Technical Appendix to

The Survival of the Welfare State[1]

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1 Not for publication.

1.1 Efficient redistribution

The following Proposition characterizes the set of efficient sequences of benefits.

Proposition 1 *A social planner whose objective function is given by*

$$\max_{\{b_t\}_{t=1}^{\infty}} \left\{ \lambda_{0s} (1-u_1) \hat{V}^{os} (b_1, b_2, u_1) + \lambda_{0u} u_1 \hat{V}^{ou} (b_1, b_2, u_1) + \sum_{t=1}^{\infty} \lambda_t \hat{V}^{y} (b_t, b_{t+1}, b_{t+2}, u_t) \right\},$$

$$\textit{subject to } b_t \in [0,1] \quad \forall t$$

will choose

$$b_t = 0 \;\; \forall t \geq 2$$

$$b_1 = \min \left\{ \max \left\{ 2 \left(\frac{\lambda_{0u}}{\lambda_{0s}(1-u_1) + \lambda_{0u} u_1} - 1 \right) u_1 + \left(1 - \frac{\lambda_1}{\lambda_{0s}(1-u_1) + \lambda_{0u} u_1} \right) \left(u_1 - \frac{(1-\beta)}{2} \right), 0 \right\}, 1 \right\}$$

provided $\lambda_t > 0 \; \forall t \geq 1$.

[1] Not for publication

Proof. Substituting for the individuals' objective function (4), the planner's objective function can be rewritten as

$$\begin{aligned}
&\max_{\{b_t\}_{t=1}^{\infty}} \left\{ \lambda_{0s}\left(1-u_1\right)V^{os}\left(b_1,b_2,u_1\right)+\lambda_{0u}u_1V^{ou}\left(b_1,b_2,u_1\right)+\sum_{t=1}^{\infty}\lambda_tV^{y}\left(b_t,b_{t+1},b_{t+2},u_t\right)\right\}\\
=&\max_{\{b_t\}_{t=1}^{\infty}} \left\{ \lambda_{0s}\left(1-u_1\right)+\lambda_{0u}u_1b_1-\left(\lambda_{0s}\left(1-u_1\right)+\lambda_{0u}u_1\right)\frac{(1-\beta)+2u_1}{4}b_1\right.\\
&-\lambda_1\frac{1}{2}\left(u_1-\frac{(1-\beta)}{2}\right)b_1-\left(\lambda_{0s}\left(1-u_1\right)+\lambda_{0u}u_1\right)\frac{1}{4}b_1^2\\
&-\left(\left(\lambda_{0s}\left(1-u_1\right)+\lambda_{0u}u_1\right)\beta+\lambda_2\right)\frac{1}{4}b_1b_2-\sum_{t=2}^{\infty}\left(\lambda_{t+1}+\lambda_{t-1}\beta^2\right)\frac{1}{4}b_tb_{t+1}\\
&\left.-\sum_{t=2}^{\infty}\left(\lambda_t+\lambda_{t-1}\right)\frac{\beta}{4}b_t^2+\sum_{t=1}^{\infty}\lambda_t\frac{(1+\beta)^2}{4}\right\}
\end{aligned}$$

As the objective function is strictly decreasing in b_t $\forall t\geq 2$, regardless of b_1 and the planner weights $\{\lambda_t\}_{t=1}^{\infty}$, it follows directly that efficient benefits are given by $b_t^*=0$ $\forall t\geq 2$. Now reconsider the planner's problem, substituting in the optimal solution for $t\geq 2$:

$$\begin{aligned}
&\max_{b_0\in[0,1]} V\left(b_1,u_1|\left\{b_t^*\right\}_{t=2}^{\infty}\right)\\
=&\max_{b_0\in[0,1]}\left\{\lambda_{0s}\left(1-u_1\right)+\lambda_{0u}u_1b_1-\left(\lambda_{0s}\left(1-u_1\right)+\lambda_{0u}u_1\right)u_1b_1\right.\\
&\left.+\left(\lambda_{0s}\left(1-u_1\right)+\lambda_{0u}u_1-\lambda_1\right)\frac{1}{2}\left(u_1-\frac{(1-\beta)}{2}\right)b_1-\left(\lambda_{0s}\left(1-u_1\right)+\lambda_{0u}u_1\right)\frac{1}{4}b_1^2\right\}.
\end{aligned}$$

