Hydrodynamics and transport properties of the Quark-Gluon Plasma

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1. **Formalism**

2. **Numerical implementation and results**
Starting point: The conservation laws

\[ \partial_\mu N^\mu = 0 \] charge conservation

\[ \partial_\mu T^{\mu\nu} = 0 \] energy-momentum conservation

\[ \partial_\mu S^\mu \geq 0 \] 2nd law of thermodynamics
Ideal fluid decomposition

Ideal fluid dynamics $\iff$ local thermal equilibrium $f(x,p) = f_{eq}(x,p)$ $\iff$ collision time scale $\ll$ macroscopic time scales $\iff$ strong coupling

\[
N^\mu = \frac{1}{(2\pi)^3} \int \frac{d^3p}{E} p^\mu f(x,p) = n u^\mu \quad n = \text{(net) charge density}
\]

\[
T^{\mu\nu} = \frac{1}{(2\pi)^3} \int \frac{d^3p}{E} p^\mu p^\nu f(x,p) = (e + p) u^\mu u^\nu - p g^{\mu\nu}
\]

\[
= e u^\mu u^\nu - p \Delta^{\mu\nu}
\]

\[
S^\mu = s u^\mu
\]

First law of thermodynamics: \[ Ts = p - \mu n + e \]

\[
\partial_\mu N^\mu = \partial_\mu T^{\mu\nu} = 0 \implies \partial_\mu S^\mu = 0
\]

(in absence of shock discontinuities, entropy is conserved)
Ideal fluid equations (in comoving frame)

Convective and transverse derivative:
\[
\partial_\mu = u_\mu D + \nabla_\mu
\]
\[
D \equiv u_\nu \partial_\nu, \quad \nabla_\mu \equiv \Delta_\mu^\nu \partial_\nu
\]

\[
\dot{n} = -n \theta
\]
\[
\dot{e} = - (e + p) \theta
\]
\[
\dot{u}_\mu = \frac{\nabla_\mu p}{e + p} = \frac{c_s^2}{1 + c_s^2} \frac{\nabla_\mu e}{e}
\]
\[
p = p(n, e)
\]

\[
\dot{f} = u_\mu \partial_\mu f \equiv D f = \text{time derivative in local rest frame}
\]
\[
\theta \equiv \partial \cdot u = \text{local expansion rate}
\]
\[
c_s^2(T) = \frac{\partial p}{\partial e} = (\text{speed of sound})^2
\]

6 equations for 6 unknowns: \(n, e, p, u_\mu\)
Non-ideal fluid decomposition

\[ f(x, p) = f_{\text{eq}}(x, p) + \delta f(x, p) \]

\[
\begin{align*}
N^\mu &= n u^\mu + V^\mu \\
&= N_{\text{eq}}^\mu + \delta N^\mu \\
T^{\mu\nu} &= e u^\mu u^\nu - p \Delta^{\mu\nu} - \Pi \Delta^{\mu\nu} + \pi^{\mu\nu} \\
&\quad + W^\mu u^\nu + W^\nu u^\mu \\
&= T_{\text{eq}}^{\mu\nu} + \delta T^{\mu\nu} \\
S^\mu &= s u^\mu + \Phi^\mu \\
&= S_{\text{eq}}^\mu + \delta S^\mu \\
n &= u_\mu N^\mu \\
V^\mu &= \Delta^{\mu\nu} N_\nu = \text{charge flow in l.r.f.} \\
e &= u_\mu T^{\mu\nu} u_\nu \\
\Pi &= -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} - p = \text{viscous bulk pressure} \\
W^\mu &= u^\nu T_{\nu\lambda} \Delta^{\lambda\mu} = \text{energy flow in l.r.f.} \\
&= q^\mu + \frac{e+p}{n} V^\mu \\
q^\mu &= \text{heat flow in l.r.f.} \\
\pi^{\mu\nu} &= T^{(\mu\nu)} \\
&\equiv \left[ \frac{1}{2} (\Delta^{\mu\sigma} \Delta^{\nu\tau} + \Delta^{\mu\tau} \Delta^{\nu\sigma}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\tau\sigma} \right] T_{\tau\sigma} \\
&= \text{viscous shear pressure tensor} \quad (\pi^\mu_\mu = 0) \\
s &= u_\mu S^\mu \\
\Phi^\mu &= \Delta^{\mu\nu} S_\nu = \text{entropy flow in l.r.f.}
\end{align*}
\]
Frame choice and matching conditions

The local equilibrium distribution $f_{eq}(x, p)$ (with local temperature $T(x)$ and chemical potential $\mu(x)$) that best matches the non-equilibrium $f(x, p)$ is defined by the matching conditions

$$u_\mu \delta T^{\mu\nu} u_{\nu} = u_\mu \delta N^\mu = 0$$

Local rest frame is ambiguous:

Eckart frame: $u^\mu = \frac{N^\mu}{\sqrt{N.N}}$ : $V^\mu = 0, \ q^\mu = W^\mu$
Landau frame: $u^\mu = \frac{T^{\mu\nu} u_\nu}{\sqrt{u_\alpha T_{\alpha\beta} u_\beta}}$ : $W^\mu = 0, \ q^\mu = -\frac{e+p}{n} V^\mu$

(Intermediate frames also possible.)

