Mean value coordinates in 3D

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Abstract: Trivariate barycentric coordinates can be used both to express a point inside a tetrahedron as a convex combination of the four vertices and to linearly interpolate data given at the vertices. In this paper we generalize these coordinates to convex polyhedra and the kernels of star-shaped polyhedra. These coordinates generalize in a natural way a recently constructed set of coordinates for planar polygons, called mean value coordinates.

Key words: barycentric coordinates, parameterization, mean value theorem.

1. Introduction

Mean value coordinates were introduced recently in [**2**] as a way of expressing a point in the kernel of a star-shaped polygon as a convex combination of the vertices. It was shown there that these coordinates can be successfully used to compute good parameterizations for surfaces represented as triangular meshes, based on convex combination maps. The coordinates can also be used to construct convex combination maps between pairs of planar regions, and to morph pairs of compatible triangulations, as in [**6**]. Mean value coordinates can also be used to smoothly interpolate piecewise linear height data given on the boundary of a convex polygon.

Since these coordinates already have several concrete applications, it seems worthwhile generalizing these coordinates to $\mathbb{R}^3$. The main purpose of this paper is to give an explicit formula for the natural generalization of mean value coordinates to star-shaped polyhedra with triangular facets. These coordinates are well-defined and positive everywhere in the kernel of the polyhedron. For a convex polyhedron, they are therefore well-defined and positive everywhere in the interior, and we further show that they then continuously extend in the expected way to the boundary. In fact we establish this latter property for *all* barycentric coordinates over convex polyhedra with triangular facets.

2. Mean value coordinates in $\mathbb{R}^3$

Let $\Omega \subset \mathbb{R}^3$ be a polyhedron, viewed as a closed region of $\mathbb{R}^3$, with triangular facets and vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^3$. Let $K \subset \Omega$ be the *kernel* of Ω, the open set consisting of all points $\mathbf{v}$ in the interior $\mathrm{Int}(\Omega)$ with the property that for all $i = 1, \ldots, n$, the only intersection between the line segment $[\mathbf{v}, \mathbf{v}_i]$ and the boundary $\partial\Omega$ is $\mathbf{v}_i$. If K is non-empty we say that Ω is *star-shaped*. Given such a polyhedron, we are interested in expressing any point $\mathbf{v}$ in K as a convex combination of the vertices. Thus we want to construct *barycentric coordinates*, that is, non-negative functions $\lambda_1, \ldots, \lambda_n : K \to \mathbb{R}$ such that for all $\mathbf{v} \in K$,

$$\sum_{i=1}^{n} \lambda_i(\mathbf{v}) = 1 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i(\mathbf{v})\mathbf{v}_i = \mathbf{v}. \tag{2.1}$$

For most potential applications, it is also preferable that these coordinate functions are as smooth as possible.

One solution would be to take the λ_i to be the coordinates of Sibson [**5**]. These coordinates are however only C^1 and are complicated to compute. Another solution would be to use the barycentric coordinates recently constructed by Warren et al. [**8**], which generalize Wachspress's coordinates [**7**]. However, these coordinates are only well-defined (and positive) when Ω is convex.

To overcome both the above drawbacks, i.e., lack of smoothness and limitation to convex polyhedra, we will instead derive a generalization of the mean value coordinates of [**2**], orginally derived for the kernels of star-shaped polygons in the plane; see Figure 1(a). If $\Omega \subset \mathbb{R}^2$ is a star-shaped polygon, we may assume that its vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are ordered, as in Figure 1(b). Then if, for $\mathbf{v} \in K$, $r_i(\mathbf{v}) = \|\mathbf{v}_i - \mathbf{v}\|$ and $\alpha_i(\mathbf{v})$ is the angle of the triangle $[\mathbf{v}, \mathbf{v}_i, \mathbf{v}_{i+1}]$ at $\mathbf{v}$, as in Figure 1(b), the mean value coordinates $\lambda_i : K \to \mathbb{R}$ are defined by

$$\lambda_i = w_i \Big/ \sum_{j=1}^{n} w_j, \qquad w_i = \frac{1}{r_i}\big(\tan\frac{\alpha_i}{2} + \tan\frac{\alpha_{i-1}}{2}\big),$$

which are C^∞ in K.

