Cosmology I

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Room: 104

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Lecture 1

• General Class Issues:
  • Change Lecture: Monday <-> Tuesday?
  • Availability: Monday to Friday: 15h-16h, email me otherwise
  • See year of 2008 for program, topics covered in lectures and what will be evaluated in the exams.
• Bibliography:
  • Oystein Elgaroy Lecture Notes (http://folk.uio.no/mota/Cosmology_I.pdf)
  • A. Liddle, An Introduction to Modern Cosmology, Wiley
  • M. Roos, Introduction to Cosmology, John Wiley & Sons
  • Ned’s Write Lecture Notes (http://www.astro.ucla.edu/~wright/cosmolog.htm)
  • R. Kolb, M. Turner, Early Universe, Addison Wesley
  • S. Dodelson, Modern Cosmology, Academic Press (Cosmology II)
• Roadmap to Cosmology
  • What is Cosmology
  • Basic properties of our Universe
  • The Course syllabus
What is Cosmology?

- The study of the Universe as a whole

- What is the Universe size, shape, age?

- How did the Universe started and what is its future?

- What is the Universe made of?

- How is matter distributed in the Universe and why is it like that?

- In order to do Cosmology we need:
  - Lots of Mathematics (Differential Equations, Differential Geometry, Topology, Group Theory)
  - Computational Skills (analyze observational data, solve equations numerically and test theory with experiments)
  - Lots of Physics (General Relativity, Particle Physics, Thermodynamics and Statistical Physics)
  - Most importantly: Observe how it looks like!
Observing what is around us

- Observation: There is matter in the Universe
- Observation: The local Universe is highly inhomogeneous
- Question: How did the planets, stars, molecules, atoms form?
• Observation: Our Sun is not unique. There are many stars, planets, etc

• Observation: The solar system is not in a privileged position (nor our galaxy)
  • leads to the Copernican Principle: humans are not privileged observers of the universe!
• Observation: Stars agglomerate into Galaxies (which have specific shapes)

• Observation: Galaxies agglomerate into Clusters
Observation: Clusters form filaments and Voids (quite specific shapes)
Lesson: At VERY large scales the Universe is Homogeneous and Isotropic!
Observation: The Universe is bathed by a Microwave Radiation
Observation: The Universe is Expanding!

Hubble Law
recession speed = $H_0 \times \text{distance}$
Sensorial Experience: Matter ‘lives in’ space and time

Question: How does one describe space and time?
What Observations Tell Us about our Universe structure?

- The most important feature of our Universe is its large scale Homogeneity and Isotropy (Observations show it is homogeneous on scales over 100 Mpc)
  - Leads to (assumption): Cosmological Principle.
  - Means: Observations made from our single vantage point are representative of the Universe as a whole (allow us to test cosmological models!)
  - At small scales (< 100 Mpc) is highly inhomogeneous (galaxies, clusters, superclusters)
  - (Inflation) Theory tell us the Universe is highly homogeneous up to 3000 Mpc (those scales are NOT observed today). Above those scales it becomes inhomogeneous again.
  - The Universe is expanding!
What Observations Tell Us about its Content?

- It is pervaded by thermal Microwave Background Radiation with temperature $T \sim 2.73$ K
  - This is highly homogeneous (fluctuations of order $10^{-5}$)
    - Implying: there were only small fluctuations in the Universe energy density when the Universe was 1000 times smaller
- There is baryonic matter, roughly one baryon per $10^9$ Photons, but no substantial amount of anti-matter
- The chemical composition of baryonic matter is about 75% of hydrogen, 25% helium plus traces of heavier elements
- Baryons contribute only to a small amount of the total matter/energy of the Universe.
  - The rest is dark matter 25% (seems to be pressureless) and dark energy
Questions that arise

- What portion of the Universe is like the part we find ourselves?
- How did the Universe started?
- How will it evolve (expanding, contracting, static)?
- What is its shape (Sphere, plane)?
- How did galaxies and clusters formed?
The Course Syllabus

- Cosmological Models
  - Spacetime Description
  - Tools to perform Observations: Cosmological Distances
  - Description of Matter in the Universe
  - Possible Universe Evolutions. Shapes, etc
- The Early and Hot Universe
  - Primordial Soup
  - Formation of First Atoms and Nucleosynthesis
- Inflation
  - Before Matter
  - Origins of Matter Fluctuations
- Structure Formation
  - Non-relativistic structure formation of clusters and galaxies

Monday, September 7, 2009
Lecture 2: The Spacetime

- Introduction to Line Elements (Special Relativity)
  - Lecture Notes 1.1 and 1.2

- Curved spacetimes (General Relativity)
  - Lecture Notes 1.3
Where do things happen? **In space ...**

When do things happen? **In time ...**

Sensorial Experience: Matter ‘lives in’ space/time

Question: How does one describe space/time?
The nature of Space and Time

- How does one describe space and time: Clocks and Rulers?
  - Aristotle's Clock and Ruler (space and time) are absolute (to the ‘prime mover’).
  - Galileo / Newton’s Ruler (space) is relative, but clock (time) is absolute.
    - Velocities are relative: Speed = Space (Relative) / Time (invariant)

- However, Michelson and Morley, 1887, showed: the speed of light does not depend on the Earth’s motion (observer independent!)
  - \( c = \text{Space} / \text{Time} \Rightarrow \text{not only space, but also time must be observer dependent!} \)

- Space and Time are relative!

- If one cannot trust our clocks, how does one then measures time?

Monday, September 7, 2009
How do we measure time?

Imagine a clock that measures time by how long it takes light to bounce back and forth ...

Relativity of distance -> relativity of time!
So what?

From the point of view of Jack, lightening struck both train cars at the same time.

From the point of view of John, lightening struck car B first and then car A later.

Simultaneity is relative - things occurring at the same time at one place may occur at different times in another place!

So... what is the invariant quantity? Einstein claimed it is the spacetime as a unity!
Special Relativity

- Special Relativity (No gravity / accelerations)

- The speed of light in empty space, $c$, is the same for all observers

- The Laws of physics are the same for all inertial observers

- The invariant quantity (the same for all inertial observers) is the spacetime interval

- In 4D, two events separated by time interval $dt$ and coordinate distances $dx, dy, dz$ (Cartesian coordinates):

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$
Consequences of line interval invariance

- Long rod of Length: $L$ (as measured by $S$ at rest)
- $S'$ is an observer moving at speed $v$
- $S$ will see $S'$ reaching the end of the Rod after: $dt = \frac{L}{v}$
- Observer $S'$ time elapsed is: $T$
- From the line element invariance: $ds^2 = \left(\frac{L}{v}\right)^2 - L^2 = c^2T^2 - 0^2$
- Hence: $T = \frac{L}{v} \sqrt{1 - v^2/c^2} < dt$ (Moving clocks run at slower rate!)
- The length of the rod for $S'$ is:

\[ L' = vT = L\sqrt{1 - v^2/c^2} < L \] (Length Contraction!)
Do we already know everything about spacetime?