Taking the first-order condition and implementing the constraint $b_1\in[0,1]$, the optimality of b_1 follows. ■

1.2 Complete characterization of the anti-welfare equilibrium

Proposition 2 *Suppose that $\beta\geq\beta_M$ and suppose beliefs are given by $\theta\in\left[\theta_d\left(\beta\right),\beta\right]$, where $\beta_M\approx 0.555$ is the real solution to the equation $\theta_d\left(\beta\right)=\beta$ and $\theta_d\left(\beta\right)$ is defined below. Then an AWE, $\langle B^{aw},U^{aw}\rangle$, exists and has the following characteristics:*

Case 1 Suppose that $\beta\geq\frac{\sqrt{5}-1}{2}\simeq 0.618$ and that beliefs are given by $\theta\in\left[\theta_a\left(\beta\right),\beta\right]$, where

$\theta_a(\beta) \equiv \left(1+\beta-\sqrt{\beta(1+\beta)}\right)$. *Then there exists an AWE implying "switch in one period"; namely:*

$$B^{aw}(u_t;\theta) = \begin{cases} \theta & \text{if } u_t > 1/2 \\ 0 & \text{if } u_t \in \left[0, \frac{1}{2}\right] \end{cases}$$

$$U^{aw}(b_t;\theta) = \begin{cases} \frac{1}{2}(1-\beta+\beta\theta+b_t) & \text{if } b_t > \theta \\ U^{pl}(b_t) & \text{else} \end{cases}$$

where $U^{pl}(b_t)$ *is defined in Proposition 1.*

Case 2 Suppose $\beta \geq \underline{\beta}$ *and suppose beliefs are such that* $\theta \in [\theta_b(\beta), \min\{\beta, \theta_a(\beta)\})$, *where* $\theta_b(\beta) \equiv \max\left\{\left(1+\sqrt{\beta}\right)^{-1}, \frac{1}{2(2+\beta)}\left(3+4\beta-\sqrt{\left(1+4\beta+8\beta^2\right)}\right)\right\}$, *and* $\underline{\beta} \simeq .570$ *is the real solution to the equation* $\theta_b(\beta) = \beta$, *(which occurs at* $\left(1+\sqrt{\beta}\right)^{-1} = \beta$*). Then there exists an AWE involving "switch in two periods"; namely:*

$$B^{aw}(u_t;\theta) = \begin{cases} \theta & \text{if } u_t \geq u^a(\beta,\theta) \\ 1 & \text{if } u_t \in (0.5, u^a(\beta,\theta)) \\ 0 & \text{if } u_t \in \left[0, \frac{1}{2}\right] \end{cases}$$

$$U^{aw}(b_t;\theta) = \begin{cases} \frac{1}{2}(1-\beta+\beta\theta+b_t) & \text{if } b_t > \theta \\ U^{pl}(b_t) & \text{else} \end{cases}$$

where $u^a(\beta,\theta) \equiv \frac{1}{2}(2+\beta-(2\beta+3-\theta)\theta)\cdot(1-\theta)^{-1}$.

Case 3 Suppose $\beta \in \left[\beta_M, 2^{-2/3}\right] \approx [0.555, 0.630]$ *and suppose beliefs are such that* $\theta \in \left[\theta_d(\beta), \min\left\{\beta, \left(1+\sqrt{\beta}\right)^{-1}\right\}\right)$, *where*

$$\theta_d(\beta) \equiv 2(1+\beta) / \left(3+2\beta+2\sqrt{\beta}\right).$$

Then there exists an AWE involving "switch in two periods"; namely:

$$B^{aw}(u_t;\theta) = \begin{cases} \theta & \textit{if} \quad u_t \in \left[u^d(\beta,\theta), 1\right] \\ \frac{3}{2} + \beta(1-\theta)/2 - u_t & \textit{if} \quad u_t \in \left[u^c(\beta,\theta), u^d(\beta,\theta)\right) \\ 1 & \textit{if} \quad u_t \in (0.5, u^c(\beta,\theta)) \\ 0 & \textit{if} \quad u_t \in \left[0, \frac{1}{2}\right] \end{cases}$$

$$U^{aw}(b_t;\theta) = \begin{cases} \frac{1}{2}(1-\beta+\beta\theta+b_t) & \textit{if } b_t > \theta \\ U^{pl}(b_t) & \textit{else} \end{cases}$$

where $u^c(\beta,\theta) \equiv \frac{1}{2} + \frac{\beta(1-\theta)}{2}$, $u^d(\beta,\theta) \equiv \frac{3+\beta}{2} - \left(\frac{2+\beta}{2} + \sqrt{\beta}\right)\theta$, *and* β_M *and* $\underline{\beta}$ *are defined above, and* $2^{-2/3}$ *is the real solution to the equation* $\left(1+\sqrt{\beta}\right)^{-1} = \theta_d(\beta)$.

