1. Introduction

1.1. Orbit equivalence.

Definition 1.1. Let $Y$ be a topological space, $B_Y$ the $\sigma$-algebra of the Borel sets of $Y$. When $Y$ is a separable complete metric space, $(Y, B_Y)$ (or, by abuse of language, $Y$) is said to be a standard Borel space (standard $\sigma$-algebra).

Remark 1.2. When $X$ is a standard Borel space, $X$ is either (at most) countable or isomorphic to the closed interval $[0, 1]$. 

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Definition 1.3. A standard Borel space with a Borel probability measure is said to be a (standard) probability space. A point \( x \) of a probability space \((X, \mu)\) is said to be an atom of \((X, \mu)\) when \( \mu(x) > 0 \). A probability space \((X, \mu)\) is said to be diffuse when it has no atom.

Example 1.4. (Examples of standard probability spaces)

1. The infinite product \( (\prod_{n \in \mathbb{N}} \{0, 1\}, \otimes_n \mu_n) \), where \( \mu_n \) is a probability measure on \( \{0, 1\} \) for each \( n \in \mathbb{N} \) is standard.

2. When \( G \) is a separable compact group, the normalized Haar measure on \( G \) makes \( G \) into a standard probability space.

When \((X, \mu)\) is a probability space, we obtain a \((w^*\text{-})\) separable von Neumann algebra \( L^\infty X \) and a normal state (also denoted by \( \mu \)) on it. To each isomorphism \( \phi: (X, \mu) \to (Y, \nu) \) of probability spaces, we obtain an isomorphism \( \phi_\ast: L^\infty Y \to L^\infty X \), \( f \mapsto f \circ \phi \) satisfying \( \mu \circ \phi_\ast = \nu \).

Theorem 1.5. (von Neumann)

1. When \((X, \mu)\) and \((Y, \nu)\) are diffuse probability spaces, there is an isomorphism \( (L^\infty (X, \mu), \mu) \simeq (L^\infty (Y, \nu), \nu) \).

2. For each isomorphism \( \sigma: L^\infty Y \to L^\infty X \) with \( \mu \sigma = \nu \), there exists a Borel isomorphism \( \phi: X \to Y \) such that \( \phi^\ast \mu = \nu \) and \( \phi \sigma = \sigma \).

Proof. (Outline): (1) We may assume that \( Y = \prod_{n \in \mathbb{N}} \{0, 1\}, \mu = \otimes_n (\frac{1}{2}, \frac{1}{2}) \). Since \( X \) is diffuse, we have a decomposition \( X = X_0 \coprod X_1 \) by Borel sets with \( \mu(X_0) = \frac{1}{2} \). We can continue this procedure as \( X_0 = X_{00} \coprod X_{01}, \mu(X_{00}) = \frac{1}{4} \), so on. The partition by \( X_{n+1} \) can be made fine enough because there is a separating family \( (B_n)_{n \in \mathbb{N}} \) in \( B_X \), which will imply the desired isomorphism between \( L^\infty X \) and \( L^\infty Y \) compatible with the normal states.

(2) Let \( \lambda \) denote the Lebesgue measure on the closed interval \([0, 1]\). Since there exists an isomorphism \( (L^\infty Y, \nu) \simeq (L^\infty [0, 1], \lambda) \), we may assume that \( Y = [0, 1] \) and \( \nu = \lambda \) here. For each \( r \in \mathbb{Q} \cap [0, 1] \), put \( E_r = \sigma(\chi_{[0,r)}) \). Define a mapping \( \phi: X \to [0, 1] \) by \( \phi(x) = \inf \{ r : x \in E_r \} \). The inverse image of \([0, t]\) under \( \phi \) is equal to \( \cup_{r \leq t} E_r \). The latter is obviously Borel, which means that \( \phi \) is a Borel map. By \( \sigma(\chi_{[0,r)}) = \phi^\ast(\chi_{[0,r)}) \) for \( r \in \mathbb{Q} \cap [0, 1] \), we have \( \sigma = \phi^\ast \) and \( \phi \sigma = \sigma \).

It remains to replace \( \phi \) by a Borel isomorphism. Let \( (B_n)_{n \in \mathbb{N}} \) be a separating family of \( X \). For each \( n \), there exists \( F_n \in B_Y \) such that \( \phi \chi_{X F_n} = \chi_{B_n} \). Thus \( N = \cup_n (B_n \triangle \phi^{-1} F_n) \) is a null set. On \( X \setminus N \), the condition \( x \in B_n \) is equivalent to \( \phi(x) \in F_n \). If \( x \) and \( y \) are distinct points of \( X \setminus N \), there exists an integer \( n \) such that \( x \in B_n \) while \( y \notin B_n \). Thus \( \phi(x) \neq \phi(y) \) and \( \phi \) is injective on \( X \setminus N \). We may assume that \( Y \setminus \phi(X \setminus N) \) is uncountable so that there is an isomorphism of \( N \) to \( Y \setminus \phi(X \setminus N) \).

Let \( \Gamma \curvearrowright (X, \mu) \) be a measure preserving action by a discrete countable group. (We may assume that it acts by Borel isomorphisms.) Let \( s \) be an element of \( \Gamma \). When \( f \) is a complex Borel function defined on \( X \), put \( \alpha_s(f): x \mapsto f(s^{-1}x) \). This induces a \( \mu \)-preserving \( * \)-automorphism on \( L^\infty X \). This way we obtain an action \( \alpha: \Gamma \curvearrowright L^\infty (X, \mu) \) preserving the state \( \mu \).

Definition 1.6. Two actions \( \Gamma \curvearrowright (X, \mu) \) and \( \Gamma \curvearrowright (Y, \nu) \) are said to be conjugate when there exists an probability space isomorphism \( \phi: (X, \mu) \to (Y, \nu) \) which is a.e.
\( \Gamma \)-equivariant. This is equivalent to the existence of a \( \Gamma \)-equivariant state preserving isomorphism \( \sigma : L^\infty(Y, \nu) \to L^\infty(X, \mu) \).

**Definition 1.7.** Let \( \Gamma \acts (X, \mu) \) be an action by measure preserving Borel isomorphisms. The subset \( \mathcal{R}_{\Gamma \acts (X, \mu)} = \{(sx, x) : s \in \Gamma \} \) of \( X \times X \) is called the orbit equivalence relation of the action.

**Definition 1.8.** Two actions \( \Gamma \acts (X, \mu) \) and \( \Lambda \acts (Y, \nu) \) are said to be orbit equivalent when there exists a measure preserving Borel isomorphism \( \phi : Y \to X \) satisfying \( \Gamma \phi(y) = \phi(\Lambda y) \) for a.e. \( y \in Y \).

**Definition 1.9.** A partial Borel isomorphism on \( X \) is a triple \((\phi, A, B)\) consisting of \( A, B \subseteq B_X \) and a Borel isomorphism \( \phi \) of \( A \) onto \( B \).

**Definition 1.10.** A measure preserving standard orbit equivalence is a subset \( \mathcal{R} \) of \( X \times X \) satisfying the following conditions:

1. \( \mathcal{R} \) is a Borel subset with respect to the product space structure.
2. \( \mathcal{R} \) is an equivalence relation on \( X \).
3. For each \( x \in X \), the \( \mathcal{R} \)-equivalence class of \( x \) is at most countable.
4. Any partial Borel isomorphism \( \phi \) whose graph is contained in \( \mathcal{R} \), \( \phi \) preserves measure.

**Theorem 1.11.** (Lusin) Let \( X, Y \) be standard spaces.

1. When \( \phi : X \to Y \) is a countable-to-one Borel map, \( \phi(X) \) is Borel. Moreover there exists a Borel partition \( X = \bigsqcup X_n \) such that \( \phi|_{X_n} \) is a Borel isomorphism onto \( \phi(X_n) \).
2. When \( \mathcal{R} \) is a standard orbit equivalence, \( \mathcal{R} = \bigsqcup_n \mathcal{R}(\phi_n) \) where \( \phi_n \) is a partial Borel isomorphism for each \( n \).

**Lemma 1.12.** Let \( A \) be a subset of a standard space \( X \). \( \phi \) a mapping of \( A \) into \( X \). \( \phi \) and \( A \) are Borel if and only if the graph \( \mathcal{G}(\phi) = \{(\phi(x), x) : x \in A\} \) of \( \phi \) is Borel in \( X \times X \).

**Proof.** \( \Leftarrow \) is an immediate consequence of Theorem 1.11. 

\( \Rightarrow \) Let \( \{B_n\}_{n \in \mathbb{N}} \) be a separating family of \( X \). The condition \( y \neq \phi(x) \) equivalent to \( (y, x) \in \cup_n (\overline{CB_n}) \times \phi^{-1}(B_n) \). Thus \( \mathcal{G}(\phi) = \overline{\cup_n (\overline{CB_n})} \times \phi^{-1}(B_n) \).

### 1.2 Preliminaries on von Neumann algebras.

Let \( H \) be a Hilbert space, \( B(H) \) the involutive Banach algebra of the continuous endomorphisms of \( H \), \( A \) a *-subalgebra of \( B(H) \). (Typically \( A \) generates a von Neumann algebra \( \mathcal{M} \) of our interest.) In the following \( A \) is often assumed to admit a cyclic tracial vector \( \xi_\tau \in H \), i.e. \( ||\xi_\tau|| = 1 \), \( A\xi_\tau \) is dense in \( H \), and that the vector state \( \tau(a) = \langle a \xi_\tau, \xi_\tau \rangle \) is tracial.

**Remark 1.13.** A state \( \tau \) is tracial means that by definition the two sesquilinear forms \( \tau(ab^*) \) and \( \tau(b^*a) \) in \( (a, b) \) are same. To check this property, by polarization it is enough to show \( \tau(aa^*) = \tau(a^*a) \). Under the assumption above \( \xi_\tau \) becomes a separating vector for \( A^\prime \). Indeed, \( a\xi_\tau = 0 \) implies \( \tau(bc^*a) = 0 \) for \( b, c \in A \), which means \( \tau(c^*ab) = 0 \) and in turn \( \langle aH, H \rangle = 0 \).

**Notation.** Let \( a\hat{\cdot} \) denote \( a\xi_\tau \). (Hence we have \( \langle a\hat{\cdot}, \hat{\cdot} \rangle = \tau(ab^*) \). )
Remark 1.14. We have a conjugate linear map $J: H \to H$ determined by $a \mapsto \hat{a}$. Then we have $JaJb = b\hat{a}$ which implies $JAJ \subset A'$ and $JA''J \subset A'$. On the other hand, for any $x \in A'$ and $a \in A$

$$\langle Jx\xi, a\xi \rangle = \langle Ja\xi, x\xi \rangle = \langle a^*\xi, x\xi \rangle = \langle x^*\xi, a\xi \rangle.$$ 
Thus $Jx\xi = x^*\xi$, thence $\xi$ is a cyclic tracial vector for $A'$. The $J$-operator for $(A', \xi)$ is exactly equal to the original $J$. Doing the same argument as above, we obtain $JA'J \subset A''$.

Remark 1.15. The map $A'' \to A', a \mapsto JaJ$ is a conjugate linear $*$-algebra isomorphism.

1.3. Crossed products. Let $\Gamma \act (X, \mu)$ be a measure preserving action of a discrete group on a standard probability space $X$. Recall that we have an action $\Gamma \act L^\infty(X)$ induced by $\alpha_s(f) = f(s^{-1} \cdot \cdot \cdot)$ for $s \in \Gamma$.

On the other hand, we get a unitary representation $\pi: \Gamma \act L^2(X, \mu)$ by the same formula $\pi_s f = \alpha_s f$ as the one on $L^\infty(X)$. Note that $\pi_s f \pi_s^* = \alpha_s(f)$ for $s \in \Gamma$ and $f \in L^\infty(X)$.

Definition 1.16. Let $\lambda: \Gamma \act B(\ell_2 \Gamma)$ denote the regular representation. The von Neumann algebra $L^\infty(X) \rtimes \Gamma$ on $L^2(X) \otimes \ell_2 \Gamma$ is generated by the operators $\pi \otimes \lambda(s)$ for $s \in \Gamma$ and $f \otimes 1$ for $f \in L^\infty(X)$ is called the crossed product of $L^\infty(X)$ by $\alpha$.

Let $A$ denote $\{\sum f \otimes 1: f \in L^2(X)\} \subset L^\infty(X) \rtimes \Gamma$. By abuse of notation, in the following $f$ stands for $f \otimes 1$ and $\lambda(s)$ for $\pi \otimes \lambda(s)$. Now $\xi_\tau = 1 \otimes \delta_\epsilon \in L^2(X) \otimes \ell_2 \Gamma$ is a cyclic tracial vector for $A$. Indeed, it is obviously cyclic, while $\tau(f \lambda(s)) = \delta_{s,e}\mu(f)$ implies the tracial property:

$$\tau(f \lambda(s)g \lambda(t)) = \delta_{s,t}f \alpha_s(g) = \delta_{s,e}\alpha_t(f)g = \tau(g \lambda(t)f \lambda(s)).$$

Note that the above expressions are nonzero only if $s = t^{-1}$.

Let $V$ denote the isometry $L^2(X) \to L^2(X) \otimes \ell_2 \Gamma$, $f \mapsto f \otimes \delta_\epsilon$. Then the contraction $E: L^\infty(X) \rtimes \Gamma \to B(L^2(X))$, $a \mapsto V^*aV$ has image $L^\infty(X)$, i.e. $E$ is a conditional expectation (see Definition 2.6) of $L^\infty(X) \rtimes \Gamma$ onto $L^\infty(X)$. Note that $\tau = \mu \circ E$.

1.4. von Neumann algebras of orbit equivalence. Let $\mathcal{R}$ be a standard orbit equivalence on $X$. Hence it is a countable disjoint union $\bigsqcup_n \mathcal{G}(\phi_n)$ of the graphs of partial isometries. We may assume that $\phi_0 = \text{Id}_X$. We will define a “Borel probability measure” on $\mathcal{R}$.

Observe that when $f: \mathcal{R} \to \mathbb{C}$ is a Borel function, $X \to \mathbb{C}$, $x \mapsto \sum f(y, x) = \sum_n f(\phi_n x, x)$ is also Borel. Define a measure $\nu$ on $\mathcal{R}$ by putting

$$\int_{\mathcal{R}} \xi d\nu = \int_{X} \sum_{y \neq x} \xi(y, x) d\mu(x)$$

for each Borel function $\xi$ on $\mathcal{R}$. Thus when $B$ is a Borel subset of $\mathcal{R}$, $\nu(B) = \int \pi_{B}^{-1}(x) \cap B d\mu(x)$ for the second projection $\pi_{B}: \mathcal{R} \to X$, $(y, x) \mapsto x$.

We get a pseudogroup $\llbracket \mathcal{R} \rrbracket$ whose underlying set is $\{\phi: \text{partial Borel isomorphism}, \mathcal{G}(\phi) \subset \mathcal{R}\}$. The composition $\phi \circ \psi$ of $\phi$ and $\psi$ is defined as the composition of the maps on $\psi^{-1}\text{dom}(\phi)$. In particular, the identity maps of the Borel sets are the units of $\llbracket \mathcal{R} \rrbracket$, and $\phi \in \llbracket \mathcal{R} \rrbracket$ implies $\phi^{-1} \in \llbracket \mathcal{R} \rrbracket$. 
For each $\phi \in \mathcal{A}$, define a partial isometry $v_\phi \in B(L^2(\mathcal{A}, \nu))$ by $v_\phi \xi(y, x) = \xi(\phi^{-1}y, x)$. Thus $v_\phi^* v_\phi = v_\phi \phi$. On the other hand, the set $\{ \chi_{\mathcal{A}}(\phi) : \phi \in \mathcal{A} \}$ is total in $L^2(\mathcal{A}, \nu)$ and $v_\phi \chi_{\mathcal{A}} = \chi_{\mathcal{A}} \phi$. Moreover, we have

$$\langle v_\phi \chi_{\mathcal{A}}, \chi_{\mathcal{A}} \rangle = \int \mathcal{G}(\phi) \cap \mathcal{G}(\theta) d\nu = \mu \{ x : \phi \psi x = \theta x \} = \langle \chi_{\mathcal{A}}, v_\phi^{-1} \chi_{\mathcal{A}} \rangle,$$

which implies $v_\phi^* = v_\phi^{-1}$.

**Definition 1.17.** The von Neumann algebra $\mathcal{V} \mathcal{N} \mathcal{A}$ on $L^2(\mathcal{A}, \nu)$ generated by $\{ v_\phi : \phi \in \mathcal{A} \}$ is called the von Neumann algebra of $\mathcal{A}$.

