee reduces the burden of servicing debt; (ii) the intervention mitigates the hold-up problem in reform effort. However, the fiscal austerity requirement limits the possibility of borrowing to smooth consumption. Moreover, inefficient terminations can occur inducing losses for the Trojka and fluctuations in consumption and effort for the sovereign.

Panel b of Figure 6 plots the welfare losses arising from the introduction of the austerity program starting from a competitive equilibrium. When the program starts, the Trojka purchases the outstanding debt at market value. Thereafter, it offers the guarantee described above. Since a costly termination may occur in equilibrium, the Trojka makes losses in expectations when entering the

[32] Note that for sufficiently high initial debt levels, this adjustment is not feasible because the benchmark debt value exceeds the maximum debt revenue that can be raised under no renegotiation. This imposes an upper bound on initial debt for which we can show the welfare losses associated with cost-neutral changes.

Alternatively, the welfare costs could be illustrated by keeping the utility of the sovereign constant and calculating the associated expected profit losses for the lenders. This is shown in Panel a of Figure 8 in Appendix B.

[33] See, e.g., Spiegel online July 17, 2015, http://www.spiegel.de/international/germany/schaeuble-pushed-for-a-grexit-and-backed-merkel-into-a-corner-a-1044259.html.

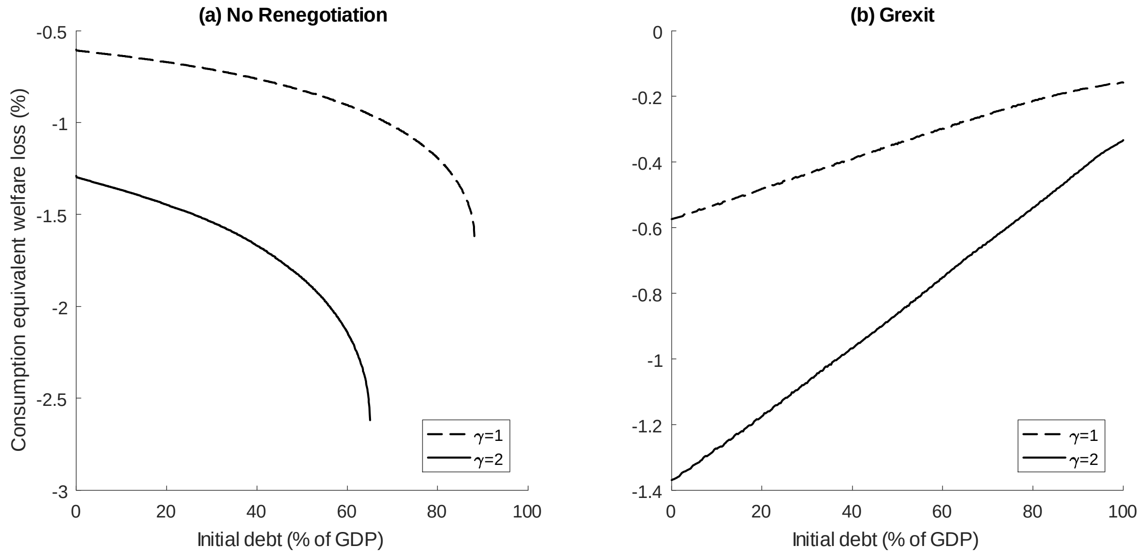

Figure 6: Panel a plots the welfare losses of ruling out renegotiation, relative to remaining in the benchmark economy. Panel b plots the welfare losses of imposing an "Austerity cum Grexit" policy, relative to remaining in the benchmark economy.

agreement. Therefore, the initial debt of the sovereign must be increased in order for the intervention to be cost neutral. Similar to the no-renegotiation case, there is an upper bound on initial debt, above which it is not feasible to achieve cost neutral interventions (since the maximum debt revenue is lower under Grexit). Therefore, the figure only displays welfare losses in the range below this upper bound.

In the plotted range, the welfare loss of Grexit is decreasing in the outstanding debt, ranging from 1.37% at zero debt to 0.33% at a debt 100% of GDP.[34] Recall that in the benchmark economy the moral hazard in reform effort is increasing in debt. In the presence of a debt guarantee the price of debt does not respond to the level of debt, in which case the moral hazard problem is mitigated. Therefore, the higher the debt the smaller is the disadvantage of the Trojka guarantee. Note that we could not prove that a Grexit-style austerity program is necessarily worse in welfare terms than the benchmark competitive equilibrium. However, we have not been able to find welfare gains for any debt level or any risk aversion.

In summary, the two last sections establish that two institutional arrangements proposed in the policy debate as instruments for allegedly improving *ex-ante* incentives – commitment not to renegotiate and fiscal austerity – may actually be inefficient. In our calibrated economies both policies are dominated in welfare terms by the laissez-faire equilibrium, and a *fortiori* by an assistance program that allows repeated renegotiations, reminiscent of the de facto policy pursued by the Trojka.

6 Conclusions

This paper presents a theory of sovereign debt dynamics under limited commitment. A sovereign country issues debt to smooth consumption during a recession whose duration is uncertain and endogenous.

[34] Panel b of Figure 8 in Appendix B shows the expected profits net of the initial value of debt when keeping the utility of the sovereign constant. There one can see that in a range of high debt levels the profit starts falling which implies that the welfare loss associated with Grexit are in general non-monotonic.

The expected duration of the recession depends on the intensity of (costly) structural reforms. Both elements – the risk of repudiation and the need for structural reforms – are salient features of the recent European debt crisis.

The competitive equilibrium, assumed to be Markovian, features recurrent debt renegotiations. Renegotiations are more likely to occur during recessions and when the country has accumulated a high level of debt. As a recession drags on, the country has an incentive to go deeper into debt. A higher level of debt in turn may obstruct rather than encourage economic reforms.

The theory bears normative predictions that are relevant for events such as the European crisis. The competitive equilibrium is inefficient for two reasons. On the one hand, due to the lack of commitment of market institutions, structural reforms are subject to a hold up problem. The intuitive reason is that the short-run cost of reforms is borne entirely by the country, while future benefits of reforms accrue in part to the creditors in the form of an *ex-post* increased price of debt, due to a reduction in the probability of renegotiation. On the other hand, the limited commitment to honor debt induces high risk premia and excess consumption volatility. A well-designed intervention by an international institution endowed with commitment power can improve welfare. The optimal policy entails an assistance program whereby an international organization provides the country with a constant transfer flow, deferring the repayment of debt to the time when the recession ends. The optimal contract takes into account that this payment is itself subject to renegotiation risk.

The result that institutions endowed with commitment power can improve on the competitive equilibrium hinges on reforms being observable, an assumption that is also maintained in the competitive equilibrium. If institutions cannot observe reform (even imperfectly), institutional commitment is powerless, and the assistance program cannot improve on the competitive equilibrium. Arguably, in the recent debt crises, many reforms were by-and-large observable (e.g., labor market reforms, or the establishment of a property registry in Greece), suggesting that commitment issues played an important role.

A second implication is that, when the sovereign credibly threatens to renege on an existing agreement, concessions should be made to avoid an outright repudiation. Contrary to a common perception among policy makers, a rigid commitment to enforce the terms of the original agreement is not optimal. Rather, the optimal policy entails the possibility of multiple renegotiations, which are reflected in the terms of the initial agreement. Likewise, we show that shutting down renegotiations is not useful, and induces instead additional welfare losses.

To retain tractability, we make important assumptions that we plan to relax in future research. First, in our theory the default cost follows an exogenous stochastic process. In a richer model, this would be part of the equilibrium dynamics. Strategic delegation is a potentially important extension. In the case of Greece, voters may have an incentive to elect a radical sovereign with the aim of delegating the negotiation power to an agent that has or is perceived to have a lower default cost than voters do (cf. Rogoff 1985). In our current model, however, the stochastic process governing the creditor's outside option is exogenous, and is outside of the control of the sovereign and creditors.

Second, again for simplicity, we assume that renegotiation is costless, that creditors can perfectly coordinate and that they have full bargaining power in the renegotiation game. Each of these assumptions could be relaxed. For instance, in reality the process of negotiation may entail costs. Moreover, as in the recent contention between Argentina and the so-called vulture funds, some creditors may hold out and refuse to accept a restructuring plan signed by a syndicate of lenders. Finally, the country may retain some bargaining power in the renegotiation. All these extensions would introduce interesting additional dimensions, and invalidate some of the strong efficiency results (for instance, the result that the market economy attains the constrained optimum in the absence of income fluctuations).

However, we are confident that the gist of the results is robust to these extensions.

Finally, by focusing on a representative agent, we abstract from conflicts of interest between different groups of agents within the country. Studying the political economy of sovereign debt would be an interesting extension. We leave the exploration of these and other avenues to future work.

References

Abraham, Arpad, Eva Carceles-Poveda, and Ramon Marimon (2014). "On the optimal design of a financial stability fund," Mimeo, European University Institute.

Aguiar, Mark, and Manuel Amador (2013). "Take the Short Route: How to Repay and Restructure Sovereign Debt with Multiple Maturities," NBER Working Paper No. 19717.

Aguiar, Mark, and Manuel Amador (2014). "Sovereign debt: a review," *Handbook of International Economics* 4, 647-87.

Aguiar, Mark, and Gita Gopinath (2006). "Defaultable Debt, Interest Rates, and the Current Account," *Journal of International Economics* 69 (1), 64–83.

Alvarez, Fernando, and Urban J. Jermann (2000). "Efficiency, equilibrium, and asset pricing with risk of default," *Econometrica* 68(4), 775–797.

Angeletos, George-Marios (2002). "Fiscal Policy with Non-Contingent Debt and the Optimal Maturity Structure," *Quarterly Journal of Economics* 117(3), 1105-1131.

Arellano, Cristina (2008). "Default risk and income fluctuations in emerging economies," *American Economic Review* 98(3), 690–712.

Arellano, Cristina, Lilia Maliar, Serguei Maliar, and Viktor Tsyrennikov (2014). "Envelope Condition Method with an Application to Default Risk Models," Mimeo, http://dx.doi.org/10.2139/ssrn.2470009.

Asonuma, Tamon, and Christoph Trebesch (2016). "Sovereign debt restructurings: preemptive or post-default," *Journal of the European Economic Association* 14(1) 175–214.

Atkeson, Andrew (1991). "International Lending with Moral Hazard and Risk of Repudiation," *Econometrica* 59(4), 1069-1089.

Atkinson, Anthony B. (1970). "On the Measurement of Inequality," *Journal of Economic Theory* 2(3), 244-263.

Barro, Robert and David Gordon (1983). "Rules, discretion, and reputation in a model of monetary policy," *Journal of Monetary Economics* 12(1), 101-121.

Benjamin, David, and Mark L. J. Wright (2009). "Recovery Before Redemption: A Theory of Delays in Sovereign Debt Renegotiations," Mimeo, State University of New York Buffalo.

Benveniste, Larry M., and Jose A. Scheinkman (1979). "On the Differentiability of the Value Function in Dynamic Models of Economics," *Econometrica* 47(3), 727–732.

Blanchard, Olivier, and Francesco Giavazzi (2003). "Macroeconomic Effects of Regulation and Deregulation in Goods and Labor Markets," *Quarterly Journal of Economics* 118(3), 879–907.

Boeri, Tito (2005). "Reforming labor and product markets: Some lessons from two decades of experiments in Europe," IMF WP/05/97.

Bolton, Patrick, and Olivier Jeanne (2007). "Structuring and restructuring sovereign debt: the role of a bankruptcy regime," *Journal of Political Economy* 115(6), 901–924.

Borensztein, Eduardo, and Ugo Panizza (2009). "The costs of sovereign default," IMF Staff Papers 56 (4), 683-741.

Broner, Fernando A., Guido Lorenzoni, and Sergio L. Schmukler (2013). "Why Do Emerging Economies Borrow Short Term?" *Journal of the European Economic Association* 11(1), 67–100.

Broner, Fernando A., Alberto Martin, and Jaume Ventura (2010). “Sovereign risk and secondary markets,” *American Economic Review* 100(4), 1523-1555.

Broner, Fernando A., and Jaume Ventura (2011). “Globalization and risk sharing,” *Review of Economic Studies* 78(1), 49-82.

Brutti, Filippo, and Philip U. Sauré (2016). “Repatriation of Debt in the Euro Crisis,” *Journal of the European Economic Association* 14(1), 145–174.

Bulow, Jeremy, and Kenneth Rogoff (1989). “A constant recontracting model of sovereign debt,” *Journal of Political Economy* 97(1), 155–178.

Bulow, Jeremy, and Kenneth Rogoff (2015). “Why sovereigns repay debts to external creditors and why it matters,” Vox CEPR’s Policy Portal, June 10, 2015.

Chatterjee, Satyajit and Burcu Eyigungor, (2012). "Maturity, Indebtedness, and Default Risk," American Economic Review, 102(6), 2674-2699.

Clausen, Andrew, and Carlo Strub (2013). “A General and Intuitive Envelope Theorem,” Mimeo. University of Edinburgh, URL:https://andrewclausen.net/Clausen_Strub_Envelope.pdf.

Cole, Harold L., and Timothy J. Kehoe (2000). “Self-fulfilling debt crises,” *Review of Economic Studies* 67(1), 91–116.

Collard, Fabrice, Michel Habib, and Jean-Charles Rochet (2015). “Sovereign Debt Sustainability in Advanced Economies,” *Journal of the European Economic Association* 13(3), 381-420.

Conesa, Juan Carlos, and Timothy J. Kehoe (2015). “Gambling for redemption and self-fulfilling debt crises,” Research Department Staff Report 465, Federal Reserve Bank of Minneapolis.

Cruces, Juan J., and Christoph Trebesch (2013). “Sovereign Defaults: The Price of Haircuts,” *American Economic Journal: Macroeconomics* 5(3), 85-177.

Dovis, Alessandro (2016). “Efficient Sovereign Default,” Mimeo, Pennsylvania State University.

Eaton, Jonathan, and Raquel Fernandez (1995). “Sovereign debt,” *Handbook of International Economics* 3, 1243-2107.

Eaton, Jonathan, and Mark Gersovitz (1981). “Debt with potential repudiation: Theoretical and empirical analysis,” *Review of Economic Studies* 48(2), 289–309.

Fernandez, Raquel, and Robert W. Rosenthal (1989). “Sovereign-debt renegotiations revisited,” Working Paper No. 2981, National Bureau of Economic Research.

Hatchondo, Juan Carlos, Leonardo Martinez, and César Sosa Padilla (2014). “Voluntary sovereign debt exchanges,” *Journal of Monetary Economics* 61 32–50

Hopenhayn, Hugo and Ivan Werning (2008). “Equilibrium default,” Mimeo, MIT.

Ilzkovitz Fabienne, and Adriaan Dierx (2011). “Structural Reforms: A European Perspective,” Reflets et perspectives de la vie économique 3/2011 (Tome L), 13–26.

Jeanne, Olivier (2009). “Debt Maturity and the International Financial Architecture,” *American Economic Review* 99(5), 2135–2148.

Kehoe, Patrick J., and Fabrizio Perri (2002). “International business cycles with endogenous incomplete markets,” *Econometrica* 70(3), 907–928.

Kocherlakota, Narayana R. (1996). “Implications of efficient risk sharing without commitment,” *Review of Economic Studies* 63(4), 595–609.

Krueger, Dirk, and Harald Uhlig (2006). “Competitive risk sharing contracts with one-sided commitment,” *Journal of Monetary Economics* 53(7), 1661–1691.

Krugman, Paul (1988). “Financing vs. forgiving a debt overhang,” *Journal of Development Economics* 29(3), 253–268.

Ljungqvist, Lars, and Thomas J. Sargent (2012). *Recursive Macroeconomic Theory*, Third Edition, Cambridge, MA: MIT Press.

Mendoza, Enrique G., and Vivian Z. Yue (2012). "A General Equilibrium Model of Sovereign Default and Business Cycles," *Quarterly Journal of Economics* 127(2), 889–946.

Müller, Andreas, Kjetil Storesletten, and Fabrizio Zilibotti (2016). "The Political Color of Fiscal Responsibility," *Journal of the European Economic Association* 14(1), 252–302.

Panizza, Ugo, Federico Sturzenegger, and Jeromin Zettelmeyer (2009). "The economics and law of sovereign debt and default," *Journal of Economic Literature* 47(3), 651–698.

Prescott, Edward C., and Robert M. Townsend (1984). "General Competitive Analysis in an Economy with Private Information," *International Economic Review* 25(1), 1–20.

Reinhart, Carmen M., and Kenneth S. Rogoff (2010). "From Financial Crash to Debt Crisis," NBER Working Paper 15795.

Reinhart, Carmen M., and Christoph Trebesch (2016). "Sovereign debt relief and its aftermath," *Journal of the European Economic Association* 14(1), 215–251.

Renner, Philipp, and Karl Schmedders (2015). "A polynomial optimization approach to principal-agent problems," *Econometrica* 83(2), 729–769.

Rogoff, Kenneth (1985). "The optimal degree of commitment to an intermediate monetary target," *Quarterly Journal of Economics* 100(4), 141–182.

Song, Zheng, Kjetil Storesletten, and Fabrizio Zilibotti (2012). "Rotten Parents and Disciplined Children: A Politico-Economic Theory of Public Expenditure and Debt," *Econometrica* 80(6), 2785–2803.

Spear, Stephen E., and Sanjay Srivastava (1987). "On repeated moral hazard with discounting," *Review of Economic Studies* 54 (4), 599–617.

Stokey, Nancy L., and Robert E. Lucas, with Edward C. Prescott (1989). *Recursive Methods in Economic Dynamics*. Publisher: Harvard University Press, Cambridge, Mass.

Sturzenegger, Federico, and Jeromin Zettelmeyer (2008). "Haircuts: estimating investor losses in sovereign debt restructurings, 1998–2005," *Journal of International Money and Finance* 27(5), 780–805.

Thomas, Jonathan, and Tim Worrall (1988). "Self-enforcing wage contracts," *Review of Economic Studies* 55(4), 541–554.

Thomas, Jonathan, and Tim Worrall (1990). "Income fluctuation and asymmetric information: An example of a repeated pincipal-agent model," *Journal of Economic Theory* 51(2), 367–390.

Tomz, Michael, and Mark L.J. Wright (2007). "Do countries default in bad times?," *Journal of the European Economic Association* 5(2-3), 352–360.

Yue, Vivian Z. (2010). "Sovereign default and debt renegotiation," *Journal of International Economics* 80(2), 176–187.

Online Appendixes of
"Sovereign Debt and Structural Reforms"

Andreas Müller, Kjetil Storesletten and Fabrizio Zilibotti

A Appendix A: Proofs of lemmas, propositions, and corollaries.

Proof of Proposition 1. Perfect insurance implies constant consumption (always) and effort (during recession). The efficient solution maximizes the discounted utility, $W = u(c)/(1-\beta) - X(p)/(1-\beta(1-p))$ with respect to c and p, subject to the following intertemporal budget constraint:

$$\frac{1}{1-\beta(1-p)}(\underline{w}-c) + \frac{\beta}{1-\beta}\frac{p}{1-\beta(1-p)}(\bar{w}-c) = b. \tag{44}$$

Note that since insurance is provided in actuarially fair markets the budget constraint holds in expected terms. Writing the Lagrangian, and differentiating it with respect to c yields $u'(c) = \lambda$. Differentiating the Lagrangian with respect to p, and simplifying terms, yields

$$X'(p)\left(\frac{1-\beta}{\beta}+p\right) - X(p) = (\bar{w}-\underline{w}) \times u'(c), \tag{45}$$

which is identical to Equation (1) in the text. Equation (44) defines a positively sloped locus in the plane (p,c), while Equation (45) defines a negatively sloped locus in the same plane. Unless the solution for effort is a corner, the two equations pin down a unique interior solution for p and c. Consider the comparative statics with respect to b (note that b only features in Equation (44)). An increase in b yields a decrease in c and an increase in p. This concludes the proof. ∎

Proof of Lemma 1. The first part follows from the definitions of Φ and $\hat{b}$. For the second part, note that $\Phi(b,w) = W(0,w) - W(b,w)$ such that if $W(b,w)$ is strictly decreasing in b, then $\Phi(b,w)$ must be strictly increasing in b. Thus, the inverse function of $\Phi(b,w)$ exits and is given by the expressions stated in Lemma 1. ∎

Proof of Proposition 2. We prove that the value function W is a fixed-point of a monotone mapping following Stokey and Lucas (1989, Theorem 17.7).

Let $\Gamma(\underline{w})$ be the space of bounded, continuous, and decreasing functions defined over $[\underline{b},\tilde{b}]$. Moreover, let d_∞ denote the supremum norm such that (Γ, d_∞) is a complete metric space. Let $z \in \Gamma(\underline{w})$ and γ be a real constant representing the outside option under outright default. $T(z;\gamma)$ is similar to the Bellman equation of the Markov equilibrium in the text, but differs in that the value of outright default in the recession state is *exogenously* given by $\gamma - \phi$. We first establish the existence of an equilibrium for an exogenous γ, and then extend the argument to an endogenous outside option as in the Markov equilibrium.

Define the following mapping:

$$T\left(z;\gamma,w\right)\left(b\right)=\max_{b'\in[\underline{b},\tilde{b}]}u(Q(b';z,\gamma)b'-b+w)+Z(b';z,\gamma,w), \tag{46}$$

where $\Phi\left(x;z,\gamma\right)=\gamma-z(x)$, and $z\left(\hat{b}\left(\phi;z,\gamma\right)\right)=\gamma-\phi$. In addition, let $T^0(z;\gamma)=z$, $T^n(z;\gamma)=T(T^{n-1}(z;\gamma);\gamma)$, $n=1,2,\ldots$, and $z^n\equiv T^n(z;\gamma)$. Moreover, when $w=\bar{w}$,

$$\begin{aligned} Z(x;z,\gamma,\bar{w}) &= \beta E\left[\max\left\{z\left(x\right),\gamma-\phi'\right\}\right], \text{ and} \\ Q(x;z,\gamma,\bar{w})x &= R^{-1}E\left[\min\left\{x,\hat{b}(\phi;z,\gamma)\right\}\right]. \end{aligned}$$

Correspondingly, when $w=\underline{w}$

$$\begin{aligned} Z(x;z,\gamma,\underline{w}) &= -X(\Psi(x;z,\gamma))+\beta\left[\begin{array}{c}\Psi(x;z,\gamma)E\left[\max\left\{W\left(x,\bar{w}\right),W\left(0,\bar{w}\right)-\phi'\right\}\right] \\ +(1-\Psi(x;z,\gamma))E\left[\max\left\{z\left(x\right),\gamma-\phi'\right\}\right]\end{array}\right], \\ X'(\Psi(x;z,\gamma)) &= \beta\left[\begin{array}{c}\left[1-F(\Phi\left(x,\bar{w}\right))\right]W(x,\bar{w}) \\ -\left[1-F(\Phi(x;z,\gamma))\right]z(x)\end{array}\right], \\ Q(x;z,\gamma,\underline{w})x &= R^{-1}E\left[\begin{array}{c}\Psi(x;z,\gamma)\min\left\{x,\hat{b}\left(\phi,\bar{w}\right)\right\} \\ +(1-\Psi(x;z,\gamma))\min\left\{x,\hat{b}\left(\phi;z,\gamma\right)\right\}\end{array}\right]. \end{aligned}$$

Note that the mapping during recession, $T\left(z;\gamma,\underline{w}\right)$, takes as given the existence of the equilibrium functions $W\left(b,\bar{w}\right)$, $\Phi\left(b,\bar{w}\right)$ and $\hat{b}\left(\phi,\bar{w}\right)$. This is legitimate because one can prove existence recursively, first for normal times and then in recession.