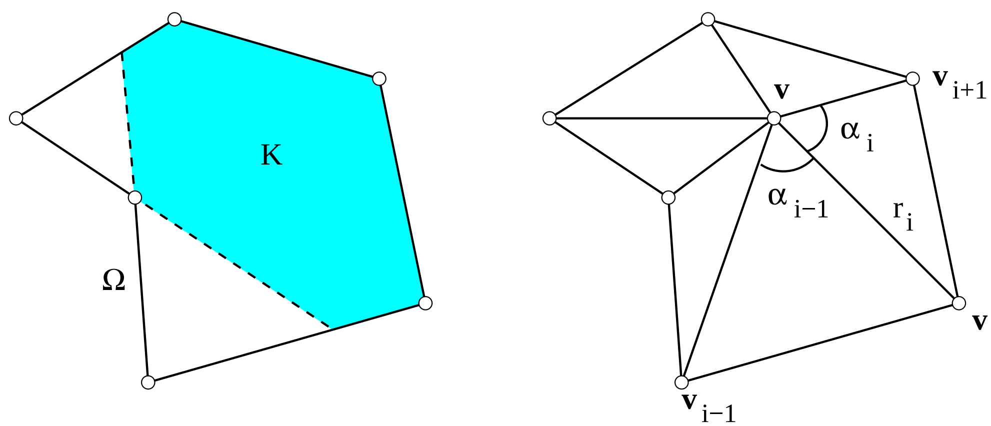


Fig 1. (a) Kernel of a polygon, (b) Mean value coordinates

Consider now again the case that Ω is a star-shaped polyhedron in $\mathbb{R}^3$ with triangular facets. For example, Ω could be convex, as in Figures 3 and 4. The boundary of Ω is a mesh $\mathcal{T}$ of triangles. For any $\mathbf{v} \in K$, each oriented triangle $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k] \in \mathcal{T}$ defines a tetrahedron $[\mathbf{v}, \mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k]$ with positive volume; see Figure 2(a). These tetrahedra could form part of a larger tetrahedral mesh of a region of $\mathbb{R}^3$.

The basic approach in [**2**] to deriving the coordinates in the planar case was the observation that the integral of all unit normals around a circle is zero. We will derive coordinates in $\mathbb{R}^3$ in an analagous way by noticing that the integral of all unit normals over a sphere is zero. Define the unit vectors $\mathbf{e}_i = (\mathbf{v}_i - \mathbf{v})/r_i$, where $r_i = \|\mathbf{v}_i - \mathbf{v}\|$, and the new points $\hat{\mathbf{v}}_i = \mathbf{v} + \mathbf{e}_i$. The points $\hat{\mathbf{v}}_i$ lie on a sphere S of unit radius centred at $\mathbf{v}$, and the projection through $\mathbf{v}$ of each triangle $T = [\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k]$ onto this sphere is a spherical triangle $\hat{T}$ with vertices $\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j, \hat{\mathbf{v}}_k$; see Figure 2(b).

Since the outward unit normal $\mathbf{n}(\mathbf{p})$ to S at any point $\mathbf{p} \in S$ is simply $\mathbf{p} - \mathbf{v}$, we have

$$\mathbf{0} = \int_S \mathbf{n}(\mathbf{p}) = \int_S (\mathbf{p} - \mathbf{v}) = \sum_{T \in \mathcal{T}} \int_{\hat{T}} (\mathbf{p} - \mathbf{v}). \tag{2.2}$$