Classical Physics

• Existence of Inertial Reference Frames (IRF)
• Relativity Principle (Hey man, physics gotta be the same in any IRF!)
• Invariance of length and time intervals

\[
x' = x - vt \\
 t' = t
\]

Special Relativity

• Existence of Inertial Reference Frames (IRF)
• Relativity Principle (Hey man, physics gotta be the same in any IRF!)
• Invariance of \(c\)

\[
x' = \gamma (x - vt) \\
 t' = \gamma (t - vx / c^2)
\]

All is OK with Kinematics, but there is something strange with gravity Dynamics

Monday, September 7, 2009
Why Wonder @ Gravity?

1) Mercury’s “extra” precession
   Newton - 531 arcsecs
   Actual   - 574 arcsecs

2) No such thing as coincidence
   Balance b/w inertia~acceleration

3) How fast is gravity?
   “instant propagation” (Newton)
   or
   “delayed information” (Einstein)
If gravitation does not depend on the characteristics of a body then it can be ascribed to spacetime. It is a spacetime property.

The gravitational acceleration is unique: it does not depend on the characteristic of the body!

\[ \vec{a} = \frac{d\vec{x}}{dt^2} \quad \text{and} \quad a=g \]

Could the gravitational phenomena then be ascribed to a spacetime property?

---

**Electric field**

\[ F = qE \]
\[ F = m(i)a \]
\[ qE = m(i)a \]
\[ a = E \frac{q}{m(i)} \]

Depending on particle charge

---

**Gravity field**

- \[ P = m(g) g \]
- \[ P = m(i) a \]
- \[ m(g)g = m(i)a \]
- \[ a = g \frac{m(g)}{m(i)} \]
- One for all bodies

---

*The gravitational acceleration is unique: it does not depend on the characteristic of the body!*

Since an acceleration is just \[ \vec{a} = \frac{d\vec{x}}{dt^2} \] and \[ a=g \] Could the gravitational phenomena then be ascribed to a spacetime property?
Gravitational Force or a SpaceTime ‘Consequence’?

• Two people starting at the equator and each heading due north. Meet at the north pole. Is this due to some mysterious force of attraction?

But, wait... we still feel the earth attraction! Do we really? How do we know?
Equivalence principle

One cannot distinguish a uniform acceleration from a uniform gravitational field!
Step back and recognize that “gravity” is not a force, but an experience. It is a collection of observed actions and behaviors and a set of personal experiences. The role of science is to provide a theory to coherently explain what is behind all these observations and experiences.

Newton provided one explanation -- a propagating force inherent in mass.

Einstein provided an alternative explanation -- mass follows the curvature of spacetime.
But why gravity $\iff$ spacetime curvature?

Newtonian physics massless objects do not feel gravity. However, every object travels in spacetime. If one claims gravity equivalent to curvature of spacetime, then even massless objects should feel that curvature! (That is not the case in Newtonian gravity)

In General Relativity even light feels the ‘gravitational effect’ (spacetime curvature)
Gravitational Lens
Galaxy Cluster 0024+1654
Hubble Space Telescope - WFPC2
The concept of elementary interaction

Newton  
Action at a distance

Faraday
Field concept

Maxwell
Quantum Fields  
(Gravity (spacetime curvature)
(field quanta exchange))
Matter follows the structure of spacetime. Where spacetime is curved by a mass, other masses will follow that curve.
Einstein replaced the idea of force with the idea of geometry. To him the space through which objects move has an inherent shape to it and the objects are just travelling along the straightest lines that are possible given this shape (J. Allday).

Understanding gravitation requires understanding spacetime geometry.

*This is what we will do next: Study geometry of curved spacetimes!*
In the previous lecture we saw that the important thing was to have an invariant quantity (the distance in spacetime).

Remarkably the distance in spacetime involves changing how we add up the distance in space with the distance in time.

\[ ds^2 = dt^2 - dx^2 \]

Actually there are many ways we can add distance depending on the coordinates that we use.
The point is, on a curved surface how you measure distance may not be as simple as we’ve seen so far.

There are many things that change once we are on a curved space.

- Polar coordinates

$$d^2 = \delta r^2 + r^2 \delta \theta^2$$

- Suppose we restrict ourselves to the circle.
- Distances on the circle would be given by theta only but the actual distance would be given by:

$$d^2 = r^2 \delta \theta^2$$

The point is, on a curved surface how you measure distance may not be as simple as we’ve seen so far.
To understand geometry we need to understand what makes a straight line on a curved space.

A straight line between two points is given by the shortest distance between those two points along the curved surface.

On the surface of a sphere the shortest distance between two points always lies on a great circle.

This is what we mean by a straight line!

*Straight lines are called Geodesics*
Geometry: study of the properties of space.

Euclidean geometry: based on postulates

- example: given an infinitely long line L and a point P, which is not on the line, there is only one infinitely long line that can be drawn through P that is not crossing L at any other point.

Some consequences:

• The angles in a triangle when added together sum up to 180°
• The circumference of a circle divided by its diameter is a fixed number: \( \pi \)
• In a right angled triangle the lengths of the sides are related by \( c^2 = a^2 + b^2 \) (Pythagoras Theorem)
Non-Euclidean Geometry: Consequences

However there are spaces that do not obey Euclid axioms. Spaces having a non-Euclidean geometry.

We will consider the (2-dimensional) example of the surface of a sphere.

Now, suppose we choose A as a point and we draw from B the parallel to A. They meet at the North Pole!!!! (Euclid axiom does not hold)

Another consequence: the sum of the angles of a triangle is higher than 180°
Quantitative Measure of Curvature

Circle $(c)$: set of points on the surface which all lie at a given distance $s$ (measured on the surface!) from a central point $P$

Euclidean (flat) space: $c = 2\pi r$

On the surface of a sphere you will measure:
$$c = 2\pi s$$

How different is it a circle in flat and curved space?

$$r = a \sin \theta \quad \quad c = 2\pi r = 2\pi a \sin \left( \frac{s}{a} \right)$$

$$\theta = \frac{s}{a} \quad \quad = 2\pi s \left( 1 - \frac{s^2}{6a^2} + \ldots \right) < 2\pi s$$
Gaussian Curvature

Gaussian curvature is the limiting difference between the circumference of a geodesic circle and a circle in the plane

\[ K \equiv \frac{3}{\pi} \lim_{s \to 0} \left( \frac{2\pi s - c}{s^3} \right) \]

In the case of the surface of a sphere:

\[ K \equiv \frac{3}{\pi} \lim_{s \to 0} \left( \frac{2\pi s - 2\pi r}{s^3} \right) \]

\[ = \frac{3}{\pi} \lim_{s \to 0} \left( 2\pi s - 2\pi s + \frac{2\pi s^3}{6a^2} - \ldots \right) = \frac{1}{a^2} \]

The Gaussian curvature of a sphere is positive. That is a general feature of positively curved spaces.

There are also negatively curved spaces!

The surface of an hyperboloid: the circumference of a circle is greater than \( c = 2\pi s \)
On small scales, space is “dimpled” by massive objects such as stars, galaxies, or clusters of galaxies. On large scales, however, where the assumptions of homogeneity and isotropy apply, space must have the same average curvature everywhere.