$\xi_\tau = \chi_{\mathcal{A}}(\text{Id}_X)$ is a cyclic tracial vector for $\mathcal{V} \mathcal{N} \mathcal{A}$: in fact,

$$\tau(v_\phi) = \mu(\{ x : \phi \circ \psi(x) = x \})$$

$$= \mu(\{ y : \psi \circ \phi(y) = y \}) \quad (y = \phi^{-1}x)$$

$$= \tau(v_\phi).$$

Note that $L^\infty X$ is contained “in the diagonal” of $\mathcal{V} \mathcal{N} \mathcal{A}$, subject to the relation $v_\phi f = (f \circ \phi^{-1}) v_\phi$. We have a conditional expectation $E : \mathcal{V} \mathcal{N} \mathcal{A} \to L^\infty X, a \mapsto V^* a V$ implemented by the “diagonal inclusion” isometry $V : L^2 X \to L^2 \mathcal{A}$. We have $E(v_\phi) = \chi_{\{ x : \phi(x) = x \}}$.

**2. Elementary theory of orbit equivalence**

2.1. **Essentially free action of countable discrete groups.** Suppose we are given a measure preserving action $\Gamma \curvearrowright (X, \mu)$ by a discrete group on a standard probability space. As in the last section we get two inclusions of von Neumann algebras:

1. $L^\infty X \subset L^2 X \times \Gamma$ in $\mathcal{B}(L^2 X \otimes \ell_2 \Gamma)$.
2. $L^\infty X \subset \mathcal{V} \mathcal{N}(\mathcal{A}, \cap (X, \mu))$ in $\mathcal{B}(L^2 \mathcal{A})$.

In general these are different, e.g. when the action is trivial.

**Definition 2.1.** An action $\Gamma \curvearrowright (X, \mu)$ is said to be essentially free when the fixed point set of $s$ has measure 0 for any $s \in G \setminus \{e\}$.

**Theorem 2.2.** When the action $\Gamma \curvearrowright (X, \mu)$ is essentially free, the above two inclusions of von Neumann algebras are equal.

**Remark 2.3.** $Jv_\phi^* = v_\phi^{-1}$ implies $J\xi(y, x) = \overline{\xi(x, y)}$.

**Proof of the theorem.** Identification of the representation Hilbert spaces is given by $U : L^2 X \otimes \ell_2 \Gamma \to L^2 \mathcal{A}, g \otimes \delta_t \mapsto g \cdot \chi_{\mathcal{A}}(t)$. When we have an equality $f \chi_{\mathcal{A}}(s) = g \chi_{\mathcal{A}}(t)$ of nonzero vectors in $L^2 \mathcal{A}$, $s$ must be equal to $t$ by the essential freeness assumption. Now,

$$U^* v_\alpha U(g \otimes \delta_t) = U^* \alpha_s(g) v_\alpha \chi_{\mathcal{A}}(t) = U^* \alpha_s(g) \chi_{\mathcal{A}(st)} = \alpha_s(g) \otimes \delta_t.$$

This shows $U^* v_\alpha U = \pi \otimes \lambda(s)$. On the other hand, $U^* f U = f \otimes 1$ is trivial. Thus, via $U$, $L^2 X \times \Gamma$ is identified to $L^2 \mathcal{A}$. 

□
Definition 2.4. Let $M$ be a finite von Neumann algebra, $A$ a von Neumann subalgebra (in the following $A$ is often assumed to be commutative). The subset $\mathcal{N}A = \{ u \in \mathcal{UM} : uAu^* = A \}$ of $\mathcal{UM}$ is called the normalizer of $A$. Likewise $\mathcal{N}^pA = \{ v \in M : \text{partial isometry, } v^*v, vv^* \in A, vAv^* = Avv^* \}$ is called the partial normalizer of $A$.

Lemma 2.5. For any $v \in \mathcal{N}^pA$, there exist $u \in \mathcal{N}A$ and $e \in \text{Proj}(A)$ such that $v = ue$. For any $\phi \in \mathcal{F}$, there exists a Borel isomorphism $\tilde{\phi}$ whose graph is contained in $\mathcal{R}$ and $\tilde{\phi}|_{\text{dom } \phi} = \phi$.

Proof. We prove the second assertion as the demonstration of the first one is an algebraic translation of it. Put $E = \text{dom } \phi$ and $F = \text{ran } \phi$. When $\mu(E \Delta F) = 0$, there is nothing to do. When $\mu(E \Delta F) \neq 0$, $\exists k > 0$ such that $\phi^k(E \setminus F) \cap (F \setminus E)$ is non-null. If not, $\phi^k(E \setminus F) \subset F \cap \bigcap (F \setminus E) = F \cap E \subset E$ up to a null set and $\phi^{k+1}$ can be defined a.e. on $E \setminus F$. Thus we would get a sequence $(\phi^k(E \setminus F))_{k \in \mathbb{N}}$ of subsets with nonzero measure. For any pair $m < n$, $\phi^m(E \setminus F) \cap \phi^n(E \setminus F)$ is equal to $\phi^n(\delta^{n-m}(E \setminus F) \cap (E \setminus F))$ which is null. This contradicts to $\mu(X) = 1$.

Now, given such $k$, put $\phi_1 = \phi \prod (\phi^{-k} |_{\phi^k(E \setminus F) \cap (E \setminus F)})$. Then we can use the maximality argument (Zorn’s lemma) to obtain a globally defined Borel isomorphism. □

2.2. Inclusion of von Neumann algebras.

Definition 2.6. Let $M \subset N$ be an inclusion of von Neumann algebras. A unital completely positive map $E : N \to M$ is said to be a conditional expectation when it satisfies $E(ab) = aE(b)$ for $a, b \in M$ and $x \in N$.

Fact. When $N$ is finite with a faithful tracial state $\tau$, there exists a unique conditional expectation $E$ that preserves $\tau$. Then we obtain an orthogonal projection $e_M : L^2N \to M \tau' \simeq L^2M$ extending $E$.

Remark 2.7. (Martingale) If we are given $N_1 \subset N_2 \subset \cdots \subset M$ with $N = \lor_i N_i$ or $M \supset N_1 \supset N_2 \supset \cdots$ with $N = \cap_i N_i$, together with conditional expectations $E_n : M \to N_n$ and $E : M \to N$, $e_n \to e$ in the strong operator topology implies $\|E(x) - E_n(x)\|_2 \to 0$.

For example, let $A \subset M$ be a finite dimensional commutative subalgebra, $e_i$ ($1 \leq i \leq n$) the minimal projections of $A$. Then $E_{A \cap M}(x) = \sum_{i=1}^n e_i x e_i$. If we have a sequence $A_1 \subset A_2 \subset \cdots \subset M$ of finite dimensional commutative subalgebras and $A = \lor A_i$, we have $E_{A_i \cap M} \to E_{A \cap M}$. The latter is equal to $E_A$ if and only if $A$ is a maximal abelian subalgebra.

Definition 2.8. A von Neumann subalgebra $A \subset M$ is said to be a Cartan subalgebra of $M$ when it is a maximal abelian subalgebra in $M$ and $\mathcal{N}(A)'' = M$. (Then we also have $M = \mathcal{N}^p(A)'$.)

Theorem 2.9. $L^\infty X \subset \mathcal{vN}\mathcal{R}$ is a Cartan subalgebra.

Proof. Since the generators $v_\phi$ are in $\mathcal{N}A$, it is enough to show that $L^\infty X$ is maximal abelian in $\mathcal{vN}\mathcal{R}$. Recall that $\mathcal{R} = \coprod \mathcal{G}(\phi_n)$ with $\phi_0 = \text{Id}_X$. Then let $a$ be an
element of the relative commutant of \( L^\infty X \). \( \hat{a} \) can be written as \( \sum_n f_n \chi_{\phi_n} \). By assumption \( fa = af \) for any \( f \in L^\infty X \). Thus,

\[
\hat{a} = \sum f_n \chi_{\phi_n}, \quad \hat{f} = JfJ\hat{a} = \sum f \circ \phi^{-1}_n \cdot f_n \chi_{\phi_n}.
\]

Hence \( ff_n = f \circ \phi_n f_n \) for any \( n \) and any \( f \), which implies \( f_n = 0 \) except for \( n = 0 \).

**Definition 2.10.** \( \mathcal{R} \) is said to be ergodic when any \( \mathcal{R} \)-invariant Borel subset of \( X \) is of measure either 0 or 1. An action \( \Gamma \curvearrowright (X, \mu) \) is said to be ergodic when \( \mathcal{R}_{\Gamma \curvearrowright X} \) is ergodic.

**Corollary 2.11.** \( vN\mathcal{R} \) is a factor if and only if \( \mathcal{R} \) is ergodic.

**Proof.** The Cartan subalgebra \( L^\infty X \) contains the center of \( vN\mathcal{R} \). The central projections are the characteristic functions of the \( \mathcal{R} \)-invariant Borel subsets.

Let \( v \in \mathcal{N}^p L^\infty \), \( E, F \in B_X \) the Borel sets (up to null sets) respectively representing the projections \( v^*v \) and \( vv^* \) in \( A \). The map \( L^\infty E \to L^\infty F, f \mapsto vf^*v \) is a *-isomorphism. Thus there exists a Borel isomorphism \( \phi_v : E \to F \) such that \( v^*f = f \circ \phi_v^{-1} \) (\( v = \sigma v_{\phi_v} \) for some \( \sigma \in UL^\infty F \)).

**Theorem 2.12.** In the notation as above, \( v\xi v^* = \xi(\phi_v^{-1}(y), x) \) \( \nu \)-a.e. for any \( v \in \mathcal{N}^p L^\infty \) and any \( \xi \in L^\infty \mathcal{R} \). In particular, \( \phi_v \in \mathcal{R} \) up to a null set. Moreover, we have \( L^\infty \vee JL^\infty J = L^\infty \mathcal{R} \).

**Proof.** Put \( A = L^\infty X \). First, \( fJgJ \in L^\infty \) for \( f, g \in A \): indeed, \( fJgJ \) is the multiplication by the function \( f(y)g(x) \) on \( \mathcal{R} \).

\[
v fJgJ v^* = vf^*vJgJ = f \circ \phi_v^{-1}JgJ \quad (JMJ = M').
\]

Hence \( v\xi v^*(y, x) = v(\phi_v^{-1}y, x) \) for \( \xi \in A \vee JA' \). It remains to show \( \chi_{\phi'(1d_x)} \in A \vee JA' \). Because, if this is satisfied, we will have \( \chi_{\phi(\phi_v)} = v\chi_{\phi(1d)}v^* \in L^\infty \mathcal{R} \).

Take an increasing sequence \( A_1 \subset A_2 \subset \cdots \) of finite dimensional algebras with \( A = \vee A_k \). The conditional expectation \( E_n : vN\mathcal{R} \to A_n \) is equal to \( \sum_k e_k^{(n)} J e_k^{(n)} J \) (as an operator on \( L^2 \mathcal{R} \)) for the minimal projections \( (e_k^{(n)})_k \) of \( A_n \). Now, \( (E_n)_n \) converges to the conditional expectation \( E_A \) onto \( A \) which is equal to the multiplication by \( \chi_{\phi(1d_x)} \) in the strong operator topology. Hence \( \chi_{\phi(1d)} \in A \vee JA' \).

**Remark 2.13.** (2-cocycle \([4]\)) Suppose we are given a map \( \sigma_{\phi, \psi} : \text{ran}(\phi \psi) \to T \) for each pair \( \phi, \psi \in \mathcal{R} \), satisfying \( \sigma_{\phi, \psi} \sigma_{\phi \psi, \theta} = (\sigma_{\psi, \theta} \circ \phi^{-1}) \sigma_{\phi, \psi} \theta \). Then \( v^*v^\sigma = \sigma_{\phi, \psi} v^\sigma_{\phi, \psi} \) determines an associative product on \( \mathbb{C}[\mathcal{R}] \) with a trace \( \tau \). The GNS representation gives an inclusion \( L^\infty X \subset vN\mathcal{R}, \sigma \subset B(L^2 \mathcal{R}) \) of von Neumann algebras.

Fact. Any Cartan subalgebra of \( vN\mathcal{R}, \sigma \) is isomorphic to \( L^\infty X \).

**Theorem 2.14.** Let \( \mathcal{R} \) (resp. \( \mathcal{I} \)) be an orbit equivalence on \( X \) (resp. \( Y \)), \( F : X \to Y \) a measure preserving Borel isomorphism. The induced isomorphism \( F_* : L^\infty X \to L^\infty Y \) can be extended to a normal *-homomorphism \( vN\mathcal{R} \to vN\mathcal{I} \) if and only if \( F\mathcal{R} \subset \mathcal{I} \) up to a \( \nu \)-null set.
Proof. For simplicity we identify $Y$ with $X$ by means of $F$. If $\|\mathcal{A}\| \leq \|\mathcal{B}\|$, the required homomorphism is induced by the isometry $L^2 \mathcal{B} \rightarrow L^2 \mathcal{A}$. Conversely, if $\pi: vN \mathcal{B} \rightarrow vN \mathcal{A}$ is an extension of $F$, for any $\phi \in \mathcal{A}$ we have
\[
\pi(v_\phi)\pi(f)\pi(v_\phi)^* = \pi(f \circ \phi^{-1}) = f \circ \phi^{-1},
\]
which implies $\pi(v_\phi) = \sigma_\phi v_\phi$ for some $\sigma_\phi \in L^\infty X$.

Let $M$ be a finite von Neumann algebra with trace $\tau$, identified to a subalgebra of $B(L^2 M)$. Suppose $A$ is a von Neumann subalgebra of $M$. Let $e_A$ be the projection onto the span of $A \xi_\tau$ and put $\langle M, A \rangle = (M \cup \{e_A\})''$.

For any $x \in M$ and $\hat{a} \in L^2 A$,
\[
e_A x \hat{a} = e_A x \hat{a} = \hat{E}_A(xa) = \hat{E}_A(x)a
\]
which implies $e_A x e_A = E_A(x)e_A$. In particular, we have
\[
\langle M, A \rangle = \left\{ \sum x_j e_A y_j + z : x_j, y_j, z \in M \right\}^{\text{wop}}.
\]
Now,
\[
e_A J x J e_A \hat{a} = e_A ax^* = E_A(ax^*) = aE_A(x^*) = JE_A(x)J \hat{a}
\]
implies $\langle M, A \rangle' = M' \cap \{e_A\}' = JA J$, consequently $\langle M, A \rangle = (JA J)'$. Note that when $A$ is commutative $e_A J a J = a^* e_A$ for $a \in A$.

We have the “canonical trace” $\text{Tr}$ on $\langle M, A \rangle$ which is a priori unbounded defined by $\sum_i x_i e_A y_i \mapsto \tau(\sum_i x_i y_i)$. Still, $\text{Tr}$ is normal semifinite, and its tracial property is verified as follows:
\[
\left\| \sum x_i e_A y_i \right\|_{L^2, \text{Tr}}^2 = \text{Tr} \left( \sum y_i^* e_A x_i^* x_i e_A y_i \right) = \sum \tau(y_i^* E_A(x_i^*) x_i y_i) = \sum \tau(E_A(x_i^* x_i)) = \left\| y_i e_A x_i^* \right\|_{L^2, \text{Tr}}^2.
\]
Suppose $A \subset M$ is Cartan. Put $\hat{A} = \{A, JAJ\}' \subset \langle M, A \rangle$.

**Example 2.15.** When $A = L^\infty X$, $M = vN \mathcal{B}$, we have $\hat{A} = L^\infty \mathcal{B}$, $e_A = \chi_\Delta$ and $\text{Tr}|_{\hat{A}} = f \, d\nu$ on $L^\infty \mathcal{B}$. Indeed,
\[
\text{Tr}(fe_A) = \tau(f) = \int_{\Delta} f \, d\mu = \int f \, d\nu \quad (f \in L^\infty X)
\]
implies
\[
\text{Tr}(uf e_A u^*) = \text{Tr}(fe_A) = \int_{\Delta} f \, d\mu = \int uf e_A u^* d\nu \quad (f \in L^\infty X, u \in \mathcal{N} A).
\]

**Remark 2.16.** When $A \subset M$ is Cartan and $p \in \text{Proj}(A)$, $A_p \subset Mp$ is also Cartan since $\mathcal{N}_{pMp}(Ap) = p\mathcal{N}_M(pA)$.

**Example 2.17.** When $Y \subset X$, the restricted equivalence $\mathcal{B}|_Y = Y \times Y \cap \mathcal{B}$ gives $vN(\mathcal{B}|_Y) = pY (vN \mathcal{B})p_Y$.