→ Need $1 + 3 + 5 = 9$ additional equations for $\Pi, q^\mu, \pi^{\mu\nu}$ from underlying transport theory.
Non-ideal fluid equations

\[\begin{align*}
\dot{n} &= -n \theta - \nabla \cdot V + V \cdot \dot{u} \\
\dot{\varepsilon} &= -(e + p + \Pi) \theta + \pi_{\mu\nu} \sigma^{\mu\nu} - \nabla \cdot W + 2 W \cdot \dot{u} \\
(e + p + \Pi) \dot{u}^{\mu} &= \nabla^{\mu} (p + \Pi) - \Delta^{\mu\nu} \nabla^{\sigma} \pi_{\nu\sigma} + \pi^{\mu\nu} \dot{u}_{\nu} \\
&\quad - [\Delta^{\mu\nu} \dot{W}_{\nu} + W^{\mu} \theta + (W \cdot \nabla) u^{\mu}] \\
\end{align*}\]

Here \(\sigma^{\mu\nu} \equiv \nabla^{\langle \mu} u^{\nu \rangle}\) is the velocity shear tensor.

Depending on frame, can set either \(V^{\mu} = 0\) or \(W^{\mu} = 0\). In Landau frame \((W^{\mu} = 0)\) and for baryon-free systems \((n = 0,\ \text{no heat conduction})\) equations simplify to:

\[\begin{align*}
\dot{\varepsilon} &= -(e + p + \Pi) \theta + \pi_{\mu\nu} \sigma^{\mu\nu} \\
(e + p + \Pi) \dot{u}^{\mu} &= \nabla^{\mu} (p + \Pi) - \Delta^{\mu\nu} \nabla^{\sigma} \pi_{\nu\sigma} + \pi^{\mu\nu} \dot{u}_{\nu} \\
\end{align*}\]

Need 6 extra equations for bulk and shear viscous pressures \(\Pi, \pi^{\mu\nu} \Rightarrow \text{different paths} \) (Navier-Stokes, Israel-Stewart, Öttinger-Grmela, BRSSS, . . . )

Here we follow Chapman-Enskog strategy: write \(f(x, p) = f_{\text{eq}}(p \cdot u(x); T(x), \mu(x)) + \delta f(x, p)\) and assume that \(\delta f \ll f\) (and thus \(\delta N^{\mu}\) and \(\delta T^{\mu\nu}\)) can be expanded in gradients of equilibrium parameters \(T, \mu, u_{\mu}\).
The second law of thermodynamics (I)

In equilibrium the identity $Ts = p - \mu n + e$ can be rewritten as

$$S_{eq}^\mu = p(\alpha, \beta)\beta^\mu - \alpha N_{eq}^\mu + \beta_\nu T_{eq}^{\mu\nu}$$

where $\alpha \equiv \mu / T$, $\beta \equiv 1 / T$, and $\beta^\mu \equiv u^\mu / T$.

The most general off-equilibrium generalization is (Israel & Stewart 1979)

$$S^\mu = p(\alpha, \beta)\beta^\mu - \alpha N^\mu + \beta_\nu T^{\mu\nu} + Q^\mu(\delta N^\mu, \delta T^{\mu\nu})$$

where $Q^\mu$ is second and higher order in the off-equilibrium deviations $\delta N^\mu$ and $\delta T^{\mu\nu}$.

The Gibbs-Duhem relation $dp = s \, dT + n \, d\mu$ can be recast as

$$\partial_\mu(p(\alpha, \beta)\beta^\mu) = N_{eq}^\mu \partial_\mu \alpha - T_{eq}^{\mu\nu} \partial_\mu \beta_\nu$$

Using also the conservation laws, the entropy production rate takes the form

$$\partial_\mu S^\mu = -\delta N^\mu \partial_\mu \alpha + \delta T^{\mu\nu} \partial_\mu \beta_\nu + \partial_\mu Q^\mu$$
The second law of thermodynamics (II)

In the Chapman-Enskog spirit, one now postulates linear relations between the off-equilibrium flows $\delta N^\mu$, $\delta T^{\mu\nu}$ and the thermodynamic forces $\partial^\mu \alpha$, $\partial^{(\mu} \beta^{\nu)}$, consistent with the second law

$$\partial_\mu S^\mu = -\delta N^\mu \partial_\mu \alpha + \delta T^{\mu\nu} \partial_\mu \beta_\nu + \partial_\mu Q^\mu \geq 0$$

These relations depend on the choice of $Q^\mu$. Standard dissipative relativistic fluid dynamics assumes $Q^\mu = 0$. In this case

$$T \partial_\mu S^\mu = \Pi X - q^\mu X_\mu + \pi^{\mu\nu} X_{\langle \mu\nu \rangle} \equiv \frac{\Pi^2}{\zeta} - \frac{q^\mu q_\mu}{2\lambda T} + \frac{\pi^\alpha\beta \pi_\alpha\beta}{2\eta} \geq 0,$$

with thermodynamic forces $X \equiv -\nabla \cdot u = -\theta$, $X^\mu \equiv \frac{\nabla^\mu T}{T} - \dot{u}^\mu = -\frac{nT}{e+p} \nabla^\mu \left( \frac{\mu}{T} \right)$ and $X_{\langle \mu\nu \rangle} \equiv \sigma_{\mu\nu} = \nabla_{\langle \mu} u_{\nu \rangle}$, can be satisfied by setting

$$\Pi = -\zeta \theta, \quad q^\mu = -\lambda \frac{nT^2}{e+p} \nabla^\mu \left( \frac{\mu}{T} \right), \quad \pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}$$

with positive transport coefficients $\zeta \geq 0$, $\lambda \geq 0$, and $\eta \geq 0$ (relativistic Navier-Stokes theory).