Proof omitted

Figure 1. Region of β and θ where AWE are sustained. The area between the 45^0 line and dashed line represents the range of β and θ considered in Proposition 3 (i.e. cases 1 and 2 only).

COMMENT:

1. Intuition for the function $\theta_b(\beta) \equiv \max\left\{\left(1+\sqrt{\beta}\right)^{-1}, \frac{1}{2(2+\beta)}\left(3+4\beta-\sqrt{\left(1+4\beta+8\beta^2\right)}\right)\right\}$. The first part, i.e. $\theta \geq \left(1+\sqrt{\beta}\right)^{-1}$, implies that the constraint $b_t \leq 1$ is strictly binding for all $u \in (0.5, u^a(\beta,\theta))$. In particular, for $\theta = \left(1+\sqrt{\beta}\right)^{-1}$and $u = u^a(\beta,\theta)$, the old unsuccessful are indifferent between setting $b = \theta$ and $b = 1$. The second part, i.e. $\theta \geq \frac{1}{2(2+\beta)}\left(3+4\beta-\sqrt{\left(1+4\beta+8\beta^2\right)}\right)$ guarantee that $U^{aw}(b;\theta) \geq u^a(\beta,\theta)$ is a rational expectations equilibrium outcome for any $b > \theta$. This restriction is necessary to ensure that $U^{aw}(b;\theta)$ satisfy rational expectations for all $b \in [0,1]$.

2. Intuition for the function $\theta_d(\beta)$. The function $\theta_d(\beta) = 2(1+\beta)/\left(3+2\beta+2\sqrt{\beta}\right)$, is defined as the solution to the equation

$$\lim_{b \to \theta^+} \{U^{aw}(b;\theta)\} = u^d(\beta, \theta),$$

i.e. the minimum beliefs θ which would guarantee $U^{aw}(b;\theta) \geq u^d(\beta,\theta)$ as a rational expectations equilibrium outcome for any $b > \theta$. This restriction is necessary to ensure that $U^{aw}(b;\theta)$ satisfy rational expectations for all $b \in [0,1]$.

1.3 Pro-welfare equilibrium with myopic voting

In order to characterize the political equilibrium under myopic majority voting, we show, as in the rational voting case, first the equilibrium under dictatorship. The following proposition characterizes the myopic dictatorship equilibria.

Proposition 3 *The myopic PL equilibrium is identical to* $\langle B^{pl}, U^{pl} \rangle$.

The myopic DP equilibrium, $\langle B^{mdp}, U^{mdp} \rangle$*, is characterized as follows;*

$$B^{mdp}(u_t) = \begin{cases} -2\left(1+\rho^{mdp}\right)\left(u_t - u^{mdp}\right) + b^{mdp} & \text{if } u_t > u^{mdp} - \frac{1-b^{mdp}}{2\left(1+\rho^{mdp}\right)} \\ 1 & \text{if } u_t \in \left[0, u - \frac{1-b^{mdp}}{2\left(1+\rho^{mdp}\right)}\right] \end{cases}$$

$$u_{t+1} = \rho^{mdp}\left(u_{t-1} - u^{mdp}\right) + u^{mdp},$$

where,

$$b^{mdp} = \frac{2+2\beta}{3+2\beta} < b^{dp},$$

$$u^{mdp} = \frac{5+3\beta}{6+4\beta} < u^{dp},$$

$$\rho^{mdp} = -\frac{1}{2\beta}\left(2+\beta-\sqrt{(4+\beta^2)}\right) \geq -\frac{1}{2}$$

The economy converges asymptotically to b^{mdp} *and* u^{mdp}*. Since* $u_{t+1} > u^{mdp} - \frac{1-b^{mdp}}{2\left(1+\rho^{mdp}\right)}$ *for all* u_t*, the constraint* $b_t \leq 1$ *never binds except possibly in the first period.*