**Exercise 2.18.** Show that when $A$ is a Cartan subalgebra of a factor $M$, $\tau p_1 = \tau p_2$ for $p_1, p_2 \in \text{Proj}(A)$ implies the existence of $v \in \mathcal{N}^p A$ such that $p_1 \sim p_2$ via $v$. This implies that given an ergodic relation $\mathcal{R}$ on $X$, subsets $Y_1$ and $Y_2$ of $X$ with the same measure, one would obtain $(A_{pY_1} \subset Mp_{Y_1}) \simeq (A_{pY_2} \subset Mp_{Y_2})$ via $v$. 
2.3. Theorem of Connes-Feldman-Weiss.

Definition 2.19. A discrete group $\Gamma$ is said to be amenable when $\ell_\infty \Gamma$ has a left $\Gamma$ invariant state.

Example 2.20. Commutative groups, or more generally solvable groups are amenable. The union of an countable increasing sequence of amenable groups are again amenable.

Definition 2.21. A cartan subalgebra $A \subset M$ is said to be amenable when there exists a state $\tilde{m} : \tilde{A} \rightarrow \mathbb{C}$ invariant under the adjoint action of $N A$. An orbit equivalence $\mathcal{R}$ on $X$ is said to be amenable when $L^\infty X \subset \nu N \mathcal{R}$ is amenable.

Remark 2.22. Let $\Gamma \curvearrowright X$ be a measure preserving essentially free action. Since $\Gamma$ is assumed to be discrete, $\mathcal{R}$ can be identified to $\Gamma \times X$ as a measurable space and an invariant measure on $\mathcal{R}$ is nothing but a product measure on $\Gamma \times X$ of an invariant measure on $\Gamma$ times an arbitrary measure on $X$. Thus, $\mathcal{R}$ is amenable if and only if $\Gamma$ is amenable.

Definition 2.23. A von Neumann algebra $M$ on $H$ is said to be injective when there exists a conditional expectation $\Phi : B(H) \rightarrow M$.

Fact. The above condition is independent of the choice of a faithfull representation $M \hookrightarrow B(H)$. Moreover, $M$ is injective if and only if it is AFD [2].

Theorem 2.24. (Connes-Feldman-Weiss [3]) Let $M$ be a factor with separable predual, $A$ a Cartan subalgebra of $M$. The following conditions are equivalent:

1. The pair $A \subset M$ is amenable.

2. This pair is AFD in the sense that for any finite subset $\mathcal{F}$ of $N A$ and a positive real number $\epsilon > 0$, there exists a finite dimensional subalgebra $B$ of $M$ such that
   - $B$ has a matrix unit consisting of elements of $N^p A$.
   - $\|v - E_B(v)\| < \epsilon$ for any $v \in \mathcal{F}$.

3. $(A, M)$ is isomorphic to $(D, \bar{\otimes} M_2 \mathbb{C})$ where $D = \bar{\otimes} D_2$ for the diagonal subalgebra $D_2 \subset M_2$. (Note that $N^p D$ is generated by the “matrix units” of $M_2^\infty = \bar{\otimes} M_2$.)

4. $M$ is injective.

Lemma 2.25. In the assertion of (2), $B$ may be assumed to be isomorphic to $M_{2^N}$ for some $N$.

Proof of the lemma. Perturbing a bit, we may assume that $\tau(e_{ij}^{(d)}) \in 2^{-N} \mathbb{N}$ for large enough $N$ where $(e_{ij}^{(d)})_{d, 1 \leq i, j \leq n_d}$ is a matrix unit of $B = \oplus_d M_{n_d}$. By taking a partition if necessary, we may assume that $\tau(e_{ii}^{(d)}) = 2^{-N}$ for any $d$ and $i$. Then, since $M$ is a factor, we have $e_{ii}^{(d)} \sim e_{jj}^{(f)}$ in $M$ for any $d, f, i$ and $j$. This means that $B$ is contained in a subalgebra of $M$ which is isomorphic to $M_{2^N}$.

Proof of (2) $\Rightarrow$ (3): Note that there is a total (with respect to the 2-norm) sequence $(v_k)_{k \in \mathbb{N}} \subset N^p A$. We are going to construct an increasing sequence of subalgebras $(B_k)_{k}$ in $M$ with compatible matrix units $(e_{ij}^{(k)})_{i,j}$ satisfying $B_k \simeq M_{2^{n_k}}$ and $\|E_{B_k}(v_l) - v_l\|^2 < \frac{1}{k}$ for $l \leq k$. 

Suppose we have constructed $B_1, \ldots, B_k$. Applying the assertion of (2) to the finite set $\mathcal{F}' = \{ e_{i,r} v_1 e_{r,1}^{(k)} \}$, we obtain a matrix units $(f_{ij})_{i,j}$ in $N^pA$ such that \[ \sum f_{ij} = e_{11}^{(k)} \text{ and} \]
\[ \| E_{\text{span} f_{ij}}(x) - x \| < \frac{1}{n(k)^2(k+1)} \]
where $n(k)$ denotes the size of $B_k$. By the assumption that $A$ is a maximal abelian subalgebra in $M$, the projections of $N^pA$ are actually contained in $A$. Thus we obtain an inclusion $D \subset A$ (hence the equality between them) under the identification $M \simeq \otimes_{n=1}^\infty M_2 = (\cup B_k)^\sigma$.

Proof of (3) $\implies$ (4): By assumption $M = (\cup B_n)^\sigma$ where $B_n$ are finite dimensional subalgebras of $M$, $M' = (\cup JB_nJ)^\sigma$. Let $\Phi_n$ denote the conditional expectation of $B(H)$ onto $(JB_nJ)^\sigma$: \[ \Phi_n(x) = \int_{JB_nJ} ^{x} \langle u, xu \rangle \, du \text{ where } du \text{ denotes the normalized Haar measure on the compact group } \mathcal{U}(JB_nJ). \]
For each $x$, the sequence $(\| \Phi_n(x) \|)_{n=1}^\infty$ is bounded above by $\| x \|$. Thus we can take a Banach limit $\Phi(x)$ of $(\Phi_n(x))$$_{n=1}^\infty$ which defines a conditional expectation of $B(H)$ onto $\cap_n (JB_nJ)^\sigma = (\cup JB_nJ)^\sigma$. \[ \Phi \text{ is a } \text{AdU}\mathcal{M} \text{-invariant state on } B(H). \]
$\mathcal{N}A$ is obviously contained in $\mathcal{U}\mathcal{M}$ and so is $\tilde{\mathcal{A}}$ in $B(H)$.

Remark 2.26. When $A \subset M$ is an amenable Cartan subalgebra and $e$ is a projection in $A$, the Cartan subalgebra $A_e \subset M_e$ is also amenable.

We are going to complete the proof of Theorem 2.24 by showing (1) $\implies$ (2).

Lemma 2.27. Let $\phi$ be a measure preserving partial Borel isomorphism on a standard probability space $(X, \mu)$. Let $E_0$ denote the fixed point set $X^\phi = \{ x \in \text{dom } \phi : \phi x = x \}$. There exist Borel sets $B_1, B_2, B_3$ of $X$ satisfying $X = \bigcup_{0 \leq i \leq 3} E_i$ and $\phi E_i \cap E_i$ is null for $i > 0$.

Proof. Take $E_1$ to be a Borel set with a maximal measure which satisfies $\phi E_1 \cap E_1 = \emptyset$. Put $E_2 = \phi E_1$. Then $\phi E_2 \cap E_2 = \emptyset$ by the injectivity of $\phi$. Finally, put $E_3 = \complement_{0 \leq i \leq 2} E_i$. Then $\phi E_3 \cap E_3$ is null by the maximality of $E_1$. \qed

Corollary 2.28. For any finite set $\mathcal{F}$ of $N^pA$, there exist projections $q_1, \ldots, q_m$ of $A$ ($m = 4^{|\mathcal{F}|}$) satisfying $\sum q_k = 1$ and $q_k v q_k$ is either 0 or in $U A_\mathcal{F}$ for any $v \in \mathcal{F}$.

Lemma 2.29. (Dye) For any finite subset $\mathcal{F} \subset \mathcal{N}A$ and $\epsilon > 0$, there exists $a \in \tilde{\mathcal{A}}_+$ with $\text{Tr}(a) = 1$ and $\sum_{u \in \mathcal{F}} \| u a u^* - a \|_{1, \text{Tr}} < \epsilon$. (Here, $\| x \|_{1, \text{Tr}} = \text{Tr}(\| x \|_1)$.)

Proof. Let $m : \tilde{\mathcal{A}} \to \mathcal{C}$ be an $\text{Ad}\mathcal{N}A$-invariant state. Since $L^1$ is $w^*$-dense in $(L^\infty)^*$, there exists a net $a_i \in \mathcal{A}_+$ satisfying $\text{Tr}(a_i) = 1$ and $\text{Tr}(a_i x) \to m(x)$ for any $x \in \tilde{\mathcal{A}}$. Then, for any $u \in \mathcal{N}A$ and $x \in \tilde{\mathcal{A}}$ \[ \text{Tr}((ua_i u^* - a_i)x) = \text{Tr}(a_i u^* xu) - \text{Tr}(a_i x) \to m(u^* xu) - m(x) = 0. \]
Thus $ua_i u^* - a_i$ is weakly convergent to 0. By Hahn-Banach’s theorem, by taking the convex closure of the sets $\{ ua_i u^* - a_i : k < i \}$, we find a sequence $(b_i)_i$ as convex combinations of the $a_i$ satisfying $\| ub_i u^* - b_i \|_{1, \text{Tr}} \to 0$ uniformly for $u \in \mathcal{F}$. \qed
Lemma 2.30. (Namioka) Let $\mathcal{F}$, $\epsilon$ be as above. There exists a projection $p$ of $\hat{A}$ satisfying $\text{Tr}(p) < \infty$ and $\sum_{u \in \mathcal{F}} \|upu^* - p\|_2^2 < \epsilon \|p\|_2^2$. 

Proof. Let $a \in \hat{A}_+$ be an element given by Lemma 2.29. For each $r > 0$ put $P_r = X_{(r, \infty)}(a)$. We have
\[
\|aua^* - a\|_{1,\text{Tr}} = \int_0^\infty \|uP_r u^* - P_r\|_{1,\text{Tr}} \, dr = 1 = \|a\|_{1,\text{Tr}} = \int_0^\infty \|P_r\|_{1,\text{Tr}} \, dr.
\]
Hence
\[
\int_0^\infty \sum_{u \in \mathcal{F}} \|uP_r u^* - P_r\|_{1,\text{Tr}} \, dr < \epsilon \int_0^\infty \|P_r\|_{1,\text{Tr}} \, dr.
\]
Thus there exists $r$ such that $p = P_r$ satisfies $\sum \|upu^* - p\|_{1,\text{Tr}} < \epsilon \|p\|_{1,\text{Tr}}$. Since the summands are differences of projections, $\|\cdot\|_{1,\text{Tr}}$ is approximately equal to $\|\cdot\|_2^2$. \hfill \Box

Lemma 2.31. (Local AFD approximation by Popa) Let $\mathcal{F}$, $\epsilon$ be as above. There exists a finite dimensional subalgebra $B \subset M$ with matrix units in $\mathcal{N}^p A$, satisfying $\|E_B(ueu) - (u - e^{-1}ue^1)\|^2 < \epsilon \|e\|^2_2$ for every $u \in \mathcal{F}$, where $e$ denotes the multiplicative unit of $B$ and $E_B$ the conditional expectation $eMe \to B$.

Proof. We may assume $1 \in \mathcal{F}$. Take $p \in \hat{A}_+$ as in Lemma 2.30. Since $\text{Tr} p < \infty$, we may assume that $p$ can be written as $\sum_{i=1}^n v_i e_A v_i^*$ for $v_i \in \mathcal{N}^p A$. By Corollary 2.28 there exist projections $(q_k)_k$ in $A$ with $\sum q_k = 1$ and each $q_k v_i^* v_j q_k$ is either 0 or is in $\mathcal{U}(Aq_k)$ for $1 \leq i, j \leq n, u \in \mathcal{F}$. Taking finer partition if necessary, we deduce that $\text{dist}(q_k v_i^* u v_j q_k, Cq_k) < \sqrt{\epsilon/n}$.

On the other hand,
\[
\sum_{u \in \mathcal{F}, k} \|\sum_{i=1}^n (u^*p - p)Jq_k J\|_2^2 = \sum_{u \in \mathcal{F}} \|u^*p - p\|_{2,\text{Tr}}^2 < \epsilon \|p\|_{2,\text{Tr}}^2 = \epsilon \sum_k \|pJq_k J\|_2^2.
\]
Hence for some $k$, $q = q_k$ satisfies $\sum \|u^*u - p\|_{2,\text{Tr}}^2 < \epsilon \|pJq J\|_2^2$. By $pJq J = \sum v_i e_A Jq_k Jv_i = \sum v_i q g A v_i^*$ since $A$ is commutative, replacing $v_i$ by $v_i q$, we may assume $v_i^* v_j = \delta_{i,j} q$ and $pJq J = p$. (Note that $p = \sum v_i e_A v_i^*$ is a projection, which means that the ranges of $v_i$ are mutually orthogonal.)

This way we obtain $\sum \|u^*u - p\|_2^2 < \epsilon \|p\|_2^2$, each $v_i u v_i^* \in A_q$ is close to a constant $z_{ij}$ by $\sqrt{\epsilon/n}$, and $(v_i)_i$ is a matrix unit in $A_q$. Put $e = \sum v_i v_i^*$. Thus
\[
\|p\|_{2,\text{Tr}}^2 = \text{Tr}(\sum v_i e_A v_i^*) = \tau(\sum v_i v_i^*) = \|e\|^2_2.
\]
Consequently,
\[
\|u^*u - p\|_2^2 = 2 \text{Tr}(p - 2 \text{Tr}(upu^*) p) = 2 \tau(e) - 2 \text{Tr}(\sum u v_i e_A v_i^* u^* v_j e_A v_j) = 2 \tau(e) - 2 \tau(ueu^*) = \|ueu - e\|_{2,\text{Tr}}^2.
\]
Hence $\sum_{u \in \mathcal{F}} \|ue - eu\|_2^2 < \epsilon \|e\|_2^2$. Now $eue = \sum v_i v_i^* u v_j v_j^* \approx \sum z_{ij} v_i v_j^* \approx \epsilon \|e\|^2_2$ in $\|\cdot\|_{2,\text{Tr}}^2$. Hence
\[
\|eue - E_B(eue)\|_{2,\text{Tr}}^2 < \epsilon \|e\|_{2,\text{Tr}}^2, \quad \|E_B(eue) - (u - e^{-1}ue^1)\|_{2,\text{Tr}}^2 < 2 \epsilon \|e\|_{2,\text{Tr}}^2.
\]
When we have a family \((B_i)\) of mutually orthogonal finite dimensional algebras satisfying the assertion of the lemma, \(e = \sum 1_{B_i}\) satisfies
\[
\|E_{\oplus B_i}(e u e) - (u - e^* u e^*)\|_{2, r}^2 < 2e \|e\|_{2, r}^2.
\]

**Lemma 2.32.** In the notation of Lemma 2.31, \(e = 1\).

**Proof.** Otherwise we can apply Lemma 2.31 to \(A_+ \subset M_+\) and \(\mathcal{F}' = e^* \mathcal{F} e^*\), to obtain a finite dimensional algebra \(B_0 \subset M_+\) satisfying the assertion of Lemma 2.31. Use the Pythagorean equality. \(\square\)

**Proof of (1) ⇒ (2):** Take \(B_1, \ldots, B_m\) satisfying \(\|\sum_{i} 1_{B_i}\|_2^2 > 1 - \epsilon\). Put \(B = \oplus_i B_i \oplus \mathbb{C}(\sum 1_{B_i})^\perp\). Then we have \(\|E_B(u) - u\|_2^2 < 3\epsilon\) for \(u \in \mathcal{F}\). \(\square\)

### 3. L²-Betti numbers

#### 3.1. Introduction

Let \(\mathfrak{F}(\Omega, X)\) denote the set of the mappings of a set \(\Omega\) into another set \(X\). Let \(\Gamma\) be a discrete group, \(\lambda\) the left regular representation of \(\Gamma\) on \(\ell_2\Gamma\). We have the “standard complex” of right \(\Gamma\) modules
\[
\begin{array}{c}
0 \rightarrow \ell_2\Gamma \xrightarrow{\partial} \mathfrak{F}(\Gamma, \ell_2\Gamma) \xrightarrow{\partial} \mathfrak{F}(\Gamma^2, \ell_2\Gamma) \rightarrow \cdots
\end{array}
\]
given by
\[
\partial(f)(s_1, \ldots, s_{n+1}) = \lambda(s_1)f(s_2, \ldots, s_{n+1}) + \sum_{1 \leq j \leq n} (-1)^j f(s_1, \ldots, s_js_{j+1}, \ldots, s_{n+1}) + (-1)^{n+1}f(s_1, \ldots, s_n).
\]

Conceptually, the above complex can be regarded as \(\text{Hom}_{\text{CT}}(P_\ast, \ell_2\Gamma)\) where \(P_\ast\) denotes the standard free resolution of the trivial left \(\Gamma\)-module \(\mathbb{C}\). For each \(n \in \mathbb{N}\), \(P_n\) is the vector space with basis \(\Gamma^{n+1}\) as a vector space over \(\mathbb{C}\). Since \(\Gamma^{n+1}\) is a left \(\Gamma\)-set by \(s(s_0, \ldots, s_n) = (s,s_0, s_1, \ldots, s_n)\), \(P_n\) has the canonically induced left action of \(\Gamma\).