Unfortunately, plugging these equations for $\Pi$, $q^\mu$, and $\pi^{\mu\nu}$ directly into the non-ideal hydro equations leads to acausal signal propagation.
The second law of thermodynamics (III)

Causal relativistic fluid dynamics requires keeping $Q^\mu$ in the entropy flux, at least up to terms of second order in the irreversible flows.

$$ S^\mu = s u^\mu + \frac{q^\mu}{T} + Q^\mu $$

Setting $q^\nu = 0$ ($n = 0$) for simplicity, we get up to second order

$$ S^\mu = s u^\mu - (\beta_0 \Pi^2 + \beta_2 \pi_{\nu\lambda} \pi^{\nu\lambda}) \frac{u^\mu}{2T} $$

This yields (after some algebra)

$$ T \partial_\mu S^\mu = \Pi \left[ -\theta - \beta_0 \dot{\Pi} - \Pi T \partial_\mu \left( \frac{\beta_0 u^\mu}{2T} \right) \right] + \pi^{\alpha\beta} \left[ \sigma_{\alpha\beta} - \beta_2 \dot{\pi}_{\alpha\beta} - \pi_{\alpha\beta} T \partial_\mu \left( \frac{\beta_2 u^\mu}{2T} \right) \right] $$

$$ = \frac{\Pi^2}{\zeta} + \frac{\pi^{\alpha\beta} \pi_{\alpha\beta}}{2\eta} \geq 0 $$

The thermodynamic forces $-\theta$, $\sigma_{\alpha\beta}$ are seen to be self-consistently modified by the irreversible flows $\Pi$, $\pi_{\alpha\beta}$. This leads to dynamical ("transport") equations for $\Pi$, $\pi_{\alpha\beta}$.
Transport equations for the irreversible flows

The resulting transport equations for \( \Pi, \pi_{\alpha\beta} \) are (Israel & Stewart 1979, Muronga 2002, 2004)

\[
\dot{\Pi} = -\frac{1}{\tau_\Pi} \left[ \Pi + \zeta \theta + \Pi \zeta T \partial_\mu \left( \frac{\tau_\Pi w^\mu}{2\zeta T} \right) \right] = -\frac{1}{\tau_\Pi'} \left[ \Pi + \zeta' \theta \right]
\]

\[
\Delta_\alpha \Delta_\beta \pi^{\mu\nu} = -\frac{1}{\tau_\pi} \left[ \pi_{\alpha\beta} - 2\eta \sigma_{\alpha\beta} + \pi_{\alpha\beta} \eta T \partial_\mu \left( \frac{\tau_\pi w^\mu}{2\eta T} \right) \right] + \text{terms that don’t generate entropy}
\]

\[
= -\frac{1}{\tau_\pi'} \left[ \pi_{\alpha\beta} - 2\eta' \sigma_{\alpha\beta} \right] + \ldots
\]

Here we introduced the relaxation times \( \tau_\Pi = \zeta \beta_0, \tau_\pi = 2\eta \beta_2 \), and \( \tau_\Pi' = \frac{\tau_\Pi}{1 + \zeta' \gamma_\Pi}, \zeta' = \frac{\zeta}{1 + \eta' \gamma_\Pi} \).

\( \zeta, \eta \) and \( \tau_\Pi, \tau_\pi \) should be calculated from the underlying microscopic theory. This has been done by KSS and BRSSS for infinitely strongly coupled SYM theory, and by AMY and YM for weakly coupled QCD in Boltzmann transport theory (see below).

The purple terms kick in wherever the expansion rate gets large and then effectively reduce the viscosities and relaxation times. The viscous pressures \( \Pi, \pi^{\mu\nu} \) relax exponentially towards their (flow-modified) Navier-Stokes limits on (flow-modified) microscopic relaxation time scales \( \tau_\Pi', \tau_\pi' \).
More second-order terms . . .

Analyzing the second law of thermodynamics misses second-order terms that don’t contribute to 2nd order entropy production but may still affect the evolution of flow.

For systems with conformal symmetry ($\Pi=0$) and vanishing chemical potentials ($q^\mu=0$) BRSSS (Baier, Romatschke, Son, Starinets, Stephanov, JHEP 04 (2006) 100) found 5 possible second-order terms:

$$
\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \tau_\pi \left[ \Delta^{\mu\alpha} \Delta^{\nu\beta} \hat{\pi}_{\alpha\beta} + \frac{4}{3} \theta \pi^{\mu\nu} \right] - \frac{\lambda_1}{2\eta^2} \pi^{\mu\alpha\nu\beta}_{\alpha} - \frac{\lambda_2}{2\eta} \pi^{\mu\alpha\omega\nu}_{\alpha}
$$

$$
-\frac{\lambda_3}{2} \omega^{\mu\nu}_{\alpha\omega\nu}_{\alpha} + \frac{\kappa}{2} \left[ R^{\mu\nu} + 2u^{\alpha} R^{\alpha\mu\nu}_{\beta} u^{\beta} \right]
$$

where $\omega^{\mu\nu} = \frac{1}{2} (\nabla_{\mu} u_{\nu} - \nabla_{\nu} u_{\mu})$ is the vorticity and $R^{\alpha\mu\nu\beta}$, $R^{\mu\nu}$ are the Riemann and Ricci tensors, respectively.

Now we have 5 second order coefficients $\tau_\pi$, $\lambda_1$, $\lambda_2$, $\lambda_3$, $\kappa$, in addition to $\eta$.