Proof. The policy function under DP maximizes utility of old unsuccessful, given b_t, solving

$$B^{mdp}(u_t) = \arg\max_{b_t} \left(b_t - \frac{(1-\beta) + (b_t + \beta b_{t+1}) + 2u_t}{4} b_t \right),$$

with a first-order condition

$$b_t = \frac{3 + \beta(1 - b_{t+1})}{2} - u_t.$$

Rearranging terms yields,

$$b_{t+1} = -\frac{2}{\beta} b_t - \frac{2}{\beta} u_t + \frac{3+\beta}{\beta}. \tag{1}$$

Optimal investment behavior implies

$$\begin{aligned} u_{t+1} &= \left(1 - \beta + b_t + \beta\left(-\frac{2}{\beta}b_t - \frac{2}{\beta}u_t + \frac{3+\beta}{\beta}\right)\right)/2 \\ &= -\frac{b_t}{2} - u_t + 2. \end{aligned} \tag{2}$$

Equations (1) and (2) form a system of linear difference equations, with a steady state given by b^{mdp} and u^{mdp} as expressed in the proposition. Both roots of the system are negative, one stable and one unstable. The stable root, denoted ρ^{mdp}, increases in β and is given by

$$\rho^{mdp} = -\frac{1}{2\beta}\left(2 + \beta - \sqrt{(4+\beta^2)}\right) \in \left[-\frac{1}{2}, -\frac{3}{2} + \frac{1}{2}\sqrt{5}\right].$$

The associated eigenvector is $\left[-2\left(1+\rho^{mdp}\right), 1\right]$ and the solution to the system is therefore given by

$$\begin{aligned} b_t &= -2\left(1+\rho^{mdp}\right)\left(u_0 - u^{mdp}\right)\left(\rho^{mdp}\right)^t + b^{mdp}, \\ u_t &= \left(u_0 - u^{mdp}\right)\left(\rho^{mdp}\right)^t + u^{mdp} \end{aligned} \tag{3}$$

Using this, we can derive the policy function $B^{mdp}(u_t) = -2\left(1+\rho^{mdp}\right)\left(\left(u_t - u^{mdp}\right)\right) + b^{mdp}$ and the law-of-motion for $u_t = \rho^{mdp}\left(u_{t-1} - u^{mdp}\right) + u^{mdp}$.

Finally, the constraint $b \in [0,1]$ is to be taken into account. While $b \geq 0$ never binds, $b \leq 1$ binds if

$$u_t < u^{mdp} - \frac{1 - b^{mdp}}{2\left(1 + \rho^{mdp}\right)}.$$

From the law-of-motion for u_t, we find that for all $t > 0$, $u_t \geq \rho^{mdp}\left(1 - u^{mdp}\right) + u^{mdp} > u^{mdp} - \frac{1-b^{mdp}}{2\left(1+\rho^{mdp}\right)}$. Thus, the constraint $b_t \leq 1$ can only bind in the initial period.

Now, consider majority voting. Clearly, if $u_0 \leq 1/2$, $b_t = 0$ for all $t \geq 0$.[2] If, instead, $u_0 > 1/2$, the welfare state will never break down under myopic voting. Under myopic voting, if $1 > u_0 > 1/2$, then $u_t > 1/2$ and $b_t > 0$ $\forall t$, i.e., the welfare state survives. Assume the opposite, then there is a period $t \geq 0$ such that $u_t > 1/2, b_t > 0$ and $u_{t+1} \leq 1/2, b_{t+1} = 0$. Then, the choice of benefits satisfies $b_t = \min\left\{1, \frac{3+\beta}{2} - u_t\right\}$,

$$\begin{aligned} u_{t+1} &= \left(1 - \beta + \min\left\{1, \frac{3+\beta}{2} - u_t\right\}\right)/2 \\ &\geq \left(1 - \beta + \min\left\{1, \frac{3+\beta}{2} - 1\right\}\right)/2 \\ &\geq 1/2, \end{aligned}$$

which contradicts the initial assumption. ■

Using the previous proposition, the next proposition immediately follows;

Proposition 4 *Under myopic voting. The equilibrium is given by*

$$B^m(u_t) = \begin{cases} B^{mdp}(u_t) & \text{if } u_t > 1/2 \\ 0 & \text{if } u_t \leq \frac{1}{2} \end{cases}$$

$$U^m(b_t) = \begin{cases} B^{mdp}(u_t) & \text{if } b_t > 0 \\ \frac{1-\beta}{2} & \text{if } b_t = 0 \end{cases}$$

[2] This is due to rational investments, since $u_{t+1} = (1 - \beta + \beta b_{t+1})/2 \leq 1/2 \forall b_{t+1}$.

