The Bott inverted infinite projective space is homotopy algebraic K-theory

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Abstract

We show a motivic stable weak equivalence between the Bott inverted infinite projective space and homotopy algebraic K-theory.

Contents

1 Introduction

The classifying space $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ of the multiplicative group scheme $\mathbf{G}_{\mathfrak{m}}$ over a noetherian base scheme S of finite Krull dimension is a simplicial presheaf on the smooth Nisnevich site of the base. When S is regular, the homotopy of $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ is determined by units and isomorphism classes of line bundles [13]. The suspension spectrum of $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ acquires a Bott element β mapping to the Bott element for the motivic spectrum KGL representing homotopy algebraic K-theory [21]. We relate KGL to the Bott inverted infinite projective space $\mathbf{P}^\infty = \mathsf{B}\mathbf{G}_{\mathfrak{m}}$.

Theorem 1.1: *Suppose S is a noetherian base scheme of finite Krull dimension. There is a natural isomorphism*

$$\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}] \xrightarrow{\cong} \mathsf{KGL}$$

in the motivic stable homotopy category.

We offer some comments to clarify the content of this result.

Since $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ is a classifying space for line bundles the motivic spectrum $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}$ can be viewed as a universal cohomology theory for which it is allowed to add line bundles. Theorem 1.1 shows that by inverting the Bott element we get K-theory which classifies all vector bundles. In addition, the orientation on Bott inverted $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ is compatible with the one on KGL.

The multiplicative group structure on the scheme $\mathbf{G}_{\mathfrak{m}}$ induces a commutative monoid structure on the motivic symmetric suspension spectrum of $\mathsf{B}\mathbf{G}_{\mathfrak{m}+}$ [9]. One expects that $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]$ gives an example of a commutative motivic symmetric ring spectrum model for K-theory.

When the base scheme is the complex numbers $\mathbf{C}$, taking points of $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]$ and KGL implies Snaith's theorem identifying $\Sigma^\infty \mathsf{B}\mathbf{C}^\times_+[\beta^{-1}] = \Sigma^\infty \mathbf{CP}^\infty_+[\beta^{-1}]$ with the periodic complex K-theory spectrum KU [18]. In contrast with this amusing observation, when the base scheme is the real numbers $\mathbf{R}$, taking points we obtain an isomorphism between contractible spectra. In effect, the spectrum with terms $\mathsf{B}\mathbf{O}$ and structure maps given by multiplication by $\eta \in \pi_1(\mathsf{B}\mathbf{O})$ is contractible because $\eta^3 = 0$ in the homotopy of $\mathsf{B}\mathbf{O}$. The fact that Bott inverted $\mathsf{B}\mathbf{Z}/2_+ = \mathbf{RP}^\infty_+$ is contractible follows also from a more general result for topological Eilenberg-MacLane spaces proven in [2].

We refer to [14] and the appendix in this paper for precursors concerning cohomology theories in motivic homotopy theory. The orientation on Bott inverted $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ alluded to in the above gives rise to a multiplicative formal group law $\mathsf{x}+\mathsf{y}-\beta\mathsf{xy}$. As shown in Section 3, it represents the universal multiplicative oriented cohomology theory. On the other hand, by applying the motivic Landweber exactness theorem [14] to the Laurent polynomial ring on the Bott element, it follows that

$$\mathsf{MGL}^{*,*}(-)\otimes_{\mathsf{L}}\mathbf{Z}[\beta,\beta^{-1}] \tag{1}$$

defines an oriented cohomology theory with a multiplicative formal group law on the full subcategory of strongly dualizable objects in the motivic stable homotopy category. Here MGL is Voevodsky's algebraic cobordism spectrum [21] and L is the Lazard ring. The formal group law $\mathsf{x}+\mathsf{y}-\beta\mathsf{xy}$ over $\mathbf{Z}[\beta,\beta^{-1}]$ induces the algebra map $\mathsf{L}\to\mathbf{Z}[\beta,\beta^{-1}]$. As remarked in the appendix $\mathsf{MGL}^{*,*}(-)$ is the universal oriented cohomology theory. This implies that (1) is the universal multiplicative oriented cohomology theory. Thus there is an isomorphism

$$\mathsf{MGL}^{*,*}(-)\otimes_{\mathsf{L}}\mathbf{Z}[\beta,\beta^{-1}]\xrightarrow{\cong}\Sigma^{\infty}\mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]^{*,*}(-), \tag{2}$$

and an induced commutative diagram of oriented motivic cohomology theories:

$$\begin{tikzcd}
\Sigma^{\infty}\mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]^{*,*}(-) \arrow[rr] & & \mathsf{KGL}^{*,*}(-) \\
& \mathsf{MGL}^{*,*}(-)\otimes_{\mathsf{L}}\mathbf{Z}[\beta,\beta^{-1}] \arrow[ul, "\cong"] \arrow[ur] &
\end{tikzcd}$$