Let \(H_i(\Gamma, \ell_2\Gamma)\) denote the \(i\)-th \((co)\)homology group of this complex. Note that this complex consists of \(RT\) modules given by the action on \(\ell_2\Gamma\), with boundary maps being \(RT\)-homomorphisms. The space of 1-cocycles
\[
Z_1 = \{b \in \mathfrak{F}(\Gamma, \ell_2\Gamma) : b(st) = b(s) + \lambda(s)b(t)\}
\]
is identified with the space of the derivations from \(\Gamma\) to \(\ell_2\Gamma\) with respect to the trivial derivations. When \(b \in Z_1\) the map
\[
s \mapsto \left(\begin{array}{cc}
\lambda(s) & b(s) \\
0 & 1
\end{array}\right)
\]
of \(\Gamma\) into \(B(\ell_2\Gamma \oplus \mathbb{C})\) becomes multiplicative. On the other hand the space of 1-coboundaries
\[
B_1 = \{b \in \mathfrak{F}(\Gamma, \ell_2\Gamma) : \exists f \in \ell_2\Gamma, b(s) = \lambda(s)f - f\}
\]
is identified with the space of the inner derivations. Note that for any \(b \in Z_1\), there is a function \(f \in \mathfrak{F}(\Gamma, \mathbb{C})\) satisfying \(b(s) = \lambda(s)f - f\) if we do not require the square summability of \(f\). Indeed, a vector system \((b(s))_{s \in \Gamma}\) is a derivation if and only if we have \(\langle b(s), \delta_t \rangle = \langle b(st) - b(t), \delta_s \rangle\) for any \(s, t \in \Gamma\), and in such a case we may put \(f(s) = \langle b(s), \delta_s \rangle\) to obtain \(b(s) = \lambda(s)f - f\).
Remark 3.1. The 0-th homology group $H_0 = Z_0$ is the space of the $\Gamma$-invariant vectors in $\ell_2\Gamma$. Thus this becomes the 0-module if and only if $\Gamma$ is infinite.

In the following we assume that $\Gamma$ admits a finite generating set $\mathcal{S}$. Let $D\Gamma$ denote the space $Z_1$ of the derivations, $\text{Inn} D\Gamma$ the space $B_1$ of the inner derivations. Let $\partial \mathcal{S}$ denote the mapping $b \mapsto (b(s))_{s \in \mathcal{S}}$ of $D\Gamma$ into $\oplus \mathcal{S} \ell_2\Gamma$. This is an injective $R\Gamma$-module map. Note that the range of $\partial \mathcal{S}$ is in $\text{Inn} D\Gamma$ if and only if $\mathcal{S}$ is an approximate kernel of $D\Gamma$. Indeed, $(f(s))_{s \in \mathcal{S}}$ is in the range of $\partial \mathcal{S}$ if and only if 

$$f(s_1) + \lambda(s_1)f(s_2) + \cdots + \lambda(s_1 \cdots s_{n-1})f(s_n) = 0$$

holds for each relation $s_1 \cdots s_n = e$ among elements of $\mathcal{S}$.

A sequence $(f_n)_{n \in \mathbb{N}}$ of unit vectors is said to be an approximate kernel of the restriction $\partial \mathcal{S}|_{\text{Inn} D\Gamma}$ when $\lambda(s)f_n - f_n$ tends to zero (in norm) for any $s \in \mathcal{S}$. $\partial \mathcal{S}|_{\text{Inn} D\Gamma}$ has an approximate kernel if and only if $\mathcal{S}$ is amenable. Thus $\partial \mathcal{S}(\text{Inn} D\Gamma)$ is closed if and only if $\mathcal{S}$ is finite or non-amenable.

Let $P$, $Q$ denote the orthogonal projections onto $\partial \mathcal{S}(D\Gamma)$ and $\partial \mathcal{S}(\text{Inn} D\Gamma)$. These commute with the diagonal action of $R\Gamma$ on $\oplus \mathcal{S} \ell_2\Gamma$, i.e. $P, Q \in M_{\mathcal{S}} L\Gamma$.

We can measure them by the trace $\tilde{\tau} = \text{Tr} \otimes \tau$. The first Betti number $\beta_1^{(2)} = \dim_{L\ell_1} H_1(\Gamma, \ell_2)$ is equal to the difference $\tilde{\tau}(P) - \tilde{\tau}(Q)$.

Example 3.2. When $\Gamma$ is a finite group, $\beta_i^{(2)} = \frac{1}{|\Gamma|}$ while $\beta_i^{(2)} = 0$ for $0 < i$ because any $C\Gamma$ module is projective. On the other hand when $\Gamma$ is equal to the free group $\mathbb{F}_n$ generated by a set $\mathcal{S}$ consisting of $n$ elements, $\text{ran} \partial \mathcal{S} = \oplus \mathcal{S} \ell_2\Gamma$ and $\beta_i^{(2)} = n - 1$.

We omit the injection $\partial \mathcal{S}$ and identify $D\Gamma$ with a subspace of $\oplus \mathcal{S} \ell_2\Gamma$. Thus $\partial^0 : \ell_2\Gamma \to \mathfrak{N}(\Gamma, \ell_2\Gamma)$ factors through $\oplus \mathcal{S} \ell_2\Gamma$ and $\partial^0 : \ell_2\Gamma \to \oplus \mathcal{S} \ell_2\Gamma$ is written as $f \mapsto (\lambda(s)f - f)_{s \in \mathcal{S}}$.

Let $\epsilon^{(2)}_1 : \oplus \mathcal{S} \ell_2\Gamma \to \ell_2\Gamma$ denote the adjoint of $\partial$. Thus $\epsilon^{(2)}_1$ is expressed as $(\xi_s)_{s \in \mathcal{S}} \mapsto \sum_{s \in \mathcal{S}} (\lambda(s^{-1}) - 1)\xi_s$ and the orthogonal complement of $\ker \epsilon^{(2)}_1$ is equal to the closure of $\text{ran} \partial = \text{Inn} D\Gamma$.

Proposition 3.3. When we identify $C\Gamma$ with the space of vectors with finite support in $\ell_2\Gamma$, we have $D\Gamma = (\ker \epsilon_1^{(2)} \cap \oplus \mathcal{S} C\Gamma)^\perp$.

Proof. The space $C\Gamma$ has $\mathfrak{N}(\Gamma, C)$ as its algebraic dual. A vector system $b \in \oplus \mathcal{S} \ell_2$ is in $D\Gamma$ if and only if there is an $f \in \mathfrak{N}(\Gamma, C)$ such that $b(s) = \lambda(s)f - f$. The latter implies

$$\forall \xi \in \ker \epsilon_1^{(2)} \cap \oplus \mathcal{S} C\Gamma, \langle \xi, b \rangle = \sum_s \langle \xi(s), b(s) \rangle = \sum_s \langle (\lambda(s^{-1}) - 1)\xi(s), f \rangle = 0.$$

Conversely, when $(b(s))_{s \in \mathcal{S}}$ is orthogonal to $\ker \epsilon_1^{(2)} \cap \oplus \mathcal{S} C\Gamma$, the functional $\langle \cdot, \cdot \rangle$ on $\oplus \mathcal{S} C\Gamma$ is induced by a functional $f$ on the kernel of the map $C\Gamma \to C$. This $f$ can be extended to a linear map on the whole $C\Gamma$, and we have $b(s) = \lambda(s)f - f$, i.e. $b \in D\Gamma$. \blacksquare

Remark 3.4. The $i$-th cohomology group $H^i(\Gamma, \ell_2\Gamma)$ is dimension isomorphic to $\text{Tor}^i_{\ell_1}(C, \ell_2\Gamma)$. This is seen by considering the exact functors $E \to E^*$ on the category of $L\Gamma$-modules and that of $L\Gamma$-bimodules, where $E^*$ denotes the dual module of the weak closure of $E$. We have functors $(A, B) \to A \otimes_{C\Gamma} B$ and $(A, B) \to
Hom_{\Gamma}(A,B) of \mathbb{C}\Gamma\text{-mod} \times \mathbb{C}\Gamma\text{-bimod} into \mathbb{C}\Gamma\text{-mod}. Then the functor equivalence 
\( (A \otimes_{\mathbb{C}\Gamma} B^*) \simeq \text{Hom}_{\mathbb{C}\Gamma}(A,B^*) \) up to dimension implies the dimension equivalence between the derived functors \( \text{Tor}_p(A,B)^* \simeq \text{Ext}^p(A,B^*) \). The case \( A = \mathbb{C} \) and \( B = \ell_2\Gamma \) describes the desired isomorphism.

For example, we have a flat resolution \( P \) of the trivial \( \Gamma \)-module \( \mathbb{C} \) with \( P_0 = \mathbb{C}\Gamma \) and \( P_1 = \mathbb{C}\Gamma \otimes_{\mathbb{C}} \mathbb{C}\mathcal{F} \), with \( d_1(a \otimes b) = ab - a \). The first torsion group \( \text{Tor}_1^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C}) \) is by definition the quotient \( \ker(id_{\ell_2\Gamma} \otimes d_1)/\ell_2\Gamma \otimes \ker d_1 \). Now \( id_{\ell_2\Gamma} \otimes d_1 = \epsilon_1^{(2)} \) implies \( \ker(id_{\ell_2\Gamma} \otimes d_1) = \text{Inn} \Delta \Gamma \) while \( \ell_2\Gamma \otimes \ker d_1 = \ker \epsilon_1^{(2)} \cap \mathcal{F} \Gamma \) implies \( \ell_2\Gamma \otimes \ker d_1 = \Delta \Gamma \).

3.2. Operators affiliated to a finite von Neumann algebra. Let \((M, \tau)\) be a finite von Neumann algebra with a faithful normal tracial state \((\tau\) is unique if \(M\) is a factor), \(L^2M\) the induced Hilbert \(M\text{-}\mathcal{M}\) module. For each \(n \in \mathbb{N}\) put \(\hat{\tau} = \tau \otimes \text{Tr} \) on \(M \otimes M_n \mathbb{C} \simeq M_n M\).

Definition 3.5. Let \(H\) be a left Hilbert module over \(M\). A densely defined closed operator \(T\) on \(H\) is said to be affiliated to \(M\), written as \(T \sim M\), when we have \(uT = Tu\) for any \(u \in \mathcal{U}(M')\). Here the equality entails the agreement of the domains, i.e. \(u \text{dom } T = \text{dom } T\).

Remark 3.6. An operator \(T\) is affiliated to \(M\) if and only if for the polar decomposition \(T = vv\) the partial isometry \(v\) and the spectral projections of \(|T|\) are in \(M\). Note that in such cases \(\tau\) takes the same value on the left support \(l(T) = vv^*\) of \(T\) and the right support \(r(T) = v^*v\).

We consider the case \(H = L^2M\). Suppose \(T \sim M\). It is said to be square integrable when \(\hat{1} \in \text{dom } T\). This condition is equivalent to

\[ \tau(|T|^2) = \|T\hat{1}\|^2 = \int t^2 d\tau(E) < \infty \]

for the spectral measure \(T = \int t dE\) of \(T\). For each \(\xi \in L^2M\) let \(L^\prime_\xi\) denote the unbounded operator defined by \(\text{dom } L^\prime_\xi = \mathcal{M} \subset L^2M\) and \(L^\prime_\xi x = \xi x\).

Proposition 3.7. The operator \(L^\prime_\xi x\) is closable and its closure \(L_\xi\) is affiliated to \(M\). Moreover we have \(L^\prime_\xi = L_\xi\). If \(T\) is affiliated to \(M\) and square integrable, \(T = L_{T\hat{1}}\).

Proof. We show the inclusion \(L^\prime_\xi \subset (L^\prime_\xi)^*\). For any elements \(x, y \in M\),

\[ \langle L^\prime_\xi \hat{x}, \hat{y} \rangle = \langle \xi x, y \rangle = \langle J\hat{y}, J(\xi x) \rangle = \langle \hat{1}y^*, x^* J\xi \rangle = \langle \hat{x}, (J\xi)y \rangle. \]

On the other hand, when \(u \in \mathcal{U}(M)\), \(uL^\prime_\xi = L^\prime_\xi u\) implies \(uL_\xi = L_\xi u\).

Next we show the inclusion \((L^\prime_\xi)^* \subset L_\xi\). Consider the polar decomposition \(L_\xi = vL_\xi|\) and the spectral decomposition \(|L_\xi| = \int_0^\infty \lambda d\mathcal{E}_\lambda\). Then \(e_\lambda v^*L_\xi = e_\lambda L_\xi|\) is bounded (i.e. is in \(M_\lambda\)) for any \(\lambda\). By definition, \(L_\xi(y\hat{1}) = (J\xi)y\) for \(y \in M\). Hence \(e_\lambda v^*L_\xi(y\hat{1}) = e_\lambda v^*(J\xi)y = (e_\lambda v^*J\xi)y\).

Putting \(y = 1\), we obtain \(e_\lambda v^*L_\xi \hat{1} = e_\lambda v^* J\xi \in M\hat{1}\) for any \(\lambda > 0\).
Thus, by definition of $(L_ξ)^*$, we have
\[
\langle (L_ξ)^* η, (e_λ v^*)^* y \tilde{} 1 \rangle = \langle η, L_ξ(e_λ v^*)^* y \tilde{} 1 \rangle = \langle η, J_ξ y(e_λ v^*)^* J_ξ \rangle = \langle η, J_ξ y(e_λ v^*)^* L_ξ \tilde{} 1 \rangle \quad \text{(by using above)}
\]
\[
= \langle η, (e_λ v^* L_ξ)^* y \tilde{} 1 \rangle.
\]
Hence $e_λ v^*(L_ξ)^* η = e_λ v^* L_ξ η = |L_ξ|e_λ η$ for any $λ > 0$. By letting $λ → ∞$, $e_λ η → η$ and $|L_ξ|e_λ η → v^*(L_ξ)^* η$. Since $|L_ξ|$ is a closed operator, $η ∈ \text{dom}(|L_ξ|) = \text{dom}(L_ξ)$. Hence $(L_ξ)^* ⊂ L_ξ$ and $|L_ξ| = v^*(L_ξ)^*$.

Finally, let us prove the last part. Let $T ∼ M$ with the polar decomposition $v|T| = T$. Note that $v^* = 1v^* ∈ \text{dom}T$, $1 ∈ \text{dom}T^*$, $T^*1 = |T|v^*$. Put $ξ = T1$, $η = T^*1$. Since $T ∼ M$, $L_ξ^0 ⊂ T$, $L_η^0 ⊂ T^*$ and we obtain $L_ξ ⊂ T ⊂ L_ξ^0$.

### 3.3. Projective modules over a finite von Neumann algebra

Let $m, n ∈ \mathbb{N}$. We have an isomorphism $\text{Mor}(M^⊕m, M^⊕n) = M_{m,n}(M)$ by multiplication of matrices on column vectors.

**Definition 3.8.** An left $M$-module $V$ is said to be finitely generated projective module when it is a projective object in the category of the $M$-modules and has a finite set generating itself.

**Remark 3.9.** Any finitely projective $M$ module is isomorphic to some $M^⊕m.P$ for a natural number $m$ and an idempotent matrix $P$ in $M_m.M$.

**Lemma 3.10.** In the above we may replace $P$ with an orthogonal projection $P^* = P$ without changing the value of $\bar{τ}(P)$.

**Proof.** Let $P_0$ be the right support of $P$. $P(P - P_0) = 0$ implies $P_0(P - P_0) = 0$. Thus $S = \text{Id} + (P - P_0)$ is invertible. With respect to the orthogonal decomposition $\text{Id} = P_0 ⊕ P_0^⊥$, these operators are expressed as

\[
P_0 = \begin{pmatrix}
\text{Id} & 0 \\
0 & 0
\end{pmatrix}, \quad P = \begin{pmatrix}
\text{Id} & 0 \\
? & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
\text{Id} & 0 \\
? & \text{Id}
\end{pmatrix}.
\]

The operator $SP_0 = SP_0S^{-1}$ is self adjoint. \(\square\)

**Remark 3.11.** When $M^⊕m.P$ and $M^⊕nQ$ are isomorphic, $\bar{τ}(P) = \bar{τ}(Q)$.

**Definition 3.12.** For each finitely projective $M$-module $V$ isomorphic to $M^⊕m.P$ where $P$ is a orthogonal projection in $M_m.M$, $\dim_M V = \bar{τ}(P)$ is called the $τ$-dimension.