1.4 A stochastic pro-welfare equilibrium

Proposition 5 *3 For all $\beta \in (0,1)$, there exists a dense compact subset of parameters and beliefs, $(p,\theta) \in [0,1] \times [0,\beta]$, including $p = 1$ and $\theta = 0$, that sustains the following "stochastic pro-welfare equilibrium", $\langle B^{spw}, U^{spw} \rangle$;*

$$B^{spw}(u_t) = \begin{cases} \frac{1}{p}\left(\frac{3}{2} - u_t\right) & \text{if } u_t > \bar{u}(\beta,p) \\ b_s - \frac{1}{2p-\beta} 2(u_t - u_s) & \text{if } u_t \in \left[\frac{3}{2} - \frac{2p}{2+\beta}, \bar{u}(\beta,p)\right] \\ 1 & \text{if } u_t \in \left(\frac{1}{2}, \frac{3}{2} - \frac{2p}{2+\beta}\right] \\ 0 & \text{if } u_t \leq 1/2 \end{cases},$$

$$U^{spw}(b_t) = \begin{cases} u_s + \frac{2p-\beta}{4p}(b_t - b_s) & \text{if } b_t \in \left[\frac{2\beta+4(1-p)}{2+\beta}, 1\right] \\ \frac{1+b_t}{2} & \text{if } b_t \in \left(\theta, \frac{2\beta+4(1-p)}{2+\beta}\right) \\ \frac{1-\beta+b_t}{2} & \text{if } b_t \leq \theta \end{cases},$$

where

$$\begin{aligned} u_s &\equiv u_{dp} + (1-p)\frac{4+\beta(5+\beta)}{3(2+\beta)(1+2p)} \\ b_s &\equiv b_{dp} + (1-p)\frac{2(4+\beta)}{3(2+\beta)(1+2p)} \\ \bar{u}(\beta,p) &= \frac{6+\beta-2(1-p)(2p-\beta)-\sqrt{2p(2p-\beta)(\beta+2(1-p))^2}}{2(2+\beta)}. \end{aligned}$$

The equilibrium law of motion is

$$u_{t+1} = \begin{cases} U^{spw}(B^{spw}(u_t)) & \text{with probability } p \\ \nu_t & \text{with probability } (1-p), \end{cases}$$

where ν_t is i.i.d. *with a p.d.f. $f(\nu_t)$ and $E(\nu_t) = 1/2$.*

Proposition 3 nests the equilibrium of the deterministic economies of Proposition 2. For $p = 1$, the stochastic pro-welfare equilibrium is identical to the PWE of Proposition 2. For

$p < 1$, an existing pro-welfare or anti-welfare majority regenerates itself with probability $(1+p)/2$ each period, whereas a change of majority occurs with probability $(1-p)/2$.

Proof.

We first prove existence of the equilibrium under DP.

Optimal effort depends on b_t and b_{t+1}, where the latter is conditional on no shocks, and satisfies

$$e_t^*(b_t, b_{t+1}) = (1 + \beta - (b_t + \beta b_{t+1}))/2.$$

Thus,

$$u_{t+1} = \begin{cases} e_t^*(b_t, b_{t+1}) & \text{with probability } p \\ \nu & \text{with probability density } (1-p) f(\nu) \end{cases}.$$

To balance the budget, $2\tau_t = (u_t + u_{t+1}) b_t / p$, yielding

$$\tau_t = \begin{cases} \frac{(1-\beta+b_t+\beta b_{t+1})+2u_t}{4} \frac{b_t}{p} & \text{with probability } p \\ \frac{\nu+u_t}{2} \frac{b_t}{p} & \text{with probability } (1-p) \end{cases},$$

and the expected tax rate being $\tau_t^e = \frac{(1-\beta+b_t+\beta b_{t+1})+2u_t}{4} b_t + (1-p) \frac{E(\nu)+u_t}{2} \frac{b_t}{p}$, with $E(\nu) = 1/2$. The equilibrium conditions can thus be written