We define upper and lower bounds for the value functions. More formally, $W_{MIN}\equiv u(\underline{w})/(1-\beta)-\phi_{\max}$, and $W_{MAX}\equiv u(\bar{w})/(1-\beta)$. It is straightforward to see that W_{MIN} and W_{MAX} are, respectively, lower and upper bounds to the present utility the country can attain in equilibrium.

We establish first that the operator $T\left(z;\gamma,w\right)\left(b\right)$ is a uniformly continuous, bounded and decreasing (in b) mapping of the function space Γ into itself. Continuity follows by the Theorem of the Maximum. Boundedness follows from the fact that utility is bounded because consumption, reform effort, the support of the default cost, and the elements of Γ are bounded. Finally, to establish that the mapping T is decreasing in b note that, for any $\Delta>0$, $T\left(z;\gamma,w\right)\left(b+\Delta\right)<T\left(z;\gamma,w\right)\left(b\right)$ since

$$T\left(z;\gamma,w\right)\left(b+\Delta\right)=\max_{b'\in[\underline{b},\tilde{b}]}u\left(Q(b';z,\gamma,w)b'+w-(b+\Delta)\right)+Z\left(b';z,\gamma,w\right)$$

$$\begin{aligned} &= u\left(Q(B\left(b+\Delta;z,\gamma,w\right);z,\gamma,w)\cdot B\left(b+\Delta;z,\gamma,w\right)+w-(b+\Delta)\right)+Z\left(B\left(b+\Delta;z,\gamma,w\right);z,\gamma,w\right) \\ &< u\left(Q\left(B\left(b+\Delta;z,\gamma,w\right);z,\gamma,w\right)\cdot B\left(b+\Delta;z,\gamma,w\right)+w-b\right)+Z\left(B\left(b+\Delta;z,\gamma,w\right);z,\gamma,w\right) \\ &\leq u\left(Q\left(B\left(b;z,\gamma,w\right);z,\gamma,w\right)\cdot B\left(b;z,\gamma,w\right)+w-b\right)+Z\left(B\left(b;z,\gamma,w\right);z,\gamma,w\right)=T\left(z;\gamma,w\right)\left(b\right), \end{aligned}$$

where $B(b;z,\gamma,w)=\arg\max_{b'\in[\underline{b},\tilde{b}]}u(Q(b';z,\gamma,w)b'-b+w)+Z(b';z,\gamma,w)$.

Next, we establish that the mapping $T\left(z;\gamma,w\right)$ is monotone in z, that its fixed-point $z^*\left(z;\gamma,w\right)\left(b\right)=\lim_{n\to+\infty}T^n\left(z;\gamma,w\right)\left(b\right)$ exists, and that $z^*\in\Gamma$. To this aim, consider $z,z^+\in\Gamma$ with $z\left(b\right)<z^+\left(b\right)$, $\forall b\in[\underline{b},\tilde{b}]$. Since z and z^+ are decreasing in b, implying that $z^+\left(\hat{b}(\phi;z^+,\gamma)\right)=\gamma-\phi>z\left(\hat{b}(\phi;z^+,\gamma)\right)$, it follows immediately that $\hat{b}(\phi;z,\gamma)<\hat{b}(\phi;z^+,\gamma)$. Consequently, $\Phi\left(b;z,\gamma\right)\geq\Phi\left(b;z^+,\gamma\right)$ for all

$b \in [\underline{b}, \tilde{b}]$, which in turn implies that $Q(b; z, \gamma, w) \leq Q(b; z^{+}, \gamma, w)$. Consider first the case of $w = \bar{w}$. We establish that $z^{+} > z \Leftrightarrow T\left(z^{+}; \gamma, \bar{w}\right)(b) > T\left(z; \gamma, \bar{w}\right)(b)$, since

$$\begin{aligned}
T\left(z^{+}; \gamma, \bar{w}\right)(b) &= \max_{b' \in [\underline{b}, \tilde{b}]} u\left(Q(b'; z^{+}, \gamma, \bar{w})b' - b + \bar{w}\right) + Z\left(b'; z^{+}, \gamma, \bar{w}\right) \\
&= u\left(Q(B\left(b; z^{+}, \gamma, \bar{w}\right); z^{+}, \gamma, \bar{w}) \cdot B\left(b; z^{+}, \gamma, \bar{w}\right) - b + \bar{w}\right) + Z\left(B\left(b; z^{+}, \gamma, \bar{w}\right); z^{+}, \gamma, \bar{w}\right) \\
&\geq u\left(Q(B\left(b; z, \gamma, \bar{w}\right); z^{+}, \gamma, \bar{w}) \cdot B\left(b; z, \gamma, \bar{w}\right) - b + \bar{w}\right) + Z\left(B\left(b; z, \gamma, \bar{w}\right); z^{+}, \gamma, \bar{w}\right) \\
&> u\left(Q(B\left(b; z, \gamma, \bar{w}\right); z, \gamma, \bar{w}) \cdot B\left(b; z, \gamma, \bar{w}\right) - b + \bar{w}\right) + Z\left(B\left(b; z, \gamma, \bar{w}\right); z, \gamma, \bar{w}\right) \\
&= \max_{b' \in [\underline{b}, \tilde{b}]} u(Q\left(b'; z, \gamma, \bar{w}\right) b' + \bar{w} - b) + Z\left(b'; z, \gamma, \bar{w}\right) = T\left(z; \gamma, \bar{w}\right)(b).
\end{aligned}$$

where the first inequality follows from the fact that $B\left(b; z, \gamma, \bar{w}\right)$ yields a lower utility relative to the optimal $B\left(b; z^{+}, \gamma, \bar{w}\right)$. The second inequality follows from the fact that $Q\left(b; z, \gamma, \bar{w}\right) \leq Q\left(b; z^{+}, \gamma, \bar{w}\right)$ for all $b \in [\underline{b}, \tilde{b}]$.

Consider, next, the case of $w = \underline{w}$. Let $E\bar{V}(b) \equiv E\left[V\left(b, \phi, \bar{w}\right)\right]$. We establish that $z^{+} > z \Leftrightarrow T\left(z^{+}; \gamma, \underline{w}\right)(b) > T\left(z; \gamma, \underline{w}\right)(b)$, since

$$\begin{aligned}
T\left(z^{+}; \gamma, \underline{w}\right)(b) &= \max_{b' \in [\underline{b}, \tilde{b}]} u\left(Q(b'; z^{+}, \gamma)b' - b + \underline{w}\right) + Z\left(b'; z^{+}, \gamma\right) \\
&= u\left(Q(B\left(b; z^{+}, \gamma\right); z^{+}, \gamma) \cdot B\left(b; z^{+}, \gamma\right) - b + \underline{w}\right) - X\left(\Psi\left(B\left(b; z^{+}, \gamma\right); z^{+}, \gamma\right)\right) \\
&\quad + \left[\begin{array}{c} \Psi\left(B\left(b; z^{+}, \gamma\right); z^{+}, \gamma\right)\beta E\bar{V}\left(B\left(b; z^{+}, \gamma\right)\right) + \\ \left(1 - \Psi\left(B\left(b; z^{+}, \gamma\right); z^{+}, \gamma\right)\right)\beta E\left[\max\left\{\gamma - \phi, z^{+}\left(B\left(b; z^{+}, \gamma\right)\right)\right\}\right] \end{array}\right] \\
&\geq u\left(Q(B\left(b; z, \gamma\right); z^{+}, \gamma) \cdot B\left(b; z, \gamma\right) - b + \underline{w}\right) - X\left(\Psi\left(B\left(b; z, \gamma\right); z, \gamma\right)\right) \\
&\quad + \left[\begin{array}{c} \Psi\left(B\left(b; z, \gamma\right); z, \gamma\right)\beta E\bar{V}\left(B\left(b; z, \gamma\right)\right) + \\ \left(1 - \Psi\left(B\left(b; z, \gamma\right); z, \gamma\right)\right)\beta E\left[\max\left\{\gamma - \phi, z^{+}\left(B\left(b; z, \gamma\right)\right)\right\}\right] \end{array}\right] \\
&> u\left(Q\left(B\left(b; z, \gamma\right); z, \gamma\right) \cdot B\left(b; z, \gamma\right) - b + \underline{w}\right) - X\left(\Psi\left(B\left(b; z, \gamma\right); z, \gamma\right)\right) \\
&\quad + \left[\begin{array}{c} \Psi\left(B\left(b; z, \gamma\right); z, \gamma\right)\beta E\bar{V}\left(B\left(b; z, \gamma\right)\right) + \\ \left(1 - \Psi\left(B\left(b; z, \gamma\right); z, \gamma\right)\right)\beta E\left[\max\left\{\gamma - \phi, z\left(B\left(b; z, \gamma\right)\right)\right\}\right] \end{array}\right] \\
&= \max_{b' \in [\underline{b}, \tilde{b}]} u(Q(b'; z, \gamma)b' - b + \underline{w}) + Z(b'; z, \gamma) = T(z; \gamma)\left(b, \underline{w}\right),
\end{aligned}$$

where the same logic as above applies.

We have established that $T(z; \gamma, w)$ is a monotone mapping with the sup norm. This mapping is an equicontinuous family (each function in Γ is uniformly continuous and the continuity is uniform for all functions in Γ). Then, Stokey and Lucas (1989, Theorem 17.7) ensures that the fixed point of $T(z; \gamma, w)$ exists, is an element of Γ and is given by $z^{*}(z; \gamma, w) = \lim_{n \to +\infty} T^{n}(z; \gamma, w)$.

Thus far, we have proven the existence of at least one fixed point of the mapping T for any exogenous outside option, $\gamma \in [W_{MIN}, W_{MAX}]$. We now use a different fixed point argument to show that, conditional on an initial z, there exists a unique fixed point that the outside options in normal times and recession are, respectively, $\gamma_{z}^{*}(\bar{w})$ and $\gamma_{z}^{*}(\underline{w})$ with the following properties: $z^{*}\left(z; \gamma_{z}^{*}(\bar{w}), \bar{w}\right)(0) = \gamma_{z}^{*}(\bar{w}) = W_{MAX}$ and $z^{*}\left(z; \gamma_{z}^{*}(\underline{w}), \underline{w}\right)(0) = \gamma_{z}^{*}(\underline{w}) \in (W_{MIN}, W_{MAX})$. To see why, note that, by the Theorem of the Maximum, $z^{*}\left(z; \gamma, w\right)(b) = \lim_{n \to +\infty} T^{n}\left(z; \gamma, w\right)(b)$ is continuous in γ. Moreover, z^{*} is bounded since $z^{*}\left(z; \gamma, w\right) \in [W_{MIN}, W_{MAX}]$. Thus, the Brouwer fixed-point theorem ensures that there exists a $\gamma_{z} \in [W_{MIN}, W_{MAX}]$ such that $z^{*}\left(z; \gamma_{z}, w\right)(0) = \gamma_{z}$. Since $z^{*}\left(z; \gamma_{z}, w\right)(0) = \gamma_{z} - \Phi\left(0; z^{*}, \gamma_{z}\right)$, this is equivalent to say that, at each fixed point,

$\Phi\left(0; z^*, \gamma_z^*\left(w\right)\right) = 0$. To prove uniqueness, we note then that $\Phi\left(0; z_\gamma^*, \gamma\right)$ is monotone increasing for all $\gamma \in \left[W_{MIN}, W_{MAX}\right]$, as the set of (potential) states of nature in which the outside option is attractive expands when γ increases. Therefore, there exists a unique fixed point $\gamma_z^*\left(w\right)$ such that $\Phi\left(0; z^*, \gamma_z^*\left(w\right)\right) = 0$. In particular in normal times $\gamma_z^*\left(\bar{w}\right) = W\left(0, \bar{w}\right) = W_{MAX}$. In recession, $\gamma_z^*\left(\underline{w}\right) = W\left(0, \underline{w}\right) \in \left(W_{MIN}, W_{MAX}\right)$.

The results proven thus far allow us to claim the existence of an equilibrium value function W such that $W\left(b, w\right) = T\left(W; W\left(0, w\right), w\right)\left(b\right)$. The definition of the remaining equilibrium functions follow from Definition 1. This establishes the existence of a Markov equilibrium. The continuity of the value function $W(b, w)$ in b follows from the Theorem of the Maximum, and implies that also the equilibrium functions Φ, Q and Ψ are continuous in b. It is also straightforward to show that W is strictly decreasing in b and, hence, that $\Phi\left(b, w\right) = W\left(0, w\right) - W\left(b, w\right)$ is strictly increasing in b. Finally, we claim that the bond revenue, $Q(b', w)b'$, is (weakly) monotone increasing in b'. This follows from Equations (6)–(8), the fact that Φ increasing in b, and from Lemma 1.

Next, we prove that B is monotone decreasing in b by applying Topkis' Theorem. To this aim, define $B(b, w) = \arg\max_{b' \in [\underline{b}, \tilde{b}]} O(b', b, w)$ where

$$O(b', b, w) = u(Q(b', w)b' - b + w) + \beta Z(b', w). \tag{47}$$

We first establish that the objective function $O(b', b, w)$ is supermodular in (b', b), i.e., if $b'_H > b'_L$ and $b_H > b_L$, then, $O(b'_H, b_H, w) - O(b'_L, b_H, w) \geq O(b'_H, b_L, w) - O(b'_L, b_L, w)$. To this aim, note that

$$\begin{aligned} & u(Q(b'_H, w)b'_H - b_H + w) - u(Q(b'_L, w)b'_L - b_H + w) \\ \geq \quad & u(Q(b'_H, w)b'_H - b_L + w) - u(Q(b'_L, w)b'_L - b_L + w). \end{aligned} \tag{48}$$

This inequality follows from the concavity of the utility function and the fact that $Q\left(b, w\right) b$ is increasing with b, implying that, for $\Delta > 0$,

$$u'(Q(x, w)x - b + w) - u'(Q(x + \Delta, w)\left(x + \Delta\right) - b + w) \geq 0.$$

Rearranging terms in (48), and adding and subtracting continuation values on both sides of the inequality yields

$$\begin{aligned} & u(Q(b'_H, w)b'_H - b_H + w) - u(Q(b'_H, w)b'_H - b_L + w) \\ & +\beta Z(b'_H, w) - \beta Z(b'_H, w) \\ \geq \quad & u(Q(b'_L, w)b'_L - b_H + w) - u(Q(b'_L, w)b'_L - b_L + w) \\ & +\beta Z(b'_L, w) - \beta Z(b'_L, w) \end{aligned}$$

that is equivalent to $O(b'_H, b_H, w) - O(b'_H, b_L, w) \geq O(b'_L, b_H, w) - O(b'_L, b_L, w)$. This establishes that $O(b', b, w)$ is supermodular in (b', b). Topkis' Theorem implies then that $B(b_H, w) \geq B(b_L, w)$. This concludes the proof of Proposition 2. ■

Corollary 1 *Let the operator T be defined as in Equation (46). If, for $w \in \{\underline{w}, \bar{w}\}$, $W\left(b, w\right) = \lim_{n \to \infty} T^n\left(W_{MIN}; W\left(b, w\right), w\right)\left(b\right) = \lim_{n \to \infty} T^n\left(W_{MAX}; W\left(b, w\right), w\right)\left(b\right)$, then the Markov equilibrium is unique (i.e., there exists a unique equilibrium value function $W\left(b, w\right)$ satisfying Definition 1).*

Proof of Corollary 1. The proof follows immediately from the Corollary to Theorem 17.7 in Stokey and Lucas (1989). ■

Proof of Proposition 3. The proof is an application of the generalized envelope theorem in Clausen and Strub (2013) which allows for discrete choices (i.e., repayment or renegotiation) and non-concave value functions. Consider the program $W(b,w) = \max_{b' \in [\underline{b},\tilde{b}]} O(b',b,w)$ where O is defined in Equation (47). Theorem 1 in Clausen and Strub (2013) ensures that if we can find a *differentiable lower support function* (DLSF) for O, then O is differentiable for all $b' \in \mathcal{B}(w)$.

We start by proving the lemma for the case of $w = \underline{w}$. The strategy of the proof involves finding DLSF for the equilibrium functions $Q(b',\underline{w})b'$ and $Z(b',\underline{w})$. To this aim, we follow the strategy of Benveniste and Scheinkman (1979), and consider the value function of a pseudo-borrower that chooses debt issuance $b' = B(x,w)$ instead of the optimal $b' = B(b,w)$,

$$\widetilde{W}(b,x,w) \equiv u\left(Q\left(B\left(x,w\right),w\right)B\left(x,w\right) - b + w\right) + Z\left(B\left(x,w\right),w\right).$$

Note that $\widetilde{W}$ is differentiable and strictly decreasing in b. Since debt issuance is chosen suboptimally, it must be the case that $\widetilde{W}(b,x,w) \leq W(b,w)$ with equality holding at $x = b$. Furthermore, let the pseudo-borrower set the default threshold at the level $\widetilde{\Phi}(b,x,w) = W(0,w) - \widetilde{W}(b,x,w)$, where $\widetilde{\Phi}(b,x,w) \geq \Phi(b,w)$. Thus, the pseudo-borrower will find it optimal to renegotiate for a range of ϕ larger than $\Phi(b,w)$. Note that $\widetilde{\Phi}(b,x,w)$ is differentiable and strictly increasing in b. Thus, the inverse function exists and is such that $\widetilde{\Phi}^{-1}_{x,w}(\phi) \leq \hat{b}(\phi,w)$ (where we define $\widetilde{\Phi}_{x,w}(b) \equiv \widetilde{\Phi}(b,x,w)$).

Consider first the case in which $w = \underline{w}$. Let

$$\widetilde{O}\left(b',b,x,\underline{w}\right) = u\left(\widetilde{Q}(b',x,\underline{w})b' - b + \underline{w}\right) + \widetilde{Z}(b',x,\underline{w}),$$

where the pseudo bond revenue function is given by

$$\begin{aligned}\widetilde{Q}\left(b',x,\underline{w}\right)b' &= \frac{1}{R}\widetilde{\Psi}(b',x)\left(\left[1 - F(\widetilde{\Phi}\left(b',x,\bar{w}\right))\right]b' + \int_0^{\widetilde{\Phi}(b',\bar{w})} \widetilde{\Phi}^{-1}_{x,\bar{w}}(\phi)dF(\phi)\right) \\ &\quad + \frac{1}{R}(1 - \widetilde{\Psi}(b',x))\left(\left[1 - F(\widetilde{\Phi}\left(b',x,\underline{w}\right)\right]b' + \int_0^{\widetilde{\Phi}(b',\underline{w})} \widetilde{\Phi}^{-1}_{x,\underline{w}}(\phi)dF(\phi)\right),\end{aligned}$$

and the continuation value is given by

$$\widetilde{Z}(b',x,\underline{w}) = -X(\widetilde{\Psi}(b',x)) + \beta\left[\begin{array}{c}\widetilde{\Psi}(b',x)E\max\left\{\widetilde{W}\left(b',x,\bar{w}\right),\widetilde{W}(0,x,\bar{w}) - \phi'\right\} \\ +(1 - \widetilde{\Psi}(b',x))E\max\left\{\widetilde{W}\left(b',x,\underline{w}\right),\widetilde{W}(0,x,\underline{w}) - \phi'\right\}\end{array}\right],$$

having defined $\widetilde{\Psi}$ as

$$\widetilde{\Psi}(b',x) = \left(X'\right)^{-1}\left(\beta\left[\begin{array}{c}E\max\left\{\widetilde{W}(b',x,\bar{w}),\widetilde{W}(0,x,\bar{w}) - \phi'\right\} \\ -E\max\left\{\widetilde{W}(b',x,\underline{w}),\widetilde{W}(0,x,\underline{w}) - \phi'\right\}\end{array}\right]\right).$$

Note that $\widetilde{Q}$, $\widetilde{Z}$ and $\widetilde{\Psi}$ are differentiable in b' since we established above that $\widetilde{W}$ and $\widetilde{\Phi}$ are differentiable.