Consider the analogy of an ant wandering over the surface of an orange. The ant will encounter small local dimples (the pores of the orange), but if the ant wanders far enough, it will discover that the orange is spherical on average. On large scales, there are three possibilities for the average curvature of space... as obvious by now... they depend on the matter content of the universe...

**Positively Curved Universe**
the sum of the vertices of a triangle is greater than 180 degrees. A sphere has a FINITE area; positively curved space has FINITE volume (but no edge)

**Negatively Curved Universe**
the sum of the vertices of a triangle is less than 180 degrees. A hyperboloid has an INFINITE area; a negatively curved space has an INFINITE volume

**Flat Universe**
the sum vertices of a triangle equals 180 degrees. A plane has infinite area; flat space has infinite volume
So what is the Universe curvature on large scale?

It's hard to tell, since we see only a limited volume within our cosmic particle horizon.

It's comparable to the difficulty that early cultures had in determining that the Earth was spherical -- positively curved -- rather than flat. Actually, it's even worse than you might think, since the local curvature due to stars, galaxies, clusters, and superclusters tends to mask the global positive or negative curvature. (Imagine trying to determine the curvature of the Earth if you were confined to Switzerland. The local curvature, due to the Alps, would totally swamp the global curvature due to the Earth's spherical shape.)

The most promising technique for determining the curvature of the Earth involves looking at the angular size of very distant objects.
Einstein’s theory replaced gravity as a force with the notion that space can have a different geometry from the Euclidean. It is a curved space. The sphere surface is 2-d and is a curved space when seen from “outside” (3-d). We live in a 4-d curved (by gravity) spacetime.

**SUMMARY**

Monday, September 7, 2009
What makes spacetime curve?
Mass and energy make spacetime curve. The more mass and energy the more the geometry of spacetime curves and is affected.

How do objects move on curved a space?
They move in straight Lines. That is they move so as to minimize the distance travelled. That is the shortest distance in between two points (and that is called a geodesic).
This is like the straight lines we had on a sphere they bend when compared to flat space.

Anything moving in spacetime will follow a geodesic path.
How do we interpret this physically?
This moving along geodesics explains how things move in a gravitational field.
Mass bends spacetime. Objects in curved spaces move on bent trajectories. Therefore objects with mass cause other things to move on curved trajectories.

In fact it is gravitation!!!!
Einstein realised in 1915 that this is what gravity is. Mass bends spacetime and objects move in spacetime along geodesics.
Thus mass affects how objects move though bending spacetime.
Light also follows geodesics.

Why don’t we see light bending and other geometrical effects?
Just like we had with special relativity where most the speeds we are used to are small, most spacetime curvatures are also small. There are places where spacetime curvatures are large, near very massive objects. These are black holes.
Exercise: Show line element for a sphere with radius $a(t)$

$$dl^2 = a^2(t) \left( \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi \right)$$

Spacetime interval: $ds^2 = c^2 dt^2 - dl^2$

Line element for a 4-spherical spacetime

$$ds^2 = c^2 dt^2 - a^2(t) \left( \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi \right)$$

This can be generalized for positive, negative and flat curvatures

$$ds^2 = c^2 dt^2 - a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi \right)$$

$k = Ka^2$  \quad K is the Gaussian Curvature, so $k = -1, 0, 1$
Robertson Walker Line Element

- The three-dimensional description can be generalized to four-dimensions by allowing the scale factor $a$ to be a function of time, $a(t)$

$$ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$

- $k=0$, clearly corresponds to flat, Minkowski space

- The radius of a sphere can be determined ($d\theta=0$, $d\phi=0$)

$$r_{\text{phys}} = a(t) \int_0^r \frac{dr}{\sqrt{1-kr^2}}$$

- therefore, for $k=+1$, Universe will be closed
- and for $k=0, -1$, Universe is open and infinite
The three plausible cosmic geometries are consistent with many different topologies. For example, relativity would describe both a torus (a doughnut like shape) and a plane with the same equations, even though the torus is finite and the plane is infinite. Determining the topology requires some physical understanding beyond relativity.
With the line element we can determine distances.

We take space (slicing) as homogeneous & isotropic.

Then there is a universal \( t \) time coordinate ("the age of the universe" = time for observers that see space slices as homogeneous & isotropic).

Galaxies are taken to be at rest (no peculiar velocity to first approximation) with respect to comoving coordinates \((\chi, \theta, \phi)\).

Their collective motion is due to the expansion of space. It is encompassed by the scale factor \( a(t) \). We take \( a_{\text{today}} = a_0 = 1 \).
Physical distance $d_p$

Actual distance from to e.g. a galaxy at a given moment of time. (For other distances, see later.)

$$d_p = a(t) \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} = a \sin^{-1}r \quad (k = 1)$$

$$= ar \quad (k = 0)$$

$$= a \sinh^{-1}r \quad (k = -1)$$

Since we took $a_0 = 1$, comoving distances are physical distances today.

Recession velocity:

$$v = \frac{\dot{d}_p}{a} = \frac{\dot{a}}{a} d_p$$

$$\equiv H d_p$$

This relation is called the Hubble law and it is exact (ie not an approximation).
Naïve expansion model (assuming $H = \text{const}$)

What we see in our Patch is consistent with isotropic and homogeneous expansion plus the “Cosmological Principle” (no privileged place in Universe!)

Homogeneous and isotropic expansion: the shape of the triangle must be preserved. Therefore

Seen from patch 1:

$$v_{12} (t) = \frac{dr_{12}}{dt} = \frac{\dot{a}}{a} r_{12} (t_0) = \frac{\dot{a}}{a} r_{12} (t) \quad v_{13} (t) = \frac{dr_{13}}{dt} = \frac{\dot{a}}{a} r_{13} (t_0) = \frac{\dot{a}}{a} r_{13} (t)$$

Seen from patch 2:

$$v_{21} (t) = \frac{dr_{21}}{dt} = \frac{\dot{a}}{a} r_{21} (t_0) = \frac{\dot{a}}{a} r_{21} (t) \quad v_{23} (t) = \frac{dr_{23}}{dt} = \frac{\dot{a}}{a} r_{23} (t_0) = \frac{\dot{a}}{a} r_{23} (t)$$

In any universe undertaking homogeneous and isotropic expansion, the velocity/distance relation must have the form

$$v(t) = \frac{\dot{a}}{a} r(t)$$

Now we see that:

$$H(t) = \frac{\dot{a}}{a}$$

You may check that the Hubble law is the only solution consistent with an homogeneous and isotropic universe, i.e. $v \propto d^2$ does not work.

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Monday, September 7, 2009
Cosmological Red Shift

Radiation is emitted from stars and other celestial bodies. This radiation has the same physical origin of the radiation we study in terrestrial laboratories (e.g. atom absorption and emission).

Stellar evolution and many other branches of astrophysics are based on such evidence. E.g. chemical composition of star surfaces are well known.

The radiation emitted by any source can be affected by the Doppler effect if there is a relative motion between the source and the receiver.

**Doppler effect (for light)**

The light of an approaching source is shifted to the blue, the light of a receding source is shifted to the red.