1) $B^{sdp}(u_t) = \arg\max_{b_t} \left\{ V^{ou}\left(b_t, u_t, B^{sdp}\left(U^{sdp}(b_t)\right)\right)\right\}$, subject to $b_t \in [0, 1]$; and

2) $U^{sdp}(b_t) = \left(1 - \beta + b_t + \beta B^{sdp}\left(U^{sdp}(b_t)\right)\right)/2$,

where

$$V^{sdp}\left(b_t, u_t, B^{sdp}\left(U^{sdp}(b_t)\right)\right) \equiv \hat{V}^{ou}(b_t, u_t)$$
$$= \frac{b_t}{p} - \left(\frac{p\left(1 - \beta + b_t + \beta B^{sdp}\left(U^{spd}(b_t)\right)\right) + 2u_t + (1-p)}{4} \frac{b_t}{p}\right).$$

Using the expressions for B^{sdp}, and U^{sdp},

$$B^{sdp}\left(U^{sdp}(b_t)\right) = \begin{cases} \frac{1}{p} + \frac{\beta}{2+\beta} - \frac{1}{2p} b_t & \text{if } b_t \in \left[\frac{2\beta+4(1-p)}{2+\beta}, 1\right] \\ 1 & \text{if } b_t \in \left[0, \frac{2\beta+4(1-p)}{2+\beta}\right) \end{cases} \tag{4}$$

where we used the fact that for $p = 1$,

$$\bar{u} - U^{sdp}(1) = \frac{1}{4}\frac{4 + 2\sqrt{2}\sqrt{(2-\beta)}\beta - \beta^2}{2+\beta} > 0,$$

implying that there is a $\bar{p} < 1$ such that any $p \in (\bar{p}, 1]$,

$$\bar{u} > U^{sdp}(1).$$

Now, we can evaluate

$$\hat{V}^{ou}(b_t, u_t) = \begin{cases} \left(\left(\frac{1}{4p}\frac{6+\beta(1+2p-\beta)}{2+\beta}\right) - \frac{1}{2p}u_t\right) b_t - \left(2 - \frac{\beta}{p}\right)\frac{b_t^2}{8} & \text{if } b_t \in \left[\frac{2\beta+4(1-p)}{2+\beta}, 1\right] \\ b_t \frac{3 - pb_t - 2u_t}{4p} & \text{else,} \end{cases},$$

which is maximized by $b_t = B^{sdp}(u_t)$. To see this in more detail, define

$$V_2 = \hat{V}^a(u_t) \equiv \max_{b_t \in \left[0, \frac{2\beta+4(1-p)}{2+\beta}\right]} b_t \frac{3 - pb_t - 2u_t}{4p}$$

$$= \begin{cases} \frac{1}{16p^2}(3 - 2u_t)^2 \equiv \hat{V}^{a,int}(u_t) \text{ if } u_t \geq \frac{1}{2}\frac{6-\beta-4(1-p)(2p-\beta)}{2+\beta} \\ (\beta + 2(1-p))\frac{6+\beta-2u_t(2+\beta)+(1-p)2(\beta-2p)}{2(2+\beta)^2 p} \equiv \hat{V}^{a,cor}(u_t) \text{ else,} \end{cases}$$

$$V_1 = \hat{V}^b(u_t) \equiv \max_{b_t \in \left[\frac{2\beta+4(1-p)}{2+\beta}, 1\right]} \hat{V}^{ou}(b_t, u_t)$$

$$= \begin{cases} \frac{1}{8}\frac{\left(\beta^2 - \beta(1+2p) + 2u_t(2+\beta) - 6\right)^2}{p(2p-\beta)(2+\beta)^2} \equiv \hat{V}^{b,int}(u_t), \text{ if } u_t \geq \frac{3}{2} - \frac{2p}{2+\beta} \\ \frac{1}{8}\frac{8+\beta(6-\beta)+(1-p)2(2-\beta)}{p(2+\beta)} - \frac{1}{2p}u_t \equiv \hat{V}^{b,cor}(u_t), \text{ else,} \end{cases}$$

which we see are generalizations of the corresponding expressions in the proof of proposition 1. The functions $\hat{V}^{a,cor}(u_t)$ and $\hat{V}^{b,cor}(u_t)$ are calculated when the constraints $b \leq$