**Lemma 3.13.** Let $V$ be a submodule of $M^⊕n$. When $V$ is closed $M^⊕n$ with respect to the $L^2$-norm ($V$ is weakly closed), $V$ is finitely generated and projective.

**Proof.** The $L^2$ completion $\tilde{V} ⊂ L^2M^⊕n$ is written as $L^2M^⊕P$ for an orthogonal projection $P$. Then $V$ is equal to $M^⊕nP$. \(\square\)

**Lemma 3.14.** For each $T ∈ \text{Mor}(M^⊕m, M^⊕n)$, its kernel and range are finitely generated projective modules.
Proof. Obviously the kernel of $T$ is weakly closed in $M^\oplus m$. On the other hand for the projection $P$ such that $\ker T = M^\oplus m P$, $T$ induces an isomorphism $MP^\perp \to \text{ran } T$.\hfill\Box

**Remark 3.15.** When a submodule $V \subset M^\oplus m$ is finitely generated, $V$ is projective. In fact, $V = M^\oplus m A$ for some $A \in M_{m,n}(M)$. Thus we have $V \simeq M^\oplus m l(A) \simeq M^\oplus m r(A) \simeq \bar{V}$.

Hence $\dim_M V = \dim_M \bar{V}$.

**Remark 3.16.** If $W \subset V$ are finitely generated projective modules, $\dim_M W \leq \dim_M V$.

**Definition 3.17.** Let $V$ be an $M$-module. Put $\dim_M V = \sup \{ \dim_M W : W \subset V, W \text{ is projective} \} \in [0, \infty]$.

**Remark 3.18.** Note that the above definition of $\dim_M$ is compatible with the previous one for finitely generated projective modules. In general, $W \subset V$ implies $\dim_M W \leq \dim_M V$ and $(V_i)_{i \in I} \uparrow V$ ($V = \cup_{i \in I} V_i$) implies $\dim_M V = \lim_i \dim_M V_i$.

**Theorem 3.19.** (Lück [6]) When 

\[
0 \longrightarrow V_0 \overset{i}{\longrightarrow} V_1 \overset{\pi}{\longrightarrow} V_2 \longrightarrow 0
\]

is exact, we have $\dim_M V_1 = \dim_M V_0 + \dim_M V_2$.

Proof. When $W \subset V_2$ is finitely generated and projective, $\pi^{-1}W$ is identified to $W \oplus \iota V_0$. Hence $\dim V_1 \geq \dim V_0 + \dim V_2$. Conversely, let $W \subset V_1$ be finitely generated projective. The weak closure $\overline{\iota V_0 \cap W}$ is closed in a finite free module, hence is projective. From the sequence $\iota V_0 \cap W \to W \to W/\iota V_0 \cap W$, we have $\dim W = \dim \iota V_0 \cap W + \dim W/\iota V_0 \cap W$. Note that there is a natural surjection $W/\iota V_0 \cap W \to W/\iota V_0 \cap W$. By the first part of the argument this implies the dimension inequality $\dim \overline{\iota V_0 \cap W} \leq \dim \iota V_0 \cap W$. On the other hand $W/\iota V_0 \cap W$ is identified to a submodule of $V_2$.\hfill\Box

**Corollary 3.20.** Let $V$ be a finitely generated $M$-module. We have a decomposition $V = V_p \oplus V_t$ where $V_p$ is projective and $\dim V = \dim V_p$. (Hence $\dim V_t = 0$.)

Proof. We have a surjection $T : M^\oplus m \to V$. Note that $\ker T$ may not be closed since we have no matrix presentation of $T$. Nonetheless, $V \simeq M^\oplus m / \ker T$ and the next lemma imply that $V_p = M^\oplus m / \ker T$ satisfies

$$\dim V = m - \dim \ker T = m - \dim \overline{\ker T} = \dim V_p.$$\hfill\Box

**Lemma 3.21.** Let $W$ be a subset of a finite free module $M^\oplus m$. We have $\dim W = \dim \overline{W}$. 

Proof. Put \( L = \{ A \in M_m M : M \oplus A \subset W \} \). This is a left ideal of \( M_m M \). We get a right approximate identity \( A_i \) of \( L \). For the orthogonal projection \( P \) such that \( \bar{W} = M \oplus P \), the right support \( r(A_i) \) converges to \( P \) in strong operator topology (for any normal representation, thus, in the ultrastrong topology). Thus for any \( \epsilon > 0 \), \( P_{\epsilon,i} - \chi_{[\epsilon, 1]}(A_i^* A_i) \) is in \( L \) and converges to \( P \) in the ultrastrong operator topology. \( \square \)

Proposition 3.22. For any \( L\Gamma \)-module \( V \), \( \dim V = 0 \) is equivalent to \( \forall \xi \in V, \epsilon > 0, \exists P \in \text{Proj} M : \tau P > 1 - \epsilon \) and \( P\xi = 0 \).

Proof. \( \Rightarrow \): Let \( \xi \in V \). Consider the exact sequence \( 0 \to L \to M \to M\xi \to 0 \) where \( L \) is the annihilator of \( \xi \). \( \dim L = \dim M \) implies the existence of projections \( P_i \) convergent to 1 in the ultrastrong topology.

\( \Leftarrow \): If \( V \supset M.Q \), \( P \) satisfies \( \tau P > 1 - \tau Q \) and \( PQ \neq 0 \). \( \square \)

Definition 3.23. A homomorphism \( \phi : V \to W \) of \( L\)-modules is said to be a dimension isomorphism when \( \dim M \ker \phi = \dim M \operatorname{cok} \phi = 0 \).

Remark 3.24. The torsion \( N \)-modules \( T = \{ V : \dim_N V = 0 \} \) form a Serre subcategory of \( N\)-mod. Analyzing \( N\)-modules up to dimension isomorphisms amounts to considering the localization \( N\)-mod/\( T \) of \( N\)-mod by \( T \). Thus, in general, when a morphism \( V_* \to W_* \) of complexes is a dimension isomorphism at each degree, the induced homomorphism between the cohomology groups is also a dimension isomorphism because it factors through an isomorphism in the localization category \( C^*(N\text{-mod}/T) \) of the \( N\)-module complex category over the torsion module category.

Lemma 3.25. The standard inclusion \( M \to L^2(M) \) is a dimension isomorphism.

Proof. Let \( \xi \in L^2 M \). We get the corresponding square integrable operator affiliated with \( M \). Put \( P_n = \chi_{[n, \infty)}(\xi^* \xi) \in \text{Proj} M \). Then \( P_n \xi \in M \) and \( P_n \to 1 \), thus \( P_n[\xi] - 0 \) in the quotient \( L^2 M/M \).

When \( H \) is a Hilbert \( M \)-module, i.e. a normal representation of \( M \) on \( H \), \( H \simeq L^2 M^\oplus n P \) for some cardinal \( n \) and an idempotent \( P \) in \( M_n M \).

Lemma 3.26. In the above notation, \( \dim_M H = \tilde{\tau}(P) \).

Proof. We have the following commutative diagram

\[
\begin{array}{ccc}
M^\oplus n & \xrightarrow{P} & L^2 M^\oplus n P \\
\downarrow \text{p.b.} & & \downarrow \\
M^\oplus n & \xrightarrow{\oplus} & L^2 M^\oplus n \xrightarrow{\text{cok}} \\
\end{array}
\]

The cokernel in the lower row has dimension 0, thus so does the one in the upper row. \( \square \)

Definition 3.27. \( \beta_n^{(2)}(\Gamma) = \dim_{L\Gamma} \operatorname{Tor}_n^{C^*}(L\Gamma, \mathbb{C}_{\text{triv}}) \) is called the \( n \)-th \( L^2 \)-Betti number of \( \Gamma \).
Remark 3.28. $\beta_n^{(2)}(\Gamma)$ is equal to $\dim_{L^\Gamma} \text{Tor}_n^{C^\Gamma}(\ell_2\Gamma, C_{\text{triv}})$.

Example 3.29. $\beta_n^{(2)}(\mathbb{F}_r) = r - 1$ when $n = 2$, $0$ otherwise. This is seen as follows: let $g_1, \ldots, g_r$ be the standard generators of $\mathbb{F}_r$. A free resolution of the trivial $C[\mathbb{F}_r]$-module $\mathbb{C}$ is given by

$$0 \longrightarrow (C[\mathbb{F}_r])^r \overset{d_1}{\longrightarrow} C[\mathbb{F}_r] \overset{\alpha}{\longrightarrow} C$$

where $d_1 : (\xi_k)_{k=1}^r \mapsto \sum (\lambda_{g} - 1)\xi_k$ and $\alpha$ is the augmentation map. Now, $d_1$ is injective: let $\chi_j \in \ell_\infty \mathbb{F}_r$ be the characteristic function of $\mathbb{F}_r g_j$. Then $(\lambda_{g} - 1)\chi_j = \delta_{j,k}\delta_{e}$ and $(\xi_k)_k \in \ker d_1$ implies

$$0 = (\sum_k (\lambda_{g} - 1)\xi_k, \chi_j) = \sum_k (\xi_k, (\lambda_{g} - 1)\chi_j) = \xi_j(e).$$

Replacing $\chi_j$ by $\chi_j = \chi_j(-t^{-1})$ for $t \in \Gamma$, we have $\xi_j(t) = 0$ for any $j$ and $t$. Thus, the torsion group is the cohomology of the complex

$$0 \longrightarrow (L^2\mathbb{F}_r)^r \overset{d_1}{\longrightarrow} L^2\mathbb{F}_r \longrightarrow 0.$$ 

Let $R$ be a ring. Recall that a right $R$-module $N$ is flat if and only if the tensor product functor $N \otimes_R -$ preserves injections $V \hookrightarrow F$ where $F$ is a finitely generated free module. The latter holds if and only if $N \otimes_R -$ preserves the injectivity of inclusion $I \hookrightarrow R$ of the left ideals.

Theorem 3.30. Let $M \hookrightarrow N$ be a trace preserving inclusion of finite von Neumann algebras. Then $N$ is flat over $M$ and $\dim_N N \otimes_M V = \dim_M V$ for any $M$-module $V$.

Proof. Recall that any finitely generated submodule of a free $M$-module is projective. (That is, $M$ is semihereditary.) To see this, let $V$ be a finitely generated submodule of a finitely generated free module $M^{\oplus m}$. $V \simeq M^{\oplus n} A$ for some $(m, n)$-matrix $A$. Then $V$ is projective, being isomorphic to $M^{\oplus n} l(A)$. Now,

$$N \otimes V \simeq N^{\oplus n} l(A) \simeq N^{\oplus m} A \hookrightarrow N \oplus M^{\oplus m}.$$ 

Hence $N$ is flat over $M$.

Let $V$ be a finitely generated $M$-module. Suppose we had an inclusion $\Phi : M^{\oplus m} P \hookrightarrow V$ of a projective module. Then $N^{\oplus m} P \hookrightarrow N \otimes V$ by the flatness of $N$. This shows that $\dim_N N \otimes_M V \leq \dim_M V$. On the other hand, for any surjection $\pi M^{\oplus n} \twoheadrightarrow V$, the induced homomorphism $\pi_* : N^{\oplus n} \twoheadrightarrow N \otimes V$ is surjective and $\dim N \otimes V = n - \dim \pi_*$, thus $\dim N \otimes V \leq \dim V$. \qed

3.4. Application to orbit equivalence.

Notation. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a probability measure preserving essentially free action. Put $A = L^\infty(X, \mu)$, $M = L\Gamma$, $N = L^\infty(X, \mu) \times \Gamma = \text{vN}(\mathcal{A}_{\Gamma \curvearrowright X})$. Let $R_0$ denote the linear span $\text{alg}(L^\infty(X, \mu), \Gamma)$ of $f(s)$ for the $f \in A, s \in \Gamma$. Let $R$ denote the linear span $\text{alg}(N(A))$ of $fv_\phi$ for the $f \in A, \phi \in [\mathcal{A}]$. 


Remark 3.31. $R_0$ is free over $\mathbb{C} \Gamma$ and $R_0 \otimes_{\mathbb{C} \Gamma} \mathbb{C} \simeq L^\infty(X)$. The induced left $R_0$-structure on $L^\infty(X)$ is given by $\sum f_i \lambda_i \cdot g = \sum f_i \alpha_i (g)$ thus $R_0 \otimes_{\mathbb{C} \Gamma} \mathbb{C} \simeq A$ and we have $\text{Tor}_n^R(N, A) \simeq \text{Tor}_n^C(N, \mathbb{C})$. The latter is isomorphic to $N \otimes_M \text{Tor}_n^C(M, \mathbb{C})$ by the flatness of $N$. Note that $\dim_N N \otimes_M \text{Tor}_n^C(M, \mathbb{C}) = \dim_M \text{Tor}_n^C(M, \mathbb{C}) = \beta^{(2)}(\Gamma)$.

Our goal is to show the equality $\dim_N \text{Tor}_n^R(N, A) = \dim_N \text{Tor}_n^R(N, A)$. Note that the latter only depends on the orbit equivalence $\mathcal{R}_{\Gamma \cap \gamma}$.

Lemma 3.32. For any $x \in R$ and $\epsilon > 0$, there is a projection $p$ in $A$ such that $\tau p > \epsilon$ and $xp^\perp \in R_0$.

Proof. When $x$ is of the form $v_\phi f$, the assertion is trivial by the expression $v_\phi = \sum \lambda(\eta_k)e_k$. The general case reduces $\tau$ the above by $\tau(p \vee q) \leq \tau p + \tau q$. \qed

For the time being let $A$ denote an arbitrary finite von Neumann algebra.

Definition 3.33. Let $V$ be a left $A$-module. For $\xi \in V$,

$$[\xi] = \inf \{ \tau p : p \in \text{Proj} A, p\xi - \xi \}$$

is called the rank norm of $\xi$.

Remark 3.34. $[\xi]$ is subadditive and scalar invariant. $V_\epsilon = \{ \xi : [\xi] = 0 \}$ is the largest submodule with $\dim_AV_\epsilon = 0$. Any $A$-module homomorphism $\phi : V \rightarrow W$ contracts $[\xi]$. Moreover for any $\eta \in \ker \phi$ and $\epsilon > 0$, there is an element $\xi \eta \in \phi^{-1}\eta$ such that $[\xi] \leq [\eta] + \epsilon$.

Definition 3.35. Let $V$ be an $A$-module. Consider a metric on $V$ defined by $d(\xi, \eta) = [\xi - \eta]$. Let $C(V)$ denote the completion of $V$ with respect to $d$.

Remark 3.36. $C(V)$ admits an left action of $A$: the continuity with respect to $d$ follows from $|a\xi| \leq \min|a|, [\xi]$: $p\xi = \xi$ implies $ap\xi = l(ap)ap\xi - l(ap)a\xi \Rightarrow |a\xi| \leq \tau(l(ap)) = \tau(r(ap))$

$C(V)$ contains $V/V_\epsilon$ as a dense subspace.

Remark 3.37. $V \subset W$ is dense if and only if for any $\xi \in W$ and $\epsilon > 0$, there exists $p \in A$ such that $\tau p < \epsilon$ such that $p^\perp\xi \in V$, which, in turn, happens if and only if $\dim_W/V = 0$.

Lemma 3.38. The functor $V \mapsto CV$ is exact.

Proof. Right exactness: consider an exact sequence $V_1 \rightarrow V_0 \rightarrow 0$. Let $\xi \in CV_0$, $(\xi_n)_{n \in \mathbb{N}} \subset V_0$ a sequence convergent to $\xi$. We may assume that $d(\xi, \xi_n) \leq 2^{-(n+1)}$. We can inductively lift $(\xi_n)$ to $(\eta_n)$ in $V_1$ such that $d(\eta_n, \eta_{n+1}) \leq 2^{-n}$.

General exactness: let

$$V_2 \xrightarrow{g} V_1 \xrightarrow{f} V_0$$

be an exact sequence, $\xi$ an element of $\ker C(f)$. Choose a sequence $(\xi_n)_n$ convergent to $\xi$. Then $f(\xi_n) \rightarrow C(f)(\xi) = 0$. This implies the existence of a sequence $(\eta_n)_n$, convergent to 0 and $f\eta_n = f\xi_n$. $\xi = \lim\xi_n - \eta_n$ is in the closure of the image of $g$, which, by the right exactness of $C$, is equal to the image of $C(g)$. \qed
Now we turn to the orbit equivalence situation: \( A \subset R_0 \subset R \subset N \). We consider \( A \)-rank metric on \( R_0 \)-modules.