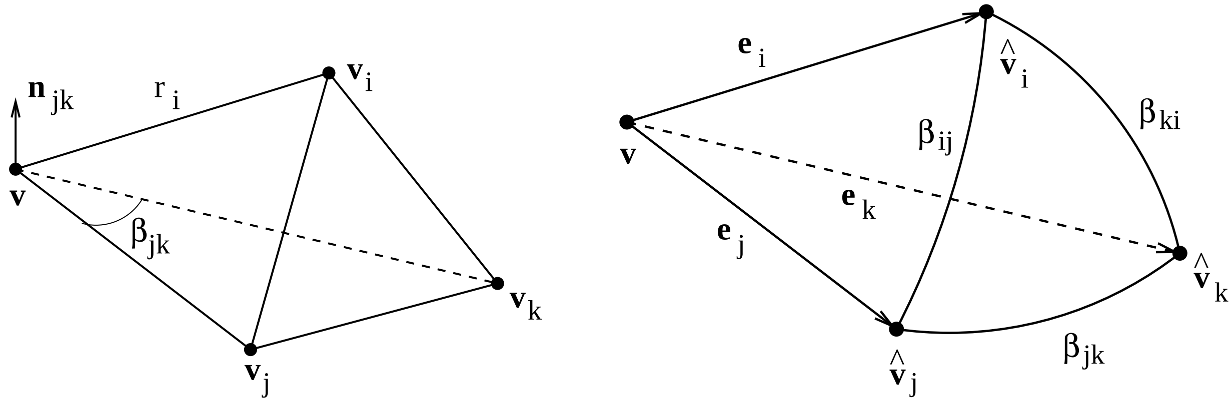


Fig 2. (a) Tetrahedron, (b) Spherical triangle

Now, suppose that the spherical triangle $\hat{T}$ has vertices $\hat{\mathbf{v}}_i$, $\hat{\mathbf{v}}_j$, $\hat{\mathbf{v}}_k$. Clearly, for any $\mathbf{p} \in \hat{T}$, we can express the vector $\mathbf{e} = \mathbf{p} - \mathbf{v}$ as

$$\mathbf{e} = \tau_i(\mathbf{e})\mathbf{e}_i + \tau_j(\mathbf{e})\mathbf{e}_j + \tau_k(\mathbf{e})\mathbf{e}_k,$$

where τ_i, τ_j, τ_k are the spherical barycentric coordinates of $\mathbf{e}$. They are positive, their sum is greater than one, and they are ratios of volumes of (non-spherical) tetrahedra:

$$\tau_i(\mathbf{e}) = \frac{\mathrm{vol}(\mathbf{e}, \mathbf{e}_j, \mathbf{e}_k)}{\mathrm{vol}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)} > 0, \quad \tau_j(\mathbf{e}) = \frac{\mathrm{vol}(\mathbf{e}_i, \mathbf{e}, \mathbf{e}_k)}{\mathrm{vol}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)} > 0, \quad \tau_k(\mathbf{e}) = \frac{\mathrm{vol}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e})}{\mathrm{vol}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)} > 0;$$

see [**1**]. It follows that

$$\int_{\hat{T}} (\mathbf{p} - \mathbf{v}) = \mu_{i,T}\mathbf{e}_i + \mu_{j,T}\mathbf{e}_j + \mu_{k,T}\mathbf{e}_k, \tag{2.3}$$

where

$$\mu_{\ell,T} = \int_{\hat{T}} \tau_\ell(\mathbf{e}) > 0, \qquad \ell \in \{i, j, k\}. \tag{2.4}$$

Thus, by grouping the sum (2.2) over the vertices rather than over the triangles, we find

$$\mathbf{0} = \sum_{i=1}^{n} \sum_{T \ni \mathbf{v}_i} \mu_{i,T}\mathbf{e}_i = \sum_{i=1}^{n} w_i(\mathbf{v}_i - \mathbf{v}),$$

where

$$w_i = \frac{1}{r_i} \sum_{T \ni \mathbf{v}_i} \mu_{i,T} > 0. \tag{2.5}$$

Since w_i, viewed as a function of $\mathbf{v} \in K$, is both positive and infinitely differentiable, we have established

Theorem 1. *The functions* $\lambda_1, \ldots, \lambda_n : K \to \mathbb{R}$ *defined by* $\lambda_i = w_i / \sum_{j=1}^n w_j$ *and equation (2.5) are barycentric coordinates which belong to* $C^\infty(K)$.

It remains to derive an explicit expression for the terms $\mu_{i,T}$. Suppose $T = [\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k]$. Referring to Figure 2, let $\beta_{rs} \in (0, \pi)$ be the angle between the two line segments $[\mathbf{v}, \mathbf{v}_r]$ and $[\mathbf{v}, \mathbf{v}_s]$, and let $\mathbf{n}_{rs}$ denote the unit normal to the face $[\mathbf{v}, \mathbf{v}_r, \mathbf{v}_s]$, pointing into the tetrahedron $[\mathbf{v}, \mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k]$, i.e., $\mathbf{n}_{rs} = (\mathbf{e}_r \times \mathbf{e}_s)/\|\mathbf{e}_r \times \mathbf{e}_s\|$.

