Suggested solutions, exam in AST4220, autumn 2003

Problem 1

a) The equation is, in units where $c = 1$,

$$\dot{\rho}_i = -3H(\rho_i + p_i).$$

Dust: In this case $p_m = 0$, and we have

$$\dot{\rho}_m = -3H\rho_m,$$

and since $H = \dot{a}/a$ we get the equation

$$\frac{d\rho_m}{dt} = -3\frac{1}{a}\frac{da}{dt}\rho_m,$$

which gives

$$\int_{\rho_{m0}}^{\rho_m} \frac{d\rho_m}{\rho_m} = -3\int_{a_0}^{a}\frac{da}{a},$$

and therefore

$$\ln\left(\frac{\rho_m}{\rho_{m0}}\right) = -3\ln\left(\frac{a}{a_0}\right) = \ln\left(\frac{a_0}{a}\right)^3,$$

and furthermore

$$\rho_m = \rho_{m0}\left(\frac{a_0}{a}\right)^3 = \rho_{m0}(1+z)^3.$$

Radiation: In this case $p_r = \rho_r/3$, and we get

$$\dot{\rho}_r = -3H\left(\rho_r + \frac{1}{3}\rho_r\right) = -4H\rho_r.$$

This equation can be solved in the same manner as in the case of dust, with the result

$$\rho_r = \rho_{r0}\left(\frac{a_0}{a}\right)^4 = \rho_{r0}(1+z)^4.$$

Cosmological constant: In this case $p_\Lambda = -\rho_\Lambda$, so that

$$\dot{\rho}_\Lambda = -3H(\rho_\Lambda - \rho_\Lambda) = 0,$$

and therefore

$$\rho_\Lambda = \rho_{\Lambda 0} = \text{ konstant}.$$

b) The number of internal degrees of freedom for a photon is equal to 2, while a neutrino only has one internal degree of freedom. In addition, neutrinos come in three different flavours, and each neutrino has a corresponding antineutrino. Here ρ is an energy density, so we find

$$\begin{aligned}
\rho_{r0}c^2 &= \frac{\pi^2}{3}\left[2 \times \frac{(k_B T_0)^4}{(\hbar c)^3} + \frac{7}{8} \times 3 \times 2 \times 1 \times \frac{(k_B T_{\nu 0})^4}{(\hbar c)^3}\right] \\
&= \frac{\pi^2}{15}\frac{(k_B T_0)^4}{(\hbar c)^3}\left[1 + \frac{21}{8}\left(\frac{4}{11}\right)^{4/3}\right] \\
&\approx 7.062 \times 10^{-14} \text{ Jm}^{-3}.
\end{aligned}$$

The critical energy density is given by

$$\rho_{c0}c^2 = 1.689 \times 10^{-9}\, h^2 \,\mathrm{J\,m^{-3}},$$

so that

$$\Omega_{r0} = \frac{\rho_{r0}c^2}{\rho_{c0}c^2} = 4.2 \times 10^{-5} h^{-2},$$

that is,

$$\Omega_{r0}h^2 = 4.2 \times 10^{-5}.$$

The energy density in dust was equal to that in radiation when

$$\Omega_{m0}(1 + z_{eq})^3 = \Omega_{r0}(1 + z_{eq})^4,$$

which gives

$$z_{eq} = \frac{\Omega_{m0}}{\Omega_{r0}} - 1 = 3.5 \times 10^3$$

for $\Omega_{m0} = 0.3$ and $h = 0.7$.

c) Starting from the first Friedmann equation we find

$$\begin{aligned}\frac{H^2}{H_0^2} &= \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 \\ &= \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r0}\left(\frac{a_0}{a}\right)^4 \\ &= \Omega_{r0}\left(\frac{a_0}{a}\right)^4\left[1+\frac{\Omega_{m0}}{\Omega_{r0}}\frac{a}{a_0}\right] \\ &= \Omega_{r0}a^{-4}\left(1+\frac{a}{a_{eq}}\right),\end{aligned}$$

where we have chosen $a_0 = 1$, as we are allowed to do for spatially flat models.

d) From the previous result we get, by taking the square root on both sides,

$$\frac{1}{aH_0}\frac{da}{dt} = \frac{\sqrt{\Omega_{r0}}}{a^2}\left(1+\frac{a}{a_{eq}}\right)^{1/2},$$

which we rearrange to

$$\frac{ada}{\sqrt{\Omega_{r0}}}\left(1+\frac{a}{a_{eq}}\right)^{-1/2} = H_0 dt.$$

We integrate on both sides,

$$H_0\int_0^t dt' = \frac{1}{\sqrt{\Omega_{r0}}}\int_0^a \frac{a'da'}{sqrt1+\frac{a'}{a_{eq}}}.$$

Introducing a new variable, $x = a'/a_{eq}$, we find

$$H_0 t = \frac{a_{eq}^2}{\sqrt{\Omega_{r0}}}\int_0^{a/a_{eq}}\frac{xdx}{\sqrt{1+x}}.$$

The integral we need is given in the text, and inserting the appropriate limits we find

$$H_0 t = \frac{a_{eq}^2}{\sqrt{\Omega_{r0}}}\left[\frac{2}{3}\left(1+\frac{a}{a_{eq}}\right)^{3/2} - 2\left(1+\frac{a}{a_{eq}}\right)^{1/2} - \frac{2}{3} + 2\right]$$

$$
\begin{aligned}
&= \frac{a_{eq}^2}{\sqrt{\Omega_{r0}}}\left[\frac{4}{3} + \left(1 + \frac{a}{a_{eq}}\right)^{1/2}\left(\frac{2}{3} + \frac{2}{3}\frac{a}{a_{eq}} - 2\right)\right] \\
&= \frac{a_{eq}^2}{\sqrt{\Omega_{r0}}}\left[\frac{4}{3} - \frac{4}{3}\left(1 + \frac{a}{a_{eq}}\right)^{1/2}\left(1 - \frac{a}{2a_{eq}}\right)\right] \\
&= \frac{4a_{eq}^2}{3\sqrt{\Omega_{r0}}}\left[1 - \left(1 - \frac{a}{2a_{eq}}\right)\left(1 + \frac{a}{a_{eq}}\right)^{1/2}\right].
\end{aligned}
$$

To find the age of the universe at $a = a_{eq}$ we insert $a = a_{eq}$ in the result above. This gives

$$H_0 t_{eq} = \frac{4}{3}\frac{a_{eq}^2}{\sqrt{\Omega_{r0}}}\left(1 - \frac{1}{\sqrt{2}}\right),$$

and if we now use $1 + z_{eq} = 1/a_{eq}$ together with the numbers given in the text, we find

$$t_{eq} = 4.8 \times 10^4 \text{ yrs.}$$

e) The redshift z_Λ is determined by the condition

$$\rho_{m0}(1 + z_\Lambda)^3 = \rho_{\Lambda 0},$$

which gives

$$1 + z_\Lambda = \left(\frac{\rho_{\Lambda 0}}{\rho_{m0}}\right)^{1/3} = \left(\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3},$$

and inserting the numbers gives

$$z_\Lambda = \left(\frac{0.7}{0.3}\right)^{1/3} - 1 = 0.3.$$

For the situation in question Friedmanns first equation becomes

$$\frac{H^2}{H_0^2} = \Omega_{m0} a^{-3},$$

(we have set $a_0 = 1$), which can be rearranged to

$$\Omega_{m0}^{-1/2} a^{1/2} da = H_0 dt.$$

We integrate this equation:

$$H_0 \int_0^{t_\Lambda} dt = \Omega_{m0}^{-1/2} \int_0^{a_\Lambda} a^{1/2} da = \frac{2}{3}\Omega_{m0}^{-1/2} a_\Lambda^{3/2},$$

and inserting numbers we get

$$t_\Lambda = 11 \times 10^9 \text{ yrs.}$$

f) See the lecture notes chapter 4.

g) In the radiation-dominated era $a \propto t^{1/2}$, so that $\dot{a}/a = 1/2t$, and with $\rho_m \approx 0$ the equation becomes

$$\frac{d^2\Delta}{dt^2} + \frac{1}{t}\frac{d\Delta}{dt} = 0.$$

Introducing $y = d\Delta/dt$ we get a first-order differential equation

$$\frac{dy}{dt} + \frac{y}{t} = 0,$$

which we rearrange to

$$\frac{dy}{y} = -\frac{dt}{t},$$

and upon integrating this we find

$$y = \frac{B_2}{t},$$

where B_2 is a constant of integration. Next we find Δ by another integration:

$$\Delta = \int y dt = B_2 \ln t + B_1,$$

where B_1 is a new constant of integration.

In a flat, matter-dominated universe $a \propto t^{2/3}$, $\dot{a}/a = 2/3t$, and

$$\rho_m = \rho_c = \frac{3H^2}{8\pi G} = \frac{1}{6\pi G t^2}.$$

Inserting this in the equation we get

$$\frac{d^2\Delta}{dt^2} + \frac{4}{3t}\frac{d\Delta}{dt} - \frac{2}{3t^2}\Delta = 0.$$

We seek solutions of the form $\Delta \propto t^n$. By substituting in the equation we get

$$n^2 - n + \frac{4}{3}n - \frac{2}{3} = 0,$$

which has two solutions, $n = -1$ and $n = 2/3$. The general solution is therefore

$$\Delta = C_1 t^{-1} + C_2 t^{2/3},$$

where C_1 and C_2 are constants.

In the last case we have $a \propto e^{H_\Lambda t}$, $\dot{a}/a = H_\Lambda$, and $\rho_m \approx 0$. The equation now becomes

$$\frac{d^2\Delta}{dt^2} + 2H_\Lambda \frac{d\Delta}{dt} = 0.$$

Again, we take $y = d\Delta/dt$, so that

$$\frac{dy}{dt} + 2H_\Lambda y = 0.$$

Rewriting this as

$$\frac{dy}{y} = -2H_\Lambda dt,$$

we find by integration

$$y = Ae^{-2H_\Lambda t}.$$

Finally, Δ is given by

$$\Delta = \int y dt = D_1 + D_2 e^{-2H_\Lambda t},$$

where D_1 and $D_2 = -A/2H_\Lambda$ are constants of integration.

h) Whether a perturbation grows or not is determined by the ratio of the timescale for gravitational collapse of a region of density ρ_m and the expansion time scale for the universe. The collapse time is

$$\tau_c \propto (G\rho_m)^{-1/2},$$

while the expansion time is

$$\tau_H \propto (G\rho_i)^{-1/2},$$

where ρ_i er komponenten som dominerer universets energitetthet. In the radiation dominated case this is ρ_r, and for the de Sitter universe it is ρ_Λ. In both cases we have $\rho_i \gg \rho_m$, which implies $\tau_c \gg \tau_H$. The universe expands faster than a matter pertubation grows, and hence there is no significant growth.

Suggested solutions, exam in AST4220, fall 2004

Problem 1

a) We have

$$H^2(z) = H_0^2 \left[\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm k0}(1+z)^2 + \Omega_{\Lambda 0} \right].$$

b) From the condition $\Omega_{\rm m0} + \Omega_{\rm k0} + \Omega_{\Lambda 0} = 1$ we find $\Omega_{\rm k0} = -3/2$ and

$$H(z) = H_0 \sqrt{\frac{1}{2}(1+z)^3 - \frac{3}{2}(1+z)^2 + 2.}$$

The derivative vanishes when

$$H'(z) = H_0 \frac{\frac{3}{2}(1+z)^2 - 3(1+z)}{\sqrt{\frac{1}{2}(1+z)^3 - \frac{3}{2}(1+z)^2 + 2}} = 0,$$

and since $z \geq 0$, $z = 1$ is the only acceptable solution. We therefore have $z_{\rm loit} = 1$. This gives $H(z_{\rm loit}) = 0$.

c) The age of the universe is given by (see the lecture notes)

$$t_0 = \int_0^\infty \frac{dz}{(1+z)H(z)}.$$

Since the Hubble parameter for the loitering model is smaller than the Hubble parameter for the flat model (see the figure) we have

$$t_0^{\rm loit} > t_0^{\rm flat}.$$

d) For a universe with dust and curvature we have

$$H(z) = H_0 \sqrt{\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm k0}(1+z)^2}.$$

Therefore

$$H'(z) = H_0 \frac{3\Omega_{\rm m0}(1+z)^2 + 2\Omega_{\rm k0}(1+z)}{\sqrt{\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm k0}(1+z)^2}},$$

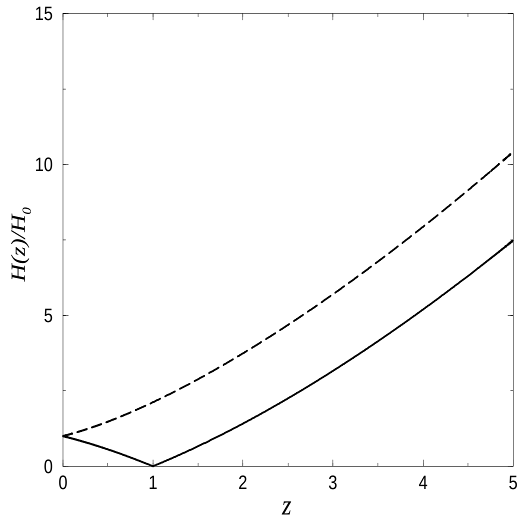

Figure 1: The Hubble parameter for the loitering model (full line) compared to the Hubble parameter for the flat model.

which vanishes for

$$1 + z_{\rm loit} = -\frac{2}{3}\frac{\Omega_{\rm k0}}{\Omega_{\rm m0}} = -\frac{2}{3}\frac{1-\Omega_{\rm m0}}{\Omega_{\rm m0}},$$

where we have used $\Omega_{\rm m0} + \Omega_{\rm k0} = 1$. We see that we must have $\Omega_{\rm m0} \geq 1$ in order to satisfy $1 + z_{\rm loit} \geq 0$. But we require $z_{\rm loit} \geq 1$, so

$$\frac{\Omega_{\rm m0} - 1}{\Omega_{\rm m0}} \geq \frac{3}{2}.$$

Since we know that $\Omega_{\rm m0} \geq 0$, this gives

$$2\Omega_{\rm m0} - 2 \geq 3\Omega_{\rm m0},$$

which gives the result $\Omega_{\rm m0} \leq -2$, and this is impossible. The conclusion is that it is impossible to have $z_{\rm loit} \geq 0$ in this model, and therefore there is no loitering phase.

e) In a universe with spatial curvature, dust and a cosmological constant, we have

$$H(z) = H_0\sqrt{\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm k0}(1+z)^2 + \Omega_{\Lambda 0}}.$$

Requiring $H'(z) = 0$ results in the same expression for $z_{\rm loit}$ as in d),

$$1 + z_{\rm loit} \geq -\frac{2}{3}\frac{\Omega_{\rm k0}}{\Omega_{\rm m0}},$$

but we now have $\Omega_{\rm k0} = 1 - \Omega_{\rm m0} - \Omega_{\Lambda 0}$. Since we at the very least must have $1 + z_{\rm loit} > 0$, we see that $\Omega_{\rm k0} < 0$, or, equivalently $\Omega_{\rm m0} + \Omega_{\Lambda 0} > 1$, and the universe is spatially closed. Furthermore, we must have $z_{\rm loit} \geq 0$, so

$$\frac{2}{3}\frac{\Omega_{\rm m0} + \Omega_{\Lambda 0} - 1}{\Omega_{\rm m0}} \geq 1,$$

which gives

$$\Omega_{\rm m0} + \Omega_{\Lambda 0} \geq \frac{3}{2}\Omega_{\rm m0} + 1,$$

that is

$$\Omega_{\Lambda 0} \geq \frac{1}{2}\Omega_{\rm m0} + 1.$$

Oppgave 2

a) For the EdS model we have $a(t) = a_0(t/t_0)^{2/3}$, $H(t) = \dot{a}/a = 2/(3t)$, and

$$\rho_{\rm m}(t) = \rho_{\rm c}(t) = \frac{3H^2}{8\pi G} = \frac{1}{6\pi G t^2}.$$

b) The equation for the time evolution of density perturbations is

$$\ddot{\Delta}_k + 2\frac{\dot{a}}{a}\dot{\Delta}_k = 4\pi G\rho_0\Delta_k.$$

The right hand side of this equation comes from Poisson's equation for the perturbations in the gravitational potential. We therefore see that ρ_0 must be the mean density of the matter that can form clumps, and this is given by $\rho_0 = (1-f_\nu)\rho_{\rm m}$. We insert this in the equation in a) and get

$$\ddot{\Delta}_k + \frac{4}{3t}\dot{\Delta}_k = 4\pi G\frac{1}{6\pi G t^2}(1-f_\nu)\Delta_k = \frac{2}{3}(1-f_\nu)\frac{\Delta_k}{t^2}.$$

c) Upon substition of the proposed solution in the equatin, we find that α must satisfy the quadratic equation

$$\alpha^2 + \frac{1}{3}\alpha - \frac{2}{3}(1-f_\nu) = 0,$$

and the positive root of this equation is (we must have $\alpha > 0$ to have a growing mode)

$$\begin{aligned}\alpha &= \frac{1}{2}\left[-\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{8}{3}(1-f_\nu)}\right] \\ &= \frac{1}{6}[\sqrt{25-24f_\nu} - 1] = \frac{1}{6}\left[5\sqrt{1-\frac{24}{25}f_\nu} - 1\right]\end{aligned}$$

For $f_\nu \ll 1$ we can use the power series $\sqrt{1-x} \approx 1 - x/2$, and this gives

$$\begin{aligned}\alpha &\approx \frac{1}{6}\left[5\left(1-\frac{12}{25}f_\nu\right) - 1\right] \\ &= \frac{2}{3}\left(1-\frac{3}{5}f_\nu\right).\end{aligned}$$

d) Since $a \propto t^{2/3}$, we can write $\Delta_k \propto a^{3\alpha/2} \propto (1+z)^{-3\alpha/2}$, and therefore

$$\begin{aligned}
\frac{\Delta_k(z=0)}{\Delta_k(z=z_{\rm eq})} &= (1+z_{\rm eq})^{3\alpha/2} = e^{\frac{3\alpha}{2}\ln(1+z_{\rm eq})} \\
&= e^{\frac{2}{3}32\left(1-\frac{3}{5}f_\nu\right)\ln(1+z_{\rm eq})} \\
&= e^{\ln(1+z_{\rm eq})}e^{-\frac{3}{5}f_\nu\ln(1+z_{\rm eq})} = (1+z_{\rm eq})e^{-\frac{3}{5}f_\nu\ln(1+z_{\rm eq})}.
\end{aligned}$$

e) For $\Omega_{\rm m} = 1$, $h = 0.5$ we get $1 + z_{\rm eq} = 5975$. The case $f_\nu = 0$ gives

$$\frac{\Delta_k(z=0)}{\Delta_k(z=z_{\rm eq})} = 1 + z_{\rm eq},$$

and for $f_\nu = 0.1$ we find

$$\frac{\Delta_k(z=0)}{\Delta_k(z=z_{\rm eq})} = (1+z_{\rm eq})e^{-\frac{3}{5}\times 0.1\times\ln 5975} = 0.59(1+z_{\rm eq}).$$

We therefore see that the growth of the perturbations is reduced by roughly 40 % if neutrinos make up 10 % of the dust.

Oppgave 3

a) For $p = w\rho$ we get the equation

$$\dot\rho = -3\frac{\dot a}{a}(1+w)\rho,$$

which gives

$$\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = -3(1+w)\int_1^a\frac{da'}{a'},$$

and thus

$$\rho = \rho_0 a^{-3(1+w)}.$$

b) Denoting the fluid by the subscript v, we have

$$\rho_v = \rho_{v0}a^{-3(1+w)} = \rho_{v0}a^{-2} = \Omega_{v0}\rho_{c0}(1+z)^2,$$

and

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left[\Omega_{\rm m0}(1+z)^3 + \Omega_{v0}(1+z)^2\right],$$

and furthermore

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho_\mathrm{m}+\rho_v+3p_v) = -\frac{4\pi G}{3}(\rho_\mathrm{m}+\rho_v-\rho_v) \\ &= -\frac{1}{2}H_0^2\Omega_{\mathrm{m}0}(1+z)^3.\end{aligned}$$

This gives

$$q = \frac{1}{2}\frac{\Omega_{\mathrm{m}0}(1+z)^3}{\Omega_{\mathrm{m}0}(1+z)^3+(1-\Omega_{\mathrm{m}0})(1+z)^2},$$

since $\Omega_{v0} = 1-\Omega_{\mathrm{m}0}$ because of spatial flatness.

c) In this case we find

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 &= H_0^2\left[\Omega_{\mathrm{m}0}(1+z)^3+\Omega_{\mathrm{k}0}(1+z)^2\right] \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\rho_{\mathrm{m}0}(1+z)^3 = -\frac{1}{2}H_0^2\Omega_{\mathrm{m}0}(1+z)^3.\end{aligned}$$

From the definition of q we now find

$$q = \frac{1}{2}\frac{\Omega_{\mathrm{m}0}(1+z)^3}{\Omega_{\mathrm{m}0}(1+z)^3+(1-\Omega_{\mathrm{m}0})(1+z)^2},$$

since $\Omega_{\mathrm{m}0}+\Omega_{\mathrm{k}0} = 1$. This is exactly the same result as in b), and it is therefore impossible to distinguish a spatially curved universe from a spatially flat one filled with a fluid with equation of state $p = -\rho/3$ based on the deceleration parameter alone. However, there are other measurements we could make in order to distinguish between the two models. For example, the angular diameter distance-redshift relationship will be different for the two models, even though they have the same Hubble parameter.

Suggested solutions, mid-term exam in AST4220, October 12 2005

Problem 1

a) The result follows from equations (1.43) and (1.44) in the lecture notes, combined with $a = 1/1 + z$.

b) We have

$$t(z = 6) = \frac{2}{3H_0} 7^{-3/2} \approx 0.50 \text{ Gyr}.$$

c) From equation (1.49) in the lecture notes we find for $z = 6$:

$$\frac{1}{7} = \frac{\Omega_{\rm m0}}{2(1 - \Omega_{\rm m0})}(\cosh u - 1),$$

and with $\Omega_{\rm m0} = 0.3$ this gives

$$u = \cosh^{-1}\left(\frac{5}{3}\right),$$

and inserting this in equation (1.50) we get

$$t(z = 6) = 0.84 \text{ Gyr}.$$

d) From equation (1.53) in the lecture notes we find

$$t(z = 6) = \frac{2}{3\sqrt{1 - \Omega_{\rm m0}}} \sinh^{-1}\left[\left(\frac{1}{a_{\rm m\Lambda} \times 7}\right)^{3/2}\right],$$

where $a_{\rm m\Lambda}^3 = \Omega_{\rm m0}/(1 - \Omega_{\rm m0})$. Inserting numbers we find $t(z = 6) \approx$ 0.89 Gyr.

e) To have a particle horizon,

$$r_{\rm PH} = \int_0^t \frac{cdt'}{a(t')} = ct_0^\alpha \left[\frac{1}{1-\alpha} t'^{1-\alpha}\right]_0^t,$$

must be finite. The condition for this is $1-\alpha > 0$, that is, $\alpha < 1$. To have an event horizon, the requirement is that

$$r_{\rm EH} = \int_t^\infty \frac{cdt'}{a(t')} = ct_0^n \left[\frac{1}{1-\alpha} t'^{1-\alpha}\right]_t^\infty ,$$

must be finite. Then $1-\alpha < 0$, that is, $\alpha > 1$. For $\alpha = 1$ both integrals diverge since $\int dt/t = \ln t$. Accelerated expansion requires $\ddot{a} \propto \alpha(\alpha-1)t^{\alpha-2} > 0$. Since the universe is expanding, we must have $\alpha > 0$, so the criterion for acceleration becomes $\alpha > 1$, which is the same as the condition for the existence of an event horizon.

f) The statement is false. A simple counterexample is provided by the de Sitter model, where $\ddot{a} > 0$, but $H = \dot{a}/a = H_0 =$ konstant. The model in the previous question is also a counterexample: $H = \dot{a}/a = \alpha/t$, which decreases with time for all $\alpha > 0$.

g) This statement is true, and it is quite easy to see this from the definition of H: $aH = a \times \dot{a}/a = \dot{a}$, and since $\ddot{a} > 0$ by definition in an accelerating universe, we have $d(aH)/dt = \ddot{a} > 0$, so that aH increases with time.

Problem 2

a) The Friedmann equations for this case are

$$\begin{aligned} \dot{a}^2 + kc^2 &= \frac{8\pi G}{3}[\rho_{\rm m0}(1+z)^3 + \rho_{\Lambda 0}], \\ \ddot{a} &= -\frac{4\pi G}{3}[\rho_{\rm m0}(1+z)^3 + \rho_{\Lambda 0} - 3\rho_{\Lambda 0}]a, \end{aligned}$$

and by using the definition of the density parameter, the two first equations in the text follow. The third follows by inserting $z = 0$ in the first equation.

b) We must have $\dot{a}(z_*) = 0$ and $\ddot{a}(z_*) \geq 0$.

c) $\dot{a}(z_*) = 0$ gir $H(z_*) = 0$, and hence

$$\Omega_{\rm m0}(1+z_*)^3 + \Omega_{\rm k0}(1+z_*)^2 + \Omega_{\Lambda 0} = 0.$$

Furthermore, $\ddot{a}(z_*) \geq 0$ gives the condition

$$\Omega_{\mathrm{m0}}(1+z_*)^3 - 2\Omega_{\Lambda 0} \leq 0.$$

We first use $\Omega_{\mathrm{k0}} = 1 - \Omega_{\mathrm{m0}} - \Omega_{\Lambda 0}$ to eliminate Ω_{k0}, insert this in the first equation, and solve for $\Omega_{\Lambda 0}$. This gives

$$\Omega_{\Lambda 0} = \frac{(1+z_*)^2(1+\Omega_{\mathrm{m0}} z_*)}{z_*(z_*+2)}.$$

If we insert this in the inequality above and simplify the expressions, we end up with the inequality we were asked to prove.

d) In this model, a has a minimum value, and this means in turn that z has a maximum value, $z = z_*$. Since this minimum value must be at least as large as the highest observed redshift, we must have $z_* \geq 6$. If we insert this in the result in c), we get

$$\Omega_{\mathrm{m0}} \leq \frac{2}{6^2 \times 9} = \frac{1}{162} \approx 0.006.$$

This is an unrealistically low value. Most observations indicate that $\Omega_{\mathrm{m0}} \sim 0.3$, and we know that the density of baryons alone is $\Omega_{\mathrm{b0}} \sim 0.04$. This model can therefore not describe the universe we live in (but it is amusing all the same!)

Suggested solutions, exam in AST4220, December 1 2005

Problem 1

a) For radiation we have $\rho c^2 \propto T^4$ and $\rho c^2 \propto a^{-4}$, which gives $T \propto a^{-1} = (1+z)$. The redshift $z_{\rm dec}$ is therefore given by

$$\frac{1+z_{\rm dec}}{1+0} = \frac{T_{\rm dec}}{T_0},$$

which gives $z_{\rm dec} = 1098$.

b) In this case, the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_{\rm m0}}{a^3} = H_0^2 \frac{8\pi G \rho_{\rm m0}}{3H_0^2} a^{-3} = H_0^2 a^{-3}.$$

From this we find

$$\frac{1}{a}\frac{da}{dt} = H_0 a^{-3/2},$$

which we can integrate and get

$$t_0 = \frac{1}{H_0}\int_0^1 a^{1/2} da = \frac{2}{3H_0} = \frac{2}{3}\frac{9.778 \text{ Gyr}}{0.7} = 9.3 \text{ Gyr}.$$

c) In the same manner as in b), we find

$$t_{\rm dec} = \frac{1}{H_0}\int_0^{a_{\rm dec}} a^{1/2} da = \frac{2}{3H_0} a_{\rm dec}^{3/2} = t_0(1+z_{\rm dec})^{-3/2} \approx 2.5 \times 10^5 \text{ yr}.$$

d) The proper distance to the particle horzion at $t_{\rm dec}$ is given by

$$\begin{aligned}
d_{\rm PH}(z_{\rm dec}) &= a(t_{\rm dec}) \int_0^{t_{\rm dec}} \frac{cdt'}{a(t')} = \frac{t_0^{2/3}}{1+z_{\rm dec}} \int_0^{t_{\rm dec}} \frac{cdt'}{t'^{2/3}} \\
&= \frac{3ct_0^{2/3}}{1+z_{\rm dec}} t_{\rm dec}^{1/3} = \frac{3ct_0^{2/3}}{1+z_{\rm dec}} t_0^{1/3}(1+z_{\rm dec})^{-1/2} \\
&= \frac{3ct_0}{(1+z_{\rm dec})^{3/2}} = \frac{2c}{H_0}(1+z_{\rm dec})^{-3/2} \\
&= 0.24 \text{ Mpc}.
\end{aligned}$$

e) The proper distance from us to $z_{\rm dec}$ is given by

$$\begin{aligned} d_P &= a(t_0)r(z_{\rm dec}) = 1 \times \int_{t_{\rm dec}}^{t_0} \frac{cdt}{a(t)} \\ &= 3ct_0^{2/3}\left(t_0^{1/3} - t_{\rm dec}^{1/3}\right) = 3ct_0\left(1 - \frac{t_{\rm dec}^{1/3}}{t_0^{1/3}}\right) \\ &= \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z_{\rm dec}}}\right) = 8.3 \times 10^3 \text{ Mpc}. \end{aligned} \tag{1}$$

f) From the definition of the angular diameter distance we have

$$d_{\rm A}(z_{\rm dec}) = \frac{d_{\rm PH}(z_{\rm dec})}{\theta_{\rm PH}} = \frac{a(t_0)r(z_{\rm dec})}{1+z_{\rm dec}},$$

and this gives

$$\begin{aligned} \theta_{\rm PH} &= \frac{d_{\rm PH}(z_{\rm dec})(1+z_{\rm dec})}{r(z_{\rm dec})} = \frac{1099 \times 0.24 \text{ Mpc}}{8.3 \times 10^3 \text{ Mpc}} \\ &= 0.032 \text{ rad} = 1.8 \text{ degrees}. \end{aligned} \tag{2}$$

Problem 2

a) We have

$$\begin{aligned} H(t) &= \frac{\dot{a}}{a} = \frac{2}{3(1+w)t} \\ \rho_\phi(t) &= \rho_{\rm c}(t) = \frac{3H^2}{8\pi G} = \frac{1}{6\pi G(1+w)^2t^2}. \end{aligned}$$

From the second Friedmann equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_\phi + 3p_\phi) = -\frac{4\pi G}{3}\rho_\phi(1+3w)$$

we see that the condition for acceleration is $w < -1/3$.

b) The energy density obeys

$$\dot{\rho_\phi} = -3H(\rho_\phi + p_\phi) = -3H\dot{\phi}^2.$$

From the given solution we find

$$\begin{aligned}\dot{\phi} &= \frac{da}{dt}\frac{d\phi}{da} = \dot{a}\sqrt{\frac{3(1+w)}{8\pi G}}\frac{1}{a} \\ &= H\sqrt{\frac{3(1+w)}{8\pi G}},\end{aligned}$$

so that

$$-3H\dot{\phi}^2 = -3H^3\frac{3(1+w)}{8\pi G} = -\frac{1}{3\pi G(1+w)^2t^3}.$$

From the result in a) we find

$$\dot{\rho}_\phi = -\frac{1}{3\pi G(1+w)^2t^3},$$

and hence $\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi)$, as desired. The potential V can be determined from the condition $p_\phi = w\rho_\phi$. In order to satisfy this condition,

$$\frac{1}{2}\dot{\phi}^2 - V(\phi) = \frac{w}{2}\dot{\phi}^2 + wV(\phi),$$

which gives

$$V(\phi) = \frac{1}{2}\frac{1-w}{1+w}\dot{\phi}^2.$$

We insert the expression we found earlier for $\dot{\phi}$ and use the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\frac{\rho_\phi^0}{a^{3(1+w)}},$$

and find

$$V(\phi) = \frac{1}{2}(1+w)\rho_\phi^0 a^{-3(1+w)}.$$

We can now use the expression for $\phi(a)$ and express a in terms of ϕ:

$$a = \exp\left[\sqrt{\frac{8\pi G}{3(1+w)}}(\phi - \phi_0)\right],$$

and therefore

$$V(\phi) = \frac{1}{2}(1+w)\rho_\phi^0 \exp\left[-\sqrt{24\pi G(1+w)}(\phi - \phi_0)\right].$$

c) We can write the potential found in b) as

$$V(\phi) = \frac{1}{2}\rho_\phi^0 \exp\left(\sqrt{24\pi G(1+w)}\phi_0\right) \exp\left(-\sqrt{3(1+w)}\phi\sqrt{8\pi G}\right),$$

which is on the required form if we let

$$V_0 = \frac{1}{2}\rho_\phi^0 \exp\left(\sqrt{24\pi G(1+w)}\phi_0\right),$$

and

$$\lambda = \sqrt{3(1+w)}.$$

Since $p_\phi + \rho_\phi = (1+w)\rho_\phi > 0$ means $w > -1$ and $\lambda > 0$ by definition, the condition $p_\phi + \rho_\phi > 0$ is satisfied if $\lambda > 0$. Acceleration requires $w < -1/3$, and this gives $\lambda < \sqrt{2}$. So the condition is

$$0 < \lambda < \sqrt{2}.$$

Oppgave 3

a) The Friedmann equation can be written

$$\frac{H^2}{H_0^2} = \frac{\Omega_{\mathrm{m0}}}{a^3} + \frac{\Omega_{\mathrm{k0}}}{a^2}.$$

Since both Ω_{m0} and Ω_{k0} are of order 1 in realistic models, the first term on the right hand side will dominate when $a \ll 1$, that is, early in the matter dominated phase.

b) From the Friedmann equation without the curvature term we find

$$\frac{1}{a}\frac{da}{dt} = H_0\Omega_{\mathrm{m0}}^{1/2}a^{-3/2},$$

and therefore

$$\int_0^a a'^{1/2}da' = H_0\Omega_{\mathrm{m0}}^{1/2}\int_0^t dt',$$

which gives

$$a(t) = \left(\frac{3}{2}H_0\Omega_{\mathrm{m0}}^{1/2}\right)^{2/3} t^{2/3}.$$

c) The equation for the time evolution of density perturbations is

$$\ddot{\Delta}_k + 2\frac{\dot{a}}{a}\dot{\Delta}_k = 4\pi G\rho_{\mathrm{m}}(t)\Delta_k.$$

We have

$$\frac{\dot{a}}{a} = \frac{2}{3t},$$

and

$$\rho_{\mathrm{m}}(t) = \frac{\Omega_{\mathrm{m}0}\rho_{\mathrm{c}0}}{a^3(t)} = \frac{1}{6\pi G t^2}.$$

The equation becomes

$$\ddot{\Delta}_k + \frac{4}{3t}\dot{\Delta}_k = \frac{2}{3t^2}\Delta_k,$$

which is exactly the same as for the EdS model. The growing mode is therefore also the same, $\Delta_k(t) \propto t^{2/3}$.

d) From a) we know that any open model will behave like a flat model at high redshifts. From c) we see that the density perturbations will grow in the same way as in the EdS model. It is therefore not a good idea to use observations at high redshifts if we want to distinguish between the two cases.

Suggested solutions, exam in AST4220, January 2006

Problem 1

a) We insert $p = w\rho$ in the given equation and find

$$\dot{\rho} = -3\frac{\dot{a}}{a}(1+w)\rho,$$

and after separating the variables we get

$$\int_{\rho_0}^{\rho} \frac{d\rho'}{\rho'} = -3(1+w)\int_1^a \frac{da'}{a'},$$

that is,

$$\ln\left(\frac{\rho}{\rho_0}\right) = -3(1+w)\ln a = \ln a^{-3(1+w)}.$$

Hence:

$$\rho = \rho_0 a^{-3(1+w)}.$$

b) For dust we have $w = 0$, for a cosmological constant $w = -1$. We therefore have

$$\begin{aligned}
\rho &= \rho_m + \rho_\Lambda = \rho_{m0}a^{-3} + \rho_{\Lambda 0} \\
&= \Omega_{m0}\rho_{c0}a^{-3} + \Omega_{\Lambda 0}\rho_{c0} \\
&= \rho_{c0}(\Omega_{\Lambda 0} + \Omega_{m0}a^{-3}) \\
&= \rho_{c0}[(1-\Omega_{m0}) + \Omega_{m0}a^{-3}],
\end{aligned}$$

where the last equality follows from $\Omega_{m0} + \Omega_{\Lambda 0} = 1$ for a spatially flat universe.

c) For the Chaplygin gas we must solve the equation

$$\frac{d\rho}{dt} = -\frac{3}{a}\frac{da}{dt}\left(\rho - \frac{A}{\rho^\alpha}\right).$$

We rewrite this as

$$\frac{d\rho}{\rho - \frac{A}{\rho^\alpha}} = -3\frac{da}{a},$$

or

$$\int_{\rho_0}^{\rho} \frac{\rho^\alpha d\rho}{\rho^{\alpha+1} - A} = -\int_1^a \frac{da}{a} = -3\ln a = \ln a^{-3}.$$

The integral on the left hand side is solved by substituting

$$x = \rho^{\alpha+1} - A,$$

so that $dx = (\alpha+1)\rho^\alpha d\rho$, and hence

$$\int_{\rho_0}^{\rho} \frac{\rho^\alpha d\rho}{\rho^{\alpha+1} - A} = \frac{1}{\alpha+1}\int_{\rho_0^{\alpha+1}-A}^{\rho^{\alpha+1}-A} \frac{dx}{x} = \frac{1}{\alpha+1}\ln\left(\frac{\rho^{\alpha+1} - A}{B}\right),$$

where we have defined $B = \rho_0^{\alpha+1} - A$. We therefore have

$$\frac{1}{\alpha+1}\ln\left(\frac{\rho^{\alpha+1} - A}{B}\right) = \ln a^{-3},$$

that is,

$$\ln\left(\frac{\rho^{\alpha+1} - A}{B}\right) = \ln a^{-3(1+\alpha)},$$

which gives

$$\rho^{\alpha+1} - A = \frac{B}{a^{3(1+\alpha)}},$$

and finally

$$\rho = \left[A + \frac{B}{a^{3(1+\alpha)}}\right]^{1/(1+\alpha)}.$$

d) For $a \ll 1$ the second term will dominate, and we get

$$\rho \approx \frac{B^{1/(\alpha+1)}}{a^3} \propto a^{-3},$$

that is, the same behaviour as for dust. For $a \gg 1$ the first term dominates, and we find

$$\rho \approx A^{1/(\alpha+1)} = \text{constant},$$

as for a cosmological constant. In this case we also get

$$p = -\frac{A}{\rho^\alpha} = -AA^{-\frac{\alpha}{\alpha+1}} = -A^{1/(\alpha+1)} = -\rho,$$

that is, the same equation of state as for a cosmological constant.

e) We can write

$$\begin{aligned}\rho &= [A + Ba^{-3(1+\alpha)}]^{1/(\alpha+1)} \\ &= (A+B)^{1/(\alpha+1)}\left[\frac{A}{A+B} + \frac{B}{A+B}a^{-3(1+\alpha)}\right]^{1/(1+\alpha)} \\ &= \rho_*[(1-\Omega_m^*) + \Omega_m^+ a^{-3(1+\alpha)}]^{1/(1+\alpha)},\end{aligned}$$

where we have used the definitions of ρ_* and Ω_m^* given in the text.

f) For $\alpha = 0$ the expression above becomes

$$\rho = \rho_*[(1-\Omega_m^*) + \Omega_m^* a^{-3}],$$

and we see that this is the same as for a spatially flat universe with dust and a cosmological constant. But note that in b) we had two components, whereas we now have only on: the Chaplygin gas.

Problem 2

a) The first Friedmann equation gives

$$H^2 = \frac{8\pi G}{3}\rho_\phi = \frac{1}{3M_{Pl}^2}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right],$$

while the continuity equation gives

$$\dot{\rho}_\phi - 3H(\rho_\phi + p_\phi).$$

We see that $\rho_\phi + p_\phi = \dot{\phi}^2$, while

$$\dot{\rho}_\phi = \dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi},$$

so that

$$\dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi} = -3H\dot{\phi}^2,$$

that is,

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}.$$

b) We start by taken the derivative of the Friedmann equation with respect to time. This gives

$$2H\frac{dH}{d\phi}\dot{\phi} = \frac{1}{3M_{Pl}^2}[V'(\phi)\dot{\phi} + \dot{\phi}\ddot{\phi}],$$

and we can cancel $\dot{\phi}$ på on both sides so that

$$2HH'\phi = \frac{1}{3M_{Pl}^2}[V'(\phi) + \ddot{\phi}].$$

From the second equation in a) we have

$$\ddot{\phi} = -3H\dot{\phi} - V'(\phi),$$

and inserting this in the equation above we get

$$2HH'(\phi) = -\frac{1}{3M_{Pl}^2}[V'(\phi) - 3H\dot{\phi} - V'(\phi)] = -\frac{1}{M_{Pl}^2}H\dot{\phi},$$

and hence

$$\dot{\phi} = -2M_{Pl}^2 H'(\phi).$$

c) We insert the result from b) in the Friedmann equation:

$$H^2 = \frac{1}{3M_{Pl}^2} = \left[V + \frac{1}{2}4M_{Pl}^4(H')^2\right] = \frac{V}{3M_{Pl}^2} + \frac{2}{3}M_{Pl}^2(H')^2,$$

and this gives immediately

$$[H'(\phi)]^2 - \frac{3}{2M_{Pl}^2}H^2(\phi) = -\frac{1}{2M_{Pl}^4}V(\phi).$$

This is the fundamental equation in the so-called *Hamilton-Jacobi-formulation* of inflation.

d) We know that H_0 is a solution of the equation in c), and that we require the same of $H_0 + \delta H$. We must therefore have

$$(H_0' + \delta H')^2 - \frac{3}{2M_{Pl}^2}(H_0 + \delta H)^2 = -\frac{1}{2M_{Pl}^4}V.$$

We drop all terms that contain δH^2 and $(\delta H')^2$, and use the fact that H_0 is a solution. We are then left with

$$2H_0'\delta H - \frac{3}{2M_{Pl}^2}2H_0\delta H = 0,$$

which gives

$$H_0'\delta H' = \frac{3}{2M_{Pl}^2}H_0\delta H.$$

e) We rewrite the differential equation as

$$H_0'\frac{d(\delta H)}{d\phi} = \frac{3}{2M_{Pl}^2}H_0\delta H,$$

and this is not too difficult to solve:

$$\int_{\delta H_i}^{\delta H}\frac{d(\delta H)}{\delta H} = \frac{3}{2M_{Pl}^2}\int_{\phi_i}^{\phi}\frac{H_0(\phi)}{H_0'(\phi)}d\phi,$$

where $\delta H_i = \delta H(\phi_i)$. The integral on the left hand side is equal to $\ln(\delta H/\delta H_i)$, and we get

$$\delta H(\phi) = \delta H(\phi_i)\exp\left[\frac{3}{2M_{Pl}^2}\int_{\phi_i}^{\phi}\frac{H_0(\phi)}{H_0'(\phi)}d\phi\right].$$

The Hubble parameter H_0 is positive (expanding universe). We have assumed $\dot{\phi} > 0$, and the result in b) then shows that $H_0'(\phi)$ must be negatvie. The integral in the exponential is therefore negative, and the larger the value of ϕ, the more negative the exponent. The perturbation will therefore die out rapidly. This is a sufficent explanation, but we can, if we wish, be more precise. From b) we have

$$H_0' = -\frac{\dot{\phi}}{2M_{Pl}^2},$$

so that

$$\frac{3}{2M_{Pl}^2}\int_{\phi_i}^{\phi}\frac{H_0}{H_0'}d\phi = -3\int_{\phi_i}^{\phi}H_0(\phi)\frac{d\phi}{\dot{\phi}}.$$

By definition $\dot{\phi} = d\phi/dt$, so we get $d\phi/\dot{\phi} = dt$, and we can transform the integral to the form

$$-3\int_{t_i}^{t}H(t)dt,$$

and from the lecture notes we see that this is the same as the number of e-foldings from t_i to t. For inflationary models of interest this number is at least 60, and so the exponent is bounded above by -180. We can therefore safely conclude that the perturbation will rapidly die out.

Suggested solutions, mid-term exam in AST4220, fall 2006

Problem 1

a) The particle horizon at $t = t_0 = 2/3H_0$ in the EdS model is given by

$$
\begin{aligned}
r_{\mathrm{PH}}(t_0) &= \int_0^{t_0} \frac{cdt}{a(t)} = ct_0^{2/3} \int_0^{t_0} t^{-2/3} dt \\
&= 3ct_0 = \frac{2c}{H_0}.
\end{aligned}
$$

The event horizon at $t = t_0$ in a dS universe is given by

$$
\begin{aligned}
r_{\mathrm{EH}}(t_0) &= \int_{t_0}^{\infty} \frac{cdt}{a(t)} = c \int_{t_0}^{\infty} e^{-H_0(t-t_0)} dt \\
&= \frac{c}{H_0}.
\end{aligned}
$$

b) Light emitted at time t, redshift z, and received today have comoving radial coordinate

$$
r = \int_t^{t_0} \frac{cdt}{a(t)} = \frac{c}{H_0}[e^{-H_0(t-t_0)} - 1].
$$

Since $1 + z = 1/a = e^{-H_0(t-t_0)}$, we get

$$
r = \frac{cz}{H_0},
$$

and equating this to $r_{\mathrm{EH}}(t_0) = c/H_0$, we find $z = 1$.

c) We denote the time at which the teacher reaches the particle horizon by t_f. Since we can treat the teacher as a ray of light(!), we find

$$
\begin{aligned}
r_{\mathrm{PH}}(t_0) &= \frac{2c}{H_0} = \int_{t_0}^{t_f} \frac{cdt}{a(t)} \\
&= 3ct_0^{2/3}(t_f^{1/3} - t_0^{1/3}) = \frac{2c}{H_0}\left[\left(\frac{t_f}{t_0}\right) - 1\right],
\end{aligned}
$$

and since $r_{\mathrm{PH}}(t_0) = 2c/H_0$ we get the equation

$$\left(\frac{t}{t_0}\right)^{1/3} - 1 = 1,$$

which gives

$$t_f = 8t_0 = \frac{16}{3H_0}.$$

d) The condition for the teacher to reach the event horzion is given by

$$r_{\mathrm{EH}}(t_0) = \frac{c}{H_0} = \int_{t_0}^{t_f} \frac{cdt}{a(t)} = \frac{c}{H_0}[1 - e^{-H_0(t_f - t_0)}],$$

which gives

$$1 = 1 - e^{-H_0(t_f - t_0)},$$

and this equation is not fulfilled for any finite value of t_f. Not surprisingly, the teacher will never reach the event horizon.

e) We start with the event horizon. From the definition we get

$$r_{\mathrm{EH}} = \int_{t_0}^{\infty} \frac{cdt}{a(t)} = \int_{1}^{\infty} \frac{cda}{a\dot{a}} = \int_{1}^{\infty} \frac{cda}{a^2 H},$$

where we have used $dt = da/\dot{a}$ and $H = \dot{a}/a$. If we introduce the redshift $1 + z = 1/a$ as an integration variable, we get

$$r_{\mathrm{EH}} = -c\int_{0}^{-1} \frac{dz}{(1+z)^2} \frac{(1+z)^2}{H(z)} = c\int_{-1}^{0} \frac{dz}{H(z)}.$$

The Friedmann equation for $\dot{a}/a$ in the case $k = 0$, with dust and vacuum energy, can be written

$$H^2(z) = H_0^2[\Omega_{\mathrm{m0}}(1+z)^3 + \Omega_{\Lambda 0}].$$

For $z = 0$ we get the condition, and inserted in the integral for r_{EH} this gives the desired result.

The calculation is the same for the particle horizon, except for the integration limits: $t = 0$ corresponds $z = \infty$, and $t = t_0$ gives $z = 1$.

f) As long as $0 < \Omega_{\mathrm{m0}} < 1$, the term with $(1+z)^3$ makes sure that the integral converges for $z \to \infty$, and the particle horizon exists. The same condition makes the integrand finite at $z = -1$, and hence the integral for the event horizon also exists.

Problem 2

a) Only the mass inside the ball of radius R contribute to the gravitational force on the particle. Let this mass be M. From Newton's second law we get

$$-\frac{GMm}{R^2} = ma = m\ddot{R},$$

that is,

$$\ddot{R} = -\frac{GM}{R^2}.$$

The universe is homogeneous, so $M = \frac{4}{3}\pi R^3 \rho$, and

$$\rho = \frac{\rho_{\mathrm{m0}}}{a^3} = \frac{\Omega_{\mathrm{m0}}\rho_{\mathrm{c0}}}{a^3} = \frac{3H_0^2}{8\pi G}\frac{\Omega_{\mathrm{m0}}}{a^3},$$

so that

$$M = \frac{H_0^2 R^3}{2G}\frac{\Omega_{\mathrm{m0}}}{a^3}.$$

If we substitute the last result in the equation of motion, we get

$$\ddot{R} = -\frac{\Omega_{\mathrm{m0}} H_0^2}{2a^3} R.$$

b) We insert the relevant results from the EdS model: $a = (t/t_0)^{2/3}$, and $H_0 t_0 = 2/3$. The equation of motion becomes

$$\ddot{R} = -\frac{2}{9}\frac{R}{t^2}.$$

We try a power-law solution, $R = At^n$, where A and n are constants, and substitute this in the equation

$$n(n-1) = -\frac{2}{9},$$

and the solutions of this quadratic equation are $n = 1/3$ og $n = 2/3$. The general solution is therefore

$$R(t) = At^{2/3} + Bt^{1/3}.$$

We apply the initial conditions $R(t_0) = R_0$ and $\dot{R}(t_0) = 0$. This gives

$$\begin{aligned} At_0^{2/3} + Bt_0^{1/3} &= R_0 \\ \frac{2}{3}At_0^{-1/3} + \frac{1}{3}Bt_0^{-2/3} &= 0. \end{aligned}$$

and these equations have $A = -R_0 t^{-2/3}$, $B = 2R_0 t^{-1/3}$ as solutions. Hence:

$$R(t) = R_0 \left[2\left(\frac{t}{t_0}\right)^{1/3} - \left(\frac{t}{t_0}\right)^{2/3} \right].$$

c) To begin with, $t = 1.1t_0$, say, we have $R(t) \approx 0.999R_0$. The particle will thus move towards the origin initially. We see that for $t \gg t_0$, we have $R(t) \approx -R_0(t/t_0)^{2/3}$. For a particle starting at $R = R_0$ ved $t = t_0$, but follows the expansion, we have $R(t) = R_0(t/t_0)^{2/3}$. The particle under consideration will therefore move towards and past us, and ends up following the expansion of the universe, but on the 'opposite side' compared to a particle that starts in the same position at the same time, but follows the expansion of the universe.

Suggested solutions, final exam in AST4220, fall 2006

Problem 1

a) We find

$$\frac{v}{c} = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = 0.92,$$

that is, $v = 2.76 \times 10^8\ \mathrm{m\,s^{-1}}$.

b) From Hubble's law we find

$$d = \frac{v}{H_0} = 1.72 \times 10^10 \text{ ly}.$$

c) The cosmic redshift is an effect of the expansion of the Universe, it is not a Doppler effect. Special relativity applies to flat spacetime, it does not take gravity into account. It can therefore not describe the Universe.

d) In an EdS universe we haev $a(t) = (t/t_0)^{2/3}$, $H_0 = 2/3t_0$. If the light from the quasar was emitted at time t_e, its proper distance from us is given by

$$\begin{aligned} d_\mathrm{P} &= \int_{t_0}^{t_0} \frac{cdt}{a(t)} = ct_0^{2/3} \int_{t_e}^{t_0} t^{-2/3} dt \\ &= 3ct_0 \left[1 - \left(\frac{t_e}{t_0} \right)^{1/3} \right] \\ &= \frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1+z}} \right), \end{aligned}$$

where we have made use of $1 + z = a(t_0)/a(t_e) = (t_0/t_e)^{2/3}$. Inserting the numbers given in the text in this expression, we find

$$d_\mathrm{P} = \frac{1.09c}{H_0} = 2.03 \times 10^{10} \text{ ly}.$$

Problem 2

a) A sufficient answer: from the Friedmann equations we find that the density parameter is given by

$$\Omega(t) - 1 \propto (aH)^{-2},$$

and the right hand side is an increasing function of time in matter- or radiation-dominated epochs. This implies that Ω must have been extremely close to 1 in the past to be of order 1 today.

b) The slow-roll parameters are given by

$$\begin{aligned}\epsilon &= \frac{E_{\mathrm{Pl}}^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{E_{\mathrm{Pl}}^2}{16\pi}\frac{p^2}{\phi^2}.\\ \eta &= \frac{E_{\mathrm{Pl}}^2}{8\pi}\frac{V''}{V} = \frac{E_{\mathrm{Pl}}^2}{8\pi}\frac{p(p-1)}{\phi^2}.\end{aligned}$$

c) The condition $\epsilon(\phi_{\mathrm{end}}) = 1$ gives

$$\phi_{\mathrm{end}} = \frac{pE_{\mathrm{Pl}}}{4\sqrt{\pi}}.$$

d) The number of e-foldings is given by

$$N = \frac{8\pi}{E_{\mathrm{Pl}}^2}\int_{\phi_{\mathrm{end}}}^{\phi_i}\frac{V}{V'}d\phi = \frac{4\pi}{pE_{\mathrm{Pl}}^2}\phi_i^2 - \frac{p}{4},$$

and the condition $N = 60$ gives

$$\phi_i = E_{\mathrm{Pl}}\sqrt{\frac{p}{4\pi}\left(60 + \frac{p}{4}\right)}.$$

0.1 Problem 3

a) The condition

$$\frac{1-\Omega_{r0}}{a^2} > \frac{\Omega_{r0}}{a^4}$$

gives

$$a^2 > \frac{\Omega_{r0}}{1-\Omega_{r0}},$$

which is equivalent to

$$a > a_m = \sqrt{\frac{\Omega_{r0}}{1-\Omega_{r0}}}.$$

b) A sligth rearrangement of the Friedmann equation gives

$$\frac{1}{a}\frac{da}{dt} = \frac{H_0\sqrt{\Omega_{r0}}}{a^2}\sqrt{1+\frac{1-\Omega_{r0}}{\Omega_{r0}}a^2} = \frac{H_0\sqrt{\Omega_{r0}}}{a^2}\sqrt{1+\frac{a^2}{a_m^2}}.$$

c) We rewrite the equation as

$$\frac{ada}{\sqrt{1+(a/a_m)^2}} = H_0\sqrt{\Omega_{r0}}dt.$$

and integrate. The integral over a is solved by using the substitution $x = (a/a_m)^2$,

$$\frac{a_m^2}{2}\int_0^{a^2/a_m^2}\frac{dx}{\sqrt{1+x}} = H_0\sqrt{\Omega_{r0}}t,$$

and since the indefinite integral involved is equal to $2\sqrt{1+x}$ we find

$$\sqrt{1+\frac{a^2}{a_m^2}} - 1 = \frac{H_0\sqrt{\Omega_{r0}}}{a_m^2}t.$$

Solving this equation with respect to a gives the desired result.

d) Taking $a = a_m$ in the result above gives

$$\frac{H_0\sqrt{\Omega_{r0}}}{a_m^2}t_m = \sqrt{2}-1,$$

and with the numbers given in the text we get $t_m \approx 8 \times 10^{-34}$ s.

e) Well into the curvature-dominated phase, $t \gg t_m$, we have

$$a \approx a_m \frac{H_0\sqrt{\Omega_{r0}}}{a_m^2} t \propto t.$$

This gives $H = \dot{a}/a = 1/t$, $\rho_c(t) = 3H^2/8\pi G = 3/8\pi G t^2 \propto t^{-2}$, and $\rho_r(t) \propto a^{-4} \propto t^{-4}$. This gives

$$\Omega_r(t) \propto \frac{t^{-4}}{t^{-2}} \propto t^{-2},$$

which means that radiation quickly becomes negligible.

f) If all of the radiation is converted to non-relativistic matter far into the curvature-dominated phase, $\Omega_{\mathrm{m}}(t) \ll 1$ because of the result in e). In the equation for the time evoluton of density perturbations,

$$\ddot{\Delta}_k + 2H\dot{\Delta}_k = 4\pi G\rho_{\mathrm{m}}(t)\Delta_k,$$

this means that the source term on the right hand side is negligible. The equation therefore reduces to

$$\ddot{\Delta}_k + \frac{2}{t}\dot{\Delta}_k = 0,$$

and by substituting $\Delta_k \propto t^n$ we find $n = 0$ and $n = -1$ as the only solutions. This means that there is no growing mode of density perturbations, which in turn means that structure formation is impossible in this model.

Suggested solutions, exam in AST4220, January 2007

Problem 1

a) We have

$$\begin{aligned}
\frac{dz}{dt} &= \frac{d}{dt}\frac{a(t)}{a(t_e)} + a(t)\frac{d}{dt_e}\left(\frac{1}{a(t_e)}\right)\frac{dt_e}{dt} \\
&= \frac{a(t)}{a(t_e)}\frac{\dot{a}(t)}{a(t)} - \frac{a(t)}{a^2(t_e)}\dot{a}(t_e)\frac{dt_e}{dt} \\
&= \frac{a(t)}{a(t_e)}\frac{\dot{a}(t)}{a(t)} - \frac{a(t)}{a(t_e)}\frac{\dot{a}(t_e)}{a(t_e)}\frac{dt_e}{dt}.
\end{aligned}$$

b) For the EdS model, $a(t) = (t/t_0)^{2/3}$, so

$$r = ct_0^{2/3}\int_{t_e}^{t}\frac{dt}{t^{2/3}} = 3ct_0^{2/3}(t^{1/3} - t_e^{1/3}).$$

Since $dr/dt = 0$, this gives

$$0 = 3ct_0^{2/3}\left(\frac{1}{3}t^{-2/3} - \frac{1}{3}t_e^{-2/3}\frac{dt_e}{dt}\right),$$

that is,

$$\frac{dt_e}{dt} = \frac{t_e^{2/3}}{t^{2/3}} = \frac{a(t_e)}{a(t)} = \frac{1}{1+z}.$$

c) Combining the results from a) and b) we get

$$\frac{dz}{dt} = (1+z)H(t) - (1+z)H(t_e)\frac{1}{1+z} = (1+z)H(t) - H(t_e).$$

d) For the EdS model $H(t) = 2/3t$, and inserted in the result from c)

$$\left(\frac{dz}{dt}\right)_{t=t_0} = \frac{2}{3t_0}\left[(1+z) - \frac{t_0}{t_e}\right] = \frac{2}{3t_0}\left[(1+z) - (1+z)^{3/2}\right],$$

and for $z = 4$ this gives

$$\frac{dz}{dt} = -\frac{4.12}{t_0} < 0.$$

e) In the de Sitter model $H(t) = H_0 = \text{konstant}$, so

$$\left(\frac{dz}{dt}\right)_{t=t_0} = (1+z)H_0 - H_0 = zH_0 = 4H_0 > 0.$$

The cosmic redshift measures the amount of linear expansion since the ligth was emitted. In the EdS model, where the expansion rate decreases with time, we see that the redshift will also decrease with time. The opposite is the case in the de Sitter model, where the universe undergoes accelerated expansion.

Problem 2

a) The horizon problem: we observe that the CMB is isotropic to within a few tenths of a milikelvin. But if we calculate the size of the particle horizon at recombination, when the radiation was released, we find that i only covers a few degrees on the sky today. Parts of the sky separated by more than a few degrees have never been in causal contact, according to this picture. It is then hard to understand why they should have almost exactly the same temperature.

b) The energy density and pressure of the scalar field are given by

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned}$$

For $V(\phi) = 0$ we have $p_\phi = \rho_\phi$. The scalar field behaves like a perfect fluid with equation of state parameter $w = 1$. In this case, there is no accelerated expansion, and therefore no inflation.

c) In this case, $p_\phi = 0$, the same as for dust, so the Universe will follow the decelerating EdS solution. No inflation in this case.

d) Using the definitions of the slow-roll parameters we have

$$\begin{aligned} \epsilon &= \frac{1}{16\pi} \\ \eta &= \frac{1}{8\pi}. \end{aligned}$$

Both are constant and less than one. Inflation will therefore never end in this model.

Problem 3

a) The equation reduces in this case to

$$H^2 = \frac{H_0^2}{a^2}.$$

The given solution has $H = \dot{a}/a = 1/t$, and by substitution we see that it satisfies the equation.

b) With the given definitions of α and β we can write

$$\begin{aligned} H^2 &= H_0^2 \frac{\Omega_{r0}}{a^4} \left(\frac{\Omega_{k0}}{\Omega_{r0}} a^2 + \frac{\Omega_{m0}}{\Omega_{r0}} + 1 \right) \\ &= H_0^2 \frac{\Omega_{r0}}{a^4} (\alpha^2 a^2 + \beta a + 1) \\ &= H_0^2 \frac{\Omega_{r0}}{a^4} \left[\left(\alpha a + \frac{beta}{2\alpha} \right)^2 + 1 - \frac{\beta^2}{4\alpha^2} \right]. \end{aligned}$$

c) For $\beta = 2\alpha$ the equation simplifies to

$$H = H_0 \frac{\sqrt{\Omega_{r0}}}{a^2} (\alpha a + 1).$$

By separation of variables and making use of

$$\frac{a}{\alpha a + 1} = \frac{1}{\alpha} - \frac{1/\alpha}{\alpha a + 1}$$

we find

$$\int_0^a \left(\frac{1}{\alpha} - \frac{1/\alpha}{\alpha a + 1} \right) da = \sqrt{\Omega_{r0}} H_0 \int_0^t dt',$$

which gives the desired result.

d) In the limit $a \gg 1$ we can neglect the logarithmic term and get

$$a \approx \alpha\sqrt{\Omega_{r0}}H_0 t,$$

as we would expect from a), since the curvature term dominates in this limit. Using the expansion of the logarithm for $\alpha a \ll 1$, we find

$$\frac{1}{2}a^2 \approx \sqrt{\Omega_{r0}}H_0 t,$$

that is, $a \propto t^{1/2}$, as in a radiation-dominated universe. This is again what we would expect, since the radiation term dominates at early times, that is, for small values of a.

Suggested solutions, mid-term exam in AST4220, fall 2007

Problem 1

a) The scale factor in the EdS model is given by $a = a_0(t/t_0)^{2/3}$, so that

$$\begin{aligned}\dot{a} &= \frac{2}{3}a_0 t_0^{-2/3} t^{-1/3} \\ \ddot{a} &= -\frac{2}{9}a_0 t_0^{-2/3} t^{-4/3}.\end{aligned}$$

Using the definition of q we get

$$q = \frac{1}{2} = \text{konstant}.$$

b) As we have seen in the lectures, the Friedmann equations for this model can be written as

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 &= H_0^2\left[\Omega_{\text{m0}}(1+z)^3 + \Omega_{\Lambda0}\right] \\ \frac{\ddot{a}}{a} &= -\frac{1}{2}H_0^2\left[\Omega_{\text{m0}}(1+z)^3 - 2\Omega_{\Lambda0}\right],\end{aligned}$$

der $\Omega_{\text{m0}} + \Omega_{\Lambda0} = 1$. From the definition of q we now find

$$\begin{aligned}q &= -\frac{\ddot{a}}{a}\left(\frac{a}{\dot{a}}\right)^2 \\ &= \frac{1}{2}\frac{\Omega_{\text{m0}}(1+z)^3 - 2\Omega_{\Lambda0}}{\Omega_{\text{m0}}(1+z)^3 + \Omega_{\Lambda0}},\end{aligned}$$

where $\Omega_{\Lambda0} = 1 - \Omega_{\text{m0}}$. For $z = 0$ we find

$$\begin{aligned}q_0 &= \frac{1}{2}\frac{\Omega_{\text{m0}} - 2\Omega_{\Lambda0}}{\Omega_{\text{m0}} + \Omega_{\Lambda0}} \\ &= \frac{1}{2}(\Omega_{\text{m0}} - 2 + 2\Omega_{\text{m0}}) \\ &= \frac{1}{2}(\Omega_{\text{m0}} - 2) \approx -0.55\end{aligned}$$

where we in the last step have substituted $\Omega_{\text{m0}} = 0.3$.

c) We use $H = \dot{a}/a$ and find

$$\frac{d}{dt}\frac{1}{H} = \frac{d}{dt}\frac{a}{\dot{a}} = \frac{\dot{a}^2 - a\ddot{a}}{\dot{a}^2} = 1 - \frac{a\ddot{a}}{\dot{a}^2} = 1 + q.$$

d) The chain rule, combined with the definition of the redshift, $1 + z = a_0/a$, gives

$$\begin{aligned}
\frac{d}{dt}\frac{1}{H} &= \frac{da}{dt}\frac{d}{da}\frac{1}{H} \\
&= \frac{da}{dt}\frac{dz}{da}\frac{d}{dz}\frac{1}{H} \\
&= \dot{a}\frac{d}{da}\left(\frac{a_0}{a} - 1\right)\left(-\frac{1}{H^2}\frac{dH}{dz}\right) \\
&= aH\left(-\frac{a_0}{a^2}\right)\left(-\frac{1}{H^2}\right)\frac{dH}{dz} \\
&= \frac{a_0}{a}\frac{1}{H}\frac{dH}{dz} = \frac{(1+z)}{H}\frac{dH}{dz}.
\end{aligned}$$

e) We equate the expressions found in c) and d) and obtain the differential equation

$$\frac{1+z}{H}\frac{dH}{dz} = 1 + q.$$

We separate the variables and integrate:

$$\int_{H_0}^{H}\frac{dH'}{H'} = \int_0^z \frac{1+q}{1+z'}dz',$$

so that

$$\ln\left(\frac{H}{H_0}\right) = \int_0^z \frac{1+q}{1+z'}dz',$$

and finally

$$H = H_0 \exp\left[\int_0^z \frac{1+q}{1+z'}dz'\right].$$

f) For $q = q_0 =$ konstant we get

$$\begin{aligned}
H(z) &= H_0 \exp\left[(1+q_0)\int_0^z \frac{dz'}{1+z'}\right] \\
&= H_0 \exp[(1+q_0)\ln(1+z)] = H_0(1+z)^{1+q_0}.
\end{aligned}$$

g) We find the luminosity distance by using the result found in f):

$$
\begin{aligned}
d_{\mathrm{L}} &= c(1+z)\int_0^z \frac{dz'}{(1+z')^{1+q_0}} \\
&= \frac{c}{H_0}(1+z)\left[-\frac{1}{q_0}(1+z')^{-q_0}\right]_0^z \\
&= \frac{c}{q_0 H_0}(1+z)\left[1-(1+z)^{-q_0}\right].
\end{aligned}
$$

h) The EdS model has $q = q_0 = 1/2$, and hence

$$
d_{\mathrm{L}} = \frac{2c}{H_0}(1+z)\left[1-\frac{1}{\sqrt{1+z}}\right] \approx 0.95\frac{c}{H_0}
$$

for $z = 0.83$. Note that this result is exact, since q truly is a constant in the EdS model. The same does not hold in the ΛCDM model, so the luminosity distance we calculate by substituting the value $q_0 = -0.55$ is an approximation to the exact result. With this value for q_0 and with $z = 0.83$ we find

$$
d_{\mathrm{L}} \approx 1.31\frac{c}{H_0}.
$$

These results indicate that a model with a cosmological constant is closer to reality than the EdS model.

i) Taylor expansion of $1/H$ about $z = 0$ gives, to first order in z

$$
\begin{aligned}
\frac{1}{H(z)} &\approx \frac{1}{H(z=0)} + z\left(\frac{d}{dz}\frac{1}{H}\right)_{z=0} \\
&= \frac{1}{H_0} - z\left(\frac{1}{H^2}\frac{dH}{dz}\right)_{z=0}.
\end{aligned}
$$

The results from c) and d) give

$$
\frac{1+z}{H^2(z)}\frac{dH}{dz} = \frac{1+q(z)}{H(z)},
$$

and for$z = 0$ we find

$$
\left(\frac{1}{H^2}\frac{dH}{dz}\right)_{z=0} = \frac{1+q_0}{H_0},
$$

so the Taylor expansion of $1/H$ can be rewritten as

$$\frac{1}{H(z)} \approx \frac{1}{H_0} - \frac{z}{H_0}(1+q_0).$$

We substitute this result in the expression for d_L and integrate:

$$\begin{aligned} d_\mathrm{L} &\approx c(1+z)\int_0^z \left[\frac{1}{H_0} - \frac{z}{H_0}(1+q_0)\right] dz \\ &= c(1+z)\left[\frac{z}{H_0} - \frac{1}{2}\frac{z^2}{H_0}(1+q_0)\right] \\ &\approx \frac{cz}{H_0}\left[1 + \frac{1}{2}(1-q_0)z\right], \end{aligned}$$

where we have neglected terms of order z^3.

j) For the EdS model we have $q_0 = 1/2$, and for our purposes we can take $q_0 = -1/2$ for the ΛCDM model. We find

$$\begin{aligned} d_\mathrm{L}(\mathrm{EdS}) &= \frac{cz}{H_0}\left(1 + \frac{1}{4}z\right) \\ d_\mathrm{L}(\Lambda\mathrm{CDM}) &= \frac{cz}{H_0}\left(1 + \frac{3}{4}z\right). \end{aligned}$$

The relative difference between the models becomes

$$\frac{\Delta d_\mathrm{L}}{d_L(\mathrm{EdS})} \approx \frac{\frac{1}{2}\frac{c}{H_0}z^2}{\frac{cz}{H_0}} = \frac{1}{2}z,$$

and we want the relative difference exceed 0.1. This gives $z > 0.2$. In practice, this estimate turns out to be on the optimistic side.

Suggested solutions, final exam in AST4220, fall 2007

Problem 1

a) From the definitions of the slow-roll parameters we find

$$\begin{aligned}\epsilon &= \frac{E_{\rm Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{E_{\rm Pl}^2}{16\pi} \\ \eta &= \frac{E_{\rm Pl}^2}{8\pi}\frac{V''}{V} = 0.\end{aligned}$$

The value of the field for which inflation ends is determined by the condition $\epsilon(\phi_{\rm end}) = 1$, which gives

$$\phi_{\rm end} = \frac{E_{\rm Pl}}{4\sqrt{\pi}}.$$

b) From the slow-roll expression for the number of e-foldings we find

$$N = 60 = \frac{8\pi}{E_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi_{\rm i}} \frac{V}{V'}d\phi.$$

We substitute this result in the expression for V and get

$$\phi_{\rm i} = \sqrt{241}\phi_{\rm end}.$$

c) We apply the result for r given in the lecture notes:

$$r = 3\sqrt{\epsilon_*},$$

where $\epsilon_* = \epsilon(\phi_*)$, and ϕ_* is determined by

$$50 = \frac{8\pi}{E_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi_*} \frac{V}{V'}d\phi.$$

This condition gives $\phi_* = \sqrt{201}\phi_{\rm end}$, and hence $\epsilon_* = 1/201$. Therefore,

$$r = 3\sqrt{\frac{1}{201}} = 0.21.$$

Oppgave 2

a) See the lecture notes, section 4.5.

b) In this case we have

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}.$$

This gives

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3(1+w)t},$$

and

$$\rho_0(t) = \rho_{c0}(t) = \frac{3H^2}{8\pi G} = \frac{1}{6\pi G(1+w)^2 t^2}.$$

We substitute this in the expression for Δ and get

$$\ddot{\Delta}_k + \frac{4}{3(1+w)t}\dot{\Delta}_k = \frac{2}{3(1+w)^2 t^2}\Delta_k.$$

We guess that the solution is on the form $\Delta_k \propto t^n$, and find that for this to be the case, n must satisfy the equation

$$n(n-1) + \frac{4}{3(1+w)}n - \frac{2}{3(1+w)^2} = 0.$$

We are interested in the growing mode, and hence we seek the positive root of this quadratic equation. This gives the result given in the text.

c) For $w = -2/3$ we have $\rho \propto 1/a$, $a \propto t^2$ and $\Delta \propto t^n \propto t^{1.372} \propto a^{0.686}$. Then $\delta\phi_k \propto a^2 \times a^{-1} \times a^{0.686} \propto a^{1.686}$, so that $\delta\phi_k$ is time-dependent.

In an EdS model, $\delta\phi_k$, and therefore also $\delta\phi$, are time-independent. This means that gravitational potential wells are of the same height when photons leave them as when they entered. Thus, there is no net change in energy for CMB photons travelling through an EdS universe. For accelerated expansion, the potential wells do change with time, and hence the CMB potons will experience a net change in in energy. This makes it possible to use careful observations of the CMB to distinguish between the two cases.

0.1 Problem 3

a) The first Friedmann equation is

$$H^2(z) = H_0^2 \left[\Omega_{\mathrm{k0}}(1+z)^2 + \sum_i \Omega_{i0}(1+z)^{3(1+w_i)} \right],$$

where w_i is the equation of state parameter for component i, and

$$\Omega_{\mathrm{k0}} + \sum_i \Omega_{\mathrm{i0}} = 1.$$

For the EdS model,

$$H(z) = H_0(1+z)^{3/2},$$

and for the dS model

$$H(z) = H_0 = \text{constant}.$$

b) In both models $\Omega_{\mathrm{k0}} = 0$. In the dS case we find

$$\int_0^z \frac{H_0 dz'}{H_0} = z$$

and

$$d_{\mathrm{A}}(z) = \frac{a_0}{1+z} \frac{c}{a_0 H_0} z = \frac{c}{H_0} \frac{z}{1+z}.$$

In the EdS case we find

$$\int_0^z \frac{H_0 dz'}{H_0(1+z')^{3/2}} = 2\left(1 - \frac{1}{\sqrt{1+z}}\right),$$

and hence

$$d_{\mathrm{A}}(z) = \frac{2c}{H_0}\left(\frac{1}{1+z} - \frac{1}{(1+z)^{3/2}}\right).$$

c) For the dS model we see that

$$\frac{d}{dz} d_{\mathrm{A}}(z) = \frac{c}{H_0} \frac{1}{(1+z)^2} > 0$$

so d_{A} is strictly increasing and has no maximum. We note, however, that it approaches c/H_0 asymptotically for large values of z. For the EdS model we find

$$\frac{d}{dz}d_{\mathrm{A}}(z) = \frac{2c}{H_0}\left[-\frac{1}{(1+z)^2} + \frac{3}{2}\frac{1}{(1+z)^{5/2}}\right],$$

which vanishes for $z = 5/4$. By considering, e.g., $d''_{\mathrm{A}}(z)$ at $z = 5/4$, we find that this is indeed a maximum, and the maximum value is $d_{\mathrm{A}}(z = 5/4) = 8c/27H_0$.

d) For $k = 0, w = -1/3$ (in the following referred to as case 1) we see that

$$H(z) = H_0(1+z),$$

while for $k = -1$ and with all $\Omega_{i0} = 0$ (case 2 in the following) we find

$$H(z) = H_0(1+z),$$

that is, the same expression. The integral we need is

$$\int_0^z \frac{H_0 dz'}{H_0(1+z')} = \ln(1+z).$$

In case 1 we find

$$d_{\mathrm{A}}(z) = \frac{c}{H_0}\frac{\ln(1+z)}{1+z},$$

while case 2 gives

$$d_{\mathrm{A}}(z) = \frac{c}{H_0(1+z)}\sinh[\ln(1+z)] = \frac{c}{2H_0}\left[1 - \frac{1}{(1+z)^2}\right],$$

where wee have used $\sinh(x) = (e^x - e^{-x})/2$. Case 1 gives

$$\frac{d}{dz}d_{\mathrm{A}}(z) = \frac{c}{H_0}\frac{1}{(1+z)^2}[1 - \ln(1+z)],$$

which vanishes at $z = e - 1$, easily shown to be a maximum. In case 2 a casual glance at the expression for $d_A(z)$ reveals that it is strictly increasing for all z, but approaches $\frac{c}{2H_0}$ asymptotically. By measuring d_A out to redshifts $z > e - 1 \approx 1.72$ we should be able to distinguish between the two cases.

Mid-term exam in i AST4220, 2008: Suggested solution

Problem 1

a) We know that

$$\begin{aligned} \rho_{\rm r} &= \rho_{\rm r0}\frac{1}{a^4} \\ \rho_\Lambda &= \rho_{\Lambda 0} = \text{ konstant,} \end{aligned}$$

so $a_{\rm eq}$ can be determined from

$$\rho_{\rm r0}\frac{1}{a_{\rm eq}^4} = \rho_{\Lambda 0}.$$

We find

$$a_{\rm eq} = \left(\frac{\rho_{\rm r0}}{\rho_{\Lambda 0}}\right)^{1/4} = \left(\frac{\Omega_{\rm r0}}{\Omega_{\Lambda 0}}\right)^{1/4}.$$

b) The first Friedmann equation becomes

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\left(\rho_{\rm r0}\frac{1}{a^4} + \rho_{\Lambda 0}\right) \\ &= H_0^2\left(\frac{\Omega_{\rm r0}}{a^4} + \Omega_{\Lambda 0}\right), \end{aligned}$$

where we also note that $\Omega_{\Lambda 0} = 1 - \Omega_{\rm r0}$ since $k = 0$.

c) We rewrite the first Friedmann equation:

$$\begin{aligned} \frac{1}{a}\frac{da}{dt} &= H_0\frac{\sqrt{\Omega_{\rm r0}}}{a^2}\left[1 + \frac{\Omega_{\Lambda 0}}{\Omega_{\rm r0}}a^4\right]^{1/2} \\ &= \frac{H_0\sqrt{\Omega_{\rm r0}}}{a^2}\left[1 + \left(\frac{a}{a_{\rm eq}}\right)^4\right]^{1/2}. \end{aligned}$$

Separation of variables in the differential equation gives

$$dt = \frac{1}{H_0\sqrt{\Omega_{\rm r0}}}\frac{a\,da}{\sqrt{1 + (a/a_{\rm eq})^4}},$$

and hence

$$t = \frac{1}{H_0\sqrt{\Omega_{r0}}}\int_0^a \frac{a'da'}{\sqrt{1+(a'/a_{eq})^4}}.$$

d) The age of the Universe at $a = a_{eq}$ is determined by

$$t_{eq} = \frac{1}{H_0\sqrt{\Omega_{r0}}}\int_0^{a_{eq}} \frac{a'da'}{\sqrt{1+(a'/a_{eq})^4}}.$$

By using the recommended substitution, we get

$$a'da' = \frac{a_{eq}^2}{2}dx,$$

so

$$t_{eq} = \frac{a_{eq}^2}{2H_0\sqrt{\Omega_{r0}}}\int_0^1 \frac{dx}{\sqrt{1+x^2}}.$$

We find the required indefinite integral in Rottmann. It is equal to $\sinh^{-1} x$, so that

$$t_{eq} = \frac{a_{eq}^2}{2H_0\sqrt{\Omega_{r0}}}\sinh^{-1}(1),$$

and by substituting the expression for a_{eq} and the given numerical values for the parameters involved, we find

$$t_{eq} \approx 4.2 \times 10^9 \text{ yrs}.$$

e) To determine when acceleration starts in this model, we write down the second Friedmann equation

$$\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\frac{\rho_{r0}}{a^4} + \rho_{\Lambda 0} + \frac{3}{c^2}\frac{1}{3}\frac{\rho_{r0}c^2}{a^4} - \frac{3}{c^2}\rho_{\Lambda 0}c^2\right) \\
&= -\frac{4\pi G}{3}\left(\frac{2\rho_{r0}}{a^4} - 2\rho_{\Lambda 0}\right) \\
&= -H_0^2\left(\frac{\Omega_{r0}}{a^4} - \Omega_{\Lambda 0}\right).
\end{aligned}$$

Transition from deceleration to acceleration occurs at the value a_{acc} of the scale factor determined by

$$\frac{\Omega_{r0}}{a_{acc}^4} - \Omega_{\Lambda 0} = 0,$$

which gives

$$a_{\rm acc} = \left(\frac{\Omega_{\rm r0}}{\Omega_{\Lambda 0}}\right)^{1/4} = a_{\rm eq}.$$

This means that acceleration starts at the same time as the densities of the two components are equal, and we found the age of the Universe at that time in d).

Problem 2

a) The scale factor in the Einstein-de Sitter model is given by

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}.$$

From this expression we find

$$\left(\frac{t}{t_0}\right)^{2/3} = \frac{a(t)}{a_0} = \frac{1}{1+z},$$

so

$$t(z) = t_0(1+z)^{-3/2}.$$

b) We find

$$\begin{aligned} t(z_1) &= t_0(1+3)^{-3/2} = \frac{1}{8}t_0. \\ t(z_2) &= t_0(1+8)^{-3/2} = \frac{1}{27}t_0. \end{aligned}$$

c) The comoving radial coordinate of an object observed today is given by

$$r = \int_{t_{\rm e}}^{t_0} \frac{cdt}{a(t)},$$

where $t_{\rm e}$ is the time at which light received at t_0 was emitted. In this case

$$\begin{aligned} r &= \frac{c}{a_0} t_0^{2/3} \int_{t_{\rm e}}^{t_0} t^{-2/3} dt \\ &= \frac{3ct_0^{2/3}}{a_0}(t_0^{1/3} - t_{\rm e}^{1/3}) \end{aligned}$$

$$\begin{aligned} &= \frac{3ct_0}{a_0}\left(1 - \frac{t_{\rm e}^{1/3}}{t_0^{1/3}}\right) \\ &= \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{1+z}}\right). \end{aligned}$$

Hence we get

$$\begin{aligned} r_1 &= r(z_1) = \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{4}}\right) = \frac{3ct_0}{2a_0}. \\ r_2 &= r(z_2) = \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{9}}\right) = \frac{2ct_0}{a_0}. \end{aligned}$$

d) Along the trajectory of the light ray we have $ds^2 = 0$. Furthermore $d\phi = d\theta = 0$, so the Robertson-Walker line element for $k = 0$ gives

$$c^2dt^2 - a^2(t)dr^2 = 0,$$

so that

$$dr = -\frac{cdt}{a(t)},$$

where we have chosen the negative sign because we consider light moving towards $r = 0$. Integration gives

$$\begin{aligned} \int_{r_2}^{r(t)} dr &= -\int_{t_{\rm e}}^{t} \frac{cdt'}{a(t')} \\ &= -\frac{ct_0^{2/3}}{a_0}\int_{t_{\rm e}}^{t}(t')^{-2/3}dt' \\ &= -\frac{3ct_0}{a_0}\left[\left(\frac{t}{t_0}\right)^{1/3} - \left(\frac{t_e}{t_0}\right)^{1/3}\right], \end{aligned}$$

and so

$$\begin{aligned} r(t) &= r_2 - \frac{3ct_0}{a_0}\left[\left(\frac{t}{t_0}\right)^{1/3} - \left(\frac{t_e}{t_0}\right)^{1/3}\right] \\ &= \frac{2ct_0}{a_0} - \frac{3ct_0}{a_0}\left[\left(\frac{t}{t_0}\right)^{1/3} - \left(\frac{t_e}{t_0}\right)^{1/3}\right]. \end{aligned}$$

e) Light reaches r_1 at the time T determined by

$$\frac{3ct_0}{2a_0} = \frac{2ct_0}{a_0} - \frac{3ct_0}{a_0}\left[\left(\frac{T}{t_0}\right)^{1/3} - \left(\frac{t_\mathrm{e}}{t_0}\right)^{1/3}\right],$$

and after cancelling common factors we get the equation

$$2 - 3\left[\left(\frac{T}{t_0}\right)^{1/3} - \left(\frac{t_\mathrm{e}}{t_0}\right)^{1/3}\right] = \frac{3}{2}.$$

We have

$$\left(\frac{t_\mathrm{e}}{t_0}\right)^{1/3} = \sqrt{\frac{a(t_\mathrm{e})}{a(t_0)}} = \frac{1}{\sqrt{1+z_2}} = \frac{1}{3},$$

and find

$$\left(\frac{T}{t_0}\right)^{1/3} = \frac{1}{2}.$$

The observer at r_1 therefore sees the object at r_2 at a redshift given by

$$1 + z_{12} = \frac{a(T)}{a(t_\mathrm{e})} = \frac{(T/t_0)^{2/3}}{(t_\mathrm{e}/t_0)^{2/3}} = \left(\frac{1/2}{1/3}\right)^2 = \frac{9}{4},$$

so, finally

$$z_{12} = \frac{5}{4}.$$

Suggested solutions to exam problems in AST4220, December 2008

Problem 1

a) The first Friedmann equation with $k = +1$ and a cosmological constant is

$$\left(\frac{\dot{a}}{a}\right) + \frac{c^2}{a^2} = \frac{\Lambda}{3}.$$

It is sufficent to substitute the expression given in the problem text and show that it is a solution of this equation. However, it is not difficult to solve this differential equation directly. We rewrite it as

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right) &= \frac{\Lambda}{3} - \frac{c^2}{a^2} \\ &= \frac{c^2}{a^2}\left(\frac{a^2}{a_\Lambda^2} - 1\right),\end{aligned}$$

where we have introduced $a_\Lambda^2 \equiv 3c^2/\Lambda$. From the last expression we note that we must have $a \geq a_\Lambda$ since the left hand side cannot be negative. Taking the square root of both sides, separating the variables and integrating, we find

$$\int_{a_\Lambda}^{a} \frac{da'}{\sqrt{\left(\frac{a'}{a_\Lambda}\right)^2 - 1}} = c\int_0^t dt' = ct.$$

We use the substitution $a' = a_\Lambda \cosh x$ and the identity $\cosh^2 x - 1 = \sinh^2 x$, and find

$$\int_0^{\cosh^{-1}(a/a_\Lambda)} \frac{a_\Lambda \sinh x dx}{\sinh x} dx = a_\Lambda \cosh^{-1}\left(\frac{a}{a_\Lambda}\right) = ct,$$

and finally,

$$a(t) = a_\Lambda \cosh\left(\frac{ct}{a_\Lambda}\right).$$

We see that a_Λ is the minimum value of the scale factor in this model, so this model does not have a singularity at $t = 0$, nor anywhere else.

b) Ten e-foldings after $t = 0$ we have

$$\ln\left(\frac{a(t)}{a_\Lambda}\right) = 10,$$

that is

$$\frac{a(t)}{a_\Lambda} = \cosh\left(\frac{ct}{a_\Lambda}\right) = e^{10},$$

or

$$\frac{ct}{a_\Lambda} = \cosh^{-1}(e^{10}) \approx 10.69,$$

which gives

$$t = 10.69\frac{a_\Lambda}{c} = 10.69\frac{\sqrt{3}}{\Lambda}.$$

The cosmological constant term in the first Friedmann equation is always equal to $\Lambda/3$. The curvature term is

$$\frac{c^2}{a^2(t)} = \frac{c^2}{a_\Lambda^2(e^10)^2} = e^{-20}c^2\frac{\Lambda}{3c^2} = e^{-20}\frac{\Lambda}{3},$$

that is, a factor $e^{-20} = 2.1\times10^{-9}$ smaller than the cosmological constant term.

c) We can write the first Friedmann equation as

$$1 + \frac{kc^2}{a^2H^2} = \frac{8\pi G}{3H^2}\rho = \frac{\rho(t)}{\rho_c(t)},$$

where $\rho_c(t)$ is the critical density at time t. We know that $\rho(t_0) > \rho_c(t_0)$, which implies $k = +1$. But k is constant, and so we see that the left hand side will always be greater than 1. It then follows that the density will always be greater than the critical density.

Problem 2

a) The Friedmann equations are

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} &= \frac{8\pi G}{3}\rho \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right).\end{aligned}$$

The condition $\rho + 3p/c^2$ implies $\ddot{a} < 0$. We rearrange the first equation to the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}.$$

Since$H_0 > 0$, the right hand side is positive today, which means that the first term is larger than the second. But since ρ is assumed to drop faster than $1/a^2$, this means that the first term will also dominate the second when we go back in time, to lower values of a. We conclude that $\dot{a} > 0$ for all $t \leq t_0$. So we have shown that $a(t)$ is strictly increasing for all $t \leq t_0$, and $\ddot{a}(t) < 0$ for all t. The graph of a function with these properties must cross the t axis somewhere, and this implies that there must have been a time when $a = 0$.

b) In this model $\ddot{a} > 0$ today. However, since $\rho_{\rm m} \propto a^{-3}$ and $\rho_\Lambda =$ constant, we have $\ddot{a} < 0$ for all times before the universe started to accelerate. The argument from the previous question then goes through for this model, too. Therefore, there must have been a time in the past when $a = 0$.

Problem 3

a) The redshift corresponding to the epoch when the size of the observable universe was 10 km is given by

$$1 + z = \frac{15 \times 10^9 \text{ ly}}{10 \text{ km}} = \frac{1.4 \times 10^{26} \text{ m}}{10^4 \text{ m}} \approx 1.4 \times 10^{22}.$$

The radiation temperature was then

$$T = T_0(1+z) = 2.73 \times 1.4 \times 10^{22} \text{ K} = 3.9 \times 10^{22} \text{ K}.$$

This corresponds to a thermal energy

$$k_{\rm B}T = 3.3 \times 10^12 \text{ MeV} = 3.3 \times 10^9 \text{ GeV}.$$

At these energies, all the degrees of freedom in the Standard Model were present, were relativistic, and in thermal equilibrium. The effective number of relativistic degrees of freedom was therefore given by (see chapter 2 in the lecture notes)

$$g_* = 28 + \frac{7}{8} \times 90 = 106.75,$$

and the time after the Big Bang in seconds was

$$t = 2.423 g_*^{-1/2} \left(\frac{k_\mathrm{B} T}{1\ \mathrm{MeV}} \right)^{-2} \mathrm{s} = 2.1 \times 10^{-26}\ \mathrm{s}.$$

b) The number of e-foldings for this potential is given by

$$\begin{aligned} N_\mathrm{tot} &= \frac{8\pi}{E_\mathrm{Pl}^2} \int_{\phi_\mathrm{end}}^{\phi_\mathrm{i}} \frac{V}{V'} d\phi \\ &= \frac{4\pi}{p E_\mathrm{Pl}^2} (\phi_\mathrm{i}^2 - \phi_\mathrm{end}^2) = \frac{4\pi \phi_\mathrm{i}^2}{p E_\mathrm{Pl}^2} \left(1 - \frac{\phi_\mathrm{end}^2}{\phi_\mathrm{i}^2} \right). \end{aligned}$$

Here ϕ_i is the value of the field at the start of inflation, and ϕ_end is its value at the end. The latter is determined by the condition $\epsilon(\phi_\mathrm{end}) = 1$, where the slow-roll-parameter ϵ is given by

$$\epsilon = \frac{E_\mathrm{Pl}^2}{16\pi} \left(\frac{V'}{V} \right)^2 = \frac{E_\mathrm{Pl}^2}{16\pi} \frac{p}{\phi^2}.$$

This gives

$$\phi_\mathrm{end} = \frac{p E_\mathrm{Pl}}{4\sqrt{\pi}}.$$

To have slow-roll, we must have $\epsilon(\phi_\mathrm{i}) \ll 1$, which gives

$$\phi_\mathrm{i}^2 \gg \frac{p^2 E_\mathrm{Pl}^2}{16\pi} = \phi_\mathrm{end}^2.$$

This means that

$$N_\mathrm{tot} \approx \frac{4\pi \phi_\mathrm{i}^2}{p E_\mathrm{Pl}^2} \gg \frac{4\pi}{p E_\mathrm{Pl}^2} \frac{p^2 E_\mathrm{Pl}^2}{16\pi} = \frac{p}{4},$$

and since $p/4$ is a number of order 1, we have proved the desired result.

Oppgave 4

a) The Jeans wavelength is the lengt scale separating balance between pressure gradient and gravitation from graviational collapse. Fourier modes of the density perturbation with $\lambda > \lambda_\mathrm{J}$ will grow with time.

b) In an EdS universe we have $a(t) = a_0(t/t_0)^{2/3}$, so that $H = 2/3t$, $\rho(t) = \rho_c(t) = 3H^2/8\pi G = 1/6\pi G t^2$, and $(1+z)^{3/2} = (a_0/a)^{3/2} = t_0/t$, $c_s^2 = c_{s0}^2(t_0/t)^2$. The equation for the time evolution of density perturbations therefore becomes

$$\ddot{\Delta}_k + \frac{4}{3t}\dot{\Delta}_k = \frac{1}{t^2}\left(\frac{2}{3} - k^2 c_{s0}^2 t_0^2\right)\Delta_k.$$

Inserting $\Delta_k \propto t^n$, we find that n must satisfy the quadratic equation

$$n^2 + \frac{1}{3}n - \left(\frac{2}{3} - k^2 c_{s0}^2 t_0^2\right) = 0,$$

and the solutions are given by the expression in the problem text:

$$n = \frac{1}{6}\left(-1 \pm \sqrt{25 - 26k^2 c_{s0}^2 t_0^2}\right).$$

c) We have three different cases to consider. The first is $25 - 36k^2c_{s0}^2t_0^2 > 0$, i.e., $kc_{s0}t_0 < 5/6$, with two real roots. To have a growing solution, we must in addition demand $n > 0$, that is, $25 - 36k^2c_{s0}^2t_0^2 > 1$, which gives the condition

$$kc_s t_0 < \frac{2}{\sqrt{6}}.$$

For $25 - 36k^2c_{s0}^2t_0^2 < 0$ we have two complex roots

$$n = \frac{1}{6}(-1 \pm i\sqrt{D}),$$

where $D = 36k^2c_{s0}^2t_0^2 - 25$. The general solution can in this case be written as

$$\begin{aligned}
\Delta_k(t) &= At^{(-1-i\sqrt{D})/6} + Bt^{(-1+i\sqrt{D})/6} \\
&= At^{-1/6}(e^{\ln t})^{-i\sqrt{D}/6} + Bt^{-1/6}(e^{\ln t})^{i\sqrt{D}/6}
\end{aligned}$$

$$
\begin{aligned}
&= t^{-1/6}[A\cos\left(\frac{\sqrt{D}}{6}\ln t\right) - i\sin\left(\frac{\sqrt{D}}{6}\ln t\right) \\
&+ B\cos\left(\frac{\sqrt{D}}{6}\ln t\right) + i\sin\left(\frac{\sqrt{D}}{6}\ln t\right)] \\
&= C_1 t^{-1/6}\cos\left(\frac{\sqrt{D}}{6}\ln t\right) + C_2 t^{-1/6}\sin\left(\frac{\sqrt{D}}{6}\ln t\right),
\end{aligned}
$$

which we recognize as oscillations with damped amplitude.

The last case is the limiting case $25 - 36k^2c_{s0}^2t_0^2 = 0$. We then have one real root, $n = -1/6$, corresponding to a decaying density perturbation.