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Doppler effect

The light of an approaching source is shifted to the blue, the light of a receding source is shifted to the red.

\[ z = \frac{\lambda' - \lambda}{\lambda} \]

blue shift    red shift

From a distant galaxy
In laboratory

Absorption lines from star
Reference lines from laboratory source
Absorption lines from star
Reference lines from laboratory source
Doppler effect

redshift:

\[ 1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \]

- \( z = 0 \): not moving
- \( z = 2 \): \( v = 0.8c \)
- \( z = \infty \): \( v = c \)
Hubble’s Law
\[ v = Hr \]
H is the Hubble constant

\[ 1 + z = \sqrt{1 + \frac{v}{c}} \]

If one knows H,
then redshift can be used as a ‘distance’ unit!
Redshift as a measure of distance and time

\[ 1 + z = \sqrt{\frac{1 + \frac{\nu}{c}}{1 + \frac{\nu}{c}}} \]

\[ d = \frac{\nu}{H_0} \]

Look Back Times

<table>
<thead>
<tr>
<th>z</th>
<th>( \frac{\nu}{c} )</th>
<th>Look-back time (10^9 years)</th>
<th>&quot;Distance&quot; (Mpc)</th>
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<tr>
<td>0.1</td>
<td>0.095</td>
<td>1.29</td>
<td>394</td>
</tr>
<tr>
<td>0.25</td>
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<tr>
<td>inf.</td>
<td>1.0</td>
<td>13.67</td>
<td>4194</td>
</tr>
</tbody>
</table>

We assume here that \( H_0 = 71 \text{ km/sec} \) (and the Universe expanded always at this rate)
The Age of the Universe

Measurements of $H_0$ give us an estimate of the Universe age

**HOWEVER:** The Hubble parameter is NOT a constant: The Universe rate of expansion has changed several times along its history
Relation between redshift and Universe scale factor

A spacetime interval in an expanding universe is given by (Robertson & Walker):

\[ ds^2 = dt^2 - a(t)^2 dx^2 \]

As a simple application, consider a photon of wavelength \( \lambda_e \) emitted by galaxy A and observed by galaxy O with wavelength \( \lambda_0 \). We can show that \( \lambda \propto a(t) \).

The comoving distance between O and A is \( \chi = \text{constant} \). Photons travel on light-cones, \( ds^2 = 0 \) or

\[ d\chi^2 = dt^2/a(t)^2 \]

Thus

\[ \chi_{\text{galaxy}} = \int_{t_e}^{t_e} \frac{dt}{a(t)} = \int_{t_e + T_e}^{t_e + T_o} \frac{dt}{a(t)} \]

where the period \( T = \lambda \) (remember \( c = 1 \)). You can use \( T_{e,o} \ll t_{e,o} \) to show that (exercise)

\[ \frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} = 1 + z \]
Solution of Previous Exercise

For an observer observing the crest of a light wave at a position \( r = 0 \) and time \( t = t_{\text{now}} \), the crest of the light wave was emitted at a time \( t = t_{\text{then}} \) in the past and a distant position \( r = R \). Integrating over the path in both space and time that the light wave travels yields:

\[
 c \int_{t_{\text{then}}}^{t_{\text{now}}} \frac{dt}{a} = \int_{R}^{0} \frac{dr}{\sqrt{1 - kr^2}}.
\]

In general, the wavelength of light is not the same for the two positions and times considered due to the changing properties of the metric. When the wave was emitted, it had a wavelength \( \lambda_{\text{then}} \). The next crest of the light wave was emitted at a time

\[
 t = t_{\text{then}} + \frac{\lambda_{\text{then}}}{c}.
\]

The observer sees the next crest of the observed light wave with a wavelength \( \lambda_{\text{now}} \) to arrive at a time

\[
 t = t_{\text{now}} + \frac{\lambda_{\text{now}}}{c}.
\]

Since the subsequent crest is again emitted from \( r = R \) and is observed at \( r = 0 \), the following equation can be written:

\[
 c \int_{t_{\text{then}}}^{t_{\text{now}} + \lambda_{\text{now}}/c} \frac{dt}{a} = \int_{R}^{0} \frac{dr}{\sqrt{1 - kr^2}}.
\]

The right-hand side of the two integral equations above are identical which means

\[
 c \int_{t_{\text{then}}}^{t_{\text{now}} + \lambda_{\text{now}}/c} \frac{dt}{a} = c \int_{t_{\text{then}}}^{t_{\text{now}}} \frac{dt}{a}.
\]

or, alternatively,

\[
 \int_{t_{\text{now}}}^{t_{\text{now}} + \lambda_{\text{now}}/c} \frac{dt}{a} = \int_{t_{\text{then}}}^{t_{\text{then}} + \lambda_{\text{then}}/c} \frac{dt}{a}.
\]

For very small variations in time (over the period of one cycle of a light wave) the scale factor is essentially a constant (\( a = a_{\text{now}} \) today and \( a = a_{\text{then}} \) previously). This yields

\[
 \frac{t_{\text{now}} + \lambda_{\text{now}}/c}{a_{\text{now}}} = \frac{t_{\text{then}} + \lambda_{\text{then}}/c}{a_{\text{then}}}
\]

which can be rewritten as

\[
 \frac{\lambda_{\text{now}}}{\lambda_{\text{then}}} = \frac{a_{\text{now}}}{a_{\text{then}}}.
\]
Redshift revisited

The ratio

\[ \frac{\lambda_0 - \lambda_e}{\lambda_e} = z \]

is what we called the redshift factor \( z \).

Thus

\[ a(t_e) = \frac{1}{1 + z} \]

The light emitted at time \( t_e \) by an object at cosmological distance is redshifted by a factor \( z \).

Conversely, the redshift gives the size scale of the universe at the time of emission.
Redshift as a measure of the size of the Universe

\[ \frac{a(t\text{then})}{a(t\text{now})} = \frac{1}{1+z} \]

- \( z=1 \Rightarrow \frac{a\text{then}}{a\text{now}} = 0.5 \)
  - at \( z=1 \), the universe had 50% of its present day size
  - emitted blue light (400 nm) is shifted all the way through the optical spectrum and is received as red light (800 nm)

- \( z=4 \Rightarrow \frac{a\text{then}}{a\text{now}} = 0.2 \)
  - at \( z=4 \), the universe had 20% of its present day size
  - emitted blue light (400 nm) is shifted deep into the infrared and is received at 2000 nm
  - most distant astrophysical object discovered so far: \( z=5.8 \)

- \( z = 1100 \Rightarrow \frac{a\text{then}}{a\text{now}} \sim 10^{-3} \) of its present size. This is the ‘Decoupling time’. We cannot ‘see’ further than that. Before that photons do not travel freely... however there are many other ways to infer earlier times (elements abundance, features in clusters and galaxies, gravitational waves, etc...)

- \( z > 10^5 \), Universe was dominated by relativistic matter.