Theorem 2.

$$\mu_{i,T} = \frac{\beta_{jk} + \beta_{ij}\mathbf{n}_{ij} \cdot \mathbf{n}_{jk} + \beta_{ki}\mathbf{n}_{ki} \cdot \mathbf{n}_{jk}}{2\mathbf{e}_i \cdot \mathbf{n}_{jk}}. \tag{2.6}$$

Proof: Instead of trying to carry out the integration in equation (2.4), we use the observation that the integral of all unit normals over any compact surface is zero. Consider again the spherical triangle $\hat{T}$ with vertices $\hat{\mathbf{v}}_i$, $\hat{\mathbf{v}}_j$, $\hat{\mathbf{v}}_k$. It defines an infinite cone

$$C = \{\mathbf{e} = \mathbf{v} + \alpha\mathbf{e}_i + \beta\mathbf{e}_j + \gamma\mathbf{e}_k : \alpha, \beta, \gamma \geq 0\},$$

which, when intersected with the solid unit sphere around $\mathbf{v}$, yields a volume V. The surface of V has four sides: the spherical triangle $\hat{T}$, bounded by three circular arcs, and three planar faces, F_{ij}, F_{jk}, F_{ki}, each bounded by two edges and one arc; see Figure 2(b). Thus, since the integral of all outward unit normals over the surface of V is zero,

$$\int_{\hat{T}} (\mathbf{p} - \mathbf{v}) = \int_{\hat{T}} \mathbf{n}(\mathbf{p}) = -\int_{F_{ij}} \mathbf{n}(\mathbf{p}) - \int_{F_{jk}} \mathbf{n}(\mathbf{p}) - \int_{F_{ki}} \mathbf{n}(\mathbf{p}).$$

The three integrals on the right hand sides are simple to compute since the outward unit normal is constant over each face. Consider for example the face F_{ij}. Noticing that its area is half the angle β_{ij} between $\mathbf{e}_i$ and $\mathbf{e}_j$, we find

$$\int_{\hat{T}} (\mathbf{p} - \mathbf{v}) = \frac{1}{2}(\beta_{ij}\mathbf{n}_{ij} + \beta_{jk}\mathbf{n}_{jk} + \beta_{ki}\mathbf{n}_{ki}).$$

Now we simply equate the right hand sides of this equation and equation (2.3) and take the scalar product with $\mathbf{n}_{jk}$, and since $\mathbf{n}_{jk} \cdot \mathbf{e}_j = \mathbf{n}_{jk} \cdot \mathbf{e}_k = \mathbf{0}$, (2.6) follows. ∎

3. Convex polyhedra

We now make a few remarks about the case when the polyhedron $\Omega \subset \mathbb{R}^3$ is convex. In this case, $K = \mathrm{Int}(\Omega)$ and our coordinates are well-defined, positive, and infinitely differentiable in $\mathrm{Int}(\Omega)$. However, equation (2.6) is not well-defined at the boundary of Ω. Nevertheless, we now show that the coordinates extend *continuously* to the boundary. In fact we prove this for *all* sets of continuous barycentric coordinates. An analogous result, Corollary 2 of [**3**], was established for convex polygons in the plane.

Theorem 3. *If Ω is convex and $\lambda_1, \ldots, \lambda_n : \mathrm{Int}(\Omega) \to \mathbb{R}$ are a set of continuous barycentric coordinates, then they have a unique continuous extension to the boundary $\partial\Omega$. The extended coordinates λ_i are linear on each facet of Ω and $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.*

Proof: First we derive a piecewise linear upper bound $L_i : \mathrm{Int}(\Omega) \to \mathbb{R}$ on each coordinate function λ_i, analogous to the upper bound derived in [**3**]. Any given $\mathbf{v}$ in $\mathrm{Int}(\Omega)$ belongs to at least one tetrahedron of the form $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell]$, where $[\mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell]$ is a triangular face of Ω not containing $\mathbf{v}_i$. Then, defining $\mathrm{vol}(\mathbf{v}, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell)$ to be the signed volume of $[\mathbf{v}, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell]$, and noticing that it is a linear function of $\mathbf{v}$, it follows from (2.1) and the non-negativity of the coordinates that

$$\mathrm{vol}(\mathbf{v}, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell) = \sum_{r=1}^{n} \lambda_r(\mathbf{v}) \mathrm{vol}(\mathbf{v}_r, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell) \geq \lambda_i(\mathbf{v}) \mathrm{vol}(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell),$$

and therefore

$$\lambda_i(\mathbf{v}) \leq \frac{\mathrm{vol}(\mathbf{v}, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell)}{\mathrm{vol}(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell)} =: L_i(\mathbf{v}).$$