$\frac{2\beta+4(1-p)}{2+\beta}$and $b \leq 1$, respectively, are binding implying that for any u_t, $\hat{V}^{a,int}(u_t) \geq \hat{V}^{a,cor}(u_t)$ and $\hat{V}^{b,int}(u_t) \geq \hat{V}^{b,cor}(u_t)$. The constraint $b_t \geq \frac{2\beta+4(1-p)}{2+\beta}$, for $\hat{V}^{b,int}(u_t)$ does not bind for $p \geq 1/2$, as is easily verified by standard algebra. We also note that $\frac{2\beta+4(1-p)}{2+\beta} < 1$ if $p > \frac{1}{2}+\frac{1}{4}\beta$, which precludes the degenerate solution $B^{sdp}\left(U^{sdp}(b_t)\right) = 1 \forall b_t$. Now, by definition, $\hat{V}^{a,int}(\bar{u}) = \hat{V}^{b,int}(\bar{u})$ and for $u_t > (<)\, \bar{u}, \hat{V}^{a,int}(u) > (<)\, \hat{V}^{b,int}(u)$ since $\frac{\partial \hat{V}^{a,int}}{\partial u} > \frac{\partial \hat{V}^{b,int}}{\partial u} \forall u \in [0,1], \beta > 0$ and $p > 1/2$. Furthermore, for $p = 1$, $\bar{u} - \frac{1}{2}\frac{6-\beta+4(1-p)\beta-8p(1-p)}{2+\beta} = \frac{1}{2}\beta\frac{2-\sqrt{(4-2\beta)}}{2+\beta} > 1$,implying, again by continuity, that for $u_t \geq \bar{u}$, $\hat{V}^{a}(u) = \hat{V}^{a,int}(u) > \hat{V}^{b,int}(u) \geq \hat{V}^{b}(u)$ for any p in a neighborhood below unity. Thus, for $u_t > \bar{u}$ $B^{sdp}(u_t)$ is the interior optimum $\frac{1}{p}\left(\frac{3}{2} - u_t\right)$.

Now, consider $u < \bar{u}$, where we need to show that $\hat{V}^{b}(u) > \hat{V}^{a}(u)$. We already know that $\hat{V}^{b,int}(u) > \hat{V}^{a,int}(u) \geq \hat{V}^{a}(u)$ when $u < \bar{u}$. Thus, it remains to be shown that for $u_t < \frac{3}{2} - \frac{2p}{2+\beta}, \hat{V}^{b}(u) = \hat{V}^{b,cor}(u) \geq \hat{V}^{a}(u)$. To see that this is the case, note that if $p > 1/2 + \beta/2$, then $\frac{1}{2}\frac{6-\beta+4(1-p)\beta-8p(1-p)}{2+\beta} > \left(\frac{3}{2} - \frac{2p}{2+\beta}\right)$, implying $\hat{V}^{a}(u) = \hat{V}^{a,cor}(u)$ for all $u < \frac{3}{2} - \frac{2p}{2+\beta}$. Now,

$$\left(\hat{V}^{b,cor}(u) - \hat{V}^{a,cor}(u)\right)|_{u_t \leq \frac{3}{2}-\frac{2p}{2+\beta}} \geq \frac{(2p-\beta)(2+\beta-4p)^2}{8(2+\beta)^2 p} > 0,$$

implying $\hat{V}^{b}(u) > \hat{V}^{a}(u)$ as required.

Finally, to show that $U^{sdp}(b_t)$ satisfies the equilibrium condition, we use (4) to substitute for b_{t+1} yielding

$$\begin{aligned} 1 - e_t^*\left(b_t, B^{sdp}\left(U^{sdp}(b_t)\right)\right) &= \begin{cases} 1 - \left(1 + \beta - \left(b_t + \beta\left(\frac{1}{p} + \frac{\beta}{2+\beta} - \frac{1}{2p}b_t\right)\right)\right)/2 & \text{if } b_t \in \left[\frac{2\beta+4(1-p)}{2+\beta}, 1\right] \\ \frac{1+b_t}{2} & \text{if } b_t \in \left[0, \frac{2\beta+4(1-p)}{2+\beta}\right) \end{cases} \\ &= U^{sdp}(b_t). \end{aligned}$$

■

Comment on proof of Proposition 3:

By plotting the different conditions for p, we can see that the strongest condition is implied by

$$\bar{u}(\beta, p) - U^{sdp}(1) \geq 0.$$

This condition is a fourth-order equation in p, given by

$$p^4 + \left(1 + \frac{2}{\beta} - \frac{\beta}{2}\right) p^3 - \left(3 + \frac{2}{\beta} + \frac{\beta}{2}\right) p^2 + \left(\frac{3}{2} + \frac{1}{2\beta} + \frac{5\beta}{8}\right) p = \frac{1}{16}(2+\beta)^2$$

where we note that the relevant root satisfies $0.5 + 0.45\beta < p < 0.6 + 0.4\beta$.