Then, $\widetilde{O}$ is a DLSF for O such that $\widetilde{O}(b', b, x, \underline{w}) \leq O(b', b, \underline{w})$ with equality (only) at $b' = x$. Thus, Theorem 1 in Clausen and Strub (2013) ensures that the objective function $O(b', b, \underline{w})$ is differentiable in b' at $b' = B(b, \underline{w})$ and that $\partial O(b', b, \underline{w})/\partial b' = \partial \widetilde{O}(b', b, B(b, \underline{w}), \underline{w})/\partial b' = 0$. In this case, a standard first-order condition yields

$$\frac{\partial u\left(Q(b', \underline{w})b' - b + \underline{w}\right)}{\partial b'} + \frac{\partial Z(b', \underline{w})}{\partial b'} = 0.$$

Moreover, Lemma 3 in Clausen and Strub (2013) ensures that the functions $W(b, \underline{w})$, $Z(b, \underline{w})$, $\Phi(b, \underline{w})$, $Q(b, \underline{w})$, and $\Psi(b)$ are differentiable and that a standard envelope condition applies, namely,

$$\begin{aligned}
\frac{\partial Z(b', \underline{w})}{\partial b'} &= \beta \left[\begin{array}{c} \Psi(b') \left[1 - F(\Phi(b', \bar{w}))\right] \frac{\partial W(b', \bar{w})}{\partial b'} \\ +(1 - \Psi(b')) \left[1 - F(\Phi(b', \underline{w}))\right] \frac{\partial W(b', \underline{w})}{\partial b'} \end{array} \right], \\
\frac{\partial W(b, \underline{w})}{\partial b} &= -u'\left(Q(B(b, \underline{w}), \underline{w})B(b, \underline{w}) - b + \underline{w}\right) < 0.
\end{aligned}$$

The proof for the case of $w = \bar{w}$ follows the same strategy and is therefore omitted. ■

Proof of Lemma 2. In normal times, the bond revenue function can be written as

$$Q(b', \bar{w})b' = R^{-1} E \min\left\{b', \hat{b}(\phi, \bar{w})\right\}.$$

Note that the minimum function in the expectation operator is concave in b'. Since the sum of concave functions is still concave, then also $E\left\{\min b', \hat{b}(\phi, \bar{w})\right\}$ must be concave in b'. This implies that the marginal bond revenue is falling in b'. Differentiating $Q(b', \bar{w})b'$ with respect to $b' \in [\underline{b}, \tilde{b}]$ yields:

$$\begin{aligned}
\frac{\partial\left(Q(b', \bar{w})b'\right)}{\partial b'} &= Q\left(b', \bar{w}\right) + \frac{\partial}{\partial b'}\left(R^{-1}\left(1 - F\left(\bar{\Phi}\left(b'\right)\right)\right) + (Rb')^{-1} \int_0^{\bar{\Phi}(b')} \bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right) \times b' \\
&= Q\left(b', \bar{w}\right) - R^{-1} b' f\left(\bar{\Phi}\left(b'\right)\right) \times \frac{\partial \bar{\Phi}\left(b'\right)}{\partial b'} \\
&\quad + b'(Rb')^{-1} \bar{\Phi}^{-1}\left(\bar{\Phi}\left(b'\right)\right) f\left(\bar{\Phi}\left(b'\right)\right) \\
&\quad \times \frac{\partial \Phi \bar{\Phi}\left(b'\right)}{\partial b'} - \underbrace{\frac{1}{R}\frac{1}{b'} \int_0^{\bar{\Phi}(b')} \left(\bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right)}_{= Q(b', \bar{w}) - \frac{1}{R}\left(1 - F\left(\bar{\Phi}(b')\right)\right)} \\
&= R^{-1}\left(1 - F\left(\bar{\Phi}\left(b'\right)\right)\right).
\end{aligned}$$

■

Proof of Proposition 4. The first-order condition of (10) for $w = \bar{w}$ yields:

$$\frac{\partial}{\partial b'}\left\{b' \times Q\left(b', \bar{w}\right)\right\} \times u'(c) + \frac{\partial}{\partial b'} \beta EV\left(b', \bar{w}\right) = 0,$$

The function V has a kink at $b' = \hat{b}(\phi, \bar{w})$. Consider, first, the range of realizations $\phi > \bar{\Phi}(b)$ implying that $b' < \hat{b}(\phi, \bar{w})$. Differentiating V in this range yields:

$$\frac{\partial}{\partial b} V(b, \phi, \bar{w}) = -u'\left[Q\left(B\left(b, \bar{w}\right), \bar{w}\right) \times B\left(b, \bar{w}\right) + \bar{w} - b\right].$$

Next, consider the region of renegotiation, $\phi < \bar{\Phi}(b)$, implying that $b > \hat{b}(\phi, \bar{w})$. In this case, $\frac{\partial}{\partial b} V(b, \phi, w) = 0$. Using the results above one obtains:

$$
\begin{aligned}
\frac{\partial}{\partial b} EV(b, \bar{w}) &= \int_0^{\bar{\Phi}(b)} \frac{\partial}{\partial b} V(b, \phi, \bar{w})\, dF(\phi) + \int_{\bar{\Phi}(b)}^{\infty} \frac{\partial}{\partial b} V(b, \phi, \bar{w})\, dF(\phi) \\
&= -\int_{\bar{\Phi}(b)}^{\infty} u' \left[Q(B(b, \bar{w}), \bar{w}) \times B(b, \bar{w}) + \bar{w} - b \right] dF(\phi) \\
&= -\left[1 - F\left(\bar{\Phi}(b)\right)\right] \times u' \left[Q(B(b, \bar{w}), \bar{w}) \times B(b, \bar{w}) + \bar{w} - b \right] \qquad (49)
\end{aligned}
$$

Plugging this expression back into the FOC, and leading the expression by one period, yields

$$
0 = \frac{\partial}{\partial b'} \left\{ b' \times Q(b', \bar{w}) \right\} \times u'(c) - \beta \left[1 - F\left(\bar{\Phi}(b')\right)\right] \times u'\left(c'|_{H,\bar{w}}\right).
$$

Finally, recall that $\frac{\partial}{\partial b}\{b \times Q(b, \bar{w})\} = \frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right)$, as established in the proof of Lemma 2. Thus, for $F\left(\bar{\Phi}(b)\right) < 1$, the above equation is equivalent to the CEE (13) in the proposition.

Although the first-order condition is also satisfied at $b' = \bar{b}$, it is possible to show that this is never an optimal solution as long as $b < \bar{b}$ (details in Appendix B).

Consider, next, the properties of the equilibrium functions C and W. Using the definition of C in Equation (11) and Lemma 2, standard algebra shows that C is continuously differentiable and decreasing, with derivative $\partial C(b, \bar{w})/\partial b = R^{-1}\left[1 - F(\bar{\Phi}(b))\right] - 1 < 0$. Since $B(b, \bar{w}) = b$, then B maps the complete domain of b into itself, $\mathcal{B}(\bar{w}) = [\underline{b}, \bar{b}]$. Proposition 3 implies then that the value function is differentiable everywhere, and that the envelope condition $\partial W(b, \bar{w})/\partial b = -u'(C(b, \bar{w}))$ applies. Differentiating this condition with respect to b yields $\partial^2 W(b, \bar{w})/(\partial b)^2 = -u''(C(b, \bar{w}))\partial C(b, \bar{w})/\partial b = u''(C(b, \bar{w}))\left[1 - R^{-1}\left[1 - F(\bar{\Phi}(b))\right]\right] < 0$. This establishes that the value function W is twice continuously differentiable and strictly concave, thereby concluding the proof of the proposition. ■

Proof of Proposition 5. We start by claiming that $c^{FB}(b, w) > C(b, w)$, where $c^{FB}(b, w)$ was defined in Proposition 1 and $C(b, w)$ denotes consumption in the market equilibrium when debt is honored. To see why, note that $\partial/\partial b\left\{W^{FB}(b, w)\right\} \geq \partial/\partial b\{W(b, w)\}$. This follows from observing that the difference between $W^{FB}(b, w)$ and $W^{FB}(b + \Delta, w)$, where $\Delta > 0$, merely reflects the utility loss from a permanent reduction in consumption $c^{FB}(b, w)$ and (in recession) a permanent increase in effort $p^{FB}(b)$. In contrast, in the market equilibrium a larger debt induces a higher volatility of consumption (recall that $\Phi(b, w)$ is monotone increasing in b – see Lemma 1 – so higher debt increases the probability of renegotiation) and (in recession) effort. Since $\partial/\partial b\left\{W^{FB}(b, w)\right\} = -u'\left(c^{FB}(b, w)\right)$ and $\partial/\partial b\{W(b, w)\} = -u'(C(b, w))$, then the claim that $c^{FB}(b, w) > C(b, w)$ follows.

Let $\phi_{\min} \equiv \left[u(\bar{w}) - u((1 - \beta)(\bar{w} - \underline{w}))\right]/(1 - \beta)$ and $b_{PV} \equiv \underline{w} + \frac{\beta}{1-\beta}\bar{w}$. The assumption in the Proposition implies that $F(\phi_{\min}) = 0$ and $b_{PV} < \bar{w}$. Consider first the range $b \leq b_{PV}$. In this range, $W(b, \bar{w}) = W^{FB}(b, \bar{w})$. To see this, note that $W(b_{PV}, \bar{w}) = u(\bar{w} - (1 - \beta) b_{PV})/(1 - \beta) = \phi_{\min}$. Since by assumption $F(\phi_{\min}) = 0$, no renegotiation is possible for $b \leq b_{PV}$, so the claim follows. The two claims above and Proposition 1 imply that $C(b, \bar{w}) = c^{FB}(b, \bar{w}) > c^{FB}(b, \underline{w}) \geq C(b, \underline{w})$ in the range $b \leq b_{PV}$. Consider next the range $b \in (b_{PV}, \bar{b}]$. In this range, $C(b, \underline{w}) \leq 0$ since debt exceeds the maximum present value of future income. In contrast, $C(b, \bar{w}) > 0$ since $b_{PV} < \bar{w}$. We have therefore established that $C(b, \bar{w}) > C(b, \underline{w})$ for all $b \leq \bar{b}$. ■

Proof of Lemma 3. If in the initial period (but not later) the country can contract on effort while issuing new debt, the problem becomes

$$\max_{b',p^*} \left\{u(c) - X(p^*) + \beta p^* \times EV\left(b', \bar{w}\right) + \beta (1-p^*) \times EV\left(b', \underline{w}\right)\right\}.$$

Note that the next-period value function V is the same as in the benchmark problem with non-contractible effort, since we are considering a one-period deviation. The first-order condition with respect to p yields

$$\begin{aligned}
0 &= \frac{d}{dp^*}\left\{Q\left(b', \underline{w}\right) b'\right\} \times u'(c) - X'(p^*) + \beta\left(EV\left(b', \bar{w}\right) - EV\left(b', \underline{w}\right)\right) \\
&\Rightarrow X'(p^*) = \left[Q\left(b', \bar{w}\right) - \hat{Q}\left(b', \underline{w}\right)\right] b' \times u'(c) \\
&\quad + \beta\left[\int_0^\infty V\left(b', \phi', \bar{w}\right) dF(\phi) - \int_0^\infty V\left(b', \phi', \underline{w}\right) dF(\phi)\right] \\
&> \beta\left[\int_0^\infty V\left(b', \phi', \bar{w}\right) dF(\phi) - \int_0^\infty V\left(b', \phi', \underline{w}\right) dF(\phi)\right],
\end{aligned} \tag{50}$$

where the last equation follows from the facts that $Q(b', \bar{w}) > \hat{Q}(b', \underline{w})$, and that

$$\begin{aligned}
\frac{d}{dp}\left\{Q\left(b', \underline{w}\right) b'\right\} &= \frac{d}{dp}\left\{\left[pQ\left(b', \bar{w}\right) + (1-p)\hat{Q}\left(b', \underline{w}\right)\right] b'\right\} \\
&= \left[Q\left(b', \bar{w}\right) - \hat{Q}\left(b', \underline{w}\right)\right] b'.
\end{aligned}$$

The right-hand side of the inequality in Equation (50) is the optimal effort in the benchmark case with non-contractible effort, given in Equation (14). This establishes the lemma. ■

Proof of Proposition 6. Consider, first, the range $b \in [0, b_1)$. Differentiate Equation (14) with respect to b',

$$\begin{aligned}
X''\left(\Psi\left(b'\right)\right)\Psi'\left(b'\right) &= \beta\left[\int_0^\infty \frac{\partial}{\partial b'} V\left(b', \phi', \bar{w}\right) dF(\phi) - \int_0^\infty \frac{\partial}{\partial b'} V\left(b', \phi', \underline{w}\right) dF(\phi)\right] \\
&= -\beta\left[1 - F\left(\bar{\Phi}\left(b'\right)\right)\right] \times u'\left[Q\left(B\left(b', \bar{w}\right), \bar{w}\right) \times B\left(b', \bar{w}\right) + \bar{w} - b'\right] \\
&\quad + \beta\left[1 - F\left(\underline{\Phi}\left(b'\right)\right)\right] \times \left[Q\left(B\left(b', \underline{w}\right)\right) \times B\left(b', \underline{w}\right) + \underline{w} - b'\right]
\end{aligned} \tag{51}$$

Taking the limit of Equation (51) as $b' \to 0$ yields

$$\begin{aligned}
X''(\Psi(0))\Psi'(0) &= \beta\left[1 - F(\underline{\Phi}(0))\right] \times \left[Q\left(B(0, \underline{w})\right) \times B(0, \underline{w}) + \underline{w} - 0\right] \\
&\quad - \beta\left[1 - F\left(\bar{\Phi}(0)\right)\right] \times u'\left[Q\left(B(0, \bar{w}), \bar{w}\right) \times B(0, \bar{w}) + \bar{w} - 0\right] \\
&= \beta\left[u'\left(Q\left(B(0, \underline{w})\right) \times B(0, \underline{w}) + \underline{w} - 0\right) - u'(\bar{w})\right] > 0,
\end{aligned} \tag{52}$$

where the last equation uses the facts that $\bar{\Phi}(0) = \underline{\Phi}(0) = F(0) = 0$ and that during normal times $c = \bar{w}$ if $b = 0$. Note that during recession, the annualized present value of income is strictly smaller than $\bar{w}$. Therefore, it can never be optimal to choose consumption during recession larger than or equal to $\bar{w}$ when $b = 0$. Since the marginal utility of consumption is larger in a recession than during normal times, the right-hand side of Equation (52) is strictly positive. Since $X'' > 0$, then $\lim_{b\to 0} \Psi'(b) = \Psi'(0) > 0$. By continuity, it follows then that $\Psi'(b)$ will be positive for a range of b close to $b = 0$, so there must exist a $b_1 > 0$ such that $\Psi'(b) > 0$ for all $b \in [0, b_1)$.

Consider, next, the range $b \in \left[b^-, \bar{b}\right)$, in which case $F(\underline{\Phi}(b)) = 1$ and $F\left(\bar{\Phi}(b)\right) < 1$. This implies that Equation (51) can be written as

$$X''(\Psi(b))\,\Psi'(b) = -\beta\left[1 - F\left(\bar{\Phi}(b)\right)\right] \times u'\left[Q\left(B(b,\bar{w}), \bar{w}\right) \times B(b,\bar{w}) + \bar{w} - b\right] < 0,$$

which establishes that $\Psi'(b) < 0$ for all $b \in \left[b^-, \bar{b}\right)$ and with strict inequality also for $b = b^-$. By continuity, it follows then that there exists a $b_2 < b^-$ such that $\Psi'(b) < 0$ for all $b \in \left(b_2, \bar{b}\right)$. Finally, in the range where $b \geq \bar{b}$, $F(\underline{\Phi}(b)) = F\left(\bar{\Phi}(b)\right) = 1$ so the right-hand side of Equation (51) becomes zero, implying that $\Psi'(b) = 0$. ■

Proof of Proposition 7. The procedure is analogous to the derivation of the CEE in normal times. The first-order condition of (10) for $w = \underline{w}$ yields

$$0 = \frac{d}{db'}\left\{Q\left(b', \underline{w}\right) \times b'\right\} \times u'(c) + \beta\left[1 - \Psi\left(b'\right)\right]\frac{d}{db'}EV\left(b', \underline{w}\right) + \beta\Psi\left(b'\right)\frac{d}{db'}EV\left(b', \bar{w}\right),$$

where a term has been cancelled by the envelope theorem. Using the same argument as in the proof of Proposition 4 we can write:

$$\frac{d}{db}EV(b, \underline{w}) = -\left[1 - F(\underline{\Phi}(b))\right] \times u'\left[Q\left(B(b,\underline{w}), \underline{w}\right) \times B(b,\underline{w}) + \underline{w} - b\right].$$

Plugging this back into the FOC (after leading the expression by one period) yields the CEE

$$\begin{aligned} 0 \;=\;& u'(c) \times \Big\{\Psi'\left(b'\right) \times R\Big[Q\left(b', \bar{w}\right) - \hat{Q}\left(b', \underline{w}\right)\Big]\, b' + \\ & + \Psi\left(b'\right) \times \left(1 - F\left(\bar{\Phi}\left(b'\right)\right)\right) + \left[1 - \Psi\left(b'\right)\right] \times \left(1 - F\left(\underline{\Phi}\left(b'\right)\right)\right)\Big\} \\ & -\beta R\left(\left[1 - \Psi\left(b'\right)\right] \times \left[1 - F\left(\underline{\Phi}\left(b'\right)\right)\right] u'\left(c'|_{H,\underline{w}}\right) + \Psi\left(b'\right) \times \left[1 - F\left(\bar{\Phi}\left(b'\right)\right)\right] u'\left(c'|_{H,\bar{w}}\right)\right), \end{aligned}$$

where the equality follows from Lemma 4. Rearranging terms yields Equation (16). ■

Proof of Lemma 4. Differentiating the bond revenue with respect to b yields

$$\begin{aligned} \frac{d}{db}\left\{Q(b,\underline{w})\, b\right\} \;&=\; \frac{d}{db}\Big\{pbQ(b,\bar{w}) + (1-p)\, b\hat{Q}(b,\underline{w})\Big\} + \Psi'(b) \times \Big(Q(b,\bar{w}) - \hat{Q}(b,\underline{w})\Big)\, b \\ &=\; \Psi(b) \times \frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1 - \Psi(b)) \times \frac{1}{R}\left(1 - F(\underline{\Phi}(b))\right) \\ &\quad +\Psi'(b) \times \Big(Q(b,\bar{w}) - \hat{Q}(b,\underline{w})\Big)\, b, \end{aligned} \tag{53}$$

where the second equality can be derived as following:

$$\begin{aligned} \frac{d}{db}\Big\{pbQ(b,\bar{w}) + (1-p)\, b\hat{Q}(b,\underline{w})\Big\} \;&=\; p\frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1-p)\,\hat{Q}(b,\underline{w}) \\ &\quad + (1-p)\left[\begin{array}{c} -\frac{b}{R} f\left(\underline{\Phi}(b)\right) \times \underline{\Phi}'(b) - \\ \frac{1}{R}\frac{1}{b}\int_0^{\underline{\Phi}(b)}\left(\underline{\Phi}^{-1}(\phi) \times f(\phi)\; d\phi\right) + \frac{1}{R} b f\left(\underline{\Phi}(b)\right)\underline{\Phi}'(b) \end{array}\right] \\ &=\; p\frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1-p)\,\frac{1}{R}\left(1 - F(\underline{\Phi}(b))\right) \end{aligned}$$

Consider, first, the case in which $\Psi(b)$ is constant, $\Psi(b) = p$. In this case, debt revenue is increasing for all $b < \bar{b}$, since, then, $p/R \times \left(1 - F\left(\bar{\Phi}(b)\right)\right) + (1-p)/R \times (1 - F(\underline{\Phi}(b))) > 0$. Moreover, it reaches a maximum at $b = \bar{b}$ (recall that $F\left(\bar{\Phi}(b)\right) < F(\underline{\Phi}(b))$ for all $b < \bar{b}$). This establishes that, if Ψ is constant, then $\bar{b}^R = \bar{b}$.

Consider, next, the general case. Proposition 6 implies that, in the range where $b \in [b_2, \bar{b}]$, $\Psi'(b) < 0$. Since $Q(b, \bar{w}) > \hat{Q}(b, \underline{w})$, then, in a left neighborhood of $\bar{b}$, $\Psi'(b) \times \left[Q(b, \bar{w}) - \hat{Q}(b, \underline{w})\right] b < 0$. This means that, starting from $\bar{b}$, one can increase the debt revenue by reducing debt, i.e., $\bar{b}^R < \bar{b}$. ■

Proof of Proposition 8. We proceed in two steps: first, we derive the CEEs (step A), and then we show that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$ (step B).

Step A: The first-order conditions with respect to $b'_{\bar{w}}$ and $b'_{\underline{w}}$ in problem (18) yields

$$0 = u'(c) \times \frac{d}{db'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + \beta\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \frac{d}{db'_{\bar{w}}} EV\left(b'_{\bar{w}}, \bar{w}\right),$$

$$0 = u'(c) \times \frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right] \frac{d}{db'_{\underline{w}}} EV\left(b'_{\underline{w}}, \underline{w}\right),$$

where $REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \equiv b'_{\underline{w}} \times Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + b'_{\bar{w}} \times Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ is the bond revenue and c is defined in the Proposition. Note that both equations have been simplified using an envelope condition. The value function has a kink at $b = \hat{b}(\phi, \underline{w})$. Consider, first, the range of realizations $\phi \in [\underline{\Phi}(b), +\infty)$, implying that $b < \hat{b}(\phi, \underline{w})$. Differentiating the value function yields:

$$\begin{aligned}
\frac{d}{db}V(b, \phi, \bar{w}) &= -u'\left[Q(B(b, \bar{w}), \bar{w}) \times B(b, \bar{w}) + \bar{w} - b\right], \\
\frac{d}{db}V(b, \phi, \underline{w}) &= -u'\left[Q_{\underline{w}}\left(B_{\underline{w}}(b), B_{\bar{w}}(b)\right) \times B_{\underline{w}}(b) + Q_{\bar{w}}\left(B_{\underline{w}}(b), B_{\bar{w}}(b)\right) \times B_{\bar{w}}(b) + \underline{w} - b\right],
\end{aligned}$$

where $B_{\underline{w}}$ and $B_{\bar{w}}$ denote the optimal issuance of the two assets, respectively. Next, consider the range of realizations $\phi < \underline{\Phi}(b)$, implying that $b \geq \hat{b}(\phi, \underline{w})$. In this case, $\frac{d}{db}V(b, \phi, \underline{w}) = 0$.

In analogy with Equation (49), we obtain:

$$\frac{d}{db} EV(b, \underline{w}) = -\left[1 - F(\underline{\Phi}(b))\right] \times u'\left(c|_{H, \underline{w}}\right). \tag{54}$$

Plugging (49) and (54) into the respective first-order conditions, and leading by one period, yields

$$u'(c) \times \frac{d}{db'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \beta\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \left[1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right] \times u'\left(c'|_{H, \bar{w}}\right),$$

$$u'(c) \times \frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) = \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right] \times \left[1 - F\left(\underline{\Phi}\left(b'_{\bar{w}}\right)\right)\right] \times u'\left(c'|_{H, \underline{w}}\right).$$

The marginal revenues from issuing recession-contingent debt is given by:

$$\begin{aligned}
&\frac{d}{db'_{\underline{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \\
&= \frac{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) - \frac{\frac{\partial\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}}}{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + b'_{\bar{w}} \times \frac{\partial}{\partial b'_{\underline{w}}} Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \\
&= \frac{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) + \frac{\partial\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}} \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right),
\end{aligned} \tag{55}$$

where, note, $\frac{\partial}{\partial b'_{\underline{w}}} Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)=\frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}} \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}$ follows from applying standard differentiation to the definition of $Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ in Equation (21). Applying the same methodology to the recovery-contingent debt, we obtain:

$$\frac{d}{d b'_{\bar{w}}} REV\left(b'_{\underline{w}}, b'_{\bar{w}}\right)=\frac{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{R}\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)+\frac{\partial \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\partial b'_{\bar{w}}} \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right).$$

The CEEs conditional on the recession continuing and ending, respectively, are then:

$$\begin{aligned}
\beta \frac{u'\left(c'|_{H, \underline{w}}\right)}{u'(c)} &= \frac{1}{R}+\frac{\partial}{\partial b'_{\underline{w}}} \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \frac{\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)\left[1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right]} \\
\beta \frac{u'\left(c'|_{H, \bar{w}}\right)}{u'(c)} &= \frac{1}{R}+\frac{\partial}{\partial b'_{\bar{w}}} \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times \frac{\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}.
\end{aligned}$$

Setting $\beta R=1$ yields Equations (22)-(23).