Therefore

$$0 \leq \lambda_i(\mathbf{v}) \leq L_i(\mathbf{v}), \qquad \mathbf{v} \in \mathrm{Int}(\Omega).$$

Now let $\mathbf{w}$ be any point in a triangular face $[\mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell]$ of Ω, not containing $\mathbf{v}_i$, and let $(\mathbf{w}_s)_{s \in \mathbb{N}}$ be any sequence of points in $\mathrm{Int}(\Omega)$ which converges to $\mathbf{w}$ as $s \to \infty$. Then we have that

$$0 \leq \lambda_i(\mathbf{w}_s) \leq L_i(\mathbf{w}_s),$$

and since $L_i(\mathbf{w}) = 0$, the sandwich lemma shows that $\lambda_i(\mathbf{w}_s) \to 0$ as $s \to \infty$.

The remaining case is that $\mathbf{w}$ belongs to a face containing $\mathbf{v}_i$, $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k]$ say. We construct a corresponding lower bound $\ell_{ijk} : \mathrm{Int}(\Omega) \to \mathbb{R}$ for λ_i as follows. Due to the convexity of Ω, we can clearly find an infinite plane P which passes through the edge $[\mathbf{v}_j, \mathbf{v}_k]$ and such that $\mathbf{v}_i$ lies to one side of P while all other vertices of Ω (apart from $\mathbf{v}_j$ and $\mathbf{v}_k$) lie on the other side. Notice that the intersection of P with Ω is a convex polygon, and so we can let $\mathbf{p}$ be any point in the interior of that polygon (so in particular, $\mathbf{p}$ will not belong to the edge $[\mathbf{v}_j, \mathbf{v}_k]$). Then for any $\mathbf{v} \in \mathrm{Int}(\Omega)$,

$$\begin{aligned}
\mathrm{vol}(\mathbf{p}, \mathbf{v}, \mathbf{v}_j, \mathbf{v}_k) &= \sum_{r=1}^{n} \lambda_r(\mathbf{v}) \mathrm{vol}(\mathbf{p}, \mathbf{v}_r, \mathbf{v}_j, \mathbf{v}_k) \\
&= \lambda_i(\mathbf{v}) \mathrm{vol}(\mathbf{p}, \mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k) + \sum_{r \neq i} \lambda_r(\mathbf{v}) \mathrm{vol}(\mathbf{p}, \mathbf{v}_r, \mathbf{v}_j, \mathbf{v}_k) \\
&\leq \lambda_i(\mathbf{v}) \mathrm{vol}(\mathbf{p}, \mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k),
\end{aligned}$$

and so

$$\lambda_i(\mathbf{v}) \geq \frac{\mathrm{vol}(\mathbf{p}, \mathbf{v}, \mathbf{v}_j, \mathbf{v}_k)}{\mathrm{vol}(\mathbf{p}, \mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)} =: \ell_{ijk}(\mathbf{v}), \qquad \mathbf{v} \in \mathrm{Int}(\Omega).$$

Now let $(\mathbf{w}_s)_{s \in \mathbb{N}}$ be any sequence of points in $\mathrm{Int}(\Omega)$ converging to $\mathbf{w}$ as $s \to \infty$. Then

$$\ell_{ijk}(\mathbf{w}_s) \leq \lambda_i(\mathbf{w}_s) \leq L_i(\mathbf{w}_s),$$

and ℓ_{ijk} and L_i clearly extend to the boundary of Ω and by construction, they agree on the face $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k]$, and in fact

$$\ell_{ijk}(\mathbf{w}) = L_i(\mathbf{w}) = \frac{\text{area}(\mathbf{w}, \mathbf{v}_j, \mathbf{v}_k)}{\text{area}(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)}.$$