Proof. Now, consider the case of majority voting and the two equilibrium conditions

1) $B^{spw}(u_t) = \arg\max_{b_t} \{V^{pw}(b_t, u_t, B^{spw}(U^{spw}(b_t)))\}$, subject to $b_t \in [0,1]$; and

2) $U^{spw}(b_t) = (1 - \beta + b_t + \beta B^{spw}(U^{spw}(b_t)))/2$,

where

$$\begin{aligned} & V^{spw}(b_t, u_t, B^{spw}(U^{spw}(b_t))) \\ = & \begin{cases} \frac{b_t}{p} - \left(\frac{p\left(1-\beta+b_t+\beta B^{sdp}\left(U^{spd}(b_t)\right)\right)+2u_t+(1-p)}{4}\frac{b_t}{p}\right) & \text{if } u_t > 1/2 \\ \frac{1}{p} - \left(\frac{p\left(1-\beta+b_t+\beta B^{sdp}\left(U^{spd}(b_t)\right)\right)+2u_t+(1-p)}{4}\frac{b_t}{p}\right) & \text{else.} \end{cases} \end{aligned}$$

Starting with condition 2), we substitute $B^{spw}(U^{spw}(b_t))$ for b_{t+1} in the optimal investment expression, giving,

$$\begin{aligned} & (1 - \beta + b_t + \beta B^{spw}(U^{spw}(b_t)))/2 \\ = & \begin{cases} 1 - \left(1 + \beta - \left(b_t + \beta\left(\frac{1}{p} + \frac{\beta}{2+\beta} - \frac{1}{2p}b_t\right)\right)\right)/2 & \text{if } b_t \in \left[\frac{2\beta+4(1-p)}{2+\beta}, 1\right] \\ (1+b_t)/2, & b_t \in \left(\theta, \frac{2\beta}{2+\beta}\right) \\ \frac{1-\beta+b_t}{2} & b_t \leq \theta \end{cases}, \\ = & U^{spw}(b_t), \end{aligned}$$

provided that, as in the non-stochastic case, $\theta \leq \beta$.

As for condition 1), consider first the range where $u_t \leq 1/2$, then $V_t^{spw}(b_t, u_t)$ is maximized by setting $b_t = 0$.

Now, consider $u_t > 1/2$. Noting that $U^{spw}(b_t) > 1/2$ for all $b_t > \theta$, we find that for any $b_t > \theta$, any $u_t > 1/2$ and any p sufficiently close to unity, $V^{spw}(b_t, u_t, B^{spw}(U^{spw}(b_t))) = V^{sdp}\left(b_t, u_t, B^{sdp}\left(U^{sdp}(b_t)\right)\right)$. We thus only need to verify that for these values of b_t, u_t and p,

$$\begin{aligned} \max_{b_t > \theta} V^{sdp}\left(b_t, u_t, B^{sdp}\left(U^{sdp}(b_t)\right)\right) &\geq \max_{b_t \leq \theta} V^{spw}(b_t, u_t, B^{spw}(U^{spw}(b_t))) \qquad (5) \\ &= \max_{b_t \leq \theta} \frac{1}{4} b_t \frac{3 + p\beta - pb_t - 2u_t}{p}. \end{aligned}$$

Since, $\max_{b_t \leq \theta} \frac{1}{4} b_t \frac{3+p\beta - pb_t - 2u_t}{p}$ tends to zero as θ approaches zero, while $\max_{b_t > \theta} V^{sdp}\left(b_t, u_t, B^{sdp}\left(U^{sdp}(b_t)\right)\right)$ $\frac{1}{16p^2} > 0$, provided that $\theta < \frac{1}{2p}$ there exists a $\bar{\theta} > 0$ such that for any $\theta < \bar{\theta}$, (5) is satisfied and an existing pro-welfare majority chooses not to induce a breakdown. ■