Nucleons in nucleus behave not like particles, moving with momenta de Broglie wavelength $\lambda = h/p$.

Because $p \sim 250$ MeV/c < $E = 939$ MeV it's behaviour is described by not Quantum Mechanics (NR QM).

1900 M. Planck – blackbody radiation: emission of el-magn waves light is not only a wave but also a particle (y-parti)

1905 A. Einstein – photoelectric effect

1924 De Broglie - particles have also a wave origine - $\lambda = h/p$ (based on Einstein photoeffect and Compton scatt)

1927 Thompson, Davission and Germer experiment showed the diff of the electrons with wavelength $\lambda = h/p$

M Born - the state of Quantum System is described by the wa (or probability amplitude) which connected with probability $P$ to find the system in volume $dV$ as
dP = |$\psi|^2 dV$ where $|$ $\psi$ $|^2$ is probability density

$P = \int \psi^* d_\psi = \int |$ $\psi$ $|^2 dV$ = total probability with normalized $\int_\psi \psi^* dV = 1$

The wave function $\psi(x,t)$ is the state vector similar to the $r(t)$ in Classical Mechanics

Superposition principle: the difference between bullets passed electrons scattered on two small slits is that one bullet always, and in the case of two opened slits the detector will detect $P_{12} = P_1 + P_2$

while electron behaves as a wave so the probability that two electrons scattered on two small slits is that one electron always, and in the case of two opened slits the detector will detect $P_{12} = P_1 + P_2$

where $P_{12} = |$ $\psi_1$ $|^2 + |$ $\psi_2$ $|^2$ i.e. in the case of microworld the superposition principle is valid

$\psi_1$ = $\psi_1 + \psi_2$

here $P_{12} = |$ $\psi_1$ $|^2 + |$ $\psi_2$ $|^2$ is the probability to find electron in the 1st slit

It means that each electron feels the existance of two slits, and the wave of one electron from 1st slit is interfering with the wave of the electron from 2nd slit. The wavefunction $\psi = \psi_1 + \psi_2 + \psi_3 + \ldots + \psi_n$

and the particle can be described by the electromagnetic wave $\rightarrow 1925$ E Schroedinger QM : particle is not classical: its momenta and position cannot be simultaneously measured exactly but with some accuracy:

1927 Heisenberg uncertainty principle:

for position x and momentum p : $\Delta x \Delta p < h/2$ , or

for energy $E$ and time $t$ $\Delta E \Delta t < h/2(E=hc)$ where $h = h/2\pi$ – Planck for angular momentum $L$ and its projection on z $l$, and on $xy$

For a long time Scientists argued about the exact nature of light.

At the beginning of the XIXth century Thomas Young proposed that light can interfere with itself.

The results of this Double slit experiment can be explained if light is made of waves.

In 1854, James Maxwell showed that an electromagnetic wave (like Radio waves)

In 1868, Heinrich Hertz confirmed Maxwell’s prediction of transmission of electromagnetic waves in Air but he also observed an unexpected phenomena: Sparks were more easily created when the wave was illuminated by ultraviolet light.

The energy of these particles of light striking a metal would expel electrons from the metal.

The predictions of Einstein’s theory were confirmed in 1915 by Max Planck and Albert Einstein.

In 1887, Albert Einstein (1879–1955) extended this idea and proposed that light was made of particles that can have only specific energy values.

In 1905, Albert Einstein (1879–1955) extended this idea and proposed that light was made of particles that can have only specific energy values.

The same phenomena was also observed in Cathode Ray Tube (CRT) a few years later and called photoelectric effect but remained unexplained.

The photoelectric effect

So what is light made of?

Einstein’s explanation of the photoelectric does not involve [an] Einstein’s theory to light is made of particles (called photons) which behave like waves.

This idea was first rejected by many scientists but further experimental evidences convinced them that this is true.

Further work by Einstein and by other scientists led to the understanding that not only light but also all forms of matter behave both as particles and as waves.

This gave birth to the Quantum Theory.