Thus the sandwich lemma shows that

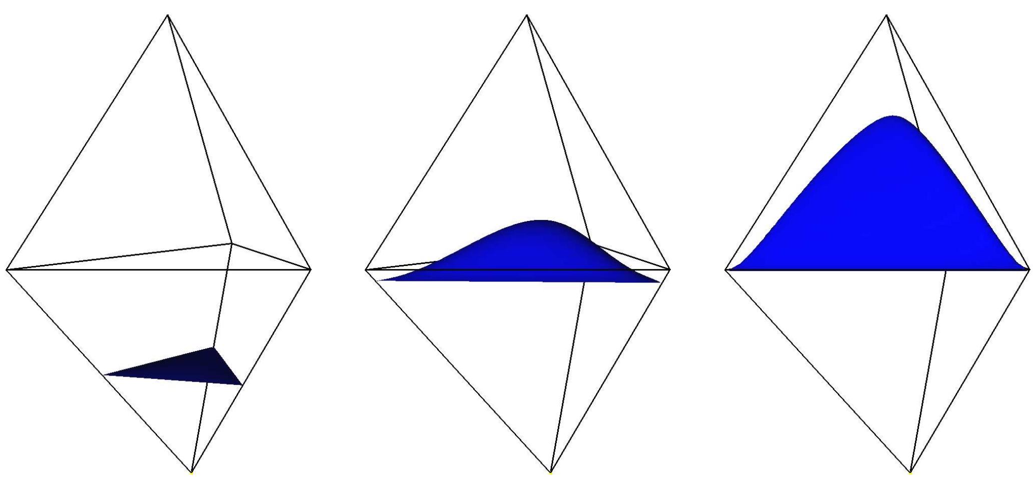
$$\lim_{s\to\infty} \lambda_i(\mathbf{w}_s) = \frac{\text{area}(\mathbf{w}, \mathbf{v}_j, \mathbf{v}_k)}{\text{area}(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)},$$

which is the barycentric coordinate of $\mathbf{w}$ with respect to $\mathbf{v}_i$ in the triangle $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k]$. ■

4. Numerical examples

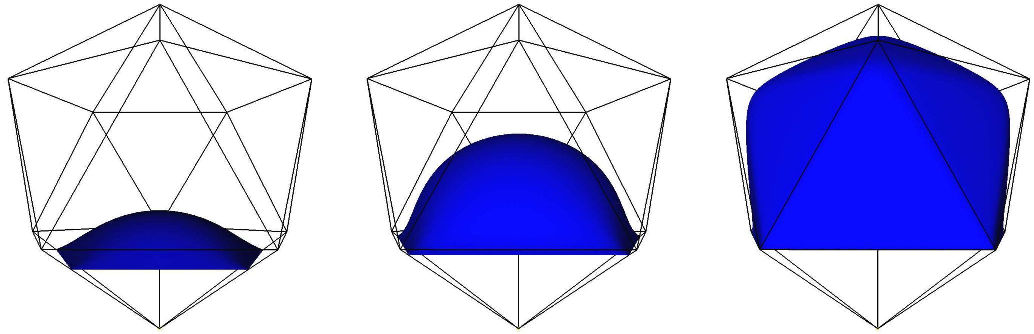
We have implemented the 3D coordinates for various polyhedra. In our implementation of (2.6) we have used the inverse cosine function to compute the angles β_{ij}. We have not found a formula which avoids using angles. Figure 3 shows a simple convex polyhedron consisting of two tetrahedra sharing a common face. There are five vertices and Figure 3 shows three iso-surfaces of the barycentric coordinate function λ_i corresponding to the lowest vertex $\mathbf{v}_i$ in the figure. These figures verify the linearity of the coordinates on the facets, established in Theorem 3. Figure 4 shows an icosahedron, and three iso-surfaces of the barycentric coordinate function λ_i where $\mathbf{v}_i$ is the lowest vertex in the figure.

We have added Figures 5 and 6 to illustrate that our coordinates appear to be well-defined both inside and outside *any* closed polyhedral mesh, of arbitrary genus, similar to what was observed by Hormann [**4**] in the planar case. Iso-surfaces are shown for the barycentric coordinate associated with the lowest vertex in each figure. The coordinates are in general no longer positive everywhere inside non-convex polyhedra, and are not all positive outside an arbitrary polyhedron, convex or not. However, we have verified numerically that they continue to satisfy (2.1) everywhere in $\mathbb{R}^3$.