Betz, Henkel and Rischke (arXiv:0812.1440 [nucl-th]) generalized this to include heat conduction and bulk viscosity $\implies$ even more coefficients . . .
Weak and strong coupling limits of shear viscosity and second order coefficients
(zero masses and chemical potentials)

<table>
<thead>
<tr>
<th>Ratio</th>
<th>pQCD ((N_f = 3)) ((\text{AMY '00,'03; YM '08}))</th>
<th>SYM ((\text{PSS '01; KSS '05; BRSSS '08}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\eta}{s})</td>
<td>(\frac{46.1}{N_c^2 g^4 \ln(4.17/g \sqrt{N_c})}) (\approx 1.7 \ (g = 2))</td>
<td>(\frac{1}{4\pi} \approx 0.08)</td>
</tr>
<tr>
<td>(\frac{(e+p)\tau_\pi}{\eta})</td>
<td>5 to 5.9</td>
<td>4 (- 2\ln 2 \approx 2.6137)</td>
</tr>
<tr>
<td>(\frac{(e+p)\lambda_1}{\eta^2})</td>
<td>5.2 to 4.1</td>
<td>2</td>
</tr>
<tr>
<td>(\frac{(e+p)\lambda_2}{\eta^2})</td>
<td>(-2\eta\frac{(e+p)\tau_\pi}{\eta}) (= -10\ \text{to} \ - 11.8)</td>
<td>(-4\ln 2 \approx -2.7726)</td>
</tr>
<tr>
<td>(\frac{(e+p)\lambda_3}{\eta^2})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{(e+p)\kappa}{\eta^2})</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

AMY = Arnold/Moore/Yaffe; YM = York & Moore; PSS = Policastro/Son/Starinets; KSS = Kovtun/Son/Starinets; BRSSS = Baier/Romatschke/Son/Starinets/Stephanov

Fortunately, terms \(\sim \lambda_{1,2,3}\) appear to be numerically unimportant for hydro evolution.
QGP – the most perfect fluid ever observed?

AdS/CFT universal lower viscosity bound conjecture:

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi}$$

Kovtun, Son, Starinets, PRL 94 (2005) 111601

Upper limit for QGP viscosity from various recent estimates are close to this bound!

But: quantitative constraint on $\eta/s$ requires viscous hydrodynamics code.
1. Formalism

2. Numerical implementation and results
Why?

Ideal hydro works well at RHIC.

But there are also indications of non-zero viscosity:
Ideal fluid dynamics breaks down at $p_T \gtrsim 1.5 - 2$ GeV/c:


- Consistent with viscous effects during early QGP stage (viscous corrections increase $\sim p_T^2$)
- Can be used to constrain QGP viscosity $\Rightarrow$ viscous hydrodynamics
Smaller, less dense collision systems: late hadronic viscosity

S. Voloshin [STAR], JPG 34 (2007) S883

HYDRO (EoS H)

HYDRO (EoS Q)

STAR Prelim., \( \nu_2(\text{FTPC})/e_{\text{par}}(2) \)

STAR Prelim., \( \nu_2(\text{ZDC})/e_{\text{std}} \)

HYDRO (EoS H) \( E_{\text{lab}}/A=11.8 \text{ GeV, Au+Au, E877} \)

HYDRO (EoS H) \( E_{\text{lab}}/A=40 \text{ GeV, Pb+Pb NA49} \)

HYDRO (EoS H) \( E_{\text{lab}}/A=158 \text{ GeV, Pb+Pb, NA49} \)

HYDRO (EoS Q)

\( 1/S \frac{dN_{\text{ch}}}{dy} \)

\( \nu_2/\epsilon \)

- viscous correction to ideal hydro \( \frac{\nu_2^{\text{measured}}}{\nu_2^{\text{hydro}}} \) scales with \( \frac{1}{S} \frac{dN_{\text{ch}}}{dy} \propto s_{\text{init}} \) (“multiplicity scaling”)

- ideal hydro limit only approached for \( e_{\text{init}} > 10 \text{ GeV/fm}^3 \)

Why? Late hadronic viscosity! (Teaney, Shuryak 2001)
Late hadronic dissipation explains reduced $v_2$ at forward rapidity and in peripheral collisions:

**3D Hydro+Cascade Model**: Ideal fluid dynamics for QGP above $T_c$, hadronic cascade with realistic cross sections (JAM) below $T_c$


- Not enough elliptic flow from perfect QGP fluid – some hadronic contribution to $v_2$ is required
- Treating the hadronic stage as ideal fluid *overpredicts* $v_2$ in peripheral collisions and at forward rapidities
- Dissipation in hadronic cascade brings theory in line with data
- $\Rightarrow$ No need for QGP viscosity!? Only if you trust Glauber!
**But: CGC gives larger initial eccentricity!**

**3D Hydro+Cascade Model:** Ideal fluid dynamics for QGP above $T_c$, hadronic cascade with realistic cross sections (JAM) below $T_c$

Hirano et al., PLB 636 (2006) 299

Lappi & Venugopalan, PRC 74 (2006) 054905

- Hadronic dissipation reduces elliptic flow buildup in peripheral collisions
- **Color Glass Condensate (CGC-KLN) model** (McLerran & Venugopalan 1994; Kharzeev, Levin, Nardi 2001) produces steeper edge of initial distribution, resulting in larger eccentricities $\epsilon$ than in Glauber model
- Ideal hydrodynamics turns larger spatial eccentricity $\epsilon$ into larger elliptic flow $v_2$
- For Glauber model initial conditions, hadronic dissipation fully explains the data; for CGC/KLN initial conditions hadronic dissipation not enough – need additional QGP viscosity!