Step B: Next, we prove that, in equilibrium, $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)>0$. To prove the claim, it is useful to define the two functions

$$\begin{aligned}
\theta_{\bar{w}}\left(b'_{\bar{w}}\right) &\equiv \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}=\frac{1}{R}\left(\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) b'_{\bar{w}}+\int_0^{\bar{\Phi}\left(b'_{\bar{w}}\right)} \bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right), \\
\theta_{\underline{w}}\left(b'_{\underline{w}}\right) &\equiv \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}}}{1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}=\frac{1}{R}\left(\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) b'_{\underline{w}}+\int_0^{\underline{\Phi}\left(b'_{\underline{w}}\right)} \underline{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right),
\end{aligned}$$

where, recall, $\underline{\Phi}(x)>\bar{\Phi}(x)$ and $F(\underline{\Phi}(x))>F\left(\bar{\Phi}(x)\right)$. Note that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)=\theta_{\bar{w}}\left(b'_{\bar{w}}\right)-\theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, where both $\theta_{\bar{w}}$ and $\theta_{\underline{w}}$ are increasing functions in the relevant range, i.e., $b'_{\bar{w}} \leq \bar{b}$ and $b'_{\underline{w}} \leq \bar{b}$. We proceed in two steps. First, we show that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}}>b'_{\underline{w}}$ (step B1). Next, we show that $b'_{\bar{w}}>b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)>0$ (step B2). Steps B1 and B2 establish jointly a contradiction ruling out that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$ (step B3).

Step B1: Suppose that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$. Then, the CEEs (22)–(23) and the assumption that $u''<0$, imply that

$$c'|_{H, \bar{w}} \leq c \leq c'|_{H, \underline{w}}. \tag{56}$$

Suppose, to derive a contradiction, that $b'_{\underline{w}} \geq b'_{\bar{w}}$. Recall that, if the recession ends and debt is honored, debt remains constant, i.e., $b''=B\left(b'_{\bar{w}}\right)=b'_{\bar{w}}$. Moreover, $Q\left(b'_{\bar{w}}, \bar{w}\right)=Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) / \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$. Thus, $c'|_{H, \bar{w}}=Q\left(b'_{\bar{w}}, \bar{w}\right) b'_{\bar{w}}+\bar{w}-b'_{\bar{w}}=\theta_{\bar{w}}\left(b'_{\bar{w}}\right)+\bar{w}-b'_{\bar{w}}$.

$$\begin{aligned}
c'|_{H, \bar{w}} &= \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \theta_{\bar{w}}\left(b'_{\bar{w}}\right)+\left(1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) \theta_{\bar{w}}\left(b'_{\bar{w}}\right)+\bar{w}-b'_{\bar{w}} \\
&\geq \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \theta_{\bar{w}}\left(b'_{\bar{w}}\right)+\left(1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) \theta_{\bar{w}}\left(b'_{\underline{w}}\right)+\bar{w}-b'_{\underline{w}} \\
&\geq \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \theta_{\bar{w}}\left(b'_{\bar{w}}\right)+\left(1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) \theta_{\underline{w}}\left(b'_{\underline{w}}\right)+\bar{w}-b'_{\underline{w}} \\
&> \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \theta_{\bar{w}}\left(b'_{\bar{w}}\right)+\left(1-\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) \theta_{\underline{w}}\left(b'_{\underline{w}}\right)+\underline{w}-b'_{\underline{w}}=c'|_{H, \underline{w}}.
\end{aligned}$$

The first inequality follows from the assumption that $b'_{\underline{w}} \geq b'_{\bar{w}}$ and the fact that $(1-p)\,\theta_{\bar{w}}(x)-x<0$ for any $p \in[0,1]$, which is due to the fact that $\theta_{\bar{w}}(x) \leq x / R<x$ for any x. The second inequality follows from the fact that $\theta_{\bar{w}}\left(b'_{\underline{w}}\right) \geq \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, see Equation (57) below. The last inequality follows from the

maintained assumption that $\bar{w} > \underline{w}$. We have therefore proven that if $b'_{\underline{w}} \geq b'_{\bar{w}}$ then $c'|_{H,\bar{w}} > c'|_{H,\underline{w}}$, which contradicts (56) and, hence, implies that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \geq 0$. We conclude from Step B1 that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}}$.

Step B2: Suppose that $b'_{\underline{w}} = b'_{\bar{w}} = x$. Then, for any x:

$$\begin{aligned}\Delta(x,x) &= \theta_{\bar{w}}(x) - \theta_{\underline{w}}(x) = \frac{1}{R}\underbrace{\int_{\bar{\Phi}(x)}^{\underline{\Phi}(x)} \left(\left(x - \underline{\Phi}^{-1}(\phi)\right) \times f(\phi)\ d\phi\right)}_{>0} \\ &\quad + \frac{1}{R}\underbrace{\int_{0}^{\bar{\Phi}(x)} \left(\left(\bar{\Phi}^{-1}(\phi) - \underline{\Phi}^{-1}(\phi)\right) \times f(\phi)\ d\phi\right)}_{>0} > 0.\end{aligned} \tag{57}$$

Since $\theta_{\bar{w}}(x)$ is an increasing function for $x \leq \bar{b}$, Equation (57) implies that $\theta_{\bar{w}}(b'_{\bar{w}}) > \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, for all $b'_{\underline{w}} < b'_{\bar{w}} \leq \bar{b}$. We conclude from Step B2 that $b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$.

Step B3: Putting together the conclusions of Step B1 and Step B2, we derive a contradiction: $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. Therefore, we must have that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. ■

Proof of Proposition 9. We write the Lagrangian,

$$\begin{aligned}\bar{\Lambda} &= \int_{\aleph}\left[\bar{w} - \bar{c}_{\phi} + \frac{1}{R}\bar{P}(\bar{\omega}_{\phi})\right] dF(\phi) + \bar{\mu}\left(\int_{\aleph}\left[u(\bar{c}_{\phi}) + \beta\bar{\omega}_{\phi}\right] dF(\phi) - \nu\right) \\ &\quad + \int_{\aleph}\bar{\lambda}_{\phi}\left[u(\bar{c}_{\phi}) + \beta\bar{\omega}_{\phi} - W(0,\bar{w}) + \phi\right] d\phi,\end{aligned}$$

with the associated multipliers $\bar{\mu}$ and $\bar{\lambda}_{\phi}$. The first-order conditions yield

$$f(\phi) = u'(\bar{c}_{\phi})\left(\bar{\mu} f(\phi) + \bar{\lambda}_{\phi}\right), \tag{58}$$

$$\left[\bar{\mu} f(\phi) + \bar{\lambda}_{\phi}\right]\beta R = -\bar{P}'(\bar{\omega}_{\phi}) f(\phi). \tag{59}$$

The envelope condition yields

$$-\bar{P}'(\nu) = \bar{\mu} \tag{60}$$

The two first-order conditions, the envelope condition and $\beta R = 1$ jointly imply that

$$u'(\bar{c}_{\phi}) = -\frac{1}{\bar{P}'(\bar{\omega}_{\phi})} \tag{61}$$

$$\bar{P}'(\bar{\omega}_{\phi}) = \bar{P}'(\nu) - \frac{\bar{\lambda}_{\phi}}{f(\phi)}. \tag{62}$$

Note that (61) is equivalent to (28) in the text. Consider, next, two cases, namely, when the PC is binding and when it is not binding.

When the PC is binding, $\bar{\lambda}_{\phi} > 0$. (62) implies then that $\bar{\omega}_{\phi} > \nu$. Then, (61) and (29) determine jointly the solution for $(\bar{c}_{\phi}, \bar{\omega}_{\phi})$. When the PC is not binding, $\bar{\lambda}_{\phi} = 0$. (62) implies then that $\bar{\omega}_{\phi} = \nu$ and $\bar{c}_{\phi} = \bar{c}(\nu)$.

What remains to be shown is that the first-order conditions are sufficient. The proof follows Thomas and Worrall (1990, Proof of Proposition 1). The details of this proof are in Proposition 16 in Appendix B. ■

Proof of Lemma 5. The Lagrangian of the planner's problem reads as

$$\begin{aligned}
\underline{\Lambda} &= \int_{\aleph} \left[\underline{w} - \underline{c}_{\phi} + \frac{1}{R} \left((1-p_{\phi}) \underline{P} \left(\underline{\omega}_{\phi} \right) + p_{\phi} \bar{P} \left(\bar{\omega}_{\phi} \right) \right) \right] dF(\phi) \\
&+ \underline{\mu} \left(\int_{\aleph} \left(u \left(\underline{c}_{\phi} \right) - X(p_{\phi}) + \beta \left((1-p_{\phi}) \underline{\omega}_{\phi} + p_{\phi} \bar{\omega}_{\phi} \right) \right) dF(\phi) - \nu \right) \\
&+ \int_{\aleph} \underline{\lambda}_{\phi} \left(u \left(\underline{c}_{\phi} \right) - X(p_{\phi}) + \beta \left((1-p_{\phi}) \underline{\omega}_{\phi} + p_{\phi} \bar{\omega}_{\phi} \right) - \underline{\nu} + \phi \right) d\phi \\
&+ \int_{\aleph} \gamma_{\phi} \left(-X(p_{\phi}) + \beta \left((1-p_{\phi}) \underline{\omega}_{\phi} + p_{\phi} \bar{\omega}_{\phi} \right) - \zeta_{dev} \right) d\phi,
\end{aligned}$$

where the Lagrange multipliers of the PC and IC must be non-negative for all ϕ, $\underline{\lambda}_{\phi} \geq 0, \gamma_{\phi} \geq 0$. The first-order conditions in combination with $\beta R = 1$ yield:

$$\begin{aligned}
f(\phi) &= u' \left(\underline{c}_{\phi} \right) \left(\underline{\mu} f(\phi) + \underline{\lambda}_{\phi} \right), & (63) \\
\underline{\mu} f(\phi) + \underline{\lambda}_{\phi} + \gamma_{\phi} &= -\underline{P}' \left(\underline{\omega}_{\phi} \right) f(\phi), & (64) \\
\underline{\mu} f(\phi) + \underline{\lambda}_{\phi} + \gamma_{\phi} &= -\bar{P}' \left(\bar{\omega}_{\phi} \right) f(\phi), & (65) \\
R^{-1} \left(\bar{P} \left(\bar{\omega}_{\phi} \right) - \underline{P} \left(\underline{\omega}_{\phi} \right) \right) f(\phi) &= \left(\underline{\mu} f(\phi) + \underline{\lambda}_{\phi} + \gamma_{\phi} \right) \left(X'(p_{\phi}) - \beta \left(\bar{\omega}_{\phi} - \underline{\omega}_{\phi} \right) \right), & (66)
\end{aligned}$$

while the envelope condition yields

$$-\underline{P}'(\nu) = \underline{\mu}. \tag{67}$$

The first-order conditions (64)–(66) imply Equations (34)-(35) in the text. Since $\underline{P}$ and $\bar{P}$ are monotonic and strictly concave, Equation (34) implies a strictly positive relationship between $\underline{\omega}_{\phi}$ and $\bar{\omega}_{\phi}$. Equation (35) yields then a strictly negative relationship between p_{ϕ} and $\underline{\omega}_{\phi}$, and a strictly increasing relationship between p_{ϕ} and $\bar{\omega}_{\phi}$. Consider, next, the IC constraint. Therefore, when the IC is binding, the binding constraint (33), (34), and (35) pin down a unique solution for p_{ϕ}, $\underline{\omega}_{\phi}$ and $\bar{\omega}_{\phi}$, denoted by $(p^*, \underline{\omega}^*, \bar{\omega}^*)$. ■

Proof of Proposition 10. We start by showing that $\nu \geq \underline{\omega}^*$ implies that the IC is never strictly binding. Combine Equations (64) and (67) to yield $\underline{P}'(\nu) \geq \underline{P}' \left(\underline{\omega}_{\phi} \right)$. Since $\underline{P}$ is strictly concave this implies that promised utility is non-decreasing, $\underline{\omega}_{\phi} \geq \nu$, conditional on staying in recession. Moreover, $\underline{\omega}_{\phi} \geq \nu \geq \underline{\omega}^*$, thus the IC is never strictly binding, $\gamma_{\phi} = 0$. When $\gamma_{\phi} = 0$, the FOCs in (64)–(66) read

$$\begin{aligned}
u' \left(\underline{c}_{\phi} \right) &= -\frac{1}{\underline{P}' \left(\underline{\omega}_{\phi} \right)} \\
\underline{P}' \left(\underline{\omega}_{\phi} \right) &= \underline{P}'(\nu) - \frac{\underline{\lambda}_{\phi}}{f(\phi)} \\
\underline{P}' \left(\underline{\omega}_{\phi} \right) &= \bar{P}' \left(\bar{\omega}_{\phi} \right) \\
X'(p_{\phi}) &= u'(c_{\phi}) R^{-1} \left(\bar{P} \left(\bar{\omega}_{\phi} \right) - \underline{P} \left(\underline{\omega}_{\phi} \right) \right) + \beta \left(\bar{\omega}_{\phi} - \underline{\omega}_{\phi} \right).
\end{aligned}$$

The solution will therefore depend on whether the PC is slack or binding:

1. When the PC is binding and the recession continues, $\phi < \tilde{\phi}(\nu)$, $\underline{\lambda}_{\phi} > 0$, $\underline{\omega}_{\phi} > \nu$, and

$$u \left(\underline{c}_{\phi} \right) - X(p_{\phi}) + \beta \left((1-p_{\phi}) \underline{\omega}_{\phi} + p_{\phi} \bar{\omega}_{\phi} \right) = W(0, \underline{w}) - \phi. \tag{68}$$

Then, (34), (35), (36) and (68) determine jointly the solution for $\left(\underline{c}_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$. In this case, there is no history dependence, i.e., ν does not matter.

2. When the PC is not binding, $\phi \geq \tilde{\phi}(\nu)$ and $\underline{\lambda}_\phi = 0$. Then, $\underline{\omega}_\phi = \nu$, and $\underline{c}_\phi = \underline{c}(\nu)$, $p_\phi = p(\nu)$, and $\bar{\omega}_\phi = \bar{\omega}_\phi(\nu)$ are determined by (36), (34), and (35), respectively. The solution is history dependent. Equation (35) and $\beta R = 1$ imply that

$$u'\left(\underline{c}\left(\nu\right)\right)\left[\bar{P}\left(\bar{\omega}\left(\nu\right)\right) - \underline{P}\left(\nu\right)\right] + \left[\bar{\omega}\left(\nu\right) - \nu\right] = \beta^{-1} X'\left(p\left(\nu\right)\right),$$

namely, the planner requires constant effort over the set of states for which the constraint is not binding: $p_\phi = p\left(\nu\right)$. Differentiating the left-hand side yields

$$\begin{aligned} & \underbrace{u''\left(\underline{c}\left(\nu\right)\right)\underline{c}'\left(\nu\right) \times \left[\bar{P}\left(\bar{\omega}\left(\nu\right)\right) - \underline{P}\left(\nu\right)\right]}_{<0} + \left[u'\left(\underline{c}\left(\nu\right)\right)\underline{P}'\left(\nu\right) + 1\right]\left(\bar{\omega}'\left(\nu\right) - 1\right) \\ = \ & u''\left(\underline{c}\left(\nu\right)\right)\underline{c}'\left(\nu\right) \times \left(\bar{P}\left(\bar{\omega}\left(\nu\right)\right) - \underline{P}\left(\nu\right)\right) < 0 \end{aligned}$$

since, recall, (36) implies that $\underline{P}'\left(\nu\right) = -1/u'\left(\underline{c}\left(\nu\right)\right)$. This implies that the right-hand side must also be decreasing in ν. Since X is convex and increasing, this implies in turn that $p\left(\nu\right)$ must be decreasing in ν. Moreover, $\underline{P}'\left(\nu\right) = -1/u'\left(\underline{c}\left(\nu\right)\right)$, implies that $\underline{c}\left(\nu\right)$ must be increasing in ν, while $\underline{P}'\left(\nu\right) = \bar{P}'\left(\bar{\omega}(\nu)\right)$ shows that also $\bar{\omega}(\nu)$ is increasing in ν.

This concludes the proof. ■

Proof of Proposition 11. This proof builds on the proof of Lemma 5 and Proposition 10. We already know that in the case $\nu < \underline{\omega}^*$ the IC is potentially binding.

1. When the IC is not binding, $\gamma_\phi = 0$, the solution $(\underline{c}_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi)$ is as described in Proposition 10.

2. When the IC and the PC is binding, $\gamma_\phi > 0$ and , $\underline{\lambda}_\phi > 0$, then $p_\phi = p^*, \underline{\omega}_\phi = \underline{\omega}^*, \bar{\omega}_\phi = \bar{\omega}^*$ as described in Lemma 5. Consumption varies with the realization of ϕ and is determined by the binding PC and IC in Equations (32) and (33) which imply Equation (38) in the proposition.

3. When the IC is binding and the PC is not binding, $\gamma_\phi > 0$ and , $\underline{\lambda}_\phi = 0$, then $p_\phi = p^*, \underline{\omega}_\phi = \underline{\omega}^*, \bar{\omega}_\phi = \bar{\omega}^*$ as described in Lemma 5. Consumption is constant across ϕ and determined by the binding promise-keeping constraint in (31). Alternatively, at the threshold realization $\tilde{\phi}(\nu)$, consumption must be the same as in Equation (38) which implies Equation (39).

Finally, we guess and verify that the threshold realization of ϕ where the IC starts binding is given by $\phi^* = \tilde{\phi}(\underline{\omega}^*) \leq \tilde{\phi}(\nu)$. Consider the promise-keeping constraint when the current promised utility is $\nu = \underline{\omega}^*$:

$$\begin{aligned} \underline{\omega}^* &= \int_0^{\phi^*} \left[u(\underline{c}_\phi) - X(p_\phi) + \beta\left[(1-p_\phi)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right]\right] dF(\phi) \\ &\quad + \int_{\phi^*}^{\phi_{\max}} \left[u(\underline{c}^*_{\phi^*}) + \zeta_{dev}\right] dF(\phi) \\ &= \int_0^{\phi^*} \left[W(0, \underline{w}) - \phi\right] dF(\phi) + \left(W(0, \underline{w}) - \phi^*\right)\left[1 - F(\phi^*)\right] \\ &= W(0, \underline{w}) - \int_0^{\phi^*} \phi dF(\phi) - \phi^*\left[1 - F(\phi^*)\right]. \end{aligned}$$

The comparison with the threshold function in (37) confirms the guess that $\phi^* = \tilde{\phi}(\underline{\omega}^*)$. ■

Proof of Proposition 12. The planner solves (30) subject to (31), (32), and (40). We write the Lagrangian,

$$\begin{aligned}
\underline{\Lambda} \;=\; & \int_{\aleph} \left[\underline{w} - \underline{c}_\phi + R^{-1}\left(\left(1-\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right)\underline{P}\left(\underline{\omega}_\phi\right) + \Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\bar{P}\left(\bar{\omega}_\phi\right)\right)\right] dF\left(\phi\right) \\
& + \underline{\mu}\left(\int_{\aleph}\left(u\left(c_\phi\right) - X\left(\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right) + \beta\left(\left(1-\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right)\underline{\omega}_\phi + \Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\bar{\omega}_\phi\right)\right) dF\left(\phi\right) - \nu\right) \\
& + \int_{\aleph} \underline{\lambda}_\phi \left(u\left(\underline{c}_\phi\right) - X\left(\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right) + \beta\left(\left(1-\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right)\underline{\omega}_\phi + \Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\bar{\omega}_\phi\right) - W(0,\underline{w}) + \phi\right) d\phi,
\end{aligned}$$

where $\underline{\mu}$ and $\underline{\lambda}_\phi$ denote the multipliers in recession. The first-order conditions with respect to $\underline{c}_\phi$, $\bar{\omega}_\phi$, and $\underline{\omega}_\phi$ yield

$$\begin{aligned}
0 &= -f(\phi) + \left[\underline{\mu} f(\phi) + \underline{\lambda}_\phi\right] u'(\underline{c}_\phi), \\
0 &= R^{-1}\left[\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\bar{P}'\left(\bar{\omega}_\phi\right) + \Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)\left[\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right]\right] f(\phi) \\
&\quad + \left[\underline{\mu} f(\phi) + \underline{\lambda}_\phi\right]\begin{bmatrix} -X'(\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi))\Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi) \\ +\beta\left[\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)1 + \Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)\left[\bar{\omega}_\phi - \underline{\omega}_\phi\right]\right]\end{bmatrix} - \\
0 &= R^{-1}\left[\left(1-\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right)\underline{P}'\left(\underline{\omega}_\phi\right) + \Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)(-1)\left[\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right]\right] f(\phi) \\
&\quad + \left[\underline{\mu} f(\phi) + \underline{\lambda}_\phi\right]\begin{bmatrix} -X'(\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi))\Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)(-1) \\ +\beta\left[\left(1-\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right)1 + \Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)(-1)\left[\bar{\omega}_\phi - \underline{\omega}_\phi\right]\right]\end{bmatrix}.
\end{aligned}$$

Moreover, the first-order condition for reform effort reads

$$X'(\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)) - \beta(\bar{\omega}_\phi - \underline{\omega}_\phi) = 0$$

and the envelope condition is given by

$$\underline{P}'\left(\nu\right) = -\underline{\mu}.$$

Combining the first-order conditions and the envelope condition yields

$$\frac{1}{u'(\underline{c}_\phi)} = -\underline{P}'\left(\nu\right) + \frac{\underline{\lambda}_\phi}{f(\phi)} \tag{69}$$

$$\beta R\frac{1}{u'(\underline{c}_\phi)} = -\left[\bar{P}'\left(\bar{\omega}_\phi\right) + \frac{\Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)}{\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)}\left[\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right]\right] \tag{70}$$

$$\beta R\frac{1}{u'(\underline{c}_\phi)} = -\left[\underline{P}'\left(\underline{\omega}_\phi\right) - \frac{\Upsilon'(\bar{\omega}_\phi - \underline{\omega}_\phi)}{1-\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)}\left[\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right]\right]. \tag{71}$$

Given that $\beta R = 1$, Equations (70) and (71) imply that (41) and (42) must hold for any realized default cost state ϕ.

For all states ϕ where the participation constraint is not binding, $\underline{\lambda}_\phi = 0$, the first-order condition in (69) satisfies

$$\frac{1}{u'(\underline{c}_\phi)} = -\underline{P}'\left(\nu\right). \tag{72}$$

In that case, the solution is history dependent, $\underline{c}_\phi = \underline{c}(\nu)$, $\underline{\omega}_\phi = \underline{\omega}(\nu)$, $\bar{\omega}_\phi = \bar{\omega}(\nu)$ and determined by (69), (41) and (42). For all states where the participation constraint is binding, $\underline{\lambda}_\phi > 0$, the participation constraint holds

$$u\left(\underline{c}_\phi\right) - X\left(\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right) + \beta \begin{pmatrix} \left(1 - \Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\right)\underline{\omega}_\phi \\ +\Upsilon(\bar{\omega}_\phi - \underline{\omega}_\phi)\bar{\omega}_\phi \end{pmatrix} = W(0, \underline{w}) - \phi.$$

In that case, the solution $\underline{c}_\phi$, $\underline{\omega}_\phi$, $\bar{\omega}_\phi$ is independent of ν and determined by the binding PC in (32), (41), and (42).