Further reading: http://en.wikipedia.org/wiki/Photoelectric_effect
QM: Schrodinger equation

- The Schrodinger equation plays the role of Newton's laws and conservation of energy in classical mechanics - i.e., it predicts the future behavior of a dynamic system. It is a wave equation in terms of the wavefunction $\psi(x,p,t)$ which predicts analytically and precisely the probability $P = |\psi(x,p,t)|^2$ of the event.
- The kinetic (T) and potential (V) energies are transformed into the Hamiltonian which acts upon the wave function to generate the evolution of the wavefunction in time and space.

$$T + V = E$$

- The Schrodinger equation gives the quantized energies of the system and gives the form of the wavefunction so that other properties may be calculated.
• **Operators in Quantum Mechanics**

  Associated with each measurable parameter in a physical system is a quantum mechanical operator. Such operators arise because in quantum mechanics you are describing nature with waves (the *wavefunction*) rather than with discrete particles whose motion and dynamics can be described with the deterministic equations of Newtonian physics. Part of the development of quantum mechanics is the establishment of the operators associated with the parameters needed to describe the system. Some of those operators are listed below.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>Any function of position, such as $x$, or potential $V(x)$</td>
</tr>
<tr>
<td>$p_x$</td>
<td>$x$ component of momentum ($y$ and $z$ same form)</td>
</tr>
<tr>
<td>$E$</td>
<td>Hamiltonian (time independent)</td>
</tr>
<tr>
<td>$KE$</td>
<td>Kinetic energy</td>
</tr>
<tr>
<td>$L_z$</td>
<td>$z$ component of angular momentum</td>
</tr>
</tbody>
</table>

  - It is part of the basic structure of quantum mechanics that functions of position are unchanged in the *Schroedinger equation*, while momenta take the form of spatial derivatives. The *Hamiltonian operator* contains both time and space derivatives.
Nonrelativistic QM equation for spinless particle-wave packet is simply the energy conservation law:
\[ \frac{p_x^2}{2m} + V(x) = E_{tot} \] - energy conservation
\[ \hat{p} = -i\hbar(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}) \]
\[ i\hbar \frac{\partial}{\partial t} \psi = (-\frac{\hbar^2}{2m} \nabla^2 + V)\psi \] - time dependent Schrödinger eq.
\[ H\psi \equiv (-\frac{\hbar^2}{2m} \nabla^2 + V)\psi = E\psi \] - if \( V=V(x) \) - time independent Schrödinger eq
\[ \psi(x,t) = \psi(x)e^{-iEt/\hbar} \] - solution of Schr. eq. for time independent case
+ conditions on wavefunction \( \psi \):
\[ \psi \text{ and } \frac{d\psi}{dx} \text{ must be continuous across any boundary:} \]
\[ \lim_{\epsilon \to 0} [\psi(a+\epsilon) - \psi(a-\epsilon)] = 0 \] - should always be fulfilled
\[ \lim_{\epsilon \to 0} \left[ \left( \frac{d\psi}{dx} \right)_{x=a+\epsilon} - \left( \frac{d\psi}{dx} \right)_{x=a-\epsilon} \right] = 0 \] - can be violated in the case of discontinuous \( V(x) \)

If we know \( \psi(x,t) \) then we know
\[ P = \int_{x_1}^{x_2} \psi^*(x,t)\psi(x,t)dx \] - probability to find particle (wave package) between \( (x_1,x_2) \)
\[ \int_{-\infty}^{\infty} \psi^*\psi dx = 1 \] - normalization condition: the total probability should be 1
\[ \langle f \rangle = \int \psi^*f(x)\psi dx \] - physical meaning of the observable \( f(x) \) is statistical average over all possible states
\[ \langle p_x \rangle = \int \psi^*\hat{p}_x\psi dx = \int \psi^*(-i\hbar \frac{\partial}{\partial x})\psi dx = -i\hbar \int \psi^*\frac{\partial\psi}{\partial x} dx \] - for momentum operator \( \hat{p}_x \)

For stationary case, \( V(x) \), \( e^{-iEt/\hbar} \) time dependent part is cancelling and \( \langle p_x \rangle \) does not depend on \( t \)
\[ j = \frac{\hbar}{2mi} (\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}) \] - particle current density
Free-particle in one dimension