⇒ To isolate effects from early viscosity, need better control over initial conditions!
At RHIC, hadronic viscosity & chemical non-equilibrium matter:

PHENIX White Paper, NPA 757 (2005) 184

All theory curves use the same hydrodynamics and EOS in QGP phase!
How we deal with the hadron phase makes all the difference . . .
The only model that simultaneously fits all data is hydro+RQMD
(Teaney & Shuryak 2001)
Hadronic dissipation effects disappear at the LHC:

T. Hirano, U. Heinz, D. Kharzeev, R. Lacey, Y. Nara, Quark Matter 2008

At LHC, all momentum anisotropy is created in QGP phase

⇒ hadronic dissipation effects become negligible

Late hadronic evolution still important for final distribution of momentum anisotropy over particle species (e.g. pions vs. protons)
Relativistic hydrodynamics

for viscous fluids
(2+1)-d viscous hydrodynamics: status

- Romatschke & Romatschke, PRL 99 (’07); Luzum & Romatschke, PRC 78 (’08) & arXiv:0901.4588:
  full Israel-Stewart eqn., EOS I, EOS L*
  \( \text{Au+Au}, T_{\text{dec}} = 150 \text{MeV} \) (EOS L* is quasiparticle EOS based on Lattice QCD)

- Song & Heinz, PLB 658 (’08), PRC 77 (’08), PRC 78 (’08), QM08, SQM2008:
  simplified I-S eqn. & full I-S eqn., EOS I, SM-EOS Q, EOS L
  \( \text{Cu+Cu & Au+Au}, T_{\text{dec}} = 130 \text{MeV} \)

- Dusling & Teaney, PRC 77 (’08):
  Öttinger-Grmela eqn., EOS I
  \( \text{Au+Au} \), kinetic decoupling by scattering cross section

- Huovinen & Molnar, QM08:
  full I-S eqn., EOS I
  comparison of viscous hydro with parton cascade

- Chaudhuri, arXiv:0704.0134, 0708.1252, 0801.3180, 0901.0460, 0901.4181, PLB 672 (’09), QM08:
  \( \text{Au+Au} \), EOS I, EOS Q

- Pratt & Vredevoogd, PRC 78 (’08):
  non-linearly modified I-S eqns., modified EOS Q, \( \text{Au+Au} \)
  viscous hydro + hadron cascade

- Denicol, Kodama et al., arXiv:0903.3595:
  simplified I-S eqn., EOS I, EOS II, EOS III (lattice motivated), \( \text{Au+Au}, T_{\text{dec}} = 130 \text{MeV} \)
  bulk viscosity only

**Code comparison and verification in progress (TECHQM)**
Viscous relativistic hydrodynamics (Israel & Stewart 1979)

Include shear viscosity $\eta$, neglect bulk viscosity (massless partons) and heat conduction ($\mu_B \approx 0$); solve

$$\partial_\mu T^{\mu\nu} = 0$$

with modified energy momentum tensor

$$T^{\mu\nu}(x) = T_0^{\mu\nu}(x) + \pi^{\mu\nu} = (e(x)+p(x))u^{\mu}(x)u^{\nu}(x) - g^{\mu\nu}p(x) + \pi^{\mu\nu}.$$  

$\pi^{\mu\nu}$ = traceless viscous pressure tensor which relaxes locally to $2\eta$ times the shear tensor $\sigma^{\mu\nu} \equiv \nabla^{(\mu} u^{\nu)}$ on a microscopic kinetic time scale $\tau_{\pi}$:

$$D\pi^{\mu\nu} = -\frac{1}{\tau_{\pi}} \left( \pi^{\mu\nu} - 2\eta \nabla^{(\mu} u^{\nu)} \right) - \frac{4}{3} \pi^{\mu\nu} \theta - \left( u^{\mu} \pi^{\nu} + u^{\nu} \pi^{\mu} \right) Du_\lambda + O(\pi^2, \pi \omega, \omega^2)$$

where $D \equiv u^\mu \partial_\mu$ is the time derivative in the local rest frame, $\theta = \partial \cdot u =$ local expansion rate, and $\omega^{\mu\nu} = \frac{1}{2} (\nabla^\nu u^\mu - \nabla^\mu u^\nu) =$ vorticity.

Kinetic theory relates $\eta$ and $\tau_{\pi}$, but for a strongly coupled QGP neither $\eta$ nor this relation are known $\implies$ treat $\eta$ and $\tau_{\pi}$ as independent phenomenological parameters. For consistency: $\tau_{\pi} \theta \ll 1$ ($\theta = \partial^\mu u_\mu =$ local expansion rate).
Azimuthally symmetric transverse dynamics with long boost invariance:

Use $(\tau, r, \phi, \eta)$ coordinates and solve

- hydrodynamic equations for $T^{\tau\tau} = (e + P)\gamma_r^2 - P$, $T^{\tau r} = (e + P)\gamma_r^2 v_r$ (with “effective pressure” $P = p - r^2 \pi^{\phi\phi} - r^2 \pi^{\eta\eta}$) together with

- kinetic relaxation equations for $\pi^{\phi\phi}$, $\pi^{\eta\eta}$:

$$\frac{1}{\tau} \partial_{\tau} \left( \tau T^{\tau\tau} \right) + \frac{1}{r} \partial_r \left( r (T^{\tau\tau} + P) v_r \right) = - \frac{p + \tau^2 \pi^{\eta\eta}}{\tau},$$

$$\frac{1}{\tau} \partial_{\tau} \left( \tau T^{\tau r} \right) + \frac{1}{r} \partial_r \left( r (T^{\tau r} v_r + P) \right) = + \frac{p + r^2 \pi^{\phi\phi}}{r},$$

$$\left( \partial_{\tau} + v_r \partial_r \right) \pi^{\eta\eta} = - \frac{1}{\gamma_r \tau_r' \pi} \left[ \pi^{\eta\eta} - \frac{2\eta'}{\tau^2} \left( \frac{\theta}{3} - \frac{\gamma_r}{\tau} \right) \right],$$

$$\left( \partial_{\tau} + v_r \partial_r \right) \pi^{\phi\phi} = - \frac{1}{\gamma_r \tau_r' \pi} \left[ \pi^{\phi\phi} - \frac{2\eta'}{r^2} \left( \frac{\theta}{3} - \frac{\gamma_r v_r}{r} \right) \right].$$