Finally, note that the threshold realization of ϕ where the constraint starts binding, $\tilde{\phi}(\nu)$, remains determined by the binding promise-keeping constraint

$$\begin{aligned} \nu &= \int_0^{\tilde{\phi}(\nu)} \left[W(0, \underline{w}) - \phi\right] dF(\phi) + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[\begin{array}{c} u\left(\underline{c}(\nu)\right) - X\left(\Upsilon(\bar{\omega}(\nu) - \underline{\omega}(\nu))\right) \\ +\beta\left(\left(1 - \Upsilon(\bar{\omega}(\nu) - \underline{\omega}(\nu))\right)\underline{\omega}(\nu) + \Upsilon(\bar{\omega}(\nu) - \underline{\omega}(\nu))\bar{\omega}(\nu)\right) \end{array} \right] dF(\phi) \\ &= W(0, \underline{w}) - \left[\int_0^{\tilde{\phi}(\nu)} \phi dF(\phi) + \left[1 - F(\tilde{\phi}(\nu))\right]\tilde{\phi}(\nu)\right]. \end{aligned}$$

Thus, $\underline{\lambda}_\phi = 0 \Leftrightarrow \phi \geq \tilde{\phi}(\nu)$ and $\underline{\lambda}_\phi > 0 \Leftrightarrow \phi < \tilde{\phi}(\nu)$. This concludes the proof of the proposition. ■

Proof of Proposition 13. We prove the proposition by deriving a contradiction. To this aim, suppose that, for $\bar{\Pi}(b) = \bar{P}(\nu)$, the planner can deliver more utility to the agent than can the market equilibrium. Namely, $\nu > EV(b, \bar{w})$. Then, since $\bar{P}$ is a decreasing strictly concave function, we must have that $\bar{P}(EV(b, \bar{w})) > \bar{P}(\nu)$ and $\bar{P}'(EV(b, \bar{w})) > \bar{P}'(\nu)$. We show that this inequality, along with the set of optimality conditions, induces a contradiction.

First, recall, that Equation (6) implies that $\bar{\Pi}(b) = RQ(b, \bar{w})\, b$. Thus,

$$\bar{P}(EV(b, \bar{w})) > \bar{P}(\nu) = RQ(b, \bar{w})\, b, \tag{73}$$

where $EV(b, \bar{w})$ is decreasing in b. Differentiating the two sides of the inequality (73) with respect to b yields

$$\bar{P}'(EV(b, \bar{w})) \times \frac{d}{db} EV(b, \bar{w}) > \frac{d}{db}\left[Q(b, \bar{w})\, b\right] \times R = 1 - F\left(\bar{\Phi}(b)\right), \tag{74}$$

where the right-hand side equality follows from the proof of Lemma 2. Next, Equation (49) implies that

$$\frac{d}{db} EV(b, \bar{w}) = -\left[1 - F\left(\bar{\Phi}(b)\right)\right] \times u'\left[C(b, \bar{w})\right],$$

where $C(b, \bar{w}) = Q(B(b, \bar{w}), \bar{w}) \times B(b, \bar{w}) + \bar{w} - b$ is the consumption level in the market equilibrium when the debt b is honored. Plugging the expression of $\frac{d}{db} EV(b, \bar{w})$ into (74), and simplifying terms, yields

$$u'(C(b, \bar{w})) > -\frac{1}{\bar{P}'(EV(b, \bar{w}))}. \tag{75}$$

Next, note that $C(b, \bar{w}) = \bar{c}(\nu)$. Equation (75) yields $u'(\bar{c}(\nu)) > -1/\bar{P}'(EV(b, \bar{w}))$, while (61) yields that $u'(\bar{c}(\nu)) = -1/\bar{P}'(\nu)$. Thus, the two conditions jointly imply that $\bar{P}'(\nu) > \bar{P}'(EV(b, \bar{w}))$ which in turn implies that $\nu < EV(b, \bar{w})$, since $\bar{P}$ is decreasing and concave. This contradicts the assumption that $\nu > EV(b, \bar{w})$.

The analysis thus far implies that $\nu \leq EV(b,\bar{w})$. We can also rule out that $\nu < EV(b,\bar{w})$, because it would contradict that the allocation chosen by the planner is constrained efficient. Therefore, $\nu = EV(b,\bar{w})$. ■

Proof of Proposition 14. The strategy of the proof is the same as that of Proposition 13. In particular, we prove that, if $p = p(EV(b,\underline{w}))$, i.e., effort is set at the constrained optimum level, then $\underline{\Pi}(b) = \underline{P}(\nu) \Leftrightarrow \nu = EV(b,\underline{w})$, where $\underline{\Pi}(b)$ is the valuation of debt conditional on staying in recession before the realization of ϕ. We prove this by deriving a contradiction. To this aim, suppose that, for $\underline{\Pi}(b) = \underline{P}(\nu)$, the planner can deliver more utility than the agent gets in the market equilibrium. Namely, $\nu > EV(b,\underline{w})$. Then, since $\underline{P}$ is a decreasing strictly concave function, we must have that $\underline{P}(EV(b,\underline{w})) > \underline{P}(\nu)$ and $\underline{P}'(EV(b,\underline{w})) > \underline{P}'(\nu)$. Note that, absent moral hazard, the price of recession-contingent debt is independent of the amount of recovery-contingent debt. It is therefore legitimate to define $\tilde{Q}_{\underline{w}}(b'_{\underline{w}}) \equiv Q_{\underline{w}}(b'_{\underline{w}}, b'_{\bar{w}})$.

First, the same argument invoked in the proof of Proposition 13 implies that $\underline{\Pi}(b) = \frac{R}{1-p}\tilde{Q}_{\underline{w}}(b)\,b$. Hence,

$$\underline{P}(EV(b,\underline{w})) > \underline{P}(\nu) = \frac{R}{1-p}\tilde{Q}_{\underline{w}}(b)\,b. \tag{76}$$

where $EV(b,\underline{w})$ is decreasing in b. Differentiating the two sides of the inequality (76) with respect to b yields:

$$\begin{aligned} & \underline{P}'(EV(b,\underline{w})) \times \frac{d}{db}EV(b,\underline{w}) \\ > \quad & \frac{R}{1-p}\frac{d}{db}\left(\tilde{Q}_{\underline{w}}(b)\,b\right) = -\left[1-F(\underline{\Phi}(b))\right], \end{aligned} \tag{77}$$

where the right-hand side equality follows from Equation (55). Next, Equation (54) implies that

$$\frac{d}{db}EV(b,\underline{w}) = -\left[1-F(\underline{\Phi}(b))\right] \times u'(C(b,\underline{w})),$$

where $C(b,\underline{w})$ is the consumption level assuming that the recession-contingent debt b is honored. Plugging in the expression of $\frac{d}{db}EV(b,\bar{w})$ allows us to simplify (77) as follows:

$$u'(C(b,\underline{w})) > -\frac{1}{\underline{P}'(EV(b,\underline{w}))}. \tag{78}$$

Next, note that $C(b,\underline{w}) = \underline{c}(\nu)$. Equation (78) yields $u'(\underline{c}(\nu)) > -\frac{1}{\underline{P}'(EV(b,\underline{w}))}$, while (72) yields that $u'(\underline{c}(\nu)) = -\frac{1}{\underline{P}'(\nu)}$. Thus, the two conditions jointly imply that $-\frac{1}{\underline{P}'(\nu)} > -\frac{1}{\underline{P}'(EV(b,\underline{w}))}$ which in turn implies that $\nu < EV(b,\underline{w})$, since $\underline{P}$ is decreasing and concave. This contradicts the assumption that $\nu > EV(b,\underline{w})$.

The analysis thus far establishes that $\nu \leq EV(b,\underline{w})$. We can also rule out that $\nu < EV(b,\underline{w})$ because it would contradict that the allocation chosen by the planner is constrained efficient. Therefore, $\nu = EV(b,\underline{w})$. ■

Proof of Proposition 15. We have already shown in Proposition 13 that the market equilibrium is equivalent to the planner solution once the economy has entered the absorbing normal time state.

Thus, we can limit the proof to the recession state. The strategy of the proof is to show that the first-order conditions of the planner problem and the market equilibrium are equivalent. Under our assumption that the first-order conditions of the planner problem are necessary and sufficient for the characterization of the optimal contract, this proves that the market equilibrium decentralizes the planner solution.

Let the solution of an interior state-contingent Markov equilibrium be denoted by the collection of functions

$$EV(b,w), C(\cdot,w), \Psi(\cdot,\cdot), B_{\bar{w}}(\cdot,w), B_{\underline{w}}(\cdot,w).$$

This solution necessarily satisfies the budget constraint in (17) and the necessary optimality conditions for reform effort and debt issuance in Equations (19), (22), and (23). We show in the following that an interior solution to the planner problem satisfies these optimality conditions and the budget constraint. Formally, we guess and verify that

$$\begin{array}{rcl}
EV(b,\underline{w}) & = & \nu \\
\underline{C}_{\phi}(b) & \equiv & C(\mathbb{B}(b,\phi,\underline{w}),\underline{w}) = \underline{c}_{\phi}(\nu) \\
\Psi_{\phi}(b) & \equiv & \Psi(B_{\underline{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w}), B_{\bar{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w})) = p_{\phi}(\nu) \\
\mathbb{B}(B_{\bar{w}}(b,\underline{w}),\phi',\bar{w}) & = & Q(B_{\bar{w}}(\mathbb{B}(B_{\bar{w}}(b,\underline{w}),\phi',\bar{w}),\bar{w})) + \bar{w} - \bar{c}_{\phi'}(\bar{\omega}_{\phi}(\nu)) \\
\mathbb{B}(b,\phi,\underline{w}) & = & \Psi_{\phi}(b) Q_{\bar{w}}(B_{\bar{w}}(\mathbb{B}(b,\phi,\underline{w}),\phi,\underline{w}),\underline{w})) \\
 & & +(1-\Psi_{\phi}(b)) Q_{\underline{w}}(B_{\underline{w}}(\mathbb{B}(b,\phi',\underline{w}),\underline{w})) + \underline{w} - \underline{C}_{\phi}(b),
\end{array}$$

can be implemented as a market equilibrium, given that the optimal contract yields zero profits in expectation to the planner

$$\begin{array}{rcl}
\underline{P}(\nu) & = & \left((1 - F(\underline{\Phi}(b)))\, b + \int_0^{\Phi(b)} \left(\underline{\Phi}^{-1}(\phi)\; dF(\phi) \right) \right) \\
 & \equiv & \underline{\Pi}(b).
\end{array}$$

Note that the zero profit condition in combination with the guess $EV(b,\underline{w}) = \nu$ implies that there must also be zero expected profits in the continuation

$$\begin{array}{rcl}
\bar{P}(\bar{\omega}_{\phi}(\nu)) & = & \bar{\Pi}\left(B_{\bar{w}}(\mathbb{B}(b,\phi,\underline{w})\right) \\
\underline{P}(\underline{\omega}_{\phi}(\nu)) & = & \underline{\Pi}\left(B_{\underline{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w})\right),
\end{array}$$

and that the continuation values are given by

$$\begin{array}{rcl}
EV(B_{\bar{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w}),\bar{w}) & = & \bar{\omega}_{\phi}(\nu) \\
EV(B_{\underline{w}}(\mathbb{B}(b,\phi,w),\underline{w}),\underline{w}) & = & \underline{\omega}_{\phi}(\nu).
\end{array}$$

By construction of the above solution candidate for the Markov equilibrium, the sovereign budget constraint in (17) and the optimality condition for reform effort in (19) are satisfied. Moreover, in the planner solution the agent is exactly indifferent of staying in the contract when

$$\left[\begin{array}{c} u(\underline{c}_{\phi}(\nu)) - X(p_{\phi}(\nu)) \\ +\beta\left(p_{\phi}(\nu)\bar{\omega}_{\phi}(\nu) + (1-p_{\phi}(\nu))\underline{\omega}_{\phi}(\nu)\right) \end{array} \right] = W(0,\underline{w}) - \phi.$$

This coincides with the indifference to renegotiate in the market equilibrium as

$$\begin{aligned}\left[\begin{array}{c} u(\underline{C}_\phi(b)) - X(\Psi_\phi(b)) \\ +\beta \left(\begin{array}{c} \Psi_\phi(b) EV(B_{\bar{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w}),\bar{w}) \\ +(1-\Psi_\phi(b)) EV(B_{\underline{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w}),\underline{w}) \end{array}\right) \end{array}\right] &= W(\mathbb{B}(b,\phi,\underline{w}),\underline{w}) \\ &= W(0,\underline{w}) - \phi, \end{aligned}$$

where the last equality follows from the indifference assumption. Thus, the threshold function of the planner solution coincides with the threshold function of the market equilibrium, $\tilde{\phi}(\nu) = \underline{\Phi}(b)$. As a consequence, the sovereign receives a value of $W(0,\underline{w}) - \phi$, for the same set of states - in the optimal contract and in the Markov equilibrium.

To show the equivalence of the dynamic optimality conditions, let us first simplify the notation by defining $b'_{\bar{w},\phi} = B_{\bar{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w})$ and $b'_{\underline{w},\phi} = B_{\underline{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w})$. The zero profit condition has two important implications:

1. The difference in profits across future states for the planner corresponds to the difference in the recovery values of the issued debt for the international investors

$$R \times \Delta(b'_{\underline{w},\phi}, b'_{\bar{w},\phi}) = \left[\bar{P}\left(\bar{\omega}_\phi(\nu)\right) - \underline{P}\left(\underline{\omega}_\phi(\nu)\right)\right],$$

where $\Delta(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})$ is defined in Equation (24) of Proposition (8) of the market equilibrium with GDP-linked debt.

2. The derivative of the promised utility with respect to the debt level is given by

$$\frac{d\underline{\omega}_\phi(\nu)}{db'_{\underline{w},\phi}} = \frac{1 - F\left(\underline{\Phi}\left(b'_{\underline{w},\phi}\right)\right)}{\underline{P}'\left(\underline{\omega}_\phi(\nu)\right)},$$

such that the derivative of the reform effort function in the market equilibrium can be written as

$$\begin{aligned}\frac{\partial \Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\partial b'_{\underline{w},\phi}} &= -\Upsilon'(\bar{\omega}_\phi(\nu) - \underline{\omega}_\phi(\nu))(-1)\frac{d\bar{\omega}_\phi(\nu)}{db_{\bar{w},\phi}} \\ &= -\Upsilon'(\bar{\omega}_\phi(\nu) - \underline{\omega}_\phi(\nu))\frac{1 - F\left(\underline{\Phi}\left(b'_{\underline{w},\phi}\right)\right)}{\underline{P}'\left(\underline{\omega}_\phi(\nu)\right)}.\end{aligned}$$

Thus, the intertemporal optimality conditions of the optimal contract in Equations (70) and (71) can be written as

$$\begin{aligned}\beta R \frac{1}{u'(\underline{c}_\phi(\nu))} &= -\left[\bar{P}'\left(\bar{\omega}_\phi(\nu)\right) + \frac{\partial \Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\partial b'_{\bar{w}}}\left(\frac{1 - F\left(\bar{\Phi}\left(b'_{\bar{w},\phi}\right)\right)}{\bar{P}'\left(\bar{\omega}_\phi(\nu)\right)}\right)^{-1} \frac{R \times \Delta(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}\right] \\ \beta R \frac{1}{u'(\underline{c}_\phi(\nu))} &= -\left[\underline{P}'\left(\underline{\omega}_\phi(\nu)\right) + \frac{\partial \Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\partial b'_{\underline{w},\phi}}\left(\frac{1 - F\left(\underline{\Phi}\left(b'_{\underline{w},\phi}\right)\right)}{\underline{P}'\left(\underline{\omega}_\phi(\nu)\right)}\right)^{-1} \frac{R \times \Delta\Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\left(1 - \Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})\right)}\right].\end{aligned}$$

We have shown in Proposition (11) that in all future states $\phi' \geq \underline{\Phi}(b'_{\underline{w},\phi})$ where the participation constraint is not binding, the optimal contract satisfies the optimality condition

$$\begin{aligned} -\underline{P}'\left(\underline{\omega}_\phi(\nu)\right) &= \frac{1}{u'(\underline{c}_{\phi'}(\underline{\omega}_\phi(\nu)))} \\ &= \frac{1}{u'(\underline{C}_{\phi'}(b'_{\underline{w},\phi},\underline{w}))}, \quad \phi' \geq \underline{\Phi}(b'_{\underline{w},\phi}), \end{aligned}$$

such that the conditional Euler Equations of the Markov equilibrium in (22) and (23) are indeed satisfied

$$\begin{aligned} \beta R \frac{u'(\bar{C}_{\phi'}(b'_{\bar{w},\phi}))}{u'(\underline{C}_\phi(b))} &= 1 + \frac{\partial \Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\partial b'_{\bar{w}}} \frac{R \times \Delta(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w},\phi}\right)\right)\right)} \\ \beta R \frac{u'(\underline{C}_{\phi'}(b'_{\underline{w},\phi}))}{u'(\underline{C}_\phi(b))} &= \left[1 + \frac{\partial \Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\partial b'_{\underline{w},\phi}} \frac{R \times \Delta(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})}{\left(1 - \Psi(b'_{\underline{w},\phi}, b'_{\bar{w},\phi})\right)\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w},\phi}\right)\right)\right)}\right], \end{aligned}$$

for $\phi' \geq \bar{\Phi}(b'_{\bar{w},\phi})$.

Finally, we verify the guess that the optimal solution to the planner problem yields the value ν in the Markov equilibrium by solving the functional equation

$$\begin{aligned} EV(b,\underline{w}) &= \int_{\aleph} \left[\begin{array}{c} u(\underline{C}_\phi(b)) - X(\Psi_\phi(b)) \\ +\beta \left(\begin{array}{c} \Psi_\phi(b) EV(B_{\bar{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w}),\bar{w}) \\ +(1-\Psi_\phi(b)) EV(B_{\underline{w}}(\mathbb{B}(b,\phi,\underline{w}),\underline{w}),\underline{w}) \end{array} \right) \end{array} \right] dF(\phi) \\ &= \int_{\aleph} \left[u\left(\underline{c}_\phi(\nu)\right) - X(p_\phi(\nu)) + \beta\left[p_\phi(\nu)\bar{\omega}_\phi(\nu) + (1-p_\phi(\nu))\underline{\omega}_\phi(\nu)\right]\right] dF(\phi) \\ &= \nu, \end{aligned}$$

where the last equality follows from the binding promise-keeping constraint of the planner problem. This confirms the initial guess that $EV(b,\underline{w}) = \nu$. Thus, if the optimality conditions of the planner problem are necessary and sufficient to characterize the optimal solution, then the Markov equilibrium with GDP-linked debt and the solution to the planner problem must be equivalent. ■

B Appendix B: Additional technical analysis

This appendix contains additional technical analysis. In particular, it provides: (i) technical details that complete the proof of Proposition 4 in Appendix A; (ii) technical details of the analysis in Section 4; (iii) an extension involving learning; (iv) the numerical algorithm used in Section 5; (v) an exact decomposition of the welfare effect (into a level effect, a volatility effect, and a discounting effect) of going from a stationary allocation to the first best; and (vi) some additional figures.

B.1 Details of the proof of Proposition 4

This section completes the proof of Proposition 4 in Appendix A. We have established above that an interior solution necessarily satisfies the FOC

$$\begin{aligned} & \left[1 - F(\bar{\Phi}(B(b,\bar{w})))\right] u'(Q(B(b,\bar{w}),\bar{w})B(b,\bar{w}) - b + \bar{w}) \\ = & \left[1 - F(\bar{\Phi}(B(b,\bar{w})))\right] u'(Q(B(B(b,\bar{w}),\bar{w}))B(B(b,\bar{w}),\bar{w}) - B(b,\bar{w}) + \bar{w}). \end{aligned}$$

Both, constant debt accumulation, $B(b,\bar{w}) = b$ and maximal debt accumulation $B(b,\bar{w}) = \bar{b}$ are obvious solution candidates. Note however, that $B(b,\bar{w}) = \bar{b}$ can only be a global maximum when the outstanding debt level is at the maximum, $b = \bar{b}$ (where the two solution candidates coincide), because otherwise the objective is strictly falling in the left neigborhood of $\bar{b}$ since

$$u'(Q(\bar{b},\bar{w})\bar{b} - b + \bar{w}) < u'(Q(\bar{b},\bar{w})\bar{b} - \bar{b} + \bar{w}), \quad b < \bar{b}.$$

Thus, we are therefore left to show that $B(b,\bar{w}) = b$ is the unique solution that satisfies the FOC. For the ease of exposition, let us rewrite the FOC as

$$\begin{aligned} & \left[1 - F(\bar{\Phi}(b'))\right] u'(Q(b',\bar{w})b' - b + \bar{w}) \\ = & \left[1 - F(\bar{\Phi}(b'))\right] u'(Q(b'',\bar{w})b'' - b' + \bar{w}). \end{aligned}$$

Suppose there existed a solution candidate where the current debt accumulation b' was *strictly reduced*, $b - b' > 0$. Because the marginal bond revenue is falling and smaller than R^{-1} this leads to a smaller reduction of today's consumption relative the increase in next period's consumption for a given b''. Therefore, b'' has to be lowered even further to equalize consumption intertemporally, $b' - b'' > b - b' > 0$. This argument can be expanded to further periods such that the equilibrium would feature accelerated *asset* accumulation and ever falling consumption which contradicts the requirement that it is a global maximum. Suppose to the contrary that current debt accumulation b' was *strictly increased*, $b' - b > 0$. Then, by the same argument as before, b'' has to be increased even further to equalize consumption intertemporally, $b'' - b' > b' - b > 0$, and the equilibrium would feature accelerated *debt* accumulation. This implies that the economy will hit the upper bound on debt accumulation $\bar{b}$ for some outstanding debt level below the maximum, $b < \bar{b}$. However, we have already shown that this cannot be optimal. Thus, $B(b,\bar{w}) = b$ is the unique maximizer of the objective function.