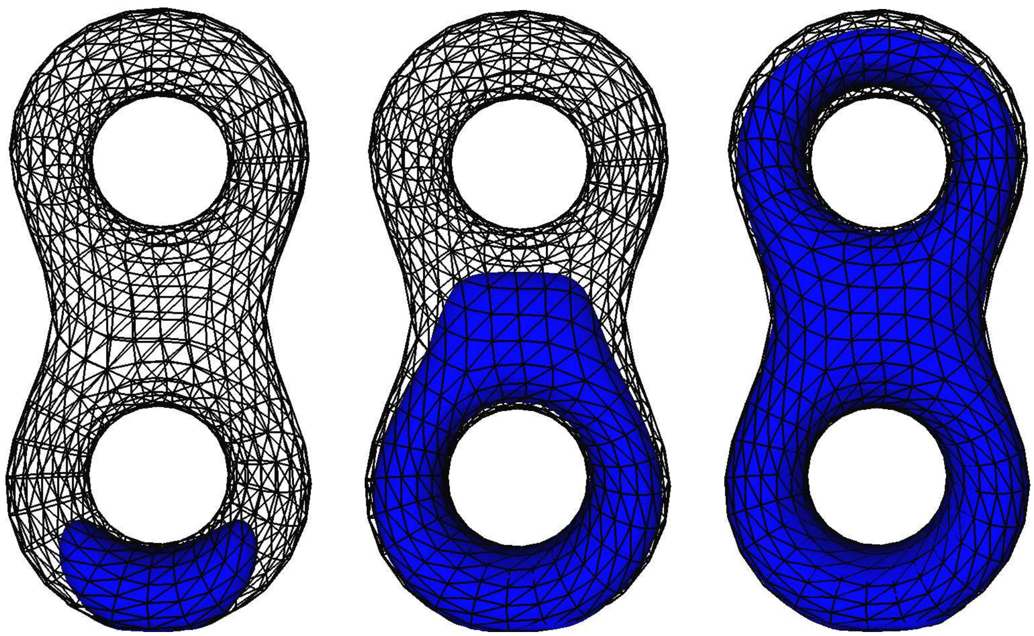
3(a) 3(b) 3(c)

Fig 3: Iso-surfaces $\{\mathbf{v} : \lambda_i(\mathbf{v}) = c\}$ for $c = 0.5, 0.05, 0.005$



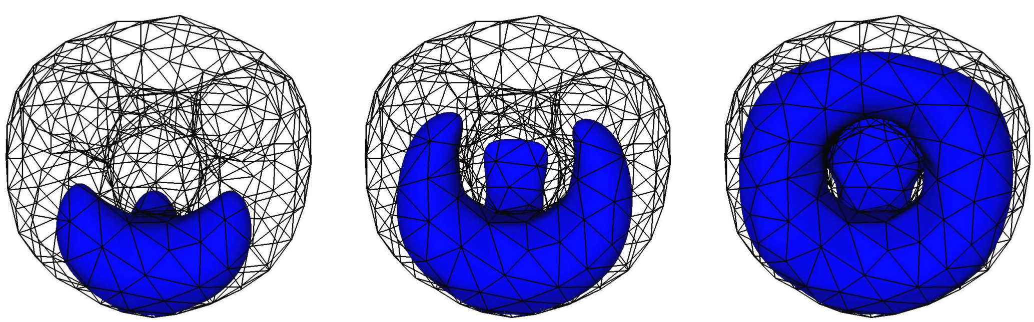
4(a) 4(b) 4(c)

Fig 4: Iso-surfaces $\{\mathbf{v} : \lambda_i(\mathbf{v}) = c\}$ for $c = 0.2, 0.05, 0.005$



5(a) 5(b) 5(c)

Fig 5: Iso-surfaces $\{\mathbf{v} : \lambda_i(\mathbf{v}) = c\}$ for $c = 0.001, 0.0001, 0.00001$



6(a) 6(b) 6(c)

Fig 6: Iso-surfaces $\{\mathbf{v} : \lambda_i(\mathbf{v}) = c\}$ for $c = 0.01, 0.0005, 0.0002$

5. Conclusions and future work

We have proposed and developed a natural extension of mean value coordinates to the kernels of star-shaped polyhedra. We have also shown that the $3D$ coordinates are well-defined everywhere in a convex polyhedron, including the boundary.

Note that we have only considered here polyhedra with triangular facets. Clearly, one way of dealing with polyhedra containing multi-sided facets, i.e., facets with more than three vertices, is first to triangulate each facet, and then apply our coordinates. Such coordinates would however depend on the choice of triangulation. One way of avoiding this dependence would be to define coordinates by taking an average of the coordinates for each triangulation. A possible alternative is to try to construct more generic mean value coordinates by starting again from the integral identity in equation (2.2). This will be a topic for future research.

We plan to make a careful mathematical study of how the $3D$ coordinates extend to arbitrary points in $\mathbb{R}^3$, even for arbitrary polyhedra, as in Figures 5 and 6, with a view to the application of interpolation over polyhedral surfaces.

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