Spectrahedra

A *spectrahedron* is a convex body that appears in applications such as optimisation and statistics. The name ‘spectra’ is used because it is defined in terms of eigenvalues, and ‘hedron’ because these sets generalise polyhedra.

A real, symmetric matrix is called *positive semidefinite* if all of its eigenvalues are nonnegative. Let $A_0$, $A_1$, $A_2$ and $A_3$ be four real, symmetric $(n \times n)$-matrices. The matrix

$$A(x) := A_0 + A_1 x_1 + A_2 x_2 + A_3 x_3$$

describes a 3-dimensional space of matrices as $x := (x_1, x_2, x_3)$ ranges over $\mathbb{R}^3$. The *spectrahedron* of this space is the set of points where $A(x)$ is positive semidefinite.

The determinant of the matrix $A(x)$ defines a surface which contains the boundary of the spectrahedron. This surface is the *symmetroid* associated to the spectrahedron.

When is the parametrisation nice?

For applications, it is important to understand the geometry of the underlying spectrahedron. In optimisation, this knowledge can be used to give bounds on the true solution. One way to study the geometry is to parametrise the boundary. For which spectrahedra is this parametrisation nice? More precisely, when can the coordinates along the boundary be expressed as a quotient of polynomials in two variables? We answer this in the case where $A(x)$ is a $(4 \times 4)$-matrix.

Singularities

On a geometric object, some points are more interesting than the rest, for instance a point of self-intersection or the tip of a spike. These points are called *singularities* and they are characterised by having more than the usual number of tangents.

The singularities of the symmetroid completely determine whether the boundary of the spectrahedron can be parametrised nicely or not.

Results

When $A(x)$ is a $(4 \times 4)$-matrix, most spectrahedra do not have a boundary that can be parametrised nicely. In this case, the general symmetroid has finitely many singularities, all of the type shown in Figure 1b.

Roughly speaking, if a spectrahedron admits a nice parametrisation, then its symmetroid is one of four types:

1. it has singularities along a line,
2. it has singularities along a conic section,
3. it passes three times through the same point,
4. it is tangent to itself at a point, in a way that locally looks like two balls touching.

There are also some subtleties concerning the rank of the matrix $A(x)$ at the singularities and the number of real singularities.

We can say more. The situation in Figure 3a is general; if the symmetroid has singularities along a line, then this line is always disjoint from the spectrahedron. Contrastingly, if the symmetroid passes three times through the same point, then this point is always on the boundary of the spectrahedron.

For symmetroids of type 2 and 4 in the above list, there is no similar statement. For these types, the special singularities can occur both on and outside of the spectrahedron.