- \( z \sim 10^{10} \) Inflation
Luminosity Distance \[ D_L \equiv \sqrt{\frac{L}{4\pi F}} \]

\[ F = \frac{L}{4\pi d^2} \]

\( L \) - luminosity, \( F \) - Flux, \( d \) - Distance

Incorporating the effects of cosmological redshift and time dilation into \( F \) one gets (Home Exercise!):

\[ D_L = a(t_0)r(1 + z) \]
**Luminosity Distance**

\[ D_L \equiv \sqrt{\frac{L}{4\pi F}} \]

\[ F = \frac{L}{4\pi d^2} \]

L - luminosity, \hspace{1cm} F - Flux, \hspace{1cm} d - Distance

Incorporating the effects of cosmological redshift and time dilation into \( F \) one gets (Home Exercise!):

\[ D_L = a(t_0)r(1 + z) \]

<table>
<thead>
<tr>
<th>( z )</th>
<th>( v/c )</th>
<th>Look-back time (10^9 years)</th>
<th>“Distance” (Mpc)</th>
<th>Luminosity Distance (Mpc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.095</td>
<td>1.29</td>
<td>394</td>
<td>455</td>
</tr>
<tr>
<td>0.25</td>
<td>0.22</td>
<td>2.92</td>
<td>895</td>
<td>1249</td>
</tr>
<tr>
<td>0.5</td>
<td>0.385</td>
<td>5.02</td>
<td>1540</td>
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<tr>
<td>1.0</td>
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</tr>
<tr>
<td>2.0</td>
<td>0.80</td>
<td>10.3</td>
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<tr>
<td>5.0</td>
<td>0.946</td>
<td>12.5</td>
<td>3826</td>
<td>47610</td>
</tr>
<tr>
<td>inf.</td>
<td>1.0</td>
<td>13.67</td>
<td>4194</td>
<td>infinity</td>
</tr>
</tbody>
</table>

We assume here that \( H_0 = 71 \) km/sec, \( \Omega_M = 0.27 \), and \( \Omega_L = 0.73 \).

**Note:** The “Distance” above is just the Look-back time multiplied by the speed of light. This is not really the distance astronomers quote for an object in the distant universe. Astronomers use the "Luminosity Distance" to tell how bright an object appears (the inverse square law). This is listed in the last column. There are two reason why these values are different, the galaxy is moving away from us (which causes photons to be redshifted and a time dilation effect) and the geometry of the Universe is not necessarily Euclidean.
The angular diameter distance to an object is defined in terms of the object's actual size, \( D \), and \( \Delta \theta \) the angular size of the object as viewed from earth.

\[
d_A = \frac{D}{\Delta \theta}
\]

The angular diameter distance to an object at redshift, \( z \), is expressed in terms of the comoving distance, \( \chi \) as:

\[
d_A = \frac{r(\chi)}{1 + z}
\]

\[
r(\chi) = \begin{cases} 
\sin \left( \sqrt{-\Omega_k H_0 \chi} \right) / \left( H_0 \sqrt{|\Omega_k|} \right) & \Omega_k < 0 \\
\chi & \Omega_k = 0 \\
\sinh \left( \sqrt{\Omega_k H_0 \chi} \right) / \left( H_0 \sqrt{|\Omega_k|} \right) & \Omega_k > 0 
\end{cases}
\]

Where \( \Omega_k \) is the curvature density

\[
\Omega_k = -\frac{k c^2}{a^2 H_0^2}
\]

and \( H_0 \) is the value of the Hubble parameter today.

Exercise: Show the relation above then show that:

\[
\frac{dL}{d_A} = (1 + z)^2
\]

Hint: Use the RW line element to compute the object actual size \( D \) (term proportional to \( \Delta \theta \)), then use relation between luminosity distance and redshift.
"I think you should be more explicit here in step two."
"You want proof? I'll give you proof!"
Equations that Describe our Universe Dynamics

B. Dynamics

The universe is considered to be filled by an homogeneous & isotropic fluid = ideal fluid.

An ideal fluid is described by

\[ p(t) \text{ its energy density and } p(t) \text{ its pressure} \]

Depending on the context, the cosmic fluid is made of elementary particles, massive (non-relativistic) or massless (or relativistic) or whole galaxies (treated as point-like objects).
Consider a spherical region of the universe, of mass \( M = \frac{4}{3}\pi \rho d^3 \), and a test galaxy of mass \( m \).

![Diagram of a spherical region of the universe with a test galaxy]

If you forget about the rest of the universe, the force felt by the test galaxy is

\[
 m \ddot{d} = -\frac{GMm}{d^2} = -\frac{4\pi G}{3c^2} m \rho d
\]

where \( G \) is Newton's constant.

(In high energy units \( G = 1/M_{\text{Planck}}^2 \) with \( M_{\text{Planck}} = 1.2 \cdot 10^{19} \) GeV)
Raychaudhuri Equation

Drop the m (principle of equivalence), drop the χ in \( \dot{a} = a(t) \chi \). You get

The Raychaudhuri equation

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \rho
\]

This is the correct equation for non-relativistic matter. If pressure (\( \equiv \) kinetic energy density) is important, then

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\rho + 3p)
\]

The effective gravitational mass-energy is thus \( \rho + 3p \). Pressure, if positive, is as attractive as matter.

Important:

If \( \rho + 3p > 0 \) (normal matter or radiation) the expansion of the universe is decelerating (slowing down).

... otherwise the universe accelerates...
General Relativity, ‘gravity’ is sourced by Mass/Potential Energy and Pressure (Kinetic Energy):

$$\nabla^2 \Phi = -\frac{4\pi G}{3} (\rho + 3P)$$

$P = \frac{1}{3} \rho$

$P = 0$

$P < -\frac{1}{3} \rho$
fluid energy conservation:
The expansion of RW universe is adiabatic (no entropy creation). Then an element of fluid of volume $V$ must satisfy $(TdS = dE + pdV)$

$$dE \equiv \rho dV + Vd\rho = -pdV$$  with  $V \propto a^3$

Then

$\dot{\rho} = -3H(\rho + p)$

Three extreme types of fluid are usually envisioned:

Dust, non-relativistic matter:

$$p = 0 \quad \rightarrow \quad \rho \propto a^{-3}$$  (Exercise!)

Radiation, relativistic matter:

$$p = \frac{\rho}{3} \quad \rightarrow \quad \rho \propto a^{-4}$$  (Exercise!)

Cosmological constant, dark energy:

$$p = -\rho \quad \rightarrow \quad \rho = \text{const.}$$  (Exercise!)
The Friedmann equation

Combining the Raychaudhuri equation and energy conservation, you get (up to an integration constant, exercise) the most important equation of cosmology (I set $c = 1$)

$$H^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2}$$

To match the integration constant with the curvature $K$, you unfortunately need General Relativity.
C. Cosmological Solutions

What is Friedmann good for?

1. The Friedmann equation relates three important observable quantities: the Hubble parameter $H$, the total energy density $\rho$ and the curvature $K$.