Close equations with EOS $p(e)$ where $e = T^{\tau\tau} - v_r T^{\tau r}$ and $v_r = T^{\tau r} / (T^{\tau\tau} + P)$.
Equations of state (EOS)

\[ p(\text{GeV/fm}^3) \text{ vs } T(\text{GeV}) \]

\[ c_s^2 = \frac{\partial p}{\partial \epsilon} \]

\( n_B = 0 \)

- EOS I
- EOS Q
- SM-EOS Q
- EOS L

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Quark Matter 2009

27(39)
(2+1)-d viscous hydro: less longitudinal work, more radial flow

\[ Cu+Cu \oplus b = 0, \text{ SM-EOS Q (Song \& Heinz, PLB 658 (‘08))} \]

\[ \tau_0 = 0.6 \frac{fm}{c}, \ e_0 = 30 \frac{GeV}{fm^3}, \ \eta = \frac{1}{4\pi}, \ \tau_\pi = 0.24 \left( \frac{200 \text{MeV}}{T} \right) \frac{fm}{c}, \ T_{\text{dec}} = 130 \text{MeV} \]

- Radial flow develops much faster, expansion turns 3-dimensional more abruptly
- Shear viscosity initially reduces the cooling due to longitudinal work, but then leads to faster cooling in the fireball center than for ideal fluid later, due to stronger radial flow (seen also by Teaney 2004, Chaudhuri 2006, 2007; Romatschke et al. 2006, 2007)
Central Cu+Cu ($b=0$): ideal vs. viscous hydro

$$\tau_0 = 0.6 \text{ fm}_c, \quad e_0 = 30 \text{ GeV fm}^{-3}, \quad \frac{n}{s} = \frac{1}{4\pi}, \quad \tau_\pi = 0.24\left(\frac{200 \text{ MeV}}{T}\right) \text{ fm}_c, \quad T_{\text{dec}} = 130 \text{ MeV}$$

- Viscous hydro smoothes out phase transition structures
- Viscous hydro cools more slowly than ideal hydro, except for the center where cooling is accelerated at late times by faster radial expansion in the viscous case
- Viscous effects increase QGP lifetime, but viscous pressure gradients in the mixed phase shorten the mixed phase lifetime
(2+1)-d viscous hydro: more radial flow $\implies$ flatter spectra

hadron $p_T$-spectra:

$$E \frac{dN}{d^3p} = \int \frac{p \cdot d^3\sigma(x)}{(2\pi)^3} \left[ f_{eq}(x, p) + \delta f(x, p) \right] = \int \frac{p \cdot d^3\sigma(x)}{(2\pi)^3} f_{eq}(x, p) \left( 1 + \frac{1}{2} \frac{p^\alpha p^\beta}{T^2(x)} \frac{\pi_{\alpha\beta}(x)}{(e+p)(x)} \right)$$

- For identical initial and freeze-out conditions, viscous evolution yields more radial flow and flatter spectra (as previously seen by Chaudhuri 2006, 2007; Romatschke 2007)
- Effect on $b = 0$ spectra can be largely absorbed by starting viscous hydro later with lower initial density (Romatschke et al., 2006, 2007)

$$\tau_0 = 0.6 \frac{fm}{c}, \; e_0 = 30 \frac{GeV}{fm^3}, \; \frac{n}{s} = \frac{1}{4\pi}, \; \tau_\pi = 0.24 \left( \frac{200 \text{ MeV}}{T} \right) \frac{fm}{c}, \; T_{dec} = 130 \text{ MeV}$$
(2+1)-d viscous hydro: less momentum anisotropy

$Cu+Cu \oplus b = 7 \text{ fm}, \text{SM-EOS Q, } \frac{\eta}{s} = \frac{1}{4\pi}$, same initial and final conditions

Spatial eccentricity and momentum anisotropy

- - - ideal hydro, --- viscous hydro

- Source eccentricity $\epsilon_x = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}$ decays initially faster, but later more slowly;

- Flow anisotropy $\epsilon_p = \frac{\langle T_{xx}^0 - T_{yy}^0 \rangle}{\langle T_{xx}^0 + T_{yy}^0 \rangle}$ develops faster initially, but soon drops significantly below ideal fluid values;

- during the first 3-4 fm/c viscous pressure components $\pi^{\mu\nu}$ contribute strong out-of-plane (i.e. negative) momentum anisotropy in the local fluid rest frame; inhibits build-up of flow anisotropy and delays local momentum isotropization

- Total momentum anisotropy $\epsilon'_p = \frac{\langle T_{xx}^0 - T_{yy}^0 \rangle}{\langle T_{xx}^0 + T_{yy}^0 \rangle}$ is reduced by almost 50% relative to ideal fluid.
Different Israel-Stewart 2nd order formalisms

Song & Heinz, PRC 78 (2008) 024902

For full Israel-Stewart system, physical observables are almost independent of 2nd-order parameter $\tau_\pi$!