By combining Theorem 1.1 and the isomorphism (2) we deduce the following motivic generalization of the classical Conner-Floyd theorem [3] relating topological K-theory and complex cobordism:

Theorem 1.2: *Suppose S is a noetherian base scheme of finite Krull dimension. There is an isomorphism of oriented cohomology theories on the full subcategory of strongly dualizable objects in the motivic stable homotopy category*

$$\mathsf{MGL}^{*,*}(-)\otimes_{\mathsf{L}}\mathbf{Z}[\beta,\beta^{-1}]\xrightarrow{\cong}\mathsf{KGL}^{*,*}(-).$$

In particular, if X is smooth projective there is an isomorphism

$$\mathsf{MGL}^{*,*}(X)\otimes_{\mathsf{L}}\mathbf{Z}[\beta,\beta^{-1}]\xrightarrow{\cong}\mathsf{KGL}^{*,*}(X).$$

We wish to point out that Theorem 1.2 has been announced by Hopkins and Morel for base schemes of characteristic zero. For arbitrary fields, Panin, Pimenov and Röndigs have shown a closely related result for the coefficient ring MGL^* [15].

An alternate proof of Theorem 1.1 has been announced by Gepner and Snaith [6]. We thank them for constructive personal communication.

2 Motivic stable homotopy theory

Fix a noetherian base scheme S of finite Krull dimension with motivic stable homotopy category $\mathbf{SH}$. The latter acquires several models based on motivic unstable homotopy of smooth schemes of finite type Sm over S, cf. [5], [7], [9], [12], [20], [21] and [22]. We consider motivic spectra with respect to the projective line pointed at $\infty\colon S \to \mathbf{P}^1$.

Recall the zero-space of KGL is $\mathbf{Z} \times \mathsf{B}\mathbf{GL}$ and $\mathbf{G}_{\mathfrak{m}} \subset \mathbf{GL}$ induces a multiplicative map $\mathsf{B}\mathbf{G}_{\mathfrak{m}} \to \{1\} \times \mathsf{B}\mathbf{GL}$ which sends a line bundle represented by a map into $\mathbf{P}^\infty$ to its class in the Grothendieck group of all vector bundles. By adjointness between motivic spaces and spectra there exists an induced map of motivic spectra

$$\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \longrightarrow \mathsf{KGL}. \tag{3}$$

A motivic spectrum E defines a cohomology theory $\mathsf{E}^{*,*}(-)$ and a homology theory $\mathsf{E}_{*,*}(-)$ on $\mathbf{SH}$ [21]: If S^1_s is the simplicial circle, F a motivic spectrum, and (p,q) a pair of integers there are abelian groups

$$\mathsf{E}^{p,q}(\mathsf{F}) \equiv \mathbf{SH}\big(\mathsf{F}, \Sigma^\infty(S_s)^{\wedge(p-2q)} \wedge \Sigma^\infty(\mathbf{P}^1)^{\wedge q} \wedge \mathsf{E}\big)$$

and

$$\mathsf{E}_{p,q}(\mathsf{F}) \equiv \mathbf{SH}\big(\Sigma^\infty(S_s)^{\wedge(p-2q)} \wedge \Sigma^\infty(\mathbf{P}^1)^{\wedge q}, \mathsf{E} \wedge \mathsf{F}\big).$$

Voevodsky has shown that $\mathsf{KGL}^{p,q}(X) = KH_{2q-p}(X)$ for $X \in \mathsf{Sm}$ where KH denotes Weibel's homotopy algebraic K-theory [21, Theorem 6.9], [23].

Next we recall the notion of an orientation on a motivic ring spectrum E as introduced by Morel, cf. [8], [17] and [19]. The unit map $\mathbf{1} \to \mathsf{E}$ yields a class $1 \in \mathsf{E}^{0,0}(\mathbf{1})$ and hence by smashing with the projective line a class $\mathsf{c}_1 \in \mathsf{E}^{2,1}(\mathbf{P}^1)$. An orientation on E is a class $\mathsf{c}_\infty \in \mathsf{E}^{2,1}(\mathbf{P}^\infty)$ – typically the first Chern class of the tautological line bundle on $\mathbf{P}^\infty$ – which restricts to c_1.