Dividing the LHS and the RHS of Friedmann by $H^2$ gives the most important equation of cosmology ;-) 

$$1 = \Omega - \frac{K}{a^2H^2}$$

where 

$$\Omega = \frac{\rho}{\rho_c}$$

with $\rho_c$ is the **critical density** defined as

$$\rho_c = \frac{3H^2}{8\pi G}$$
Friedmann cosmology - basic equations

Einstein equation:
\[ R_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} \equiv G_{\mu\nu} = -8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \]

\( G_{\mu\nu} - \) Einstein tensor, \( R_{\mu\nu} - \) Ricci tensor, \( \mathcal{R} - \) curvature scalar

Energy-momentum tensor \( T^\mu_\nu = (\rho, -p, -p, -p) \)

Description of cosmology: two equations

0-0 component (Friedmann equation):
\[ H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \]

i-i component:
\[ 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G \rho \]

Alternative forms: derived from above equations

scale factor acceleration equation:
\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) \]

stress energy conservation equation:
\[ \dot{\rho} + 3H(\rho + p) = 0 \]
The Universe is expanding into what?

It is the space itself that is expanding? Yes.

Are rulers expanding? No, only gravitationally independent systems participate in the expansion!

The Hubble law is a linear expansion law which generates an homologous expansion (it is the same as seen from every Galaxy)

The expansion looks the same as seen from A or from B

\[ V = c \cdot z = H \cdot d \quad (H = 70 \text{ km s}^{-1} \text{ Mpc}^{-1} \pm 10\%) \]

- Get Hubble’s law if the galaxy distribution expands uniformly
- No outside to the expansion

\[ H = 70 \pm 7 \text{ km/sec Mpc} \]
The Friedmann equation

Combining the Raychaudhuri equation and energy conservation, you get (up to an integration constant, exercise) the most important equation of cosmology (I set $c = 1$)

$$H^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2}$$

(Exercise!)

To match the integration constant with the curvature $K$, you unfortunately need General Relativity.
Friedmann Equation

- A classical example is useful in illustrating the meaning of $K$ in the Friedmann equation.

- Consider a point mass $m$ at distance $R$ from the Earth being attracted by the mass of within a sphere of radius $R$, with mass density $\rho$. The total attractive mass is $M = 4\pi R^3 \rho / 3$.

- The Newtonian equation of motion ($ma=F$) is

  $$m \ddot{R} = -\frac{M m G_N}{R^2}$$

- Integration yields

  $$\frac{1}{2} m \dot{R}^2 - \frac{m M G_N}{R} = \text{constant}$$
The Friedmann Equation

\[ \frac{1}{2} m \ddot{R}^2 - \frac{m M G_N}{R} = \text{constant} \quad \text{where } M = 4\pi R^3 \rho / 3 \]

- The first term of this classical solution is the kinetic energy and the second term is the potential energy, so the constant is the total energy.

- Compare this to the Friedmann equation

\[ H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G_N \rho}{3} - \frac{K c^2}{R^2} \]

- We see that the total energy is \(-\frac{1}{2} K c^2 m\), revealing the meaning of K:
  - -1, positive energy, open, expanding without bound
  - +1, negative energy, closed, recollapses
\[ E = \frac{1}{2} \dot{a}^2 - \frac{4\pi G}{3} \rho a^2 = -\frac{K}{a^2} \]

- **E > 0**: Hyperbolic
- **E = 0**: Flat
- **E < 0**: Spherical

Catastrophe potential \( V \propto a^{-1} \)

Directly analogous to Newton's escape velocity problem...
Curvature of the space-time continuum \( k \):

\[
\begin{align*}
\text{k > 0} & \quad \text{positive curvature} \\
\text{k = 0} & \quad \text{zero curvature} \\
\text{k < 0} & \quad \text{negative curvature}
\end{align*}
\]

**Closed**

- Sum of angles > \( 180^\circ \)

**Flat**

- Sum of angles = \( 180^\circ \)

**Open**

- Sum of angles < \( 180^\circ \)
Friedmann-Lemaître universe, $k = 1$: 

\[ R \]

[Image of graph showing expansion and contraction of the universe over time from the Big Bang.]
Einstein-de Sitter universe, $k = 0$: 

Time

Big Bang

$R$
Friedmann-Lemaître universe, $k = -1$: 

![Graph showing the expansion of the universe over time from the Big Bang.]
Universe started with $R=0$ at a finite time in the past (Big Bang). Space and time come into existence during Big Bang, which happens everywhere in space. Space is moving apart according to Hubble's distance-velocity relationship:

$$v = H_0 \times d$$
New interpretation of redshifts: Galaxies get further apart because the space between them is physically expanding.

As space expands, the wavelength of light expands.

Cosmological redshifts are due to the cosmological expansion of wavelength of light, not the Doppler shifts from galaxy motions.

Redshifts: \[ z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} \]

New: \[ \lambda_{\text{obs}} = \left( \frac{R_{\text{Earth}}}{R_{\text{galaxy}}} \right) \lambda_{\text{em}} \]

We obtain: \[ z = \frac{R_{\text{Earth}}}{R_{\text{galaxy}}} - 1 \]
The real (proper) distance of galaxies is

\[ d = R \times (\text{coordinate distance}) \]

or

\[ d = R(t) \times D \]

where \( D \) is the co-moving distance.

The expansion velocity is

\[ v = \dot{R} \times (\text{coordinate distance}) \]
Hubble relation:

\[ v = H_0 \times d \]

Insertion of

\[ d = R \times (\text{coordinate distance}) \]

in

\[ v = \dot{R} \times (\text{coordinate distance}) \]

gives

\[ H_0 = \frac{\dot{R}}{R} \]

or

\[ H_0 = \frac{1}{R} \times \frac{dR}{dt} \]

The Hubble constant is a scaling factor which gives the present-day expansion rate of the universe.
The Friedmann Equation can take the form

$$\left( \frac{dR}{dt} \right)^2 = \frac{8\pi G}{3} \rho R^2 - kc^2$$

which gives (dividing by $R^2$)

$$H^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{R^2}$$

Have a close look at this equation:

- $H^2$ must be positive $\Rightarrow$ right-hand side of equation must be positive.
- In case density $\rho = 0$:
  - $k$ must be negative ($k = -1$).
  This implies that empty universes have a negative curvature.
  Flat or spherical universes must contain matter.
In a flat universe \((k = 0)\) the Friedmann equations

\[
H^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{R^2}
\]

becomes

\[
H^2 = \frac{8\pi G}{3} \rho
\]

Solving for the density \(\rho\) gives

\[
\rho = \rho_{\text{crit}} = \frac{3H^2}{8\pi G}
\]

which is defined as the \textbf{critical density}. 

We can generalize this by defining the density parameter $\Omega$:

$$\Omega = \frac{\rho}{\rho_c}$$

With the density parameter $\Omega$ the Friedmann equation can be rewritten as:

$$\Omega = 1 + \frac{k c^2}{H^2 R^2}$$
Three possible cases for this equation:

\[ \Omega = \frac{\rho}{\rho_c} \]

\[ \Omega < 1: \quad (\rho < \rho_c) \]

**hyperbolic Universe**

expansion forever

\[ \Omega = 1: \quad (\rho = \rho_c) \]

**flat Universe**

expansion forever (slowing)

\[ \Omega > 1: \quad (\rho > \rho_c) \]

**spherical Universe**

expansion and collapse

Physical interpretation:

If the matter density of the Universe (matter per cubic space) is more than a certain 'critical density', gravity will lead to a re-collapse of the Universe.
Exercise: Write the Raychaudhuri Equation in terms of q

The deceleration parameter $q$ gives a measure of how fast the Universe is decelerating:

$$ q = \frac{1}{2} \Omega $$

The critical density, or matter content of the Universe, can be measured either directly by determining how much mass is in the Universe, or by measuring the deceleration parameter $q$. 
\[ q = \frac{1}{2} \Omega \]

\[ k = -1 \quad q < \frac{1}{2} \quad \Omega < 1 \quad (\rho < \rho_c) \]
open, hyperbolic Universe

\[ k = 0 \quad q = \frac{1}{2} \quad \Omega = 1 \quad (\rho = \rho_c) \]
open, flat Universe

\[ k = 1 \quad q > \frac{1}{2} \quad \Omega > 1 \quad (\rho > \rho_c) \]
closed, spherical Universe
The Hubble time $t_H = 1 / H_0$ is the cosmic time since the Big Bang in case the Universe is expanding at a constant rate.
The age of the Universe is always less than the Hubble time in standard models.
Static Universes

Einstein was a firm believer that the Universe is static (i.e., no evolution).

A static universe has to be a closed universe with positive curvature $k>0$.

A static universe has to be in an equilibrium of forces. An unknown force needs to balance gravity.

Introduction of the Cosmological Constant $\Lambda$ proposed by Einstein in 1917.

Static Einstein de Sitter Universe has a spherical space (closed and finite) and contains a force $\Lambda$ which opposes gravity.
Peculiar properties of a static universe:

- Antipode
- Observer

Static Spherical Space
Peculiar properties of a static universe:

- A body moving away from an observer gets smaller, but when half way through the antipode, it gets bigger as it recedes.

- Light from the antipode is magnified (multiple right rays). A body at the antipode appears as if very nearby.

- Because light circumnavigates the “cosmic globe”, we can see the backs of our own heads.

Assume Universe with a density of water and size of the solar system: time to circum-navigate the universe is \((4/\rho)^{1/2} = 2\) hrs (for \(\rho = 1\) g/cm\(^3\)).

You can see what you did 2hrs, 4hrs, 6hrs, etc, ago!

If density equal to atmosphere: you can see you ancestors!
Paradox: Any angle from the earth the sight line will end at the surface of a star. To understand this we compare it to standing in a forest of white trees. If at any point the vision of the observer ended at the surface of a tree, wouldn’t the observer only see white? This contradicts the darkness of the night sky and leads many to wonder why we do not see only light from stars in the night sky.

Exercise: solve this paradox!

Hint: Think about:
- Finite Speed of light,
- Finite age of the Universe
- and most importantly: Expanding Universe!
Definitions

- **Standard Cosmological Model:**
  Homogeneous and isotropic Universe that started in the Big Bang and expands ($\Lambda = 0$).

- **Critical Density $\rho_{\text{crit}}$:**
  Average density of the Universe needed to make the Universe flat.

- **Density Parameter $\Omega$:**
  Defined as $\Omega = \rho / \rho_{\text{crit}}$ determines if the critical density has been reached.

- **Cosmological Constant $\Lambda$:**
  Acts as repulsive force to gravity (set to zero in most scenarios).

- **Deceleration Parameter $q$:**
  Defined as $q = \frac{1}{2} \Omega$, gives a measure of how fast the Universe is decelerating.

- **Hubble Constant $H_0$:**
  Defined as $H_0 = \text{velocity} / \text{distance}$.

- **Hubble Time $t_H$:**
  Defined as $t_H = 1 / H_0$ is the cosmic time since the Big Bang.
Friedmann Equation

For $K = 0$, \[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G_N \rho}{3} = 2 \frac{G_N M}{R^3}
\]

\[
R = \left( \frac{9G_N M}{2} \right)^{1/3} t^{2/3}
\]

(Exercise: Repeat calculation for $k=1$, $k=-1$)

therefore, the value of the Hubble constant today:

\[ H_0^{-1} = \frac{R}{\dot{R}} = 3t_0/2 \]

\[ t_0 = \frac{1}{\sqrt{6\pi G_N \rho_0}} = \frac{2}{3} H_0^{-1} = \frac{6.6}{h_0} \text{ Gyr} = 8-11 \text{ Gyr for } h_0 = 0.7 \pm 0.1 \]
Friedmann Equation

\[ t_0 = \frac{1}{\sqrt{6\pi G_N \rho_0}} = \frac{2}{3} H_0^{-1} = \frac{6.6}{h_0} \text{ Gyr} = 8-11 \text{ Gyr for } h_0 = 0.7 \pm 0.1 \]

- but this age is a problem for other observations
  - eg. The ages of globular clusters
  - white dwarf cooling rates
  - uranium isotope dating
- these suggest \( t_0 = 10-14 \text{ Gyr} \), marginally consistent with Hubble's constant

- But, \( K \) and \( \Lambda \) need not be 0, which will change the estimate
  - current thinking based on observations is that \( \Lambda \) is not 0
Equation of State: Relation between Pressure and the Energy Density

need to relate $P$ and $\rho$

• this relation depends on the substance under consideration
• it is called the equation of state of the substance

• some useful equations of state:
  • non-relativistic gas: $P = nk_B T = \rho k_B T / \mu$ where $\mu$ is particle mass
  • since $3k_B T = \mu v^2$, we have:

$$\frac{P}{\rho} = \frac{v^2}{3c^2} \ll 1, \quad P_m \approx 0$$

• radiation (ultra-relativistic):

$$\frac{P}{\rho} = \frac{1}{3}, \quad P_r = \frac{1}{3} \rho_r$$

• Cosmological Constant($\Lambda$): as its energy density is constant with time:

$$\dot{\rho} = -3H(\rho + P) = 0 \quad \Rightarrow \quad P_\Lambda = -\rho_\Lambda$$

• this gives acceleration, since $\rho + 3P < 0$ \quad $\Rightarrow$ \quad $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$
The fluid equation revisited

\[ \dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0 \]

- **Radiation**
  - \( P_r = \frac{1}{3}\varepsilon_r \rightarrow \)
  - \( \varepsilon_r \propto a^{-4} \)

- **Nonrelativistic matter**
  - \( P_m \approx 0 \rightarrow \)
  - \( \varepsilon_m \propto a^{-3} \)

- **Λ**
  - \( P_\Lambda = -\varepsilon_\Lambda \rightarrow \)
  - \( \varepsilon_\Lambda = \text{constant} \)

- **More general form**
  - \( P = w\varepsilon \rightarrow \)
  - \( \varepsilon \propto a^{-3(w+1)} \)
Friedmann cosmology - useful solutions

Radiation dominated (RD):
\[ p = \frac{1}{3} \rho \Rightarrow \rho \propto a^{-4} \quad \text{and} \quad a(t) \sim t^{1/2} \]

Matter dominated (MD):
\[ p = 0 \Rightarrow \rho \propto a^{-3} \quad \text{and} \quad a(t) \sim t^{2/3} \]

Vacuum:
\[ p = -\rho \Rightarrow \rho = \text{constant} \quad \text{and} \quad a(t) \sim \exp(\text{constant}' \ast t) \]
Evolution of global geometry

Recast Friedmann equation: \[ \frac{k}{H^2 a^2} = \frac{\rho}{3H^2/8\pi G} - 1 \equiv \Omega - 1 \]

\[ \Omega \equiv \frac{\rho}{\rho_C}, \quad \rho_C \equiv \frac{3H^2}{8\pi G} \] ("critical density")

\[ k = +1 \quad \Rightarrow \quad \Omega > 1 \quad \text{closed} \]
\[ k = 0 \quad \Rightarrow \quad \Omega = 1 \quad \text{flat} \]
\[ k = -1 \quad \Rightarrow \quad \Omega < 1 \quad \text{closed} \]

Behavior

\[ |\Omega - 1| \ll \frac{a}{a_0} \quad \text{MD} \]
\[ \ll \left(\frac{a}{a_0}\right)^2 \quad \text{RD} \]
\[ \ll \left(\frac{a_0}{a}\right)^2 \quad \text{Vacuum} \]
**Cosmological distances**

- Proper distance between origin and object:
  - \( ds^2 = -c^2dt^2 + a^2(t)[dr^2 + x(r)^2d\Omega^2] \) (R-W metric)
  - \( d_P(t) = a(t) \int dr = a(t) r \) \((d_P \text{ is not a comoving distance})\)
  - \( r = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \) for light emitted at \( t_e \) and observed at \( t_0 \)
  - Therefore the proper distance to an object at time \( t_0 \) is \( d_P(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \)
  - If \( t_e = 0 \) we call this the **horizon distance** – it’s the furthest we can currently see
Cosmological models

\[ \dot{a}(t)^2 = \frac{8\pi G}{3c^2} \left( \frac{\varepsilon_{r0}}{a(t)^2} + \frac{\varepsilon_{m0}}{a(t)} \right) - \frac{kc^2}{R_0} + \frac{\Lambda}{3} a(t)^2 \]

- Different components of energy density just add:
  - note different \( a \) dependence
  - at small \( a \), radiation must dominate
  - matter takes over when \( a > \varepsilon_{r0}/\varepsilon_{m0} \)
  - at large \( a \), cosmological constant dominates if it exists

- Therefore sensible to consider single components

Monday, September 7, 2009
Radiation only

\[
a \, da = \sqrt{\frac{8\pi G \varepsilon_r}{3c^2}} \, dt
\]

- \( a = (t/t_0)^{1/2} \) and \( \varepsilon_r \propto t^{-2} \)
- age of universe: \( \ln a = \frac{1}{2}(\ln t - \ln t_0) \rightarrow H = 1/2t \)
  - so \( t_0 = 1/2H_0 \)
- proper distance:

\[
d_P(t_0) = c t_0^{1/2} \int_{t_e}^{t_0} \frac{dt}{\sqrt{t}} = 2ct_0 \left( 1 - \sqrt{\frac{t_e}{t_0}} \right) = \frac{c}{H_0} \left( \frac{z}{1+z} \right)
\]

\[
d_P(t_e) = d_P(t_0)/(1 + z)
\]
Matter only

\[ \sqrt{a} \, da = \sqrt{\frac{8\pi G \varepsilon m_0}{3c^2}} \, dt \]

- \( a = \left(\frac{t}{t_0}\right)^{2/3} \) and \( \varepsilon_m \propto t^{-2} \)
- age of universe: \( \ln a = \frac{2}{3}(\ln t - \ln t_0) \rightarrow H = 2/3t \)
  - so \( t_0 = 2/3H_0 \)
- proper distance:
  \[
  d_P(t_0) = c t_0^{2/3} \int_{t_e}^{t_0} \frac{dt}{t^{2/3}} = 3ct_0 \left( 1 - \left(\frac{t_e}{t_0}\right)^{1/3} \right) = \frac{2c}{H_0} \left( 1 - \frac{1}{\sqrt{1+z}} \right)
  \]
  \[
  d_P(t_e) = d_P(t_0)/(1+z)
  \]
**Curvature only**

\[ da = \sqrt{-\frac{k c^2}{R_0^2}} \, dt \]

- If \( k = 0 \), \( a = \text{constant} \): flat, static, empty universe
- If \( k = -1 \), \( a \propto t \): universe expands at constant speed
  - Milne model
  - Age = \( 1/H_0 \)
  - Proper distance \( d_P(t_0) = ct_0 \ln(1+z) \)
- \( k = +1 \) does not produce a physically viable model
\[ \frac{da}{a} = \sqrt{\frac{\Lambda}{3}} \, dt = H_0 \, dt \]

- \( a = \exp[H_0(t - t_0)] \) : universe expands exponentially
  - de Sitter model
  - infinitely old: \( a \to 0 \) only as \( t \to -\infty \)
  - proper distance \( d_P(t_0) = cz/H_0 \)
- this is a “Steady State” universe which always looks the same
Single component universes

Expansion

Proper distance

Proper distance $d(t)$

Radiation
Matter
Curvature
Lambda
Multi-component universes

\[ \dot{a}(t)^2 = H_0^2 \left( \frac{\Omega_{r0}}{a(t)^2} + \frac{\Omega_{m0}}{a(t)} + \left(1 - \Omega_{r0} - \Omega_{m0} - \Omega_{\Lambda0}\right) + \Omega_{\Lambda0} a(t)^2 \right) \]

- “This is not a user-friendly integral” (Ryden)
  - fortunately at different times different components will dominate
    - best current values: \( \Omega_{m0} = 0.27, \Omega_{\Lambda0} = 0.73, \Omega_{r0} = 8.4 \times 10^{-5} \)
    - matter-radiation equality at \( a = \Omega_{r0}/\Omega_{m0} = 0.0003 \)
    - matter-\( \Lambda \) equality at \( a = (\Omega_{m0}/\Omega_{\Lambda0})^{1/3} = 0.72 \)
  - at any given time can usually use single-component model
$\log(\text{density})$

$k=0$

- Radiation: $a(t) \sim t^{1/2}$; $\rho_{\text{rad}} \sim 1/t^2$
- Matter: $a(t) \sim t^{2/3}$; $\rho_{\text{m}} \sim 1/t^2$

$\log(t)$
Example: matter + $\Lambda$

Expansion rate

- $\Lambda$ dominated: slope +1
- Matter dominated: slope $-1/2$

Model with $\Omega_{\Lambda 0} = 0.7$, $\Omega_{m0} = 0.3$
Example: matter + $\Lambda$

- model with $\Lambda=0$
- model with $\Lambda=0.73$
Friedmann model plus cosmological constant can describe wide variety of behaviour

- expanding, recollapsing or static
- also “bouncing” and “loitering” models
- this technology all available in 1920s

However, models have free parameters

- $H_0, \Omega_m, \Omega_r, \Omega_\Lambda$
- need to determine these to see what model predicts for our universe
$K = 0$ solutions, also valid for Early Universe

**Matter**

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad H = \frac{2}{3t} \quad D_H = \frac{2}{H}$$

**Radiation**

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \quad H = \frac{1}{2t} \quad D_H = \frac{1}{H}$$

$\Lambda$

$$a(t) = a(t_i) \exp(H(t-t_i)) \quad H = \text{const} \quad D_H \to \infty$$