Ottinger-Grmela system appears to give very similar results (Dusling & Teaney '08)
Pion elliptic flow from 2D+1 viscous hydrodynamics

Dependence on system size, EOS, and 2nd order formalism used:

\[
\frac{\eta}{s} = \frac{1}{4\pi}, \quad \tau_\pi = 0.24\left(\frac{200\text{ MeV}}{T}\right) \text{ fm/c}
\]

- Elliptic flow very sensitive to even minimal shear viscosity!
- Viscous $v_2$ suppression significantly larger in smaller collision systems
- Difference in $v_2$ suppression between SM-EOS Q and EOS L $\sim 25 - 30\%$
Constraining $\eta/s$ from charged hadron elliptic flow data

Luzum & Romatschke, PRC 78 (2008) 034915

- **Largest uncertainty ($\sim 100\%$)**: initial source eccentricity!
- Others: EOS near $T_c$ ($\sim 25–30\%$); chemical comp. below $T_c$ (??); late hadronic viscous effects not subtracted
- “Conservative” upper limit: $\frac{\eta}{s} < \frac{6}{4\pi}$  (Luzum & Romatschke ’08)

Ulrich Heinz

Quark Matter 2009  34(39)
Bulk viscosity: less radial and elliptic flow

(Huichao Song, SQM2008)

\[ p \rightarrow p + \Pi \]

\[ \Pi = -\frac{1}{\tau_\Pi} (\Pi + \zeta \nabla \cdot u) + 2^{\text{nd}} \text{ order} \]

- bulk viscosity increases \( v_2 \) suppression and should not be neglected!
- how to separate shear viscosity from bulk viscosity?

![Graph showing bulk viscosity increases and \( v_2 \) suppression.](image-url)
Multiplicity scaling of elliptic flow and entropy production
Multiplicity scaling of the normalized elliptic flow $\nu_2/\varepsilon_x$ (I)

Song & Heinz, PRC 78 (2008) 024902

- Freeze-out at constant $\epsilon_{\text{dec}}$ introduces time scale, breaking the scale invariance of ideal hydro and cutting short the build-up of elliptic flow before it saturates
- At the same $\frac{1}{S} \frac{dN_{\text{ch}}}{dy}$, collisions between smaller nuclei and more peripheral collisions freeze out earlier, with less elliptic flow $\nu_2/\varepsilon_x$
- This breaks the multiplicity scaling with $\frac{1}{S} \frac{dN_{\text{ch}}}{dy}$ even for ideal hydro
- For larger than minimal $\eta/s$ this scaling is broken more strongly in viscous hydro
  At fixed $\frac{1}{S} \frac{dN_{\text{ch}}}{dy}$, smaller collision systems and more peripheral collisions show more viscous suppression of $\nu_2/\varepsilon_x$ than more central collisions or collisions of larger nuclei
General tendency of experimental data consistent with viscous effects

At low \((1/S)(dN_{ch}/dy)\) the data require more than minimal shear viscosity
(because of the highly viscous late hadron gas stage)

Search for scale-breaking effects requires more accurate data

Realistic modeling must account for \(T\)-dependence of shear and bulk viscosity, especially near \(T_c\)
Summary

- **Shear viscosity reduces** the longitudinal pressure but **increases** the transverse pressure in heavy ion collision
  ➞ slower cooling by longitudinal work initially, but faster cooling by stronger transverse expansion later

- For same initial conditions, viscous hydro leads to flatter $p_T$-spectra (**increased radial flow**)

- While viscous pressure effects on angle-averaged $p_T$-spectra (**radial flow**) can be largely absorbed by changing the initial conditions (starting the transverse expansion later and with lower initial energy density), this increases the destructive effects of shear viscosity on the buildup of **elliptic flow**.

- The effects of shear viscosity on elliptic flow are **large**! **RHIC data seem to require** $\frac{n}{s} < 0.5$. Largest uncertainty stems from initial source eccentricity.

- With full I-S system, **sensitivities** to initial values and kinetic relaxation time for $\pi^{\mu\nu}$ are **weak and can be neglected**. I-S and Ö-G appear to give very similar results.

- Viscous entropy production **roughly scales** with multiplicity per transverse area; larger viscous effects for smaller collision systems and larger impact parameters.

- **Multiplicity scaling** of normalized elliptic flow $v_2/\epsilon_x$ **weakly broken** by freeze-out in ideal hydro and slightly more strongly broken by shear viscosity in viscous hydro. Experimentally observed scaling requires **larger $\eta/s$ in hadronic matter than in QGP**.
Supplements
Transverse dynamics w/o azimuthal symmetry, but with long. boost invariance: Use \((\tau, x, y, \eta)\) coordinates and solve

- hydrodynamic equations for \(T^{\tau\tau} = (e+p)\gamma^2\tau^2 - p + \pi^{\tau\tau},\ T^{\tau x} = (e+p)\gamma^2\mu x + \pi^{\tau x},\ T^{\tau y} = (e+p)\gamma^2\nu y + \pi^{\tau y}\):

\[
\frac{1}{\tau}\partial_{\tau}(\tau T^{\tau\tau}) + \partial_x(v_x T^{\tau\tau}) + \partial_y(v_y T^{\tau\tau}) = S^{\tau\tau}[v_x, v_y, \pi^{\eta\eta}, \pi^{\tau\tau}, \pi^{\tau x}, \pi^{\tau y}]
\]

\[
\frac{1}{\tau}\partial_{\tau}(\tau T^{\tau x}) + \partial_x(v_x T^{\tau x}) + \partial_y(v_y T^{\tau x}) = S^{\tau x}[v_x, v_y, \pi^{xx}, \pi^{xy}, \pi^{\tau x}]
\]

\[
\frac{1}{\tau}\partial_{\tau}(\tau T^{\tau y}) + \partial_x(v_x T^{\tau y}) + \partial_y(v_y T^{\tau y}) = S^{\tau y}[v_x, v_y, \pi^{yy}, \pi^{xy}, \pi^{\tau y}]
\]

- kinetic relaxation equations for \(\pi^{\tau\tau}, \pi^{\tau x}, \pi^{\tau y}\), and \(\pi^{\eta\eta}\) (4, not 3!).

Close equations with EOS \(p(e)\) where \(e = M_0 - \mu M\) and \(\mu = M/(M_0 + p(e))\) (again one implicit scalar equation!), with the definitions

\((M_0, M_x, M_y) \equiv (T^{\tau\tau} - \pi^{\tau\tau}, T^{\tau x} - \pi^{\tau x}, T^{\tau y} - \pi^{\tau y})\) and \(M = \sqrt{M_x^2 + M_y^2}\),

and the relations \(v_x = M_x/M,\ v_y = M_y/M\).
Sensitivity to initial values for viscous pressure tensor

Romatschke & Romatschke 2007 seem to find much smaller viscous effects than we do. But they initialize their evolution with $\pi^{mn} = 0$. Could this be the origin of the discrepancy? No!

Green lines show results for $\pi_0^{mn} = 0$, with otherwise identical parameters

$\rightarrow$ weak sensitivity to initial conditions for viscous pressure tensor.
At the same $\frac{1}{S} \frac{dN_{ch}}{dy}$, collisions between larger nuclei and more central collisions take longer to freeze out.
Comparison between VISH2+1 and Romatschkes’ code

Evolution of total momentum anisotropy $\epsilon'_p$, Au+Au with EOS I

Au+Au, b=7fm EOSI

- Ideal hydro -- Song & Heinz, $\eta/s=0$, $1/\tau_\pi=0$
- Viscous hydro -- Song & Heinz, $\eta/s=0.004$
- Viscous hydro -- 2Romatschke, $\eta/s=0.004$
- Viscous hydro -- Song & Heinz ($\pi^{mn}=0$)
- Viscous hydro -- 2Romatschke (same viscous equations as Song & Heinz)
- Viscous hydro -- 2Romatschke (same viscous equation as Song & Heinz, smaller $dx \, dy \, dt$)
- Viscous hydro -- 2Romatschke (original full viscous equations)

$T_0=0.3$ GeV, $\tau_0=0.6$ fm/c
$\eta/s=0.08$, $\tau_\pi=6\eta/sT$
Sensitivity to parameters

$(\tau_\pi, \pi^{\mu\nu}(\tau_0))$
Sensitivity to initial values for viscous pressure tensor

Thin lines: $\pi_0^{mn} = 0$;    Thick lines: $\pi_0^{mn} = 2\eta\sigma^{mn} \equiv 2\eta\nabla\langle m u n \rangle$.

$\tau_0 = 0.6 \text{ fm}/c$, $e_0 = 30 \text{ GeV}/\text{fm}^3$, $\eta/s = \frac{1}{4\pi}$, $\tau_{\pi} = 0.24 \left(\frac{200 \text{ MeV}}{T}\right) \text{ fm}/c$, $T_{\text{dec}} = 130 \text{ MeV}$

largest viscous pressure components vs. time

$$\Sigma = \pi^{xx} + \pi^{yy}, \quad \Delta = \pi^{xx} - \pi^{yy}$$

- For fixed $\eta/s$, viscous pressure components become small at late times $\longrightarrow$ ideal hydro
- After $\tau \sim 1 \text{ fm}/c \sim 5\tau_{\pi}$, viscous pressure tensor has lost all memory of initial conditions!
- Effects of initial $\pi^{mn}$ on final $v_2$ are small
Sensitivity to kinetic relaxation time $\tau_\pi$:

Cu+Cu, $b=7$ fm, SM-EOS Q, $\pi^-$

- Faster kinetic relaxation at fixed $\eta/s$ reduces viscous effects $\rightarrow$ Janik 2007
- larger $\tau_\pi \rightarrow$ larger $\frac{\pi^{\mu\nu}}{e+p}$ at early times, and more deviation from ideal hydro!
Limits of viscous hydrodynamics:
The limits of viscous hydrodynamics

At sufficiently large $p_T$, viscous corrections become large even if $\eta/s$ is small.

\[ |\delta N(p)| > \frac{1}{2} |N_0(p)| \]

indicates breakdown of the assumptions:

- For larger initial energy densities, $p_T$-range increases where viscous hydro can be applied to describe hadron spectra.
Tests of the viscous hydro code VISH2+1

- \( \eta \rightarrow 0 \longrightarrow \) ideal fluid code AZHYDRO (test hydro evolution algorithm)

- \( \nabla \bot p = 0, \tau_\pi \rightarrow 0 \Longrightarrow \) reproduce analytic soln. of boost-invariant Navier-Stokes

- \( \eta, \tau_\pi \) small \( \Longrightarrow \) Israel-Stewart \( \rightarrow \) Navier-Stokes (tests kinetic evolution algorithm for \( \pi^{\mu\nu} \))

- \( \pi^{\mu\nu}_{\mu} = 0, u_\mu \pi^{\mu\nu} = 0 \) to better than 2%

- Evolution of \( e, u^\mu, \pi^{\mu\nu} \) by VISH2+1 tested against Romatschkes’ code:
  - excellent agreement for identical initial conditions, EOS, kinetic evolution equations
  - large difference in published \( v_2(p_T) \) due to extra terms in \( D \pi^{\mu\nu} = \ldots \) used by the Romatschkes
Viscous entropy production larger for faster-expanding fireballs

- Entropy production scales approximately with charged multiplicity density per unit area, $\frac{1}{S} \frac{dN_{ch}}{dy}$
- Entropy production fraction is larger for smaller $\frac{1}{S} \frac{dN_{ch}}{dy}$ (lower-energy and more peripheral collisions)
- At the same $\frac{1}{S} \frac{dN_{ch}}{dy}$, collisions between larger nuclei or more central collisions take longer to freeze out, generating slightly more entropy
- In full Israel-Stewart approach, $\Delta S / S_0$ is (almost) $\tau_\pi$-independent

EOS I: ideal gas of massless partons
SM-EOS Q: 1st order QGP-HRG phase transition
EOS L: smooth crossover from lattice QCD data above $T_c$ to HRG below $T_c$. 