Stationary Schrödinger eq. for free particle \( H\psi = E\psi \) with \( H = \frac{p^2}{2m} \) and \( V(x) = 0 \) looks like

\[
-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi \quad \text{and has solution:}
\]

\[
\psi(x) = A' \sin(kx) + B' \cos(kx) \quad \text{or equivalently}
\]

\[
\psi(x) = A e^{ikx} + B e^{-ikx}, \quad \text{where} \quad k^2 = 2mE / \hbar^2
\]

\[
\psi(x) = A e^{i(kx-\omega t)} + B e^{-i(kx+\omega t)} \quad \text{- time-dependent solution with} \quad E = \hbar \omega
\]

I. \( A e^{i(kx-\omega t)} \) - a wave traveling to \( x = +\infty \)
II. \( B e^{-i(kx+\omega t)} \) - a wave traveling to \( x = -\infty \)

Boundary conditions are not valid in this case because both

\( \cos^2 kx \) and \( \sin^2 kx \) diverge at \( x = \pm \infty \).

In contrast we put at \( x = -\infty \) a source (accelerator) which emits particles

with the rate \( I \) particles per second with momentum \( p = \hbar k \) in positive direction \( x = +\infty \).

Then \( B = 0 \) because there is no particles travelling in opposite direction to \( x = -\infty \).

The particle current should coincide with initial current of the emitted particles:

\[
\begin{align*}
 j &= \frac{\hbar}{2mi} (\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}) = \frac{\hbar}{2mi} (A^* e^{i(kx-\omega t)} (ik) A e^{i(kx-\omega t)} - A e^{i(kx-\omega t)} (-ik) A^* e^{-i(kx-\omega t)}) = \\
 j &= \frac{\hbar}{2mi} 2 |A|^2 ik = \frac{\hbar k}{m} |A|^2 = I, \quad \text{so}
\end{align*}
\]

\( A = \sqrt{mI / \hbar k} \).

The plane wave solution for free particle is

\[
\psi(x,t) = \sqrt{mI / \hbar k} e^{i(kx-\omega t)} \quad \text{with} \quad E = \hbar \omega \quad \text{and} \quad p = \hbar k.
\]
Infinite potential cell

Stationary Schrodinger eq. for free particle in box

\[ H\psi = E\psi \] with \( H = \frac{p^2}{2m} + V(x) \)

and \( V(x) = \begin{cases} \infty, & \text{at } x > a \text{ and } x < 0 \\ 0, & \text{at } 0 \leq x \leq a \end{cases} \)

looks like

\[ -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi \] at \( 0 < x < a \) and has solution:

\[ \psi(x) = A \sin(kx) + B \cos(kx) \]

with continuity at \( x = 0 \) and \( x = a \):

\[ \psi(0) = 0 \Rightarrow B = 0, \]

\[ \psi(a) = 0 \Rightarrow A \sin(kL) = 0, A \neq 0 \] then

\[ \sin(ka) = 0 \]

with solutions at \( kL = n\pi \) where \( n = 1, 2, 3 \ldots \) and \( k^2 = \frac{2mE}{\hbar^2} \)

\[ E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} n^2 \] - here energy \( E_n \) is quantized, only certain values are permitted

\( A \) is found from normalization condition for wave function \( \psi(x) = A \sin\left(\frac{\pi nx}{a}\right) \)

\[ \int_0^a \psi^* \psi dx = 1 \]

\[ \int_0^L |A|^2 \sin^2 \frac{\pi nx}{a} dx = 1. \text{ Using } \cos 2x = 1 - 2\sin^2 x \text{ what gives } \sin^2 x = (1 - \cos 2x) / 2 \text{ one gets} \]

\[ 1 = \frac{1}{2} \int_0^a |A|^2 \left( 1 - \cos 2 \frac{\pi n}{a} x \right) dx = \frac{1}{2} |A|^2 \left( a - \frac{1}{2\pi n} \int_0^a \cos \frac{\pi n}{a} x dx \right) = \frac{1}{2} |A|^2 \left( a - \frac{1}{2\pi n} \int_0^a \cos \frac{\pi n}{a} x dx \right) = \frac{1}{2} |A|^2 \left( a - \frac{1}{2\pi n} \right) \]

\[ 1 = \frac{|A|^2}{2} a, \text{ because } \int_0^{2\pi n} \cos x dx = \sin x \bigg|_0^{2\pi n} = \sin(2\pi n) - \sin(0) = 0 \text{ at } n = 1, 2, 3\ldots \]

\( A = \sqrt{2/a} \).