**Lemma 3.39.** When \( V \) is an \( R_0 \) (resp. \( R \)) module, \( CV \) admits an \( R_0 \) (resp. \( R \)) module structure.

**Proof.** If \( x - \sum_{n=1}^{N} v_{\phi_n} f_n \), for any \( \xi \in V \) we have the estimate \( [x\xi] \leq n[\xi] \). \( \square \)

**Lemma 3.40.** When \( V \) is an \( R_0 \) module, \( CV \) admits an \( R \)-module structure.

**Proof.** Let \( x \in R, (x_n)_n \) be a sequence in \( R_0 \) convergent to \( x \). For any \( \xi \in V, x_n\xi \) is \( A \)-rank convergent to \( x\xi \). \( \square \)

**Lemma 3.41.** When \( V \) is a left \( R_0 \)-module, \( N \otimes_{R_0} V \to N \otimes_{R_0} CV \) is a dimension isomorphism.

**Proof.** Suppose \( x = \sum a_i \otimes \xi_i \) (\( a_i \in N, \xi_i \in V \)) represents 0 in \( N \otimes_{R_0} CV \). In the tensor product over \( C \),

\[
\sum a_i \otimes \xi_i = \sum (b_j v_j \otimes \eta_j - b_j \otimes v_j \eta_j)
\]

for \( b_j \in N, v_j \in R_0, \eta_j \in CV \). For each \( j \), there is a projection \( p_j \) such that \( \tau(p_j) \sim 0 \) and \( p_j^\perp \eta_j \in V \). Thus we get a representative of \( x \) given by

\[
\sum (b_j v_j \otimes p_j \eta_j - b_j \otimes v_j p_j \eta_j) + \sum (b_j v_j \otimes p_j^\perp \eta_j - b_j \otimes v_j p_j^\perp \eta_j)
\]

The second summand becomes 0 in \( N \otimes_{R_0} V \). Now, choose the smallest projection \( p \) in \( N \) such that \( pv_j p_j = v_j p_j, p_j \leq p \). Then \( x = (1 \otimes p)x \) and \( [x]_N \sim 0 \). Hence \( N \otimes V \to N \otimes CV \) is an isometry. When \( \xi_n \in V \) converges to \( \xi \in CV, a \otimes \xi_n \) converges to \( a \otimes \xi \) in \( [-]_N \). \( \square \)

**Remark 3.42.** For any \( R \)-module \( W, N \otimes_{R_0} W \to N \otimes R \) is an \( \dim_N \)-isomorphism.

**Theorem 3.43.** \( \dim_N \operatorname{Tor}_{R_0}^n (N, A) = \dim_N \operatorname{Tor}_{R}^n (N, A) \).

**Proof.** Consider projective resolutions of \( A \): \( P_* \to A \) as an \( R_0 \)-module, \( Q_* \to A \) as an \( R \)-module. We have morphisms \( \phi_*: P_* \to Q_* \) and \( \psi_*: Q_* \to CP_* \). Thus we get a commutative diagram

\[
\begin{array}{cccccc}
\cdots & \to & P_n & \to & \cdots & \to & P_0 & \to & A \\
\downarrow \phi_n & & \downarrow \phi_0 & & \parallel \\
\cdots & \to & Q_n & \to & \cdots & \to & Q_0 & \to & A \\
\downarrow \psi_n & & \downarrow \psi_0 & & \parallel \\
\cdots & \to & CP_n & \to & \cdots & \to & CP_0 & \to & CA \\
\downarrow C\phi_n & & \downarrow C\phi_0 & & \parallel \\
\cdots & \to & CQ_n & \to & \cdots & \to & CQ_0 & \to & CA.
\end{array}
\]
By the uniqueness of projective resolution up to homotopy, compositions of two homomorphisms $\psi_n \phi_n$ and $C\phi_n \psi_n$ are homotope to the the standard inclusion isomorphisms.

Now, the standard inclusion $P_* \to CP_*$ induces a dim$_N$-isomorphism after applying the $N \otimes_{R_0} -$ functor by Lemma [3.41]. Thus, $\text{Id}_N \otimes \phi_* :$ and $\text{Id}_N \otimes \psi_*$ are inverse to each other via the identification of $N \otimes P_* \simeq N \otimes CP_*$ and $N \otimes Q_* \simeq N \otimes CQ_*$. Hence $\text{Id}_N \otimes \phi_*$ induces a dimension isomorphism on cohomology groups. □

**Corollary 3.44.** Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be essentially free probability measure preserving actions. If $\mathcal{B}_{\Gamma \curvearrowright X} \simeq \mathcal{B}_{\Lambda \curvearrowright Y}$, $\beta^2_\Lambda(\Gamma) = \beta^2_\Lambda(\Lambda)$.

**Remark 3.45.** Put $\beta^2_\Lambda(A, N) = \text{dim}_N \text{Tor}^R_2(N, A)$. For any nonzero projection $p$ in $A$, $\beta^2_\Lambda(A, N) = \tau(p) \beta^2_\Lambda(pA, pNp)$.

## 4. Derivations on von Neumann algebras

In the following we only consider normal Hilbert (bi)modules over von Neumann algebras. Examples of such modules include the identity bimodule $L^2N$ and the coarse $(M, N)$-module $L^2M \otimes L^2N$.

Let $\Gamma$ be a countable discrete group, $(\pi, H_0)$ a unitary representation of $\Gamma$. A map $b : \Gamma \to H_0$ is said to be a derivation when it satisfies $b(st) = b(s) + \pi(s)b(t)$ i.e. a derivation for the $(\pi, \text{triv})$-bimodule structure. A derivation $b$ is said to be inner when there exists $\xi \in H_0$ such that $b(s) = \pi(s)\xi - \xi$. Put

$$H^1(\Gamma, \pi) = \{\text{derivations}\} / \{\text{inner derivations}\}.$$

When $b$ is a derivation, $\phi_r(s) = e^{-r\|b(s)\|^2}$ for $r \geq 0$ determines a positive semidefinite semigroup. Our goal is to show that it extends to a semigroup $\phi_r : L\Gamma \to L\Gamma$ of $\tau$ preserving completely positive maps.

### 4.1. Densely defined derivations.

Let $M$ denote $L^2\Gamma$. Consider $H = M \otimes H_0$. A left action $M \to B(H)$ is defined by $\lambda(f) \mapsto \lambda \otimes \pi(f)$ (this is possible by the Fell absorption.) On the other hand we have a right action $M^\circ \to B(H)$ is defined by $\rho(g) \mapsto \rho(g) \otimes \text{id}$. Put $\delta(s) = \delta_\pi \otimes b(s) \in C^*\Gamma \otimes H_0$. By

$$\delta(st) = \delta_s \otimes b(s) + \pi(s)b(t) = \rho(1t^{-1})\delta(s) + \lambda \otimes \pi(s)\delta(t),$$

$\delta$ extends to a (possibly unbounded) derivation $C\Gamma \to H$ satisfying $\delta(xy) = x\delta(y) + \delta(x)y$.

**Notation.** Let $(M, \tau)$ be a finite von Neumann algebra with a faithful normal tracial state, $\mathcal{D}$ a weak*-dense $*$-subalgebra of $M$. Let $H$ be a Hilbert bimodule over $M$, $\delta : M \to H$ a derivation defined on $\mathcal{D}$ which is closable as a densely defined operator $L^2M \to H$. Let $\delta$ denote its closure.

We are going to show that the domain of $\delta$ is a $*$-subalgebra of $L(H)$ and that $\delta$ is a derivation.

**Notation.** Let $\|\cdot\|_{\text{Lip}}$ denote the 1-Lipschitz norm:

$$\|f\|_{\text{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$  

Let $\text{Lip}_0$ denote the space of 1-Lipschitz continuous functions which map 0 to 0.
For any $x \in L^2Msa$, regarded as a self-adjoint unbounded operator on $L^2M$, we can consider its functional calculus $f(x)$.

**Proposition 4.1.** When $x, y \in L^2Msa$ and $f \in \text{Lip}_0$, the functional calculus $f(x), f(y)$ is in $L^2M$ and

$$\|f(x) - f(y)\|_2 \leq \|f\|_{\text{Lip}} \|x - y\|_2.$$

**Proof.** For the spectral measure $E(t)$ of $x$, $x = \int t dE(t)$ and $\|x\|_2^2 = \int |t|^2 dE(T)$. Thus $\|f(x)\|_2^2 = \int |f(t)|^2 dE(t) \leq \|f\|_{\text{Lip}} \int |t|^2 dE(t)$ and $f(x)$ is in $L^2M$. For the second assertion, consider the bilinear map

$$C_0(\mathbb{R})^2 \ni (f, g) \mapsto \tau(f(x)g(x)) = \langle f(x) \hat{1}, f \hat{1} \rangle.$$

This defines a linear form $C_0(\mathbb{R} \times \mathbb{R}) \to \mathbb{C}$, i.e. $\tau(f(x)g(y)) = \int fg d\mu$ for some measure $\mu$ on $\mathbb{R} \times \mathbb{R}$. Thus, $\tau(\|f(x) - f(y)\|^2)$ is equal to

$$\int |f(s) - f(t)|^2 d\mu(s, t) \leq \|f\|^2_{\text{Lip}} \int |s - t|^2 d\mu(s, t) = \|f\|^2_{\text{Lip}} \|x - y\|^2_2. \quad \Box$$

**Definition 4.2.** Let $I$ be a bounded closed interval in $\mathbb{R}$, $f \in C^1(I)$. The function

$$\tilde{f}(x, y) = \begin{cases} \frac{f(x) - f(y)}{x - y} & (x \neq y) \\ f'(x) & (x = y) \end{cases}$$

is called the difference quotient of $f$.

Note that $\frac{\|\tilde{f}\|_{\infty}}{\|f\|_{\infty}} = \frac{\|f'\|_{\infty}}{\|f\|_{\infty}}$. When $a \in Msa$ and $[-\|a\|, \|a\|] \subset I$, we have $\pi_a : C(I \times I) \to B(H)$ by $\pi_a(f \otimes g)\xi = f(a)\xi g(a)$.

**Lemma 4.3.** For any $a \in \mathbb{D}$ and $f \in C^1(I)$, the operator $f(a)$ is in $\text{dom} \tilde{\delta}$ and $\tilde{\delta}(f(a)) = \pi_a(\tilde{f})\tilde{\delta}(a)$.

**Proof.** The assertion is obvious for polynomial functions. The equality for the general $C^1$-functions follows from it because it is compatible with the $C^1$-norm. \hfill $\Box$

**Remark 4.4.** When $T$ is a closed operator on $H$, $x_n \to x$ ($n \to \infty$) in $H$ and $\sup_n \|Tx_n\| < \infty$ imply that $x \in \text{dom} T$ and that $Tx \in \bigcap_{m=0}^{\infty} \text{conv} \{Tx_n : n \geq m\}$, where $\text{conv}$ denotes the closed convex span. This is because, taking a suitable subsequence if necessary, we may assume that the bounded sequence $Tx_n$ is weakly convergent to some $y$. Taking the convex closure, we can find a sequence $(z_n)_{n \in \mathbb{N}}$ such that $Tz_n \to y$ in norm and that $z_n$ is in the algebraic convex closure of $\{x_k : k \geq n\}$. By construction, $(z_n)_{n \in \mathbb{N}}$ converges to $x$.

**Lemma 4.5.** Let $x$ be an unbounded self-adjoint operator on $L^2M$ which is in $\text{dom} \tilde{\delta}$, $f \in \text{Lip}_0$. Then $f(x) \in \text{dom} \tilde{\delta}$ and $\|\tilde{\delta}(f(x))\| \leq \|f\|_{\text{Lip}} \|\tilde{\delta}(x)\|$. 

Proof. Choose a mollifier \((\phi_n)\) and set \(f_n = f \ast \phi_n\). Thus \(f_n\) is of \(C^1\) class and \(f_n \to f\) uniformly on \(I\). By
\[
|f_n(y) - f_n(z)| = \int |f(y - r) - f(z - r)|\phi_n(r)dr \leq \|f\|_{\text{Lip}} |y - z|,
\]
we have \(\|f_n\|_{\text{Lip}} \leq \|f\|_{\text{Lip}}\). Now take a sequence \((a_n)_{n \in \mathbb{N}}\) in \(\mathcal{D}_{sa}\) which is convergent to \(x\) in \(\|-\|_2\)-norm. Then
\[
\|\delta(f_n(a))\| = \left\|\pi_n(f_n)\delta(a)\right\| \leq \|f_n\|_{\text{Lip}} \|\delta a\|.
\]
This shows \(f(x) \in \text{dom } \delta\).

\textbf{Definition 4.6.} A derivation \(\delta: M \to H\) is said to be real when we have
\[
\langle \delta(x), \delta(y)z \rangle = \langle z^*\delta(y^*), x^* \rangle
\]
for any \(x, y, z \in M\).

\textbf{Remark 4.7.} We summarize a few properties of real derivations.

- When \(M\) is the group von Neumann algebra \(L\Gamma\) of a group \(\Gamma\), the above condition is equivalent to \(\langle \delta(s), \delta(t) \rangle \in \mathbb{R}\).
- In general, when we have a \(J\)-operator, \(\delta\) is real if and only if \(Jx\delta(y)z = z^*\delta(y^*)x^*\), since, by definition, \(\langle \delta(x), \delta(y)z \rangle\) is equal to \(\langle z^*J\delta(y), J\delta(x) \rangle\).

- When \(\delta\) is real, \(\text{dom } \delta\) is self adjoint.

Let \(\overline{\mathcal{D}}\) denote \(\text{dom } \delta\).

\textbf{Lemma 4.8.} Let \(\delta\) be a real derivation. When \(x \in \overline{\mathcal{D}}\), \(|x|\) is also in \(\overline{\mathcal{D}}\) and \(M \cap \overline{\mathcal{D}}\) is a \(*\)-subalgebra of \(M\).

Proof. Consider the linear map \(\delta^{(2)}: M_2 \overline{\mathcal{D}} \to M_2H \simeq H^{\oplus 4}\). Then \(\delta^{(2)} = \overline{\delta^{(2)}}\) and for any \(z \in \overline{\mathcal{D}}\),
\[
w = \begin{bmatrix} 0 & z^* \\ z & 0 \end{bmatrix} \in \text{dom } \overline{\delta^{(2)}} \Rightarrow w^2 = \begin{bmatrix} |z|^2 & 0 \\ 0 & |z^*|^2 \end{bmatrix} \in \text{dom } \overline{\delta^{(2)}}.
\]
Thus \(|z|^2\) is in \(\mathcal{D}\).

Let \(x, y \in \overline{\mathcal{D}}\). The polarization
\[
x^*y = \frac{1}{4} \sum i^k|x + iy|
\]
shows \(x^*y \in \overline{\mathcal{D}}\), and in particular \(x^* \in \overline{\mathcal{D}}\) follows from \(1 \in \mathcal{D}\). \(\square\)

\textbf{Lemma 4.9.} For any \(x \in \overline{\mathcal{D}} \cap M_{sa}\), there exists a sequence \((x_n)_{n \in \mathbb{N}}\) in \(\mathcal{D}_{sa}\) such that
\[
\|x_n - x\|_2 \to 0, \|\delta(x_n) - \delta(x)\| \to 0\text{ and }\|x\|_{\infty} \leq \|x\|_{\infty}.
\]
In particular, \(x_n \xrightarrow{\text{u.s.t.}} x\) in the ultrastrong topology.

Proof. The only nontrivial part is the last inequality. This is achieved by the functional calculus with respect to the function
\[
f(t) = \begin{cases} \|x\|_{\infty} & (\|x\|_{\infty} < t) \\ t & (|t| \leq \|x\|_{\infty}) \\ -\|x\|_{\infty} & (t < -\|x\|_{\infty}). \end{cases}
\]
\(\square\)
Theorem 4.10. The restriction of $\tilde{\delta}$ to $\mathfrak{D} \cap M$ is a derivation.

Proof. Let $x \in \mathfrak{D} \cap M$. Choose a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathfrak{D}$ weakly convergent to $x$ and $\delta(x_n) \to \delta(x)$. For each $y \in \mathfrak{D} \cap M$, we have $x_n y \to x y$ in the $\| - \|_2$-norm. Since $y$ is bounded, we have $\delta(x_n) y \to \delta y$. On the other hand, the representation of $M$ on $H$ is normal, which implies $x_n \delta(y) \to x \delta(y)$. Thus we have $\delta(xy) = x \delta(y) + \delta(x)y$. Similar approximation in $y$ shows that $\delta(xy) = x \delta(y) + \delta(x)y$. For any $y \in \mathfrak{D} \cap M$. □

4.2. Semigroup associated to a derivation. In the following we assume $M \cap \mathfrak{D} = \emptyset$. Put $\Delta = \delta^* \delta$. This is a positive self adjoint operator on $L^2 M$ satisfying $\Delta \tilde{1} = \tilde{1}$ and commutes with the $J$ operator so that we have $\{ \Delta(x^*) = (\Delta x)^* \}$. Put $\phi_t = e^{-t\Delta}$. This is a semigroup of positive contractions satisfying $\phi_t \tilde{1} = \tilde{1}$ and $\phi_t \not\sim \text{Id}$ as $t \searrow 0$. The normalized resolvents

$$\eta_\alpha = \frac{\alpha}{\alpha + \Delta}$$

for $\alpha > 0$ are again positive contractions on $L^2 M$ satisfying $\eta_\alpha \not\sim \text{Id}$ as $\alpha \searrow \infty$. These operators are related to each other as follows:

$$\begin{align*}
\Delta \quad & \xrightarrow{\text{exponential}} \quad \phi_t & \xrightarrow{\text{Laplace trans.}} \quad \eta_\alpha \\
& \xrightarrow{\text{derivation}} & \xrightarrow{\text{inverse}} & 
\end{align*}$$

where the Laplace transform is given by

$$\eta_\alpha = \alpha \int_0^\infty e^{-\alpha t} \phi_t dt = \int_0^\infty e^{-t\phi_\Delta} dt.$$ 

Recall that any unital completely positive map $\phi: M \to M$ is expressed as $V^* \pi(x)V$ for some representation $\pi: M \to B(K)$ and an isometry $V: L^2 M \to K$ (Steinespring’s theorem). When $\phi$ is normal, $\pi$ can be taken as a normal representation (we may take the normal part of a possibly non-normal $\pi$ given by Steinespring’s theorem). Thus,

(1) For any $x \in M$, $\phi(x^*x) - \phi(x^*)\phi(x) = V^*\pi x^*(1 - VV^*)\pi x V \geq 0$. When $\phi$ preserves $\tau$, $\| \phi(x) \|_2 \leq \| x \|_2$.

(2) When $\phi$ preserves $\tau$, $\| \phi(x^*y) - \phi(x^*)\phi(y) \|_2 = \| V^*\pi x(1 - VV^*)\pi y V \tilde{1} \|$ is bounded from above by

$$\| \phi(x^*x) - \phi x^*\phi x \|_\infty^2 (\tau(\phi(y^*y) - \phi y^*\phi y))^{1/2} \leq 2 \| x \|_\infty \| y - \phi(y) \|_2$$

by $\tau(\phi(y^*y) - \phi y^*\phi y) = \| y \|_2^2 - \| \phi y \|_2^2$, etc.

Fact. Consider the 1-norm $\| x \|_1 = \sup \{ \| \tau(xy) \| : \| y \|_\infty \leq 1 \}$ for $x \in M$. $x \in L^2 M$ is in $M$ if and only if $\sup \{ \| \tau(xy) \| : \| y \|_1 \leq 1, xy \in M \}$ is finite.

Theorem 4.11. (Sauvageot, [11, ?]) The contractions $\phi_t$ and $\eta_\alpha$ map $M$ into $M$, are unital completely positive and $\tau$-symmetric, i. e. $\tau(\phi_t(x)y) = \tau(x\phi_t(y))$ etc.

Proof. Observe that $\phi^{(n)}_t = e^{-t\Delta^{(n)}}$ where $\Delta^{(n)} = \delta^{(n)}*\tilde{\delta}^{(n)}$ for $\delta^{(n)}: M_n \mathfrak{D} \to M_n H$. Thus, it is enough to show that the maps are positive to conclude that they are actually completely positive. Put

$$\Delta_\alpha = \frac{\alpha \Delta}{\alpha + \Delta} = \alpha(1 - \eta_\alpha).$$
Then
\[ \phi_t = e^{-t\Delta}e = \lim_{\alpha \to \infty} e^{-t\Delta_\alpha} = \lim_{\alpha \to \infty} e^{-t\alpha} \sum_{n=0}^{\infty} \frac{t^n\eta_\alpha}{n!} \]
where the limit is taken in the strong operator topology (note: this might be the norm topology, as we are using \(c_0\) functions converging from below). The last expression is compatible with the \(\|y\|_1 \leq 1\) functional. Thus it reduces to show that \(\eta_\alpha\) restricts to a positive map on \(M\).

By scaling \(\delta\), we may assume that \(\alpha = 1\). Let \(x \in M_+\) and put \(y = (1 + \Delta)^{-1}x \in \text{dom } \Delta\). We have
\[ \|\delta y\|^2 + \|y\|^2 = \langle y, \Delta y \rangle + \langle y, y \rangle = \langle y, x \rangle \]
Then the function \(\Phi(z) = \|\delta(z)\|^2 + \|z - x\|^2\) for \(z \in \mathcal{F}_{sa}\) satisfies
\[
\|\delta(z - y)\|^2 + \|z - y\|^2 = \|\delta(z)\|^2 - 2\langle z, \Delta y \rangle + \|\delta(y)\|^2 + \|z\|^2 - 2\langle z, y \rangle + \|y\|^2
\]
\[
= \|\delta(z)\|^2 + \|z\|^2 - 2\langle z, x \rangle + \|x\|^2
\]
\[
- (\|\delta(y)\|^2 + \|y\|^2 - 2\langle y, x \rangle + \|x\|^2)
\]
\[
= \Psi(z) - \Psi(y).
\]
Consider a function
\[ f(t) = \begin{cases} \|x\|_\infty & (\|x\|_\infty < t) \\ t & (0 \leq t \leq \|x\|_\infty) \\ 0 & (t < 0) \end{cases} \]
of \(\text{Lip}_0\) class with \(\|f\|_{\text{Lip}} = 1\). Then
\[ \Psi(f(z)) = \|\delta(f(z))\|^2 + \|f(z) - f(x)\| \leq \Psi(z). \]
Take a sequence \((z_n)_{n \in \mathbb{N}}\) in \(\mathcal{F}_{sa}\) with \(\|z_n - y\|_2 \to 0\) and \(\|\delta z_n - \delta y\|_2 \to 0\). Then we have
\[ \|f z_n - y\|_2^2 \leq \Psi(f z_n) - \Psi(y) \leq \Psi(z_n) - \Psi(y) \to 0. \]
Thus \(y = \lim f z_n\) and \(0 \leq y \leq \|x\|_\infty\) and \(\eta_1\) is shown to be unital positive. \(\square\)

Let \(B\) be a von Neumann subalgebra of \(M\). Then we are interested in “when \(\phi_t\) converges uniformly on \(B_1\)?” Roughly, this means “\(\delta\) is inner on \(B\).”

**Lemma 4.12.** Let \(\Omega \subset M_1\). Then \(\phi_t \to \text{id}\) uniformly on \(\Omega\) as \(t \to 0\) if and only if \(\eta_\alpha \to \text{id}\) uniformly on \(\Omega\) as \(\alpha \to \infty\).

**Proof.** \(\Rightarrow\): We have
\[ \|x - \eta_\alpha x\|_2 \leq \int_0^\infty e^{-s} \|x - \phi_z(x)\|_2 \, ds, \]
but \(\|x - \phi_z(x)\|_2\) does not exceed 2.

\(\Leftarrow\): Suppose \(\phi_s\) did not converge uniformly on \(\Omega\). Then there is a constant \(c\) such that for any \(t\) there exists an element \(x_t\) of \(\Omega\) satisfying \(\langle x_t - \phi_t x_t, x_t \rangle \geq c\)
Lemma 4.13. For the latter convenience we record the following equalities:

1. In $B(L^2M)$,
   \[
   \eta^\frac{1}{2} = \frac{1}{\pi} \int_0^\infty \frac{t^{-\frac{1}{2}}}{1 + \xi} \eta_{\alpha(t+1)} dt.
   \]

2. In $B(L^2M)$,
   \[
   (\text{Id} - \eta_0)^{\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \frac{t^{-\frac{1}{2}}}{1 + \xi} (1 - \eta_{\alpha(t+1)}) dt = \text{Id} - \theta_0
   \]
   where $\theta_0$ restricts to a unital completely positive map on $M$.

3. $\psi_t = e^{-t\Delta^\frac{1}{2}}$ is $\tau$-symmetric and unital completely positive on $M$.

Proof. (1): we have
   \[
   s^{\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \frac{s^{\frac{1}{2}}}{s + t} t^{-\frac{1}{2}} dt \Rightarrow \eta^\frac{1}{2} = \frac{1}{\pi} \int_0^\infty \frac{\eta_0}{t + \eta_0} t^{-\frac{1}{2}} dt.
   \]

On the other hand,
   \[
   \frac{\eta_0}{t + \eta_0} = \frac{\alpha}{\alpha(1 + t) + t\Delta} = \frac{1}{1 + t} \eta_{\alpha(1 + t)}.
   \]

(3): We have $\Delta^\frac{1}{2} = \alpha^\frac{1}{2} (\text{Id} - \eta_0)^{\frac{1}{2}} = \alpha^\frac{1}{2} (\text{Id} - \theta_0)$. Thus $\psi_t$ can be written as
   \[
   \lim_{\alpha \to \infty} e^{-t\Delta^\frac{1}{2}} = \lim_{\alpha \to \infty} e^{-\alpha^\frac{1}{2} t \theta_0}. \]

Lemma 4.14. For $x, y \in \mathcal{D}$, put $\Gamma(x^*, y) = \Delta^\frac{1}{2}(x^*)y + x^*\Delta^\frac{1}{2}(y) - \delta^\frac{1}{2}(x^*y)$. Then we have
   \[
   \|\Gamma(x^*, y)\|_2 \leq 4 \|\delta(x)\| \|x\|_\infty \|\delta(y)\| \|y\|_\infty .
   \]

Proof. First we have
   \[
   \Gamma(x^*, y) = \frac{d}{dt}(\psi_t(x^*y) - \psi_t(x^*)\psi_t(y))\big|_{t=0}.
   \]

Note that $\|\psi_t x\| \leq \|x\|$. Define a sesquilinear form on $\mathcal{D} \otimes M$ by $\langle y \otimes b, x \otimes a \rangle = \tau(a^*\Gamma(x^*, y)b)$. This is positive semidefinite by
   \[
   \langle \sum x_i \otimes a_i, \sum x_i \otimes a_i \rangle = \lim_{t \to 0} \tau(\sum a_i \psi_t(x_i^*x_j) - \psi_t x_i^*\psi_t(x_j) a_j) \leq 0.
   \]

For $z = v|z| \in M$, we have
   \[
   \tau(\Gamma(x^*, y)z) = |\langle y \otimes v|z|^\frac{1}{2}, x \otimes |z|^\frac{1}{2} \rangle| \leq |\langle y \otimes v|z|^\frac{1}{2}, y \otimes v|z|^\frac{1}{2} \rangle|^\frac{1}{2} \langle x \otimes |z|^\frac{1}{2}, x \otimes |z|^\frac{1}{2} \rangle^\frac{1}{2}.
   \]
Here, $\langle x \otimes a, x \otimes a \rangle \leq \|aa^*\|_2 \|\Gamma(x^*, x)\|_2$ and
\[
\|\Gamma(x^*, x)\| \leq \left\| \Delta_{\frac{1}{2}}^\perp x^* \right\|_2 \|x\|_\infty + \|x^*\|_\infty \left\| \Delta_{\frac{1}{2}}^\perp x \right\|_2 + \left\| \Delta_{\frac{1}{2}}(x^*) \right\|_2 \\
\leq 4 \|\delta(x)\| \|x\|_\infty.
\]
(Here we used the fact that $\left\| \Delta_{\frac{1}{2}}(x^*) \right\|_2 = \|\delta(x^*)x + x^*\delta(x)\|.$) Hence we arrive at
\[
|\tau(\Gamma(x^*, y)z)|^2 \leq \|\Gamma(x^*, x)\|_2 \|z\|_2 \|(y^*, y)\| \|z\|_2,
\]
thus $\|\Gamma(x^*, y)\|_2^2 \leq \|\Gamma(x^*, x)\|_2 \|(y^*, y)\|.$

Put $\zeta_\alpha = \delta_{\alpha}^\perp$. $\Delta_{\frac{1}{2}} \zeta_\alpha = \Delta_{\alpha}^\perp (\text{Id} - \eta_\alpha)^{\frac{1}{2}}$ (hence bounded) and $\left\| \Delta_{\alpha}^\perp x \right\|_2 = \alpha \langle (\text{Id} - \eta_\alpha)x, x \rangle$. Put $\tilde{\delta}_\alpha = \alpha^{\frac{1}{2}} \delta_{\alpha}^\perp$. Thus $\left\| \tilde{\delta}_\alpha(x) \right\| = \langle (\text{Id} - \eta_\alpha)x, x \rangle$ and $\|\delta\|_\alpha (x) \rightarrow 0$ if and only if $\|x - \eta_\alpha x\|_2 \rightarrow 0$.

**Theorem 4.15.** (Peterson?) Let $\Omega \subset M_1$ and suppose $\eta_\alpha \rightarrow \text{Id}$ uniformly on $\Omega$. Then we have $\left\| \tilde{\delta}_\alpha(ax) - \zeta_\alpha(a)\tilde{\delta}_\alpha(x) \right\| \rightarrow 0$ ($\alpha \rightarrow \infty$) uniformly for $a \in \Omega$ and $x \in M_1$.

**Proof.** By assumption $\zeta_\alpha$ and $\theta_\alpha$ converge uniformly to $\text{Id}$ on $\Omega$, by e.g. $\theta_\alpha = \frac{1}{\pi} \int_0^\infty \frac{t^{\frac{1}{2}}}{1 + t} \eta_\alpha^\perp \, dt$.

In particular, $\theta_\alpha(ax) \approx \theta_\alpha(a(\theta_\alpha)(x)) \approx a\theta_\alpha(x)$ where $\approx$ means the 2-norm convergence under $\alpha \rightarrow \infty$. Now,
\[
\alpha^{\frac{1}{2}} \Delta_{\frac{1}{2}} \zeta_\alpha(ax) = \alpha^{\frac{1}{2}} (\text{Id} - \theta_\alpha)(ax) \approx \alpha^{\frac{1}{2}} a(\text{Id} - \theta_\alpha)(x) \approx \alpha^{\frac{1}{2}} \zeta_\alpha(a)(\text{Id} - \theta_\alpha)(x)
\]
\[
= \alpha^{\frac{1}{2}} \zeta_\alpha(a) \Delta_{\frac{1}{2}} \zeta_\alpha(x) \approx \alpha^{\frac{1}{2}} \Delta_{\frac{1}{2}}(\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x)
\]
where the last approximation is given by applying Lemma 4.14 to get the error estimate
\[
4 \left( \alpha^{\frac{1}{2}} \left\| \delta_{\alpha}^\perp(\zeta_\alpha(a)) \right\| \right) \left\| \alpha^{\frac{1}{2}} \delta_{\alpha}^\perp x \right\|.
\]
Here, $\alpha^{\frac{1}{2}} \left\| \delta_{\alpha}^\perp(\zeta_\alpha(a)) \right\| \rightarrow 0$ and $\left\| \alpha^{\frac{1}{2}} \delta_{\alpha}^\perp x \right\|$ is bounded by 1.

Finally we arrive at
\[
\tilde{\delta}_\alpha(ax) \approx \alpha^{\frac{1}{2}} (\delta_{\alpha}(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x) = \zeta_\alpha(a)\tilde{\delta}_\alpha(x).
\]

**Theorem 4.16.** (Haagerup) Let $M$ be a von Neumann algebra. $M$ is finite injective if and only if for any nonzero central projection $p$ of $M$, there exist $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in \mathcal{U}(pM)$ such that $\|\sum_{i=1}^n u_i \otimes u_i\|_\infty = n$.

**Proof.** (Outline) $\Rightarrow$: By Connes’ theorem, $M \otimes_{\min} \hat{M} \rightarrow B(L^2M)$ can be defined by $(a \otimes b)x = axb^*$. Now, $\left(\sum_{i=1}^n u_i \otimes u_i\right)1 = n1$ when $u_i \in \mathcal{U}M$.