B.2 Formal properties of the analysis of Section 4

Proposition 16 *There exists unique profit functions $\bar{P}$ and $\underline{P}$ that solve the programs (25) and (30), respectively. Moreover, $\bar{P}$ and $\underline{P}$ are continuously differentiable and strictly concave. Given* *the promised utility ν and the realization ϕ, (i) if $w = \bar{w}$ there exists a unique optimal pair of promised utility and consumption $\{\bar{\omega}_\phi(\nu), c_\phi(\nu)\}$; (ii) if $w = \underline{w}$ there exists a unique optimal 4-tuple of promised utilities, consumption and effort, $\{\underline{\omega}_\phi(\nu), \bar{\omega}_\phi(\nu), p_\phi(\nu), c_\phi(\nu)\}$. The first-order conditions in Propositions 9 and 10 are necessary and sufficient when the solution is interior.*

The proof strategy follows Thomas and Worrall (1990, Proof of Proposition 1), i.e., we show that the problem is a contraction mapping to establish the uniqueness and strict concavity of $\bar{P}$ and $\underline{P}$. The differentiabilty of $\bar{P}$ and $\underline{P}$ follows from an application of Lemma 1 in Benveniste and Scheinkman (1979). Finally, we prove that $\bar{P}$ and $\underline{P}$ pin down uniquely promised utilities, effort and consumption.

The arguments used to prove Proposition 16 in normal times and recession are mirror image of each other, except that the recession case is complicated by the presence of an effort choice. For this reason, we prove the results when $w = \underline{w}$ (assuming the properties of $\bar{P}$ follow the proposition), omitting the simpler proof for the case in which $w = \bar{w}$ (more precisely, the arguments are extended by setting $X(p_\phi) = 0$ and $p_\phi = 1$).

We prove the results in the form of three lemmas and one corollary. We first proof the above Proposition for the case where the planner's choice is not restricted by the incentive constraint, and then generalize to the case with the incentive constraint.

Define, first, the mapping $\underline{T}(x)(\nu)$ as the right-hand side of the planner's functional equation

$$\underline{T}(x)(\nu) = \max_{\left(\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi\in\aleph}\right)\in\underline{\Lambda}(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \begin{bmatrix} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi)x(\underline{\omega}_\phi) \end{bmatrix}\right] dF(\phi)$$

where maximization is constrained by the set $\underline{\Lambda}(\nu)$ defined by

$$\begin{aligned} \int_\aleph \left[u(c_\phi) - X(p_\phi) + \beta\left[p_\phi\bar{\omega}_\phi + (1-p_\phi)\underline{\omega}_\phi\right]\right] dF(\phi) \quad &\geq \quad \nu \\ u(c_\phi) - X(p_\phi) + \beta\left[p_\phi\bar{\omega}_\phi + (1-p_\phi)\underline{\omega}_\phi\right] \quad &\geq \quad \underline{\nu} - \phi, \quad \forall\phi\in\aleph, \\ c_\phi \in [0,\bar{w}],\ p_\phi \in [\underline{p},\bar{p}],\ \nu, \underline{\omega}_\phi \in [\underline{\nu} - E[\phi], \underline{\nu}],\ \bar{\omega}_\phi \quad &\in \quad [\bar{\nu} - E[\phi], \bar{\nu}]. \end{aligned}$$

Recall that $\underline{\nu} = W(0, \underline{w})$ and $\bar{\nu} = W(0, \bar{w})$ are the values of the outside option during recession and normal times, respectively. We take as given the uniqueness, strict concavity, and differentiability of the profit function in normal times, $\bar{P}$. Moreover, let the profit in normal times be bounded between $\bar{P}_{MIN} = 0$ and $P_{MAX} = \bar{w}/(1-\beta)$.

Lemma 6 *$\underline{T}(x)$ maps concave functions into strictly concave functions.*

Proof. Let $\nu' \neq \nu'' \in [\underline{\nu} - E[\phi], \underline{\nu}]$, $\delta \in (0,1)$, $\nu^o = \delta\nu' + (1-\delta)\nu''$, $\underline{P}_k(\nu) = \underline{T}(\underline{P}_{k-1})(\nu)$, and $\underline{P}_{k-1}$ be concave. Then,

$$\underline{P}_{k-1}(\delta\nu' + (1-\delta)\nu'') \geq \delta\underline{P}_{k-1}(\nu') + (1-\delta)\underline{P}_{k-1}(\nu'').$$

We follow the strategy of Thomas and Worrall (1990, Proof of Proposition 1), i.e., we construct a feasible but (weakly) suboptimal contract, $\left\{c^o_\phi, p^o_\phi, \bar{\omega}^o_\phi, \underline{\omega}^o_\phi\right\}_{\phi\in\aleph}$, such that even the profit generated by the suboptimal contract $\underline{P}^o_k(\delta\nu' + (1-\delta)\nu'') \leq \underline{P}_k(\delta\nu' + (1-\delta)\nu'')$ dominates the linear combination of maximal profits $\delta\underline{P}_k(\nu') + (1-\delta)\underline{P}_k(\nu'')$. Define the weights $\underline{\delta}, \bar{\delta} \in (0,1)$ and the 4-tuple $(c^o_\phi, p^o_\phi, \underline{\omega}^o_\phi, \bar{\omega}^o_\phi)$

such that

$$\begin{aligned}
\underline{\delta} &\equiv \frac{\delta\left[1-p_\phi(\nu')\right]}{\delta(1-p_\phi(\nu'))+(1-\delta)(1-p_\phi(\nu''))} \equiv \delta\frac{1-p_\phi(\nu')}{1-p^o_\phi(\nu^o)} \\
\bar{\delta} &\equiv \frac{\delta p_\phi(\nu')}{\delta p_\phi(\nu')+(1-\delta)p_\phi(\nu'')} \equiv \delta\frac{p_\phi(\nu')}{p^o_\phi(\nu^o)} \\
\underline{\omega}^o_\phi(\nu^o) &= \underline{\delta}\underline{\omega}_\phi(\nu')+(1-\underline{\delta})\underline{\omega}_\phi(\nu'') \\
\bar{\omega}^o_\phi(\nu^o) &= \bar{\delta}\bar{\omega}_\phi(\nu')+(1-\bar{\delta})\bar{\omega}_\phi(\nu'') \\
c^o_\phi(\nu^o) &= u^{-1}\left[\delta u(c_\phi(\nu'))+(1-\delta)u(c_\phi(\nu''))\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
(1-p^o_\phi(\nu^o))\underline{\omega}^o_\phi(\nu^o) &= \delta\left(1-p_\phi(\nu')\right)\underline{\omega}_\phi(\nu')+(1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'') \\
p^o_\phi(\nu^o)\bar{\omega}^o_\phi(\nu^o) &= \delta p_\phi(\nu')\bar{\omega}_\phi(\nu')+(1-\delta)p_\phi(\nu'')\bar{\omega}_\phi(\nu'')
\end{aligned}$$

By construction the suboptimal allocation satisfies

$$c^o_\phi \in [0,\bar{w}],\ p^0_\phi \in [\underline{p},\bar{p}],\ \underline{\omega}^o_\phi \in [\underline{\nu}-E\left[\phi\right],\underline{\nu}],\ \bar{\omega}^o_\phi \in [\bar{\nu}-E\left[\phi\right],\bar{\nu}],$$

and, given the promised-utility ν^o, is also consistent with the promise-keeping constraint

$$\begin{aligned}
&\int_\aleph \left[u\left(c^o_\phi(\nu^o)\right)-X(p^o_\phi(\nu^o))+\beta\left[(1-p^o_\phi(\nu^o))\underline{\omega}^o_\phi(\nu^o)+p^o_\phi(\nu^o)\bar{\omega}^o_\phi(\nu^o)\right]\right]dF(\phi) \\
=\ &\int_\aleph \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right)+(1-\delta)\left(c_\phi(\nu'')\right)-X\left(\delta p_\phi(\nu')+(1-\delta)p_\phi(\nu'')\right) \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu')+(1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu')+(1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right]dF(\phi) \\
>\ &\int_\aleph \left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right)+(1-\delta)\left(c_\phi(\nu'')\right)-\left[\delta X(p_\phi(\nu')+(1-\delta)X(p_\phi(\nu''))\right] \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu')+(1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu')+(1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right]dF(\phi) \\
=\ &\delta\nu'+(1-\delta)\nu''=\nu^o.
\end{aligned}$$

The fact that $X\left(\delta p_\phi(\nu')+(1-\delta)p_\phi(\nu'')\right)<\delta X(p_\phi(\nu')+(1-\delta)X(p_\phi(\nu''))$ follows from the convexity of X. Moreover, the participation constraint for any ϕ yields

$$\begin{aligned}
&u\left(c^o_\phi(\nu^o)\right)-X(p^o_\phi(\nu^o))+\beta\left[(1-p^o_\phi(\nu^o))\underline{\omega}^o_\phi(\nu^o)+p^o_\phi(\nu^o)\bar{\omega}^o_\phi(\nu^o)\right] \\
=\ &\left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right)+(1-\delta)\left(c_\phi(\nu'')\right)-X(\delta p_\phi(\nu')+(1-\delta)p_\phi(\nu'')) \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu')+(1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu')+(1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right] \\
>\ &\left[\begin{array}{c} \delta u\left(c_\phi(\nu')\right)+(1-\delta)\left(c_\phi(\nu'')\right)-\left[\delta X(p_\phi(\nu')+(1-\delta)X(p_\phi(\nu''))\right] \\ +\beta\left[\delta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu')+(1-\delta)(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')\right] \\ +\beta\left[\delta p_\phi(\nu')\bar{\omega}_\phi(\nu')+(1-\delta)p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \end{array}\right] \\
=\ &\delta\left[u\left(c_\phi(\nu')\right)-X(p_\phi(\nu')+\beta(1-p_\phi(\nu'))\underline{\omega}_\phi(\nu')+\beta p_\phi(\nu')\bar{\omega}_\phi(\nu')\right] \\
&+(1-\delta)\left[c_\phi(\nu'')-X(p_\phi(\nu''))+\beta(1-p_\phi(\nu''))\underline{\omega}_\phi(\nu'')+\beta p_\phi(\nu''))\bar{\omega}_\phi(\nu'')\right] \\
\geq\ &\delta\left(\underline{\nu}-\phi\right)+(1-\delta)\left(\underline{\nu}-\phi\right)=\underline{\nu}-\phi,
\end{aligned}$$

where we used again the strict convexity of the cost function X. Thus, we have proven that the suboptimal allocation $\left\{c^o_\phi, p^o_\phi, \underline{\omega}^o_\phi, \bar{\omega}^o_\phi\right\}_{\phi\in\aleph}$ is feasible. Namely, it satisfies the participation constraints and delivers at least the promised utility ν^o. The profit function evaluated at the optimal contract $\left\{c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right\}_{\phi\in\aleph}$ then implies the following inequality,

$$
\begin{aligned}
& \delta \underline{P}_k(\nu') + (1-\delta)\underline{P}_k(\nu'') \\
= \;& \delta \underline{T}(\underline{P}_{k-1})(\nu') + (1-\delta)\underline{T}(\underline{P}_{k-1})(\nu'') \\
= \;& \int_\aleph \left[\begin{array}{c} \underline{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] + \\ \beta\left[\delta(1-p_\phi(\nu'))\underline{P}_{k-1}(\underline{\omega}_\phi(\nu')) + (1-\delta)(1-p_\phi(\nu''))\underline{P}_{k-1}(\underline{\omega}_\phi(\nu''))\right] \\ \beta\left[\delta p_\phi(\nu')\bar{P}(\bar{\omega}_\phi(\nu')) + (1-\delta)p_\phi(\nu'')\bar{P}(\bar{\omega}_\phi(\nu''))\right] \end{array} \right] dF(\phi) \\
= \;& \int_\aleph \left[\begin{array}{c} \underline{w} - [\delta c_\phi(\nu') + (1-\delta)c_\phi(\nu'')] + \\ \beta(1-p^o_\phi(\nu^o))\left[\underline{\delta}\underline{P}_{k-1}(\underline{\omega}_\phi(\nu')) + (1-\underline{\delta})\underline{P}_{k-1}(\underline{\omega}_\phi(\nu''))\right] \\ \beta p^o_\phi(\nu^o)\left[\bar{\delta}\bar{P}(\bar{\omega}_\phi(\nu')) + (1-\bar{\delta})\bar{P}(\bar{\omega}_\phi(\nu''))\right] \end{array} \right] dF(\phi) \\
< \;& \int_\aleph \left[\begin{array}{c} \underline{w} - u^{-1}\left(\delta u(c_\phi(\nu')) + (1-\delta)u(c_\phi(\nu''))\right) + \\ \beta(1-p^o_\phi(\nu^o))\underline{P}_{k-1}(\underline{\delta}\underline{\omega}_\phi(\nu') + (1-\underline{\delta})\underline{\omega}_\phi(\nu'')) \\ \beta p^o_\phi(\nu^o)\bar{P}(\bar{\delta}\bar{\omega}_\phi(\nu') + (1-\bar{\delta})\bar{\omega}_\phi(\nu'')) \end{array} \right] dF(\phi) \\
= \;& \int_\aleph \left[\underline{w} - c^o_\phi(\nu^o) + \beta\left[p^o_\phi(\nu^o)\bar{P}(\bar{\omega}^o_\phi(\nu^o)) + (1-p^o_\phi(\nu^o))\underline{P}_{k-1}(\underline{\omega}^o_\phi(\nu^o))\right]\right] dF(\phi) \\
\equiv \;& \underline{P}^o_k(\nu^o) \le \underline{P}_k(\nu^o) = \underline{P}_k(\delta\nu' + (1-\delta)\nu'').
\end{aligned}
$$

The first inequality follws from the strict concavity of the utility function and the profit function in normal times, along with the assumed concavity of $\underline{P}_{k-1}$. The second inequality, $\underline{P}_k(\nu^o) \ge \underline{P}^o_k(\nu^o)$, follows from the fact that the optimal allocation delivers (weakly) larger profits than the suboptimal one. We conclude that $\underline{P}_k(\delta\nu' + (1-\delta)\nu'') > \delta\underline{P}_k(\nu') + (1-\delta)\underline{P}_k(\nu'')$, i.e., $\underline{P}_k$ is strictly concave. This concludes the proof of the lemma. ■

Let $\underline{\Omega}$ denote the space of continuous functions defined over the interval $[\underline{\nu} - E[\phi], \underline{\nu}]$ and bounded between $\underline{P}_{MIN} = -(\bar{w} - \underline{w})/(1-\beta)$ and $P_{MAX} = \bar{w}/(1-\beta)$. Moreover, let d_∞ denote the supremum norm, such that $(\underline{\Omega}, d_\infty)$ is a complete metric space.

Lemma 7 *The mapping $\underline{T}(x)$ is an operator on the complete metric space $(\underline{\Omega}, d_\infty)$, $\underline{T}(x)$ is a contraction mapping with a unique fixed-point $\underline{P} \in \underline{\Omega}$.*

Proof. By the Theorem of the Maximum $\underline{T}(x)(\nu)$ is continuous in ν. Moreover, $\underline{T}(x)(\nu)$ is bounded between $\underline{P}_{MIN}$ and P_{MAX} since even choosing zero consumption for any realization of ϕ would induce profits not exceeding P_{MAX}

$$
\begin{aligned}
\underline{w} + \beta\int_\aleph \left[p_\phi\bar{P}(\bar{\omega}_\phi) + (1-p_\phi)x(\underline{\omega}_\phi)\right] dF(\phi) &< \bar{w} + \beta/(1-\beta)\bar{w} \\
&= \bar{w}/(1-\beta) = P_{MAX},
\end{aligned}
$$

and choosing the maximal consumption of $\bar{w}$ for any ϕ would induce profits no lower than $\underline{P}_{MIN}$,

$$
-(\bar{w} - \underline{w}) + \beta\int_\aleph x(\omega_\phi)dF(\phi) \ge \underline{P}_{MIN}.
$$

Thus, $\underline{T}(x)(\nu)$ is indeed an operator on $(\underline{\Omega}, d_\infty)$.

According to Blackwell's sufficient conditions $\underline{T}$ is a contraction mapping (see Lucas and Stokey (1989, Theorem 3.3) if: (i) $\underline{T}$ is monotone, (ii) $\underline{T}$ discounts.

1. Monotonicity: Let $x, y \in \underline{\Omega}$ with $x(\nu) \geq y(\nu)$, $\forall \nu \in [\underline{\nu} - E[\phi], \underline{\nu}]$. Then

$$\begin{aligned}
\underline{T}(x)(\nu) &= \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi\in\aleph}) \in \underline{\Lambda}(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi) x(\underline{\omega}_\phi) \end{array}\right]\right] dF(\phi) \\
&\geq \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi\in\aleph}) \in \underline{\Lambda}(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi) y(\underline{\omega}_\phi) \end{array}\right]\right] dF(\phi) \\
&= \underline{T}(y)(\nu).
\end{aligned}$$

2. Discounting: Let $x \in \underline{\Omega}$ and $a \geq$ be a real constant. Then

$$\begin{aligned}
\underline{T}(x+a)(\nu) &= \max_{(\{c_\phi, p_\phi, \bar{\omega}_\phi, \underline{\omega}_\phi\}_{\phi\in\aleph}) \in \underline{\Lambda}(\nu)} \int_\aleph \left[\underline{w} - c_\phi + \beta \left[\begin{array}{c} p_\phi \bar{P}(\bar{\omega}_\phi) \\ +(1-p_\phi) \left(x(\underline{\omega}_\phi) + a\right) \end{array}\right]\right] dF(\phi) \\
&\leq \underline{T}(x)(\nu) + \beta a \\
&< \underline{T}(x)(\nu) + a,
\end{aligned}$$

since $\beta < 1$.

Thus, $\underline{T}$ is indeed a contraction mapping and according to Banach's fixed-point theorem (see Lucas and Stokey (1989, Theorem 3.2)) there exists a unique fixed-point $\underline{P} \in \underline{\Omega}$ satisfying the stationary functional equation,

$$\underline{P}(\nu) = \underline{T}(\underline{P})(\nu).$$

■

Corollary 2 *The profit function $\underline{P}(\nu)$ is strictly concave in $\nu \in [\underline{\nu} - E[\phi], \underline{\nu}]$.*

This follows immediately from Lucas and Stokey (1989, Corollary 1). Since the unique fixed-point of $\underline{T}$ is the limit of applying the operator n times $\underline{T}^n(x)(\nu)$ starting from any (and, in particular the concave ones) element x in $\underline{\Omega}$, and the operator $\underline{T}$ maps concave into strictly concave functions the fixed-point $\underline{P}$ must be strictly concave.

Lemma 8 *The profit function $\underline{P}(\nu)$ is continuously differentiable in $\nu \in [\underline{\nu} - E[\phi], \underline{\nu}]$.*

Proof. The proof is an application of Benveniste and Scheinkman (1979, Lemma 1). Recall that $\bar{P}$ and $\underline{P}$ are strictly concave. Consider the pseudo profit function

$$\begin{aligned}
\underline{\widetilde{P}}(\tilde{\nu}, \nu) \equiv & \int_0^{\tilde{\phi}(\tilde{\nu})} \left[\underline{w} - \tilde{c}_\phi(\tilde{\nu}) + \beta\left[p_\phi(\nu)\bar{P}(\bar{\omega}_\phi(\nu)) + (1-p_\phi(\nu))\underline{P}(\underline{\omega}_\phi(\nu))\right]\right] dF(\phi) \qquad (79) \\
& + \int_{\tilde{\phi}(\tilde{\nu})}^\infty \left[\underline{w} - \tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}) + \beta\left[p_\phi(\nu)\bar{P}(\bar{\omega}_\phi(\nu)) + (1-p_\phi(\nu))\underline{P}(\underline{\omega}_\phi(\nu))\right]\right] dF(\phi)
\end{aligned}$$

where the triplet $(p_\phi(\nu), \underline{\omega}_\phi(\nu), \bar{\omega}_\phi(\nu))$ is the same as in the optimal contract given an initial promise ν, and the consumption function, $\tilde{c}_\phi(\tilde{\nu})$, is defined implicitly by the condition

$$\begin{aligned}\tilde{\nu} &= \int_0^{\tilde{\phi}(\tilde{\nu})} \left[u\left(\tilde{c}_\phi(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) + p_\phi(\nu)\bar{\omega}_\phi(\nu)\right]\right] dF(\phi) \\ &+ \int_{\tilde{\phi}(\tilde{\nu})}^{\infty} \left[u\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[(1-p_\phi(\nu))\underline{\omega}_\phi(\nu) + p_\phi(\nu)\bar{\omega}_\phi(\nu)\right]\right] dF(\phi). \qquad (80)\end{aligned}$$

Note that for $\tilde{\nu} = \nu$, Equation (80) is equivalent to the promise-keeping constraint. Moreover, for all states $\phi \leq \tilde{\phi}(\tilde{\nu})$ such that the (pseudo-)participation constraint is binding,

$$u\left(\tilde{c}_\phi(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[\begin{array}{c}(1-p_\phi(\nu))\underline{\omega}_\phi(\nu)\\ +p_\phi(\nu)\bar{\omega}_\phi(\nu)\end{array}\right] = \underline{\nu} - \phi. \qquad (81)$$

Otherwise, when $\phi > \tilde{\phi}(\tilde{\nu})$ then consumption and promised utility are history dependent, implying that

$$u\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right) - X(p_\phi(\nu)) + \beta\left[\begin{array}{c}(1-p_\phi(\nu))\underline{\omega}_\phi(\nu)\\ +p_\phi(\nu)\bar{\omega}_\phi(\nu)\end{array}\right] = \underline{\nu} - \tilde{\phi}(\tilde{\nu}). \qquad (82)$$

Substituting in the right hand-side of (81) and (82), respectively, in the first and second line of (80), pins down the threshold $\tilde{\phi}(\tilde{\nu})$ that separates states in which the participation constraint is binding from states with in which it is not binding:

$$\tilde{\nu} = \underline{\nu} - \int_0^{\tilde{\phi}(\tilde{\nu})} \phi dF(\phi) - \left(1 - F\left(\tilde{\phi}(\tilde{\nu})\right)\right) \times \tilde{\phi}(\tilde{\nu}).$$

Differentiating Equation (81) with respect to $\tilde{\nu}$ shows that, for $\phi \leq \tilde{\phi}(\tilde{\nu})$, $u'\left(\tilde{c}_\phi(\tilde{\nu})\right)\tilde{c}'_\phi(\tilde{\nu}) = 0$. Differentiating Equation (80) shows that the consumption function, $\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})$, is also continuously differentiable when $\phi > \tilde{\phi}(\nu)$. In particular,

$$1 = (1 - F(\tilde{\phi}(\nu)))u'\left(\tilde{c}_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu})\right)\tilde{c}'_{\tilde{\phi}(\tilde{\nu})}(\tilde{\nu}). \qquad (83)$$

Recall that the function $\underline{\widetilde{P}}$ has the properties that $\underline{\widetilde{P}}(\tilde{\nu}, \nu) \leq \underline{P}(\nu)$ with $\underline{\widetilde{P}}(\nu, \nu) = \underline{P}(\nu)$. Thus, Lemma 1 in Benveniste and Scheinkman (1979) implies that the profit function $\underline{P}(\nu)$ is continuously differentiable in $\nu \in [\underline{\nu} - E[\phi], \underline{\nu}]$, with derivative

$$\begin{aligned}\underline{P}'(\nu) &= \underline{\widetilde{P}}_{\tilde{\nu}}(\nu, \nu) = -(1 - F(\tilde{\phi}(\nu)))\tilde{c}'_{\tilde{\phi}(\nu)}(\nu) \\ &= -1/u'\left[(c(\nu))\right] < 0.\end{aligned}$$

The value of $\underline{\widetilde{P}}_{\tilde{\nu}}$ follows from the differentiation of (79) using standard methods. The last equality follows from (83) and from the fact that $\tilde{c}_{\tilde{\phi}(\nu)}(\nu) = c(\nu)$. This establishes that the profit function $\underline{P}(\nu)$ is continuously differentiable, concluding the proof.

We can now establish that the constrained allocation is unique. ■

Lemma 9 *The constrained-optimal allocation is characterized by a unique 4-tuple of state-contingent promised utilities, consumption and effort levels,* $\left\{\underline{\omega}_\phi(\nu), \bar{\omega}_\phi(\nu), p_\phi(\nu), c_\phi(\nu)\right\}$.

Proof. Lemma 6 implies that there cannot be two optimal contracts with distinct promised-utilities. Suppose not, so that there exists a 4-tuple of promised utilities $\left\{\underline{\omega}'_\phi, \underline{\omega}''_\phi, \bar{\omega}'_\phi, \bar{\omega}''_\phi\right\}$ such that either $\underline{\omega}'_\phi(\nu) = \underline{\omega}''_\phi(\nu)$ or $\bar{\omega}'_\phi(\nu) \neq \bar{\omega}''_\phi(\nu)$ (or both). Then, from the strict concavity of $\underline{P}$ and $\bar{P}$, it would be possible to construct a feasible allocation that dominates the continuation profit implied by the proposed optimal allocations, i.e., either $\underline{P}(\delta\underline{\omega}'_\phi + (1-\underline{\delta})\underline{\omega}''_\phi) > \underline{\delta}\underline{P}(\underline{\omega}'_\phi) + (1-\underline{\delta})\underline{P}(\underline{\omega}''_\phi)$, or $\bar{P}(\bar{\delta}\bar{\omega}'_\phi + (1-\bar{\delta})\bar{\omega}''_\phi) > \bar{\delta}\bar{P}(\bar{\omega}'_\phi) + (1-\bar{\delta})\bar{P}(\bar{\omega}''_\phi)$ (or both). This contradicts the assumption that the proposed allocations are optimal, establishing that the optimal contract pins down a unique pair of promised utilities, $\left\{\underline{\omega}_\phi, \bar{\omega}'_\phi\right\}$.

Finally, we show that a unique pair of promised utilities pins down uniquely effort and consumption. More formally, the first order conditions imply that

$$\begin{aligned} X'\left(p_\phi\right) &= \beta\left(-\underline{P}'\left(\underline{\omega}_\phi\right)^{-1}\left(\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right) + \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right), \\ -\underline{P}'\left(\underline{\omega}_\phi\right)^{-1} &= u'(c_\phi), \end{aligned}$$

implying that, given ν and ϕ, effort and consumption are uniquely determined. ■

B.2.1 Incentive constraint

The proof of the profit function's strict concavity and differentiability when the planner problem includes the incentive constraint for reform effort provision is by-and-large a corollary of the case without the additional incentive constraint. Moreover, we have already shown that the optimal allocation is unique when the incentive constraint is binding.

We know from Proposition 10 that for $\nu > \underline{\omega}^*$ the additional incentive constraint is never relevant, thus strict concavity and the differentiability of the profit function follows immediately from the above analysis. On the other hand, if $\nu \leq \underline{\omega}^*$, than the profit function evaluated at the optimal contract reads as

$$\begin{aligned} \underline{P}(\nu) &= \int_0^{\tilde{\phi}(\underline{\omega}^*)} \left[\underline{w} - \underline{c}_\phi + \beta\left[(1-p_\phi)\underline{P}(\underline{\omega}_\phi) + p_\phi\bar{P}(\bar{\omega}_\phi)\right]\right] dF(\phi) \\ &+ \int_{\tilde{\phi}(\underline{\omega}^*)}^{\tilde{\phi}(\nu)} \left[\underline{w} - \underline{c}^*_\phi + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^*\bar{P}(\bar{\omega}^*)\right]\right] dF(\phi) \\ &+ \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[\underline{w} - \underline{c}^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^*\bar{P}(\bar{\omega}^*)\right]\right] dF(\phi). \end{aligned}$$

Note that the promised-utility ν only enters the last two terms such that the first derivative of the profit function is given by (we will prove differentiability of the profit function below)

$$\begin{aligned} \underline{P}'(\nu) &= \left[\underline{w} - \underline{c}^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^*\bar{P}(\bar{\omega}^*)\right]\right] f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu) \\ &- \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{d\underline{c}^*_{\tilde{\phi}(\nu)}}{d\tilde{\phi}(\nu)} \tilde{\phi}'(\nu) dF(\phi) \\ &- \left[\underline{w} - \underline{c}^*_{\tilde{\phi}(\nu)} + \beta\left[(1-p^*)\underline{P}(\underline{\omega}^*) + p^*\bar{P}(\bar{\omega}^*)\right]\right] f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu) \\ &= -\int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{d\underline{c}^*_{\tilde{\phi}(\nu)}}{d\nu} dF(\phi) < 0, \end{aligned}$$

where Equation (39) implies that $d\underline{c}^*_{\tilde{\phi}(\nu)}/d\nu = -u'(\underline{c}^*_{\tilde{\phi}(\nu)})^{-1}\tilde{\phi}'(\nu) > 0$ as the threshold $\tilde{\phi}(\nu)$ is decreasing in the promised utility. The negative second derivative follows immediately

$$\underline{P}''(\nu) = -\left[\int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \frac{d^2\underline{c}^*_{\tilde{\phi}(\nu)}}{d\nu^2} dF(\phi) + \frac{d\underline{c}^*_{\tilde{\phi}(\nu)}}{d\nu} f(\tilde{\phi}(\nu))(-\tilde{\phi}'(\nu))\right] < 0,$$

because

$$\begin{aligned} d^2\underline{c}^*_{\tilde{\phi}(\nu)}/d\nu^2 &= \left[-u''(\underline{c}^*_{\tilde{\phi}(\nu)})\tilde{\phi}'(\nu)^2 + u'(\underline{c}^*_{\tilde{\phi}(\nu)})\tilde{\phi}''(\nu)\right] \\ &\quad \times \left[u'(\underline{c}^*_{\tilde{\phi}(\nu)})^{-1}\tilde{\phi}'(\nu)\right]^{-2} \\ &> 0. \end{aligned}$$

The positive sign of the second derivative is based on the fact that $\tilde{\phi}''(\nu) > 0$. This can be verified from totally differentiating Equation (37) with respect to ν

$$\begin{aligned} 1 &= -\left[\tilde{\phi}(\nu)f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu)\right] - \left[\tilde{\phi}'(\nu)\left[1 - F(\tilde{\phi}(\nu))\right] - \tilde{\phi}(\nu)f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu)\right] \\ &= -\tilde{\phi}'(\nu)\left[1 - F(\tilde{\phi}(\nu))\right] \quad \Rightarrow \tilde{\phi}'(\nu) = -\left[1 - F(\tilde{\phi}(\nu))\right]^{-1} < 0. \\ \tilde{\phi}''(\nu) &= -f(\tilde{\phi}(\nu))\tilde{\phi}'(\nu)/\left[1 - F(\tilde{\phi}(\nu))\right]^2 > 0. \end{aligned}$$

Finally, as $\tilde{\phi}(\nu)$ is continuously differentiable, so is consumption, $\underline{c}^*_{\tilde{\phi}(\nu)}$, and the profit function, $\underline{P}(\nu)$. This concludes the proof of Proposition 16.

B.3 Extension: learning

In this section we consider an extensions of the theory. In our theory, renegotiation is unambiguously good for the borrower. On the one hand, consumption always increases upon renegotiation, in line with the empirical evidence documented by Reinhart and Trebesch (2016). On the other, renegotiations do not affect the terms at which the country can borrow in future. In particular, conditional on the debt level, the risk premium is independent of the country's credit history. In this section, we sketch an extension where bond prices depend on the frequency of previous renegotiations. We assume that there is imperfect information about the distribution from which countries draw their realizations of ϕ. In particular, there are two types of countries, creditworthy (CW) and not creditworthy (NC), that draw from different distributions.[35] In particular, $F_{NC}(\phi) \geq F_{CW}(\phi)$, with strict inequality holding for some ϕ, implying that the NC country is more likely to have lower realizations of the default cost. We assume that priors are common knowledge, and denote by π the belief that the borrower is CW. Beliefs are updated according to Bayes' rule:

$$\pi' = \frac{f_{CW}(\phi)}{f_{CW}(\phi) \times \pi + f_{NC}(\phi) \times (1-\pi)}\pi \equiv \Gamma(\phi, \pi).$$

Moreover, we define

$$F(\phi|\pi) \equiv \pi F_{CW}(\phi) + (1-\pi) F_{NC}(\phi),$$

[35] One could assume that the distributions have a common support in order to rule out perfectly revealing realizations. However, this is not essential.

and restrict attention to market equilibria during normal times ($w = \bar{w}$). For the ease of exposition, normal times variables will be indicated with a bar on top for the rest of this section.

In the new environment, the price of debt depends on the prior about the country's type, i.e., $\bar{Q}(b', \pi)$. No arbitrage implies the following bond price:

$$\bar{Q}(b', \pi) = \frac{1}{R} \left(\begin{array}{c} (1 - F(\Phi^*(b', \pi) | \Gamma(\Phi^*(b', \pi), \pi))) + \\ \frac{\pi}{b'} \int_0^{\Phi^*(b',\pi)} \hat{b}(\phi, \Gamma(\phi, \pi)) \ dF_{CW}(\phi) + \frac{1-\pi}{b'} \int_0^{\Phi^*(b',\pi)} \hat{b}(\phi, \Gamma(\phi, \pi)) \, dF_{NC}(\phi) \end{array} \right)$$

where $\Phi^*(b', \pi)$ denotes the threshold ϕ' such that debt will be honored next period if and only if $\phi' \geq \Phi^*(b', \pi)$. More formally, Φ^* is the unique fixed point of the following equation

$$\Phi^* = \bar{\Phi}\left(b', \Gamma(\Phi^*, \pi)\right).$$

The function Φ^* takes into account that the realization of ϕ' will itself alter next-period beliefs, which in turn affect the country's incentive to renegotiate.[36] The bond price is falling in b' and increasing in π.

Consider, next, the consumption-savings decision. The CEE yields (formal derivation below):

$$\begin{aligned} 1 - F\left(\Phi^*\left(\bar{B}(b, \pi), \pi\right) | \Gamma(\Phi^*, \pi)\right) &= \pi \int_{\Phi^*(b,\pi)}^{\infty} \frac{u'\left[\bar{C}\left(\Gamma(\phi', \pi), \bar{B}(b, \pi)\right)\right]}{u'\left[\bar{C}(b, \pi)\right]} dF_{CW}(\phi') \\ &\quad + (1 - \pi) \int_{\Phi^*(b,\pi)}^{\infty} \frac{u'\left[\bar{C}\left(\Gamma(\phi', \pi), \bar{B}(b, \pi)\right)\right]}{u'\left[\bar{C}(b, \pi)\right]} dF_{NC}(\phi'). \end{aligned} \tag{84}$$

If next-period consumption conditional on honoring the debt did not depend on ϕ', then the CEE would boil down to Equation (13). However, in this extension, the realized consumption growth depends on ϕ' because creditors learn over time about the borrowers' types. For example, take two realization of ϕ', say ϕ'_h and ϕ'_l, such that $\phi'_h > \phi'_l$, neither inducing renegotiation. Here, consumption will be larger under ϕ'_h because the larger realization has a stronger positive effect on the belief that the country is CW. This improves the terms of borrowing, and hence consumption. Note that renegotiation might be associated with a fall in consumption – for example if the realized ϕ is just below $\Phi^*(b', \pi)$ the effect of a very small renegotiation is more than offset by that of Bayesian updating. Conversely, a country experiencing a sequence of large ϕ's which induces it to honor debt for a long time will enjoy an increasing consumption.

In summary, this simple extension shows that our theory can incorporate learning effects through which countries prone to renegotiation are punished by the market with high interest rates.

B.3.1 Formal derivation of Equation (84).

Let the sovereign's value functions be denoted by $\bar{V}(b, \phi, \pi)$ and $\bar{W}(b, \pi)$. Since outright default is never observed in equilibrium, the value function simplifies to

$$\bar{V}(b, \phi, \pi) = \max_{b' [\underline{b}, \tilde{b}]} \left\{ u\left(\bar{Q}\left(b', \pi\right) \times b' + \bar{w} - \mathbb{B}(b, \phi, \bar{w}, \pi)\right) + \beta \times EV\left(b', \pi\right) \right\}. \tag{85}$$

[36] Note that some functions must be redefined to take into account their dependence on public beliefs. Apart from $\bar{Q}(b', \pi)$, defined in the text, $\hat{b}(\phi, \pi)$ is the renegotiated debt given ϕ and π. Moreover, $\bar{\Phi}(b', \pi)$ denotes the threshold that makes the country indifferent between honoring the debt level b' and defaulting, conditional on the realized belief π.

where $EV(b',\pi) \equiv E\bar{V}\left(b',\phi',\Gamma\left(\phi',\pi\right)\right)$.

The function $\bar{\Phi}$ is such that $\bar{\Phi}(b,\pi) = \bar{W}(0,\pi) - \bar{W}(b,\pi)$. Given a debt issuance of b' and a current prior of π, debt will be honored next period if $\phi' \geq \Phi^*(b',\pi)$ where Φ^* is the unique fixed point of the following equation

$$\Phi^* = \bar{\Phi}\left(b',\Gamma\left(\Phi^*,\pi\right)\right).$$

The probability of renegotiation is

$$\begin{aligned}
E\left\{F\left(\bar{\Phi}\left(b',\pi'\right)|\pi'\right),\pi\right\} &= \pi F_{CW}\left(\bar{\Phi}\left(b',\Gamma\left(\Phi^*,\pi\right)\right)\right) + (1-\pi) F_{NC}\left(\bar{\Phi}\left(b',\Gamma\left(\Phi^*,\pi\right)\right)\right) \\
&= \pi F_{CW}\left(\Phi^*\left(b',\pi\right)\right) + (1-\pi) F_{NC}\left(\Phi^*\left(b',\pi\right)\right) \\
&= F\left(\Phi^*\left(b',\pi\right),\Gamma\left(\Phi^*,\pi\right)\right),
\end{aligned}$$

and an arbitrage argument then implies the following bond price

$$\bar{Q}\left(b',\pi\right) = \frac{1}{R}\left(\begin{array}{c} 1 - F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right) + \\ \frac{\pi}{b'}\int_0^{\Phi^*(b',\pi)} \hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)\, dF_{CW}(\phi) + \frac{1-\pi}{b'}\int_0^{\Phi^*(b',\pi)} \hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right) dF_{NC}(\phi) \end{array}\right).$$

In what follows, we assume that the relevant equilibrium functions are differentiable in b. Then, differentiating $b \times \bar{Q}(b,\pi)$, with respect to b yields

$$\begin{aligned}
\frac{d}{db}\left\{b \times \bar{Q}(b,\pi)\right\} &= \bar{Q}(b,\pi) + b \times \frac{d}{db}\bar{Q}\left(b',\pi\right) \\
&= \bar{Q}(b,\pi) - \frac{b}{R}\left(\frac{\partial F\left(\Phi^*(b,\pi)|\Gamma\left(\Phi^*,\pi\right)\right)}{\partial\phi}\frac{\partial\Phi^*(b,\pi)}{\partial b}\right) + \\
&\quad + \frac{b}{R}\frac{\pi}{b}\hat{b}\left(\Phi^*(b,\pi),\Gamma\left(\Phi^*(b,\pi),\pi\right)\right) f_{CW}\left(\Phi^*(b,\pi)\right)\frac{\partial\Phi^*(b,\pi)}{\partial b} \\
&\quad - \frac{1}{R}\frac{\pi}{b}\int_0^{\Phi^*(b,\pi)} \hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right)\, dF_{CW}(\phi) \\
&\quad + \frac{b}{R}\frac{1-\pi}{b}\hat{b}\left(\Phi^*(b,\pi),\Gamma\left(\Phi^*(b,\pi),\pi\right)\right) f_{NC}\left(\Phi^*(b,\pi)\right)\frac{\partial\Phi^*(b,\pi)}{\partial b} \\
&\quad - \frac{1}{R}\frac{1-\pi}{b}\int_0^{\Phi^*(b,\pi)} \hat{b}\left(\phi,\Gamma\left(\phi,\pi\right)\right) dF_{NC}(\phi),
\end{aligned}$$

such that

$$\begin{aligned}
\frac{d}{db}\left\{b \times \bar{Q}(b,\pi)\right\} &= \bar{Q}(b,\pi) - \underbrace{\frac{b}{R}\left(\frac{\partial F\left(\Phi^*(b,\pi)|\Gamma\left(\Phi^*,\pi\right)\right)}{\partial\phi}\frac{\partial\Phi^*(b,\pi)}{\partial b}\right)}_{A+B} + \\
&\quad \underbrace{+\pi\frac{b}{R} f_{CW}\left(\Phi^*(b,\pi)\right)\frac{\partial\Phi^*(b,\pi)}{\partial b}}_{A} + \underbrace{(1-\pi)\frac{b}{R} f_{NC}\left(\Phi^*(b,\pi)\right)\frac{\partial\Phi^*(b,\pi)}{\partial b}}_{B} \\
&\quad - \bar{Q}(b,\pi) + \frac{1}{R}\left(1 - F\left(\Phi^*(b,\pi)|\Gamma\left(\Phi^*,\pi\right)\right)\right) \\
&= \frac{1}{R}\left(1 - F\left(\Phi^*(b,\pi)|\Gamma\left(\Phi^*,\pi\right)\right)\right).
\end{aligned}$$

Next, consider the consumption-savings decision. The first-order condition of (85) reads

$$\frac{d}{db'}\left\{\bar{Q}\left(b',\pi\right)b'\right\}\times u'\left[\bar{Q}\left(b',\pi\right)\times b'+\bar{w}-\mathbb{B}\left(b,\phi,\bar{w},\pi\right)\right]+\frac{d}{db'}\beta EV\left(b',\pi\right)=0.$$

The value function has a kink at $b=\hat{b}\left(\phi,\pi\right)$. Consider, first, the range where $b<\hat{b}\left(\phi,\pi\right)$. Differentiating the value function yields

$$\frac{d}{db}\bar{V}\left(b,\phi,\pi\right)=-u'\left[\bar{Q}\left(\bar{B}\left(b,\pi\right),\pi\right)\times\bar{B}\left(b,\pi\right)+\bar{w}-b\right],$$

where $\bar{B}$ denotes the optimal issuance of new bonds. Next, consider the region of renegotiation, $b>\hat{b}\left(\phi,\pi\right)$. In this case, $\frac{d}{db}\bar{V}\left(b,\phi,\pi\right)=0$.

Using the results above one obtains

$$\begin{aligned}
\frac{d}{db}EV\left(b,\Gamma\left(\phi,\pi\right)\right) &= \pi\int_{\aleph}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{CW}\left(\phi\right)+\left(1-\pi\right)\int_{\aleph}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\\
&= \pi\begin{pmatrix}\int_{0}^{\Phi^*(b,\pi)}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{CW}\left(\phi\right)\\+\int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{CW}\left(\phi\right)\end{pmatrix}\\
&\quad+\left(1-\pi\right)\begin{pmatrix}\int_{0}^{\Phi^*(b,\pi)}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\\+\int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\end{pmatrix}\\
&= \pi\int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{CW}\left(\phi\right)\\
&\quad+\left(1-\pi\right)\int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF_{NC}\left(\phi\right)\\
&= \int_{\Phi^*(b,\pi)}^{\infty}\frac{d}{db}\bar{V}\left(b,\phi,\Gamma\left(\phi,\pi\right)\right)dF\left(\phi|\pi\right)\\
&= -\int_{\Phi^*(b,\pi)}^{\infty}u'\left[\bar{Q}\left(\bar{B}\left(b,\Gamma\left(\phi,\pi\right)\right),\Gamma\left(\phi,\pi\right)\right)\times\bar{B}\left(b,\Gamma\left(\phi,\pi\right)\right)+\bar{w}-b\right]dF\left(\phi|\pi\right).
\end{aligned}$$

Plugging this expression back into the FOC, and leading the expression by one period, yields

$$\begin{aligned}
0 &= \frac{1}{R}\left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)\times u'\left[\bar{Q}\left(b',\pi\right)\times b'+\bar{w}-\mathbb{B}\left(b,\phi,\bar{w},\pi\right)\right]\\
&\quad-\beta\int_{\Phi^*(b,\pi)}^{\infty}u'\left[\bar{Q}\left(\bar{B}\left(b',\Gamma\left(\phi,\pi\right)\right),\Gamma\left(\phi,\pi\right)\right)\times\bar{B}\left(b',\Gamma\left(\phi,\pi\right)\right)+\bar{w}-b\right]dF\left(\phi|\pi\right)
\end{aligned}$$

thus

$$\beta R=\left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)\left(\int_{\Phi^*(b,\pi)}^{\infty}\frac{u'\left[\bar{C}\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[\bar{C}\left(\pi,b\right)\right]}dF\left(\phi|\pi\right)\right)^{-1},$$

where the first step uses the fact that $\frac{d}{db'}\left\{\bar{Q}\left(b',\pi\right)b'\right\}=\frac{1}{R}\left(1-F\left(\Phi^*\left(b',\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right)\right)$, as shown above. Since $\beta R = 1$, then

$$
\begin{aligned}
& 1 - F\left(\Phi^*\left(\bar{B}\left(b,\pi\right),\pi\right)|\Gamma\left(\Phi^*,\pi\right)\right) \\
= & \left(\int_{\Phi^*(b,\pi)}^{\infty} \frac{u'\left[\bar{C}\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[\bar{C}\left(\pi,b\right)\right]} dF\left(\phi|\pi\right)\right) \\
= & \pi \int_{\Phi^*(b,\pi)}^{\infty} \frac{u'\left[\bar{C}\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[\bar{C}\left(\pi,b\right)\right]} dF_{CW}\left(\phi\right) + (1-\pi) \int_{\Phi^*(b,\pi)}^{\infty} \frac{u'\left[\bar{C}\left(\Gamma\left(\phi,\pi\right),\bar{B}\left(b,\pi\right)\right)\right]}{u'\left[\bar{C}\left(\pi,b\right)\right]} dF_{NC}\left(\phi\right).
\end{aligned}
$$

B.4 Numerical algorithm

In this section, we discuss the numerical algorithms used in Section 5.

B.4.1 Market equilibrium

We solve for the market equilibrium described in Section 3 with an augmented value function iteration algorithm. Let $b = (b_1, b_2, ..., b_N)$ denote the equally spaced and inreasingly ordered grid for sovereign debt. Let $\phi = (\phi_1, \phi_2, ...\phi_S)$ be the increasingly ordered grid for the default cost, where the location of any grid point, ϕ_s, is chosen such that the cumulative weighted sum, $\widetilde{F}(\phi_s) \equiv \sum_{k=1}^{s} 1/S$, approximates the CDF of the default cost shock $F(\phi_s)$. We choose $N = 5'000$ and $S = 600$ to get a solution with high accuracy.[37]

1. Guess the default threshold, $\Phi_0(b, w) \in \phi$, for both aggregate states w and guess the reform effort, $\Psi_0(b)$, over the debt grid b. Compute the associated bond revenue,

$$
\begin{aligned}
Q_0(b, \bar{w})b &= \hat{Q}_0(b, \bar{w})b \\
Q_0(b, \underline{w})b &= \Psi_0(b)\hat{Q}_0(b, \bar{w})b + (1 - \Psi_0(b))\hat{Q}_0(b, \underline{w})b,
\end{aligned}
$$

   where the discounted recovery values are given by

$$
\hat{Q}_0(b, w)b = R^{-1}\left((1 - \widetilde{F}(\Phi_0(b, w)))b + \sum_{\phi_s \in \phi_1, ..., \Phi_0(b,w)} \hat{b}_0(\phi_s, w)/S\right)
$$

   and $\hat{b}_0(\phi, w) \in b$ is the inverse function of $\Phi_0(b, w)$.

2. Guess the value functions conditional on honoring the debt, $W_{0,0}(b, w)$. For any given debt level, $b_n \leq \hat{b}_0(\phi_S, \bar{w})$, on the debt grid update the value function in normal times according to

$$
\begin{aligned}
W_{i+1,0}(b_n, \bar{w}) = & \max_{b' \in (b_1, ..., \hat{b}_0(\phi_S, \bar{w}))} u\left(Q_0(b', \bar{w})b' + \bar{w} - b_n\right) \\
& + \beta\left[(1 - \widetilde{F}(\Phi_0(b', \bar{w})))W_{i,0}(b', \bar{w}) + \sum_{\phi_s \in \phi_1, ..., \Phi_0(b', \bar{w})} W_{i,0}\left(\hat{b}_0(\phi_s, \bar{w}), \bar{w}\right)/S\right],
\end{aligned}
$$

[37] We use the conditional Euler equations of the market equilibrium to evaluate the accuracy of the solution in terms of market consumption. The mean Euler equation error across all states is 6.7×10^{-4}. Thus, there is a \$6.7 error on average for each \$10'000. Note that we approximate and evaluate the accuracy of the solution globally over the full debt grid. Moreover, the conditional Euler equation in recession involves the derivative of an equilibrium function (reform effort) and non-smooth debt accumulation. Thus, we consider the accuracy of the solution to be high. When increasing the number of grid points to $N = 10'000$ and $S = 1'000$, the mean Euler equation error can be further reduced to 2.0×10^{-4} at the usual cost of computational time.

until convergence. For the remaining grid points, $b_n \geq \hat{b}_0(\phi_S, \bar{w})$, set $W_{i+1,0}(b_n, \bar{w}) = W_{i+1,0}(\hat{b}_0(\phi_S, \bar{w}), \bar{w})$. In recession, for any given debt level, $b_n \leq \hat{b}_0(\phi_S, \underline{w})$, on the debt grid, update the value function according to

$$\begin{aligned} W_{i+1,0}(b_n, \underline{w}) &= \max_{b' \in (b_1, \ldots, \hat{b}_0(\phi_S, \bar{w}))} u\left(Q_0(b', \underline{w})b' + \underline{w} - b_n\right) \\ &+ \beta\left(1 - \Psi_0(b')\right) \left[\begin{array}{c} (1 - \widetilde{F}(\Phi_0(b', \underline{w})))W_{i,0}(b', \underline{w}) \\ + \sum_{\phi_s \in \phi_1, \ldots, \Phi_0(b', \underline{w})} W_{i,0}\left(\hat{b}_0(\phi_s, \underline{w}), \underline{w}\right)/S \end{array} \right] \\ &+ \beta \Psi_0(b') \left[\begin{array}{c} (1 - \widetilde{F}(\Phi_0(b', \bar{w})))W_{\infty,0}(b', \bar{w}) \\ + \sum_{\phi_s \in \phi_1, \ldots, \Phi_0(b', \bar{w})} W_{\infty,0}\left(\hat{b}_0(\phi_s, \bar{w}), \bar{w}\right)/S \end{array} \right], \end{aligned}$$

until convergence. For the remaining grid points, $b_n \geq \hat{b}_0(\phi_S, \underline{w})$, set $W_{i+1,0}(b_n, \underline{w}) = W_{i+1,0}(\hat{b}_0(\phi_S, \underline{w}), \underline{w})$. $W_{\infty,0}(b', w)$ denotes the converged value function conditional on the guess for the threshold and the reform effort.

3. Update the default threshold and the reform effort according to

$$\Phi_{j+1}(b, w) = W_{\infty,j}(0, w) - W_{\infty,j}(b, w),$$

and Equation (14). Go back to step 1 and iterate until convergence.

B.4.2 Optimal contract with one-sided commitment

We solve for the second-best allocation with an augmented function iteration algorithm. Consider the same grid $\phi = (\phi_1, \phi_2, \ldots \phi_S)$ for the default cost that we used above. Let $\nu_w = (\nu_w(\phi_1), \ldots, \nu_w(\phi_S))$ denote the grid for promised utility, where

$$\begin{aligned} \nu_{\bar{w}}(\phi_s) &= \bar{\nu} - \sum_{k=1}^{s} \phi_k/S - \phi_s\left(1 - \widetilde{F}(\phi_s)\right) \\ \nu_{\underline{w}}(\phi_s) &= \underline{\nu} - \sum_{k=1}^{s} \phi_k/S - \phi_s\left(1 - \widetilde{F}(\phi_s)\right). \end{aligned}$$

Note that given a promised utility, $\nu_w(\phi_s)$, the default cost realization $\phi_s = \tilde{\phi}(\nu_w(\phi_s))$ corresponds to the state s where the participation constraint of the debtor starts binding. It turns out to be convenient to set the promised utility for a continued recession, $\bar{\omega}_{\bar{w}} = \nu_{\bar{w}}$, and the promised utility for a continued recession, $\underline{\omega}_{\underline{w}} = \nu_{\underline{w}}$.

1. Guess the reform effort, $p_{\underline{w},0}(\nu_{\underline{w}})$, over the grid $\nu_{\underline{w}}$.
2. Guess the future consumption, $c'_{w,0,0}(\nu_w)$, and the promised utilitiy, $\bar{\omega}_{\underline{w},0,0}(\nu_{\underline{w}})$, over the grids $\nu_{\bar{w}}$ and $\nu_{\underline{w}}$.
3. Compute current consumption from the Euler Equations (which holds for all states s where the participation constraint is not strictly binding)

$$c_{w,0,0}(\nu_w) = \left(u'\right)^{-1}\left[u'\left(c'_{w,0,0}(\nu_w)\right)\beta R\right],$$

and the initial promised utility, $\tilde{\nu}_{w,0}\left(\nu_w\right)$, implicitly defined by

$$\begin{aligned}
\bar{\nu} - \tilde{\phi}(\tilde{\nu}_{\bar{w},0,0}\left(\nu_{\bar{w}}\right)) &= u(c_{\bar{w},0,0}(\nu_{\bar{w}})) + \beta\nu_{\bar{w}} \\
\underline{\nu} - \tilde{\phi}(\tilde{\nu}_{\underline{w},0,0}\left(\nu_{\underline{w}}\right)) &= u(c_{\underline{w},0,0}(\nu_{\underline{w}})) - X(p_{\underline{w},0}(\nu_{\underline{w}})) + \beta\left[p_{\underline{w},0}(\nu_{\underline{w}})\bar{\omega}_{\underline{w},0,0}(\nu_{\underline{w}}) + (1 - p_{\underline{w},0}(\nu_{\underline{w}}))\nu_{\underline{w}}\right],
\end{aligned}$$

4. Update the guess for the future consumption function by interpolating ν_w on the pairs $(\tilde{\nu}_{w,0,0}, c_{w,0,0})$ to yield $c'_{w,1,0}(\nu_w)$ Update the guess for promised utility by interpolating $c'_{\underline{w},1,0}(\nu_{\underline{w}})$ on the pairs $\left(c_{\underline{w},0,0}, \tilde{\nu}_{\underline{w},0,0}\right)$ to yield $\bar{\omega}_{\underline{w},1,0}(\nu_{\underline{w}})$. Go back to step 3 and iterate until convergence. Let $c_{w,\infty,0}(\nu_w)$ and $\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}})$ denote the converged functions given the guess on the reform effort.

5. Guess the profit functions, $\bar{P}_0(\nu_{\bar{w}})$ and $\underline{P}_0(\nu_{\underline{w}})$. Update the profit function in normal times according to

$$\begin{aligned}
\bar{P}_{i+1,0}(\tilde{\nu}_{\bar{w},\infty,0}\left(\nu_{\bar{w}}\right)) &= \left(1 - \widetilde{F}(\tilde{\phi}(\tilde{\nu}_{\bar{w},\infty,0}\left(\nu_{\bar{w}}\right)))\right)\left[\bar{w} - c_{\bar{w},\infty,0}(\nu_{\bar{w}}) + R^{-1}\bar{P}_{i,0}(\nu_{\bar{w}})\right] \\
&\quad + \sum_{\phi_s \in \phi_1,\ldots,\tilde{\phi}(\tilde{\nu}_{\bar{w},\infty,0}(\nu_{\bar{w}}))} \left[\bar{w} - c_{\bar{w},\infty,0}(\nu_{\bar{w}}(\phi_s)) + R^{-1}\bar{P}_{i,0}(\nu_{\bar{w}}(\phi_s))\right]/S,
\end{aligned}$$

until convergence, $\bar{P}_{\infty,0}(\nu_{\bar{w}})$. In recession, update according to

$$\begin{aligned}
\underline{P}_{i+1,0}(\tilde{\nu}_{\underline{w},\infty,0}\left(\nu_{\underline{w}}\right)) &= \left(1 - \widetilde{F}(\tilde{\phi}(\tilde{\nu}_{\underline{w},\infty,0}\left(\nu_{\underline{w}}\right)))\right)\left[\begin{array}{c} \underline{w} - c_{\underline{w},\infty,0}(\nu_{\underline{w}}) \\ +R^{-1}\left[\begin{array}{c} p_{\underline{w},0}(\nu_{\underline{w}})\bar{P}_{i,0}(\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}})) \\ +(1 - p_{\underline{w},0}(\nu_{\underline{w}}))\underline{P}_{i,0}(\nu_{\underline{w}}) \end{array}\right] \end{array}\right] \\
&\quad + \sum_{\phi_s \in \phi_1,\ldots,\tilde{\phi}(\tilde{\nu}_{\underline{w},\infty,0}(\nu_{\underline{w}}))} \left[\begin{array}{c} \underline{w} - c_{\underline{w},\infty,0}(\nu_{\underline{w}}(\phi_s)) \\ +R^{-1}\left[\begin{array}{c} p_{\underline{w},0}(\nu_{\underline{w}}(\phi_s)\bar{P}_{i,0}(\bar{\omega}_{\underline{w},\infty,0}(\nu_{\underline{w}}(\phi_s))) \\ +(1 - p_{\underline{w},0}(\nu_{\underline{w}}(\phi_s)))\underline{P}_{i,0}(\nu_{\underline{w}}(\phi_s)) \end{array}\right] \end{array}\right]/S.
\end{aligned}$$

6. Update the reform effort function according to

$$p_{\underline{w},j+1}(\nu_{\underline{w}}) = \left(X'\right)^{-1}\left[\begin{array}{c} u'(c_{\underline{w},\infty,j}(\nu_{\underline{w}}))R^{-1}\left(\bar{P}_{\infty,j}(\bar{\omega}_{\underline{w},\infty,j}(\nu_{\underline{w}})) - \underline{P}_{\infty,j}(\nu_{\underline{w}})\right) \\ +\beta\left(\bar{\omega}_{\underline{w},\infty,j}(\nu_{\underline{w}}) - \nu_{\underline{w}}\right) \end{array}\right].$$

Go back to step 3 and iterate until convergence.

B.5 An exact decomposition of welfare effects

This section illustrates a case when the welfare gain decomposition proposed in section 5.4.1 is exact. Namely, that the welfare effect of going from a stationary competitive equilibrium allocation to the first best can be decomposed into a level effect, a volatility effect, and a discounting effect.

For simplicity, we abstract from reform effort and default costs and assume that consumption in the competitive equilibrium is log normal with $\ln(c) \sim N\left(\ln\left(\bar{C}_{ce}\right) - \frac{v}{2}, v\right)$. Thus, the average consumption is $\bar{C}_{ce}$ and the variance of $\ln(c)$ is v. Since c is log normal, the expected utility is

$$V = \sum_{t=0}^{\infty}\beta^t E\frac{(c_t)^{1-\gamma}}{1-\gamma} = \begin{cases} \frac{1}{1-\beta}\frac{1}{1-\gamma}\cdot\left(\bar{C}_{ce}\right)^{1-\gamma}\exp\left(\gamma(\gamma-1)\frac{v}{2}\right) & \text{for } \gamma \neq 1 \\ \frac{1}{1-\beta}\left(\ln\left(\bar{C}_{ce}\right) - \frac{v}{2}\right) & \text{for } \gamma = 1. \end{cases}$$

Calculate now the discounted utility of a first best allocation where the present value of consumption is $R/(R-1)\cdot\bar{C}_{FB}$. The optimal consumption sequence is given by

$$c_t = (\beta R)^{\frac{t}{\gamma}}\, c_0.$$

Calculating the present value of consumption yields an expression for c_0,

$$\begin{aligned}
\frac{R}{R-1}\cdot\bar{C}_{FB} &= \sum_{t=0}^{\infty}\frac{c_t}{R^t} = \sum_{t=0}^{\infty}\frac{(\beta R)^{\frac{t}{\gamma}}\, c_0}{R^t} = c_0\sum_{t=0}^{\infty}\left((\beta R)^{\frac{1}{\gamma}}\, R^{-1}\right)^t = \frac{c_0}{1-(\beta R)^{\frac{1}{\gamma}}\, R^{-1}} \\
&\Rightarrow \\
c_0 &= \left(1-(\beta R)^{\frac{1}{\gamma}}\, R^{-1}\right)\frac{R}{R-1}\cdot\bar{C}_{FB}.
\end{aligned}$$

When $\gamma \neq 1$ the discounted utility is

$$V_{FB} = \sum_{t=0}^{\infty}\beta^t\frac{\left(c_t^{FB}\right)^{1-\gamma}}{1-\gamma} = \frac{\left(1-(\beta R)^{\frac{1}{\gamma}}\, R^{-1}\right)^{1-\gamma}}{1-\beta\,(\beta R)^{\frac{1-\gamma}{\gamma}}}\left(\frac{R}{R-1}\right)^{1-\gamma}\frac{\left(\bar{C}_{FB}\right)^{1-\gamma}}{1-\gamma},$$

and in the log case ($\gamma = 1$),

$$V_{FB} = \sum_{t=0}^{\infty}\beta^t\ln\left(c_t^{FB}\right) = \frac{1}{1-\beta}\ln\left(\bar{C}_{FB}\right) + \frac{1}{1-\beta}\ln\left((1-\beta)\frac{R}{R-1}\right) + \ln(\beta R)\frac{\beta}{(1-\beta)^2}.$$

Calculate the welfare gain χ of going from the competitive equilibrium V to the first best V_{FB} in the log case

$$\begin{aligned}
E\sum_{t=0}^{\infty}\beta^t\ln(c_t) &= \sum_{t=0}^{\infty}\beta^t\ln\left((1+\chi)\, c_t^{FB}\right) \\
&\Rightarrow
\end{aligned}$$

The welfare gain χ can then be decomposed as follows,

$$\ln(1+\chi) = \underbrace{-\ln\left(\frac{(1-\beta)\,R}{R-1}\right)-\ln(\beta R)\frac{\beta}{(1-\beta)}}_{\text{Discounting effect}}\quad\underbrace{-\frac{v}{2}}_{\text{Volatility effect}}\quad+\underbrace{\ln\left(\bar{C}_{FB}\right)-\ln\left(\bar{C}_{ce}\right)}_{\text{Level effect}}$$

Finally, calculate the welfare gain χ of going from the competitive equilibrium V to the first best V_{FB} in the case with $\gamma \neq 1$:

$$\begin{aligned}
E\sum_{t=0}^{\infty}\beta^t\frac{(c_t)^{1-\gamma}}{1-\gamma} &= E\sum_{t=0}^{\infty}\beta^t\frac{\left((1+\chi)\, c_t^{FB}\right)^{1-\gamma}}{1-\gamma} \\
&\Rightarrow
\end{aligned}$$

$$\frac{1}{1-\beta}\exp\left(\gamma(\gamma-1)\frac{v}{2}\right)\left(\bar{C}_{ce}\right)^{1-\gamma} = (1+\chi)^{1-\gamma}\frac{\left(1-(\beta R)^{\frac{1}{\gamma}}\, R^{-1}\right)^{1-\gamma}}{1-\beta\,(\beta R)^{\frac{1-\gamma}{\gamma}}}\left(\frac{R}{R-1}\right)^{1-\gamma}\left(\bar{C}_{FB}\right)^{1-\gamma}$$

which implies that the welfare gain χ can be decomposed as follows,

$$
\begin{aligned}
& \ln(1+\chi) \\
= \quad & \underbrace{\frac{1}{1-\gamma} \ln\left(\frac{1-\beta(\beta R)^{\frac{1-\gamma}{\gamma}}}{1-\beta}\right) - \ln\left(\left(1-(\beta R)^{\frac{1}{\gamma}} R^{-1}\right)\left(\frac{R}{R-1}\right)\right)}_{\text{Discounting effect}} \\
& \underbrace{-\gamma \cdot \frac{v}{2}}_{\text{Volatility effect}} \quad + \underbrace{\log\left(\bar{C}_{FB}\right) - \log\left(\bar{C}_{ce}\right)}_{\text{Level effect}}.
\end{aligned}
$$

B.6 Additional figures

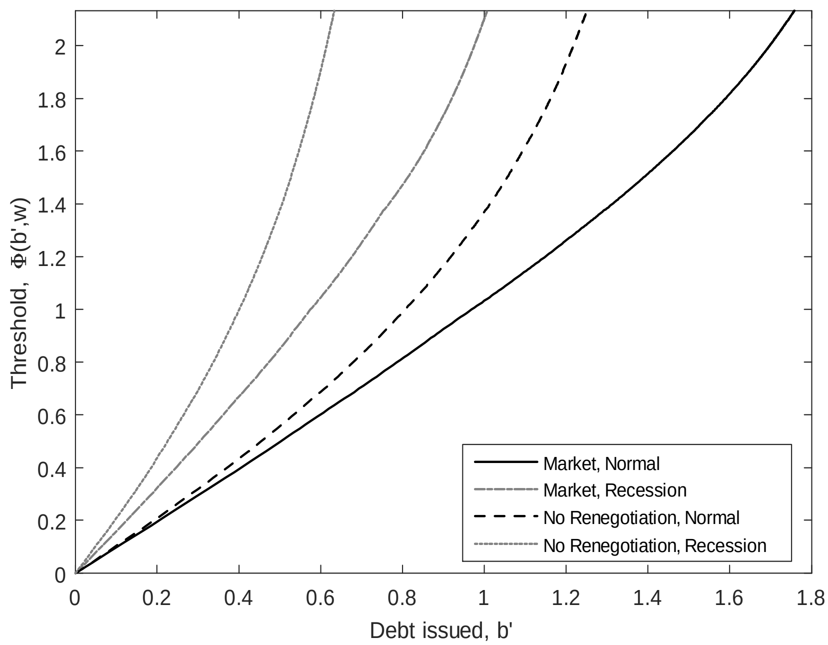

Figure 7: Comparison of threshold functions in normal times and recession between the competitive equilibrium of Section 3 (*market*) and the no-renegotiation equilibrium analyzed in Section 5.5.

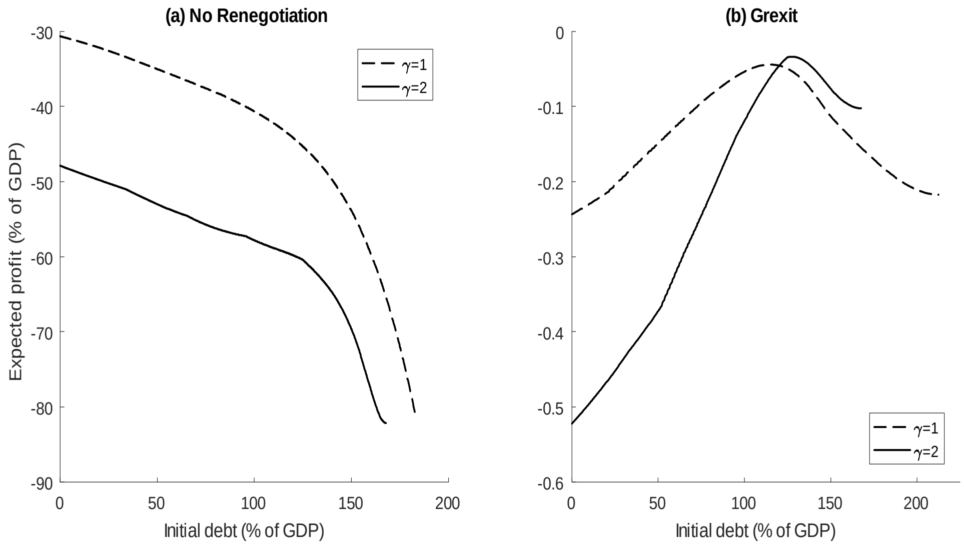

Figure 8: Panel a plots the expected profits (net of the initial debt value) for the lenders when ruling out renegotiation, where the debt is adjusted so as to keep the sovereign indifferent between this alternative economy and remaining in the benchmark economy. Panel b plots the equivalent expected profits when imposing an "Austerity cum Grexit" policy, relative to remaining in the benchmark economy. Negative profits are equivalent to a welfare loss.