The algebraic cobordism spectrum MGL has the same universal property as complex cobordism MU, meaning there is a one-to-one correspondence between orientations on E and motivic ring maps $\mathsf{MGL} \to \mathsf{E}$ (the results in [8], [17] and [19] hold over S).

Applying the bar construction to the multiplication map on $\mathbf{G}_{\mathfrak{m}}$ yields an H-space structure $\mathfrak{m}\colon \mathsf{B}\mathbf{G}_{\mathfrak{m}} \times \mathsf{B}\mathbf{G}_{\mathfrak{m}} \to \mathsf{B}\mathbf{G}_{\mathfrak{m}}$. If $(\mathsf{E}, \mathsf{c}_\infty)$ is oriented, then

$$\mathfrak{m}^*(\mathsf{c}_\infty) \in \mathsf{E}^{2*,*}(\mathbf{P}^\infty \times \mathbf{P}^\infty) = \mathsf{E}^*[[\mathsf{x}, \mathsf{y}]]$$

defines a one-dimensional formal group law $\mathsf{F}_{\mathsf{E},\mathsf{c}_\infty}(\mathsf{x}, \mathsf{y})$ over the commutative subring $\mathsf{E}^* \equiv \oplus \mathsf{E}^{2*,*}$, where $\mathsf{x} \equiv \mathsf{pr}_1^*(\mathsf{c}_\infty)$, $\mathsf{y} \equiv \mathsf{pr}_2^*(\mathsf{c}_\infty)$ and $\mathsf{pr}_i\colon \mathsf{B}\mathbf{G}_{\mathfrak{m}} \times \mathsf{B}\mathbf{G}_{\mathfrak{m}} \to \mathsf{B}\mathbf{G}_{\mathfrak{m}}$ denotes the projection on the ith factor.

Example 2.1: *The Thom spaces $\mathsf{Th}(\gamma_n)$ of the tautological vector bundles γ_n over $\mathsf{B}\mathbf{GL}_n$ together with the structure maps obtained from the maps $\mathbf{A}^1 \times \gamma_n \to \gamma_{n+1}$ given by $(s,t) \mapsto \big(st_1 + \mathsf{sh}_{\mathbf{A}^\infty}(t)\big)$ for the shift map $\mathsf{sh}_{\mathbf{A}^\infty}(t_0, t_1, \ldots) = (0, t_0, t_1, \ldots)$ on $\mathbf{A}^\infty$ comprise the algebraic cobordism spectrum [21]. The zero-section of the line bundle γ_1 yields a weak equivalence $\mathsf{B}\mathbf{G}_{\mathfrak{m}} \to \mathsf{Th}(\gamma_1)$ and an orientation on MGL determined by the composite*

$$\gamma_\infty\colon \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \longrightarrow \Sigma^\infty \mathsf{Th}(\gamma_1) \longrightarrow \mathbf{P}^1_+ \wedge \mathsf{MGL},$$

where $(\mathbf{P}^1)^{\wedge n} \wedge \mathsf{Th}(\gamma_1) \to \mathbf{P}^1 \wedge \mathsf{Th}(\gamma_n)$ is defined using the structure maps of MGL.

If S is a field of characteristic zero, $\mathsf{L} \to \mathsf{MGL}^*$ is an isomorphism by unpublished work of Hopkins and Morel [1], [10], i.e. $\mathsf{F}_{\mathsf{MGL},\gamma_\infty}(\mathsf{x}, \mathsf{y})$ is the universal formal group law. Recall that L denotes the Lazard ring.

Example 2.2: *Let $\beta \in \mathsf{KGL}^{-2,-1}$ be the KGL Bott element [11], [21]. One verifies easily that the class $\beta^{-1}\big(1 - [\mathcal{O}_{\mathbf{P}^\infty}(-1)]\big) \in \mathsf{KGL}^{2,1}(\mathbf{P}^\infty)$ yields an orientation on KGL and the corresponding multiplicative formal group law is given by*

$$\mathsf{F}_{\mathsf{KGL},\beta^{-1}(1-[\mathcal{O}_{\mathbf{P}^\infty}(-1)])}(\mathsf{x}, \mathsf{y}) = \mathsf{x} + \mathsf{y} - \beta \mathsf{x}\mathsf{y}.$$

3 The Bott inverted infinite projective space

In what follows we introduce the motivic spectrum $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]$ in the title and show that it represents the universal multiplicative oriented cohomology theory.

The identity map of $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}$ defines cohomology classes

$$\xi_\infty \in \big(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}\big)^{0,0}(\mathbf{P}^\infty), \;\; \xi_1 \equiv \xi_{\infty_{|\mathbf{P}^1}} \in \big(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}\big)^{0,0}(\mathbf{P}^1).$$

Clearly ξ_∞ pulls back to the identity element of $\big(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}\big)^{0,0}$. The class $1-\xi_1$ is send to $1-[\mathcal{O}_{\mathbf{P}^1}(-1)]$ in $K_0(\mathbf{P}^1)$ under the map

$$\big(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}\big)^{0,0}(\mathbf{P}^1) \longrightarrow \mathsf{KGL}^{0,0}(\mathbf{P}^1).$$

Now consider the image of the class $1-\xi_1$ in the reduced cohomology of $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ under the identification

$$\big(\widetilde{\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}}\big)^{0,0}(\mathbf{P}^1) \cong \big(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}\big)_{2,1}.$$

There exists a corresponding map $\Sigma^\infty \mathbf{P}^1 \to \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}$ and taking the adjoint of

$$\Sigma^\infty \mathbf{P}^1 \wedge \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \longrightarrow \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \wedge \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \longrightarrow \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}$$

we get

$$\beta\colon \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \longrightarrow \Sigma^{-2,-1}\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}. \tag{4}$$

In (4) we desuspend using the standard notation $S^{p,q} = S_s^{p-q} \wedge \mathbf{G}_{\mathfrak{m}}^{\wedge q}$ for mixed motivic spheres [13, §3.2].

Definition 3.1: *Bott inverted* $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ *is the homotopy colimit or mapping telescope*

$$\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}] \equiv \text{hocolim}\,\big(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \xrightarrow{\beta} \Sigma^{-2,-1}\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \xrightarrow{\Sigma^{-2,-1}\beta} \cdots\big).$$

There is an evident motivic ring structure on $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]$ and a corresponding cohomology theory. Note that β is send to the Bott element of KGL which classifies the virtual vector bundle $1-[\mathcal{O}_{\mathbf{P}^1}(-1)]$. In what follows we denote by the same letter the image of the class ξ_∞ under the map induced by $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \to \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]$.

Lemma 3.2: *The cohomology class*

$$\beta^{-1}(1-\xi_\infty) \in (\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}])^{2,1}(\mathbf{P}^\infty)$$

defines an orientation on Bott inverted $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$*. The corresponding formal group law is*

$$\mathsf{F}_{\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}],\beta^{-1}(1-\xi_\infty)}(\mathsf{x},\mathsf{y}) = \mathsf{x}+\mathsf{y}-\beta\mathsf{x}\mathsf{y}.$$

Proof. The map

$$\mathfrak{m}^* \colon (\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}])^{*,*}(\mathbf{P}^\infty) \longrightarrow (\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}])^{*,*}(\mathbf{P}^\infty \times \mathbf{P}^\infty)$$

sends $\beta^{-1}(1-\xi_\infty)$ to $\beta^{-1}(1-\xi_\infty \otimes \xi_\infty)$. By definition $\xi_\infty \otimes 1 = 1-\beta\mathsf{x}$ and $1 \otimes \xi_\infty = 1-\beta\mathsf{y}$, so that $\xi_\infty \otimes \xi_\infty = (1-\beta\mathsf{x})(1-\beta\mathsf{y}) = 1-\beta(\mathsf{x}+\mathsf{y}-\beta\mathsf{x}\mathsf{y})$. Hence we get

$$\beta^{-1}(1-\xi_\infty \otimes \xi_\infty) = \mathsf{x}+\mathsf{y}-\beta\mathsf{x}\mathsf{y}.$$

□

Definition 3.3: *An oriented motivic spectrum or cohomology theory* $(\mathsf{E}, \mathsf{c}_\infty)$ *is called multiplicative if*

$$\mathsf{F}_{\mathsf{E},\mathsf{c}_\infty}(\mathsf{x},\mathsf{y}) = \mathsf{x}+\mathsf{y}-u\mathsf{x}\mathsf{y},$$

where $u \in \mathsf{E}_{2,1}$ *is invertible.*

Next we formulate an important universal property of Bott inverted $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$.

Theorem 3.4: *The Bott inverted infinite projective space represents the universal multiplicative oriented cohomology theory.*

A proof of Theorem 3.4 follows by combining the next three lemmas. Throughout $(\mathsf{E}, \mathsf{c}_\infty)$ denotes a multiplicative oriented cohomology theory.

Lemma 3.5: *The classes* $u \in \mathsf{E}_{2,1}$ *and* $\mathsf{c}_\infