Modelling Subsurface Flow – not what you see, but what you imagine

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FEMLAB conference, Oslo October 13. 2005
Outline:

• teaching
• how to make COMSOL MULTIPHYSICS behave according to my concepts
• examples

Why focus on groundwater?
Important for society

Groundwater resources of the world

Major part of the world population are depending on groundwater resources
The global water balance

~ 30 times more groundwater than water in the lakes

Important for nature

<table>
<thead>
<tr>
<th>Storage Compartments of Water</th>
<th>Vol. (thousands of km$^3$)</th>
<th>Percentage of Total Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ocean</td>
<td>1,230,000</td>
<td>97.2</td>
</tr>
<tr>
<td>Glaciers and ice caps</td>
<td>28,600</td>
<td>2.2</td>
</tr>
<tr>
<td>Shallow groundwater</td>
<td>4,000</td>
<td>0.3</td>
</tr>
<tr>
<td>Lakes</td>
<td>123</td>
<td>0.009</td>
</tr>
<tr>
<td>Atmosphere</td>
<td>12.7</td>
<td>0.001</td>
</tr>
<tr>
<td>Rivers</td>
<td>1.2</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Groundwater flow is a boundary value problem => measurements
Why focus on groundwater?

Groundwater: the liquid of life!

interface to human society and science

groundwater flow

contaminant transport

heterogeneity

concept of geology

structure

numerical simulation
Concept of flow:

Henri-Philibert-Gaspard Darcy

Darcy's experiments (1856):

Darcy's law: governs the laminar (nonturbulent) flow of fluids in homogeneous, porous media

\[ Q \propto A \frac{\Delta h}{\Delta x} \]

\[ \Delta x \]

\[ \Delta h \]

\[ h_1 - h_2 \]

\[ Q \]

→ Conservation of momentum
1. Conservation of momentum:

\[ \mathbf{q} = - K \nabla h, \]
where \( h = \frac{p}{\rho g} + z \)

2. Conservation of mass:

\[ \frac{\partial (S_0 h)}{\partial t} = - \nabla \cdot \mathbf{q} + N, \]
where \( S_0 \) is specific storage \([1/L]\),
and \( N \) is a sink/source term

3. Constitutive relations:

e.g. if \( S_0 \) is a function of \( h \) → Richards’ equation (not today!)
or if \( \rho \) is a function of concentration of solute (later)

Here: Dupuit-Forchheimer assumption:

\[ \frac{\partial h}{\partial z} = 0, \quad \text{but} \quad q_z \neq 0 \]
which means 3D → 2D

4. Boundary conditions and (if transient) initial conditions
Cross Section of the Gardermoen Delta

After K. Tuttle (1997)
From 1,2 we have:

\[ \frac{\partial (S_0 h)}{\partial t} = \nabla \cdot \left( K \nabla h \right) + N. \]

From 3, life gets easier:

\[ \frac{\partial (mS_0 h)}{\partial t} = \nabla \cdot \left( mK \nabla h \right) + N, \]

where \( m \) is thickness of water saturated zone
\( mS_0 = S \) [-] is called storage coefficient
\( mK = T \) [L^2/T] is called transmissivity

For a confined aquifer:

\[ m = \text{top} - \text{bottom of aquifer} \]

\[ m = \text{const.} \]

For an unconfined aquifer:

\[ m = h - \text{bottom of aquifer} \]
Remember (2): \( Q = Aq \ (L^3/T) \)
By using boundary integration in postprocessing, you get the water flow directly!

Remember (1): \( h(t_0) \neq 0 \) if bottom = 0
Challenge: estimate effective parameters

\[ T = \int_{0}^{m} T(z) \, dz \]

in a way that the response \( h \) is reproduced according to the observations

\[ \frac{\partial (S \, h)}{\partial t} = \nabla \cdot (T \, \nabla h) - N(t) \]

The Boussinesq equation

where:
- \( T \) effective transmissivity (L\(^2\)/T)
- \( S \) storage coefficient(-)
- \( h \) hydraulic head (L)
- \( t \) time (T)
- \( N(t) \) source/sink (L/T), infiltration or pumping
Example: The Gardermoen Delta

The inland glacier
~10500 B.P.
(Andersen, B.G., 2000)

Sea level about 200 m higher than present sea level.
Average paleo-discharge: ~3000m²/s
(Tuttle, K., 1998)

The Gardermoen paleo-delta
The Gardermoen Delta Today:
Paleo-distribution Channels Show **Radial Flow**: 

- Trandum Delta Paleo-portal
- Helgebostad Delta Paleo-portal

**x-profile**
"The Gardermoen Doughnut"

Steady state model for \( R_1 \leq r \leq R_2 \):

i) Darcy’s law:

a) \[ \frac{Q_r}{m} = q = -k \frac{dh}{dr} \]

b) \[ Q_r = - \frac{d(\frac{1}{2}kh^2)}{dr} \]

if \( m=h \), then unconfined aquifer

ii) Balance of mass:

\[ Q_A = N\pi l^2 - N\pi r^2 \]

\[ Q_r = - \frac{Q_A}{2\pi r} \]

i and ii): \[ d(\frac{1}{2}kh^2) = \frac{N}{2} \left( \frac{l^2}{r} dr - r dr \right) \]

Repeat for \( l \leq r \leq R_2 \) and eliminate \( l \) to get one closed form expression for \( h(r) \)
Plot observations of groundwater head as a function of radial distance from center:

let \( k = k(r) \)

or \( H = H(r) \)
\[ dh^2 = N \left( \frac{l^2}{rk} \, dr - \frac{r}{k} \, dr \right) \]

where \( k = k_1 - a(r-R_1) \)

Two (simple) integrals:

\[ \int \frac{1}{r(k_0-ar)} \, dr \quad (1) \]

\[ \int \frac{r}{(k_0-ar)} \, dr \quad (2) \]

Implement the solution as a function in MATLAB:

```matlab
function [h] = funk_h_lin(radius, R1, R2, h1, h2, N, k_1, k_2)
```
minimum$(h_{\text{obs}} - h_{\text{calc}})$ gives $k_1$ at $R_1$ and $k_2$ at $R_2$
Radial Duit-Forchheimer Solution where $k = k(r)$

Interpolation Functions:

$R_1$, $R_2$, $k_1$, $k_2$
Steady State Numerical Solution and Analytical Solution:

Transient Simulation

Precipitation event

\[ S = 0.01 \]
OK!

But, what about the real aquifer?

Precipitation event

<table>
<thead>
<tr>
<th>mm/d</th>
<th>0.00</th>
<th>2.00</th>
<th>4.00</th>
<th>6.00</th>
<th>8.00</th>
<th>10.00</th>
<th>12.00</th>
<th>14.00</th>
<th>16.00</th>
<th>18.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>days from 27.09.2000</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

Transient boundary condition:

\[ H_1(t) = H_1 \sin(t/\text{day}) \]

Conclusion:
1) minor changes in h with N(t)
2) not very sensitive to \( H_1 \)

OK!

But, what about the real aquifer?
Digitize boundary by using: `GINPUT` Graphical input from mouse.

`[X,Y] = GINPUT(N)` gets N points from the current axes and returns the X- and Y-coordinates in length N vectors X and Y.
Spline interpolation
Make solid object by:
\[ s1 = \text{geomcoerce('solid',{c})}; \]
geomplot(s1);
and import geometry in Comsol Multiphysics
What about hydraulic conductivity?

Two radial structures with linearly decreasing $k$ superimposed on each other.

What is best average?
arithmetic mean?

Glacial portals
geometric mean?
harmonic mean?

\[ \frac{1}{T_1} \frac{1}{T_2} \]

Glacial portals
Difference in hydraulic conductivity between harmonic - and geometric mean.
Hydraulic head
Input: hydraulic head as **arithmetic mean** between the Trandum and Li deltas
Hydraulic head
Input: hydraulic head as **geometric mean** between the Trandum and Li deltas
Hydraulic head
Input: hydraulic head as **harmonic mean** between the Trandum and Li deltas
Hydraulic head gradient,
minimum gradient at ground water divide
max. gradient in ravine areas (landslide) and glacial boundary (kettle lake area)
Hydraulic head gradient

With terminal and railway culvert (drawdown)

To protect the preservation area, the water balance of the area has to be maintained because the hydraulic gradient is the driving force of the gully processes.
Hydraulic head gradient

With terminal and railway culvert (drawdown) and two injection points of water from the culvert.
The Comsol meshing routines makes life more practical!
Plot cross-section before and after Airport construction
Note change in head gradient towards ravine area
The majority of the world population depends on groundwater resources. Groundwater resources of the world are important for society. Coastal aquifers: possible contamination of groundwater by seawater intrusion.
Density driven flow: 2-way coupling between flow & transport

- Density dependent fluid flow - Darcy’s Law expresses in terms of pressure $p$:

$$\rho[\xi(1-\theta) + \zeta\theta] \frac{\partial p}{\partial t} + \theta \frac{\partial \rho}{\partial c} \frac{\partial c}{\partial t} + \nabla \cdot [-\rho \frac{\kappa}{\eta} \nabla (p + \rho g D)] = 0$$

compressibility = 0

- Salt concentration – Saturated solute transport

$$\theta \left( \frac{\partial c}{\partial t} \right) + \nabla \cdot \left[ -\theta D \nabla c + u C \right] = 0$$

- $\rho$ varies with $c$

$$\rho = \rho + \gamma(c - c_0)$$

where:

$$\gamma = \frac{\rho_t - \rho_0}{c_s - c_0}$$

from the presentation by Leigh Soutter
The Henry Problem

Porosity, $\theta = 0.35$
Seawater concentration, $C_s = 35$ kilograms per cubic meter
Fluid density of seawater, $\rho_s = 1,025$ kilograms per cubic meter
Fluid density of freshwater, $\rho_f = 1,000$ kilograms per cubic meter
Inflow rate, $Q_{in} = 5.702$ cubic meters per day per meter
Inflow concentration, $C_{in} = 0.0$ kilograms per cubic meter
Equivalent freshwater hydraulic conductivity, $K_f = 864$ meters per day
Longitudinal and transverse dispersivity, $\alpha_L = \alpha_T = 0.0$ meter
Molecular diffusion, $D_m = 1.62925$ square meters per day (case 1)
Molecular diffusion, $D_m = 0.57024$ square meter per day (case 2)

Figure 12. Boundary conditions and model parameters for the Henry problem.
Dm = 1.6295 m²/d
$D_m = 0.57024 \text{ m}^2/\text{d}$
Decrease influx of fresh water from 5.702 m³/s to 0.05702 m³/s
What is the effect of tidal changes of 0.1 times thickness of aquifer?
Conclusion

COMSOL MULTIPHYSICS is useful for

  teaching
  and business

  and

  it’s fun!
Thank you!