$\Leftarrow$: The minimal tensor product $M \otimes_{\min} \hat{M}$ acts on $H \otimes H$ i.e. the Hilbert-Schmidt space of $H$. For any finite set $F \subset \mathcal{U}M$ containing 1 and $\left\| \sum_{a \in F} u \otimes a \right\| = \left| F \right|$, there exists $T \in HS(H)$ of 2-norm 1, $\left\| \sum_{a \in F} u Tu^* \right\| \approx \left| F \right|$. Then $uTu^* \approx T$. Now, define $\phi_F(x) = \text{Tr}(T^*xT)$. Then $\phi_F(uxu^*) = \phi_x(a) u \phi(x)$ for $u \in F$. We obtain
an ultrafilter convergence $\phi_F \to \phi \in S(B(H))$ such that $\phi(uxu^*) = \phi(x)$ for any $u \in UM$. This holds under any central projection, which means $M$ is injective. □

Recall that we are investigating closable real derivations on $M$. Thus, $H$ is an $M$-bimodule with a $J$-operator: $J(a\delta(x)b) = b^*\delta(x^*)a^*$. We have the operators

$$
\eta_\alpha = \frac{\alpha}{\alpha + \delta^*}, \zeta_\alpha = \eta_\alpha^{\frac{1}{2}}, \bar{\delta}_\alpha = \alpha^{-\frac{1}{2}}\delta \zeta_\alpha : M \to H.
$$

As $\alpha \to \infty$, we have $\left\|\bar{\delta}_\alpha(a)\right\|^2 = \left\|(Id - \eta_\alpha)\right\|^2 = \tau((a - \eta_\alpha a)a^*) \leq 0$.

**Theorem 4.17.** Let $(M; r)$ be a finite von Neumann algebra, $H = (L^2 M \otimes L^2 M)^{\otimes \mathbb{N}}$. Suppose $Q \subset M$ is a von Neumann subalgebra without injective summand. Then $\phi_t \to Id$ uniformly on $(Q' \cap M)_1$.

**Proof.** It is enough to show that for any nonzero central projection $p \in Q$, there exists a central projection $q \leq p$ in $Q$ such that $\phi_t \to Id$ on $q(Q' \cap M)_1$. In fact, then by the maximal argument we would get a family $(p_i)_{i \in I}$ of nonzero central projections such that $\sum_{i \in I} p_i = 1$ and $\phi_t \to Id$ on $p_i(Q' \cap M)_1$ for each $i$. Taking a finite subset $I_0 \subset I$ such that $\tau(\sum_{i \in I_0} p_i) < \frac{\varepsilon}{3}$, we find $t_0$ such that $t > t_0$ implies $\left\|\phi_t(a) - a\right\|_2 < \frac{\varepsilon}{3}$ for any $a \in p_{I_0}(Q' \cap M)_1$. On the other hand, for any $a \in p_{I_0}(Q' \cap M)_1$ we have $\tau(a - p_{I_0}a) < \frac{\varepsilon}{3}$.

Thus we are going to prove the negation of the above claim leads to that $pQ$ is injective. Let $q \leq p$ be a nonzero central projection in $Q$, $u_1, \ldots, u_n \in \mathcal{U}(qQ)$. As $\phi_t$ does not converge uniformly on $q(Q' \cap M)_1$, there exists $x_\alpha \in q(Q' \cap M)_1$ for any $\alpha$ such that $\liminf \left\|\delta_\alpha(x_\alpha)\right\| > 0$.

Applying Theorem 4.15 to the finite subset $\Omega = \{u_1, \ldots, u_n\}$ on which $\phi_t$ is uniformly convergent, for any $x \in q(Q' \cap M)_1$, as $\alpha \to \infty$,

$$
\sum_i \zeta_\alpha(u_i)\bar{\delta}_\alpha(x)\zeta_\alpha(u_i^*) \approx \sum_i \delta_\alpha(u_i x u_i^*) = n\delta_\alpha(x).
$$

Thus, $\left\|\sum_i \zeta_\alpha(u_i) \otimes \zeta_\alpha(u_i^*)\right\|_{\min} \to n$ as $\alpha \to \infty$. On the other hand, since $\zeta_\alpha$ is a normal unital completely positive map, $\left\|\sum_i \zeta_\alpha(u_i) \otimes \zeta_\alpha(u_i^*)\right\|_{\min}$ is always bounded by $\left\|\sum_i u_i \otimes \bar{u}_i\right\|$, which shows that $\left\|\sum_i u_i \otimes \bar{u}_i\right\| = n$. Thus we have the injectivity of $pQ$ by Theorem 4.16. □

**Remark 4.18.** If a 1-cocycle $b: \mathbb{F}_r \to \ell_2 \mathbb{F}_r^{\otimes n}$ satisfies $\|b(s)\|_2^2 = |s|$, we obtain a derivation $\delta$ on $\ell_2 \mathbb{F}_r \otimes \ell_2 \mathbb{F}_r^{\otimes n}$ given by $\delta(s) = \delta_{\Delta} \otimes b$ where $\delta_{\Delta}$ is the “diagonal” operator on $\ell_2 \mathbb{F}_r$ which multiplies the standard base $\delta_s$ by $|s|$. The semigroup $\phi_t$ associated to this derivation is written as $\phi_t(\lambda(s)) = e^{-t|s|}\lambda(s)$, thus it is in $\mathcal{K}(L^2 M)$.

When $B$ is a von Neumann subalgebra of $L\mathbb{F}_r$, $\phi_t \to Id$ uniformly on $B_1$ if and only if $B$ is a direct sum $\oplus M_{n_i}$ of finite dimensional algebras.

**Corollary 4.19.** Let $Q$ be a von Neumann subalgebra of $L\mathbb{F}_r$ without injective summand. Then the relative commutant $Q' \cap L\mathbb{F}_r$ is completely atomic. In particular, $Q \otimes L^\infty[0, 1] \nsubseteq L\mathbb{F}_r$. 
Theorem 4.20. Let \( (M; \tau) \) be a finite von Neumann algebra, \( H = (L^2 M \otimes L^2 M) \), \( \delta \) a closable real derivation. If \( B \subset M \) is diffuse (i.e. without minimal projection) von Neumann subalgebra such that \( \phi_t \) converges to \( \text{Id} \) uniformly on \( B_t \), one has \( \phi_t \rightarrow \text{Id} \) uniformly on \( N(B)_1 \).

Proof. Since \( B \) is diffuse, there exists a sequence \((v_n)_{n \in \mathbb{N}}\) in \( UB \) ultraweakly convergent to \( 0 \) (e.g. \( e^{2\pi i \text{int}} \in L^\infty [0, 1] \) for \( n \in \mathbb{N} \)). For any \( u \in N(B) \),

\[
\|\delta_\alpha(u)\| \leq \liminf \|\delta_\alpha(u) - \zeta_\alpha(v_n)\delta_\alpha(u)\|_{\text{c}} - \|\delta_\alpha(u) - \delta_\alpha(v_n u u^* v_n^*)\| = 0 \quad (n \rightarrow \infty)
\]

The convergence holds uniformly for \( u \). It remains to apply the following lemma to \( N(B) = G \).

Lemma 4.21. When \( \phi_t \to \text{Id} \) uniformly on \( G \subset UM \), we have the uniform convergence \( \phi_t \to \text{Id} \) on \( G''_1 \).

Proof of the lemma. Let \( \phi: M \to M \) be a \( \tau \)-symmetric unital completely positive map (hence a contraction). Consider the Stinespring construction on \( M \otimes_{alg} L^2 M \) by \( (a \otimes x, b \otimes y) = (\phi(b^* a) x, y) \). This is positive semi definite by the unital completely positivity. The \( M \)-action \( a.(c \otimes x), b = ac \otimes xb \) is bounded and induces an \( M \)-bimodule structure on the completion.

Now, for \( \xi_0 = 1 \otimes 1 \in M \otimes L^2 M \),

\[
\|a\xi_0 - \xi_0 a\|^2 = \tau(\phi(aa^*)) + \tau(aa^*) - 2\Re(\tau(\phi(aa^*))) = 2(\tau((a - \phi(a))a^*)).
\]

On the other hand,

\[
\frac{1}{2}\|a - \phi(a)\|^2 \leq \|a\xi_0 - \xi_0 a\| \leq 2\|a - \phi(a)\| \|a\|_2.
\]

Thus, if \( \|u - \phi(u)\| \leq \epsilon \), we have \( \|\xi_0 - u\xi_0 u^*\| \leq \sqrt{2}\epsilon \). By taking the circumcenter of \( \{u\xi_0 u^* : u \in G\} \), we get a \( G \)-invariant vector \( \eta_0 \) satisfying \( \|\xi_0 - \eta_0\| \leq \sqrt{2}\epsilon \) (this is possible by the Ryll-Nardzewski’s fixed point theorem). Thus we obtain \( \|a\xi_0 - \xi_0 a\| \leq 2\sqrt{2}\epsilon \) for \( a \in (G'')_1 \).

Appendix A. Embeddability of subalgebras

Let \( A \subset M \) be an inclusion of finite von Neumann algebras with a trace \( \tau \) on \( M \). Recall that we have the associated Jones projection \( e_A \in B(L^2 M) \), the orthogonal projection onto \( L^2 A = A1 \) and the basic extension \( (M, A) \) of \( M \):

\[
\langle M, A \rangle = \text{vN} \{M, e_A\} = \left\{ \sum_{i \in \text{finite}} x_i e_A y_i : x_i, y_i \in M \right\}
\]

and the semifinite trace \( \text{Tr}(\sum x_i e_A y_i) = \sum \tau(x_i y_i) \) on \( (M, A) \).

Theorem A.1. (Popa) Let \( A \subset M \) be an inclusion of separable finite von Neumann algebras, \( p \) a nonzero projection in \( M \), \( B \subset pM p \) a von Neumann subalgebra. The following are equivalent:

1. There are no sequence \( (w_n)_n \) in \( UB \) such that \( \|E_A(y^* w_n x)\| \to 0 \) for any \( x, y \in M \).
(2) There exists a nonzero positive element $d \in \langle M, A \rangle$ of finite trace such that $0 \notin \overline{\text{conv}}w \{wdw^* : w \in UB \}$.

(3) There exists a closed nonzero $B$-A submodule $H$ of $pL^2M$ such that dim$_A H_A$ is finite.

(4) There exists a projection $e$ in $A$, another $0 \neq f$ in $B$ and a normal $*$-homomorphism $\theta : fBf \rightarrow eAe$ such that there exists a nonzero partial isometry $v \in M$ satisfying $xv = v\theta(x)$ for any $x \in fBf$, and $ve^* \in (fBf)' \cap FMf$, $v^*v \in \theta(fBf)' \cap eMe$.

Proof. (1) $\Rightarrow$ (2): By assumption there exits a finite set $\mathcal{F} \subset M$ and $\epsilon > 0$ such that
\[
\inf_{w \in UB} \sum_{x,y \in \mathcal{F}} \|E_A(w^*wx)^2\|_2^2 \geq \epsilon.
\]
Now, put $d = \sum_{y \in \mathcal{F}} ye\hat{A}y^* \in \langle M, A \rangle_+$. By definition $\text{Tr}(d) < \infty$ and we have
\[
\sum_{x \in \mathcal{F}} \langle w^*dw\hat{x}, \hat{x} \rangle = \sum_{x,y \in \mathcal{F}} \langle e\hat{A}w^*wx, y^*wx \rangle = \sum_{x,y \in \mathcal{F}} \|E_A(w^*wx)^2\|_2^2 \geq \epsilon
\]
for any $w \in UB$.

(2) $\Rightarrow$ (3): Let $\mathcal{C}$ denote the closed convex hull of $\{wdw^* : w \in UB\}$ in $L^2(M, A)$. We can take the circumcenter $d_0$ of $\mathcal{C}$ which is not equal to zero by (2). Then $d_0$ is in $B' \cap p(M, A)p$ and $\text{Tr}(d_0) \leq \text{Tr}(d) < \infty$. Thus we can take a nonzero spectral projection $q$ of $d_0$ such that $\text{Tr}(q) < \infty$. Now, $H = qL^2M$ is a $B$-A submodule with dim$_A H_A = \text{Tr}(q)$.

(3) $\Rightarrow$ (4): Fact. When $H$ is a $B$-A module with dim$_A H_A < \infty$, there exists a nonzero projection $f$ of $B$, an $fBf$-A module $K \subset fH$ such that $K \hookrightarrow L^2A_A$ as a right $A$-module.

Thus, let $V$ denote such an injection $K_A \rightarrow L^2A_A$. When $x \in fBf$, $VxV^* \in \text{End}_A(L^2A_A) = A$. Thus $\theta(x) = VxV^*$ defines a normal $*$-homomorphism (since $V$ is injective) $\theta$ of $fBf$ into $eAe$ for $e = VV^*$. Put $\xi = V^*1 \in K$. Since $V\xi = VV^*1 = \hat{e}$, $\xi \neq 0$. On the other hand, for any $x \in fBf$,
\[
x\xi = V^*VxV^*\hat{1} = V^*\theta(x)\hat{1} = V^*\hat{1}\theta(x) (\theta(x) \in eAe) = \xi\theta(x).
\]
Now we are going to investigate
\[
\xi \in K \subset fH \subset pL^2M \subset L^2M
\]
as a square integrable operator affiliated with $M$. By above we have $xL\xi = L\xi\theta(x)$ for any $x \in U(fBf)$. Let $vL\xi$ be the polar decomposition of $L\xi$. Then
\[
|L\xi|^2 = (xL\xi)^*xL\xi = (L\xi\theta(x))^*L\xi\theta(x) = \theta(x)^*|L\xi|^2\theta(x)
\]
for $x \in U(fBf)$. Thus $|L\xi|$ commutes with $\theta(fBf)$. In particular $v^*v = s(|L\xi|) \in \theta(fBf)' \cap eMe$. Finally,
\[
xv|L\xi| = xL\xi = L\xi\theta(x) = v|L\xi|\theta(x) = v\theta(x)|L\xi|,
\]
which implies $xvv^*v = v\theta(x)v^*v$, i.e. $xv = v\theta(x)$ for any $x \in fBf$.

(4) $\Rightarrow$ (1): Take $e, f, v$ as in (4). Let $E_\theta$ denote the conditional expectation $eMe \rightarrow \theta(fBf)$. Then $0 \neq E_\theta(v^*v) \in Z(\theta(fBf))$, $vE_\theta(v^*v)^2v^* \in (fBf)' \cap FMf$. 

Let \((f_i)_{i \in I}\) be a maximal family of mutually orthogonal nonzero projections satisfying \(f_0 = f\) and \(f_i \not< f\) in \(B\). Thus, \(\sum f_i\) is equal to the central support \(z_B(f)\) of \(f\) in \(B\). Put \(u_0 = f\). For each \(i\), take a partial isometry \(u_i\) satisfying \(u_i^*u_i = f_i\) and \(u_i^*u_i \leq f\). Put \(v_i = u_i v\). Now we have, for \(w \in UB\),

\[
\sum_i \|E_A(v_i^*wv_0)\|^2 \geq \sum_i \|wv^*E_\theta(v_i^*wv_0)\|^2 = \cdots = \tau(E_\theta(v^*v))^3 > 0.
\]

Since \(\sum \|v_i^*\|^2 \leq 1\) and \(\|E_A(v_i^*wv_0)\|_2 \leq \|v_i^*\|_2\), there exists a finite subset \(\mathcal{F}\) of \(\{v_i : i \in I\}\) containing \(v_0\) and \(\sum_{v_i \notin \mathcal{F}} \|v_i^*\|^2 < \tau(E_\theta(v^*v)^3)/2\).

**Definition A.2.** Let \(A\) and \(B\) be von Neumann subalgebras of \(M\). \(B\) is said to embed into \(A\) inside \(M\) when the equivalent conditions of Theorem A.1 hold for \(B\) and \(A\).

**Corollary A.3.** If \(B\) does not embed into \(A\) inside \(M\), there exists a commutative von Neumann subalgebra \(B_0\) of \(B\) which does not embed into \(A\) inside \(M\). Equivalently, if any commutative subalgebra of \(B\) embeds into \(A\), \(B\) also embeds into \(A\).

**Remark A.4.** The above theorem is useful when we have \(\tau\)-symmetric unital completely positive maps \(\phi_i : M \to M\) which restrict to the identity map on \(A\), giving \(\hat{\phi}_i \in \langle M, A \rangle \cap A'\). Often one has \(\hat{\phi}_i \in \mathbb{K}(M, A) = C^*(xe_{A}y : x, y \in M)\).

\(B \subset M\) is said to be rigid when \(\phi_i \rightarrow \text{Id}\) uniformly on the unit ball of \(B_1\). Then, taking \(\phi = \hat{\phi}_{i_0}\) that satisfies

\[
\|\phi(b) - b\| < \frac{1}{3} \quad (\forall b \in B_1),
\]

\(d = \chi_{[\frac{1}{4}, 1]}(\hat{\phi})\) satisfies \(\text{Tr}(d) < \infty\) and

\[
\|w\phi w^* - \hat{1}\| \leq \frac{1}{2} + \|w\hat{\phi} w^* - \hat{1}\| = \frac{1}{2} + \|\phi(w^*) - w^*\|_2 \leq \frac{5}{6}.
\]

Hence \(\text{conv}^2\{wdw^*\}\) does not contain 0 and \(B\) embeds into \(A\) inside \(M\).

**References**