The plane wave solution for free particle in the infinite box is

\[ \psi_n(x,t) = \sqrt{\frac{2}{a}} \sin \frac{\pi nx}{a} \text{ for } E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2 \text{ and } n = 1, 2, 3\ldots \]
**Potential Barrier, \(E > V_0\) and (tunnelling effect) \(E < V_0\)**

\[
\begin{align*}
E > V_0 & \quad \rightarrow & \quad E < V_0 \\
\psi_1 x \approx A_1 e^{k_1 x} + B_1 e^{-k_1 x} & \quad & \psi_5 x \approx F e^{k_1 x} \\
\psi_2 x \approx A_2 e^{k_2 x} + B_2 e^{-k_2 x} & \quad & \psi_6 x \approx Fe^{-k_1 x} \\
\psi_3 x \approx \bar{F} e^{k_1 x} & \quad & \psi_7 x \approx Ce^{k_1 x} + D e^{-k_1 x}
\end{align*}
\]

Stationary Schrodinger eq. \(E\) for step potential \(V_0\):
\[
\frac{p^2}{2m} + V(x) \psi = E \psi
\]

- \(V_0\) if \(E > V_0\)
- \(E < V_0\)
- \(E = V_0\)

System of linear equations:
\[
A + B = C + D \quad \text{(1)}
\]

\[
| A | - | B | = |-i( k^2 - k^2 \sinh k x)\rangle
\]

Transmission coefficient for barrier \(V_0 > E\):
\[
T = 1 - \left| \frac{2k_2 \cosh k_2 - i( k^2 - k^2 \sinh k x)}{2} \right|^2
\]

Where \(k_2 a \ll 1\), \(T \approx 1\)

\(k_2 a \gg 1\), \(T \approx \left| \frac{2k_2 \cosh k_2 - i( k^2 - k^2 \sinh k x)}{2} \right|^2\)

For the case \(V_0 < E\), \(k_2 a \approx \frac{1}{2m} h V_0 / E\)

The WKB approximation can be used for the case of arbitrary smooth potential \(V(x)\):
\[
T = e^{-2\sum k \exp(k)} \rightarrow at \Delta x \rightarrow 0 \exp \left[ -2\int dx \left( \frac{2m}{h^2} [V(x) - E] \right)^{1/2} \right] = WKB approximation - Wentzel – Kramers – Brillouin
A.2. Density of the states and Three dimensional box

In the case of the well with infinite potential barrier the solution of the Schrodinger eq.

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t),$$

are the standing waves vanishing at the ends of the well

$$\psi_n(x,t) = \left\{ \begin{array}{ll} A\sin(k_n x)e^{-i\omega_nt}, & 0 < x < L, \quad |A| = \frac{\sqrt{2}}{L}, \\ 0, & \text{otherwise,} \end{array} \right.$$  

where $n = \{1, 2, 3, 4, \ldots \}$, corresponding to discrete energy levels always $> 0$ because of uncertainty principle: $\Delta p \sim 1/L \rightarrow \Delta(E = p^2/2m) \sim 1/L^2$.

For **3 dimensional well** the energy and the wavefunction are

$$E_n = \frac{n^2\hbar^2\pi^2}{2mL^2} = \frac{n^2\hbar^2}{8mL^2},$$  

$$E_{n_x,n_y,n_z} = \frac{n_x^2\hbar^2}{2mL_x^2} + \frac{n_y^2\hbar^2}{2mL_y^2} + \frac{n_z^2\hbar^2}{2mL_z^2}.$$

The number of the states is proportional to the number of lattice points distanced from each other by ($\pi/L$): $(L/\pi)^3$

Or in the Fermi sphere it will be 1/8 of the sphere (because $nxyz > 0$)

$$n(k_0) = \frac{1}{8} \cdot \frac{4}{3} \pi k_0^3 \cdot (L/\pi)^3 = \frac{V}{(2\pi)^3} \frac{4}{3} \pi k_0^3$$

for all $k < k_0$ while for $k_0 < k < k_0 + dk_0$:

$$dn(k_0) = \frac{V}{(2\pi)^3} \frac{4}{3} \pi k_0^2 dk_0,$$

$$dn(p) = \frac{V}{(2\pi)^3} \frac{4}{3} \pi p^2 dp,$$

$$dn(E) = \frac{V}{(2\pi)^3} \frac{4}{3} \pi k_0^2 dk_0$$

where $\rho(k_0) = dn(k_0)/dk_0 = \frac{V}{(2\pi)^3} \frac{4}{3} \pi k_0^2$ is density of states for $k_0 < k < k_0 + dk_0$

$$\rho(p) = dn(p)/dp = \frac{V}{(2\pi)^3} \frac{4}{3} \pi p^2$$

is density of states for momenta $p < p < p + dp : k = \hbar p$

$$\rho(E) = \rho(p)dp/dE = \frac{V}{(2\pi)^3} \frac{4}{3} \pi p^2v = \frac{\sqrt{2\pi\hbar^2}}{2mp}$$

- for $E < E < E + dE$, $E = p^2/2m, p = mv$

$$\rho(E) = \frac{V}{(2\pi)^3} \frac{p^2v}{vd\Omega}$$

for solid angle $\Omega$, $d^3p = p^2dpd\Omega$ 

$$dE = p/mdp = vdp$$

For particles with spin spin multiplicity factor should be included:

for spin=1/2 according to the Pauli principle such factor is 2.
A.3 Perturbation theory and the Second Golden Rule

In perturbation theory the Hamiltonian at any time $t$ may be written as $H(t) = H_0 + V(t)$, where $H_0$ is unperturbed Hamiltonian and $V(t)$ is small.

The solution for eigenfunctions of $H$ starts by expanding in the terms of the complete set of energy eigenfunctions $|u_n> \text{ of } H_0 : H_0 |u_n> = E_n |u_n>$

$|\psi(t)> = \sum c_n(t) |u_n> \exp{-iE_n t/\hbar}$, where $E_n$ are the corresponding energies.

If $|\psi(t)>$ is normalized to unity $<\psi(t)|\psi(t)> = 1$ then $|c_n(t)|^2$ is probability that at time $t$ the system will be in the state $|u_n>$. Substituting it in Schrodinger eq $i\hbar \frac{\partial c_f(t)}{\partial t} = \sum V_{fn}(t) c_n(t) \exp{-i\omega_{fn} t}$, where matrix element $V_{fn}(t) = <u_f|V(t)|u_n>$ and angular frequency $\omega_{fn} = (E_f - E_n)/\hbar$.

Initial state of the system $|u_i>$ at $t=0$ then $c_n(0) = \delta_{ni}$ and $c_i(t) = c_i(0) + 1/i\hbar \int_0^t V_{ii}(t') dt'$

Final state of the system $f \neq i$ $c_f(t) = 1/i\hbar \int_0^t V_{fi}(t') e^{i\omega_{fi} t'} dt'$ general, for $V(t)$

For const $V = 0$ and then $V0$ $c_i(t) = V_{fi}/(\hbar \omega_{fi})(1-e^{i\omega_{fi} t})$

Probability for transition from state $i$ to state $f$:

$P_{fi} = |c_f(t)|^2 = 4|V_{fi}|^2/\hbar^2 [\sin^2(1/2 \omega_{fi} t)/\omega_{fi}^2]$. At large $t$ it is valid only if $\Theta | \omega_{fi} | = |E_f - E_i| < 2\pi \hbar/ t$ (uncertainty principle) then $[\sin^2(1/2 \omega_{fi} t)/\omega_{fi}^2] \rightarrow 1/2 \pi \hbar t \delta(E_f - E_i)$.

$P_{fi} = 2\pi t/\hbar |V_{fi}|^2 \delta(E_f - E_i) - \text{probability}$

$dP_{fi}/dt = 2\pi/\hbar |V_{fi}|^2 \delta(E_f - E_i) - \text{transition probability per unit time} - \text{valid for discrete final states}$.

$dT_{fi}/dt = \int dP_{fi}/dt \rho(E_f) dE_f = 2\pi/\hbar [|V_{fi}|^2 \rho(E_f)]_{E_f=E_i} - \text{transition probability per unit time} - \text{for continuous spectra}$
Dirac Delta Function

1 Definition

Dirac's delta function is defined by the following property

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

with

$$\int_{t_1}^{t_2} dt \delta(t) = 1$$

if $0 \in [t_1, t_2]$ (and zero otherwise). It is “infinitely peaked” at $t = 0$ with the total area of unity. You can view this function as a limit of Gaussian

$$\delta(t) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi \sigma}} e^{-t^2/2\sigma^2}$$

or a Lorentzian

$$\delta(t) = \lim_{\sigma \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + t^2}.$$

The important property of the delta function is the following relation

$$\int dt f(t) \delta(t - t_0) = f(t_0)$$

for any function $f(t)$. This is easy to see. First of all, $\delta(t)$ vanishes everywhere except $t = 0$. Therefore, it does not matter what values the function $f(t)$ takes except at $t = 0$. You can then say $f(t)\delta(t) = f(0)\delta(t)$. Then $f(0)$ can be pulled outside the integral because it does not depend on $t$, and you obtain the r.h.s. This equation can easily be generalized to

$$\int dt f(t) \delta(t - t_0) = f(t_0).$$

2 Fourier Transformation

It is often useful to talk about Fourier transformation of functions. For a function $f(t)$, you define its Fourier transform as

$$\tilde{f}(s) = \int_{-\infty}^{\infty} dt \frac{e^{ist}}{\sqrt{2\pi}} f(t).$$

This transform is reversible, i.e., you can go back from $\tilde{f}(s)$ to $f(t)$ by

$$f(t) = \int_{-\infty}^{\infty} ds \frac{e^{-ists}}{\sqrt{2\pi}} \tilde{f}(s).$$

You may recall that the patterns from optical or X-ray diffraction are Fourier transforms of the structure. For example, Laue determined the crystallographic structure of solid by doing inverse Fourier transform of the X-ray diffraction patterns.

If you set $f(t) = \delta(t)$ in the above equations, you find

$$\tilde{\delta}(s) = \int_{-\infty}^{\infty} dt \frac{e^{ist}}{\sqrt{2\pi}} \delta(t) = \frac{1}{\sqrt{2\pi}},$$

$$\delta(t) = \int_{-\infty}^{\infty} ds \frac{e^{-ists}}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} ds \frac{e^{-ists}}{2\pi}.$$}

In other words, the delta function and a constant $1/\sqrt{2\pi}$ are Fourier-transform of each other.

Another way to see the integral representation of the delta function is again using the limits. For example, using the limit of the Gaussian Eq. (3),

$$\delta(t) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi \sigma}} e^{-t^2/2\sigma^2}$$

$$= \lim_{\sigma \to 0} \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} e^{-\omega^2/2\sigma^2} e^{-i\omega t}$$

$$= \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} e^{-i\omega t}.\quad (11)$$
In Ernest Rutherford’s laboratory, Hans Geiger and Ernest Marsden (a 20 yr old undergraduate student) carried out experiments to study the scattering of \textit{alpha particles} by thin metal foils.

In 1909 they observed that alpha particles from radioactive decays occasionally scatter at angles greater than 90°, which is physically impossible unless they are scattering off something more massive than themselves. This led Rutherford to deduce that the positive charge in an atom is concentrated into a small compact nucleus.

During the period 1911-1913 in a table-top apparatus, they bombarded the foils with high energy alpha particles and observed the number of scattered alpha particles as a function of angle.

Based on the Thomson model of the atom, all of the alpha particles should have been found within a small fraction of a degree from the beam, but Geiger and Marsden found a few scattered alphas at angles over 140 degrees from the beam.

Rutherford’s remark “It was quite the most incredible event that ever happened to me in my life. It was almost as incredible as if you had fired a 15-inch shell at a piece of tissue paper and it came back and hit you.” The scattering data was consistent with a small positive nucleus which repelled the incoming positively charged alpha particles. Rutherford worked out a detailed formula for the scattering (Rutherford formula), which matched the Geiger-Marsden data to high precision.

Finally, we need to calculate the differential cross-section. If the initial flux of projectile particles crossing a plane perpendicular to the beam direction is \( J \), then the intensity of particles having impact parameters between \( b \) and \( b + db \) is \( 2\pi bJ \ dv \) and this is equal to the rate \( dW \) at which particles are scattered into a solid angle \( d\Omega = 2\pi \sin\theta \ d\theta \) between \( \theta \) and \( \theta + d\theta \). Thus

\[
\begin{align*}
\frac{dW}{d\Omega} &= 2\pi J \sin\theta \ d\theta.
\end{align*}
\]  

\[
\begin{align*}
\frac{d\sigma}{d\Omega} &= \frac{b}{\sin^2\theta} \
\frac{db}{d\Omega}.
\end{align*}
\]  

\[
\begin{align*}
\frac{\Delta \sigma}{d\Omega} &= \frac{1}{16\pi \beta^2 \cos^4\theta/2}.
\end{align*}
\]  

\[
\begin{align*}
\frac{d\sigma}{d\Omega} &= \left( \frac{ze^2}{16\pi \beta^2 \cos^4\theta/2} \right)^2 \cos\theta/2.
\end{align*}
\]  

This is the final form of the Rutherford differential cross-section for nonrelativistic scattering.
Appendix C 2. Rutherford scattering CM versus QM

**Classical mechanics**: Rutherford's model enables us to derive a formula for the angular distribution of scattered α-particles.

**Basic Assumptions:**

- a) Scattering is due to Coulomb interaction between a- particle and positively charged atomic nucleus.
- b) Target is thin enough to consider only single scattering (and no shadowing)
- c) The nucleus is massive and fixed.
- d) Scattering is elastic

### C.2 Quantum Mechanics

While (C.10) is adequate to describe the α-particle scattering experiments, in the case of electron scattering we need to take account of both relativity and quantum mechanics. This may be done using the general formula for the differential cross-section in terms of the scattering potential that was derived in Chapter I. We will neglect spin factors.

The starting equation is (1.29), which in the present notation is

\[
\frac{d\sigma}{d^2 \alpha} = \frac{1}{4\pi^2} \frac{\rho^2}{uv} |\mathcal{M}(q)|^2,
\]

where \(v\) and \(p\) are the velocity and momentum respectively of the projectile (which for convenience we take to have a unit negative charge), because the target is assumed to be heavy, with \(v = |v|, p = \gamma p\) and the primes refer to the final-state values. The matrix element is given by

\[
\mathcal{M}(q) = \int V(r) e^{i\mathbf{q} \cdot \mathbf{r}} d^3 r,
\]

where \(V(r)\) is the Coulomb potential

\[
V(r) = V_0(r) = -\frac{Z e^2 r}{r},
\]

where \(Z\) is the charge of the target nucleus. Inspection of the integrand in (C.15) shows that it diverges at large \(r\). However, in practice, charges are always screened at large distances by intervening matter and so we will integrate the integral as

\[
\mathcal{M}_c(q) = \lim_{\lambda \to 0} \int \left( -\frac{Ze^2 r}{r} \right) e^{i\mathbf{q} \cdot \mathbf{r}} d^3 r.
\]

To evaluate this, take \(q\) along the \(z\) axis, so that in spherical polar coordinates \(q = q\hat{r}\). The angular integration may then be done and yields

\[
\mathcal{M}_c(q) = \frac{2 \pi e^2 Z e \lambda \hbar \lambda}{q} \int_0^\infty \int_0^\pi \sin(q \lambda) \theta \theta \, d\theta.
\]

The remaining integral may be done by parts (twice) and taking the limit \(\lambda \to 0\) gives

\[
\mathcal{M}_c(q) = -\frac{4 \pi e^2 Z e \hbar}{q^2}.
\]

Finally, substituting (C.19) into (C.14) gives

\[
\frac{d\sigma}{d^2 \alpha} = 4\pi^2 \frac{\rho^2}{u'c'q^2} = 4\pi^2 \frac{\rho^2}{ucq^2},
\]

which is the general form of the Rutherford differential cross-section. To see that this is the same as (C.13) in the nonrelativistic limit, we may substitute the approximations

\[
p' = p'' = 2mE_{in}, \quad u = u' = \sqrt{2E_{in}/m},
\]

where \(E_{in}\) is the energy of the incoming particle.
Finite potential cell

Stationary Schrodinger eq. for free particle in box

\[ H\psi = E\psi \] with \( H = \frac{\hbar^2}{2m} + V(x) \)

\[ V(x) = \begin{cases} V_0, & \text{at } x > a \text{ and } x < 0 \\ 0, & \text{at } 0 \leq x \leq a \end{cases} \]

looks like

\[- \frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \] at \( 0 < x < a \) and has solution:

\[ \psi(x) = A \sin(kx) + B \cos(kx) \] with continuity at \( x = 0 \) and \( x = a \):

\[ \psi(0) = 0 \Rightarrow B = 0, \]

\[ \psi(a) = 0 \Rightarrow A \sin(kL) = 0, A \neq 0 \text{ then} \]

\( \sin(ka) = 0 \) with solutions at \( kL = \pi n \) where \( n = 1, 2, 3 \ldots \) and \( k^2 = \frac{2mE}{\hbar^2} \)

\[ E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} n^2 \] - here energy \( E_n \) is quantized, only certain values are permitted

\( A \) is found from normalization condition for wave function \( \psi(x) = A \sin\left(\frac{\pi nx}{a}\right) \)

\[ \int_0^a |\psi|^2 dx = 1 \]

\[ \int_0^L |A|^2 \sin^2\left(\frac{\pi nx}{a}\right) dx = 1. \] Using \( \cos2x = 1 - 2\sin^2x \) what gives \( \sin^2x = (1 - \cos2x)/2 \) one gets

\[ 1 = \frac{1}{2} \int_0^a |A|^2 (1 - \cos2\frac{\pi n}{a}x) dx = \frac{1}{2} |A|^2 \left(a - \frac{2\pi n}{a} \int_0^a \cos2\frac{\pi n}{a} x dx\right) = \frac{1}{2} |A|^2 \left(a - \frac{a}{2\pi n} \int_0^{2\pi n/a} \cos2\frac{\pi n}{a} x dx\right) = \frac{|A|^2}{2} a \]

\[ 1 = \frac{|A|^2}{2} a , \text{ because } \int_0^{2\pi n/a} \cos x dx = \sin x \bigg|_0^{2\pi n} = \sin(2\pi n) - \sin(0) = 0 \text{ at } n = 1, 2, 3... \]

\[ A = \sqrt{\frac{2}{a}} \]

The plane wave solution for free particle in the infinite box is

\[ \psi_n(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi nx}{a}\right) \quad \text{for} \quad E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2 \text{ and } n = 1, 2, 3,... \]
5 coefficients $A, B, C, D, F$ can be defined from 4 boundary conditions, it means that one - for example, $A$, is defined from somewhere else.

$$\psi_{-}^{0}\omega^{0} \psi_{-}^{0} \omega^{0}, \quad \psi_{-}^{0} \omega^{0} \psi_{-}^{0} \omega^{0}$$

System of linear equations

$$A + B = C + D \quad (1)$$
$$B + C = D + 0 \quad (2)$$
$$A + a C + a D = 0 \quad (3)$$
$$e^{\text{a}D} - e^{\text{a}D} = 0 \quad (4)$$

where $\sinh x = e^x - e^{-x} / 2, \cosh x = e^x + e^{-x} / 2$

$$T = \begin{bmatrix} 2k_{x} & k_{x}^{2} \\ k_{x} & 2k_{x}^{2} \end{bmatrix}$$

$$4k_{x}^{2}e^{aD} + e^{aD} + 27k_{x}^{2}e^{aD} = 0$$

$$4k_{x}^{2}e^{aD} + e^{aD} + 27k_{x}^{2}e^{aD} = 0$$

Transmission coefficient for barrier $V_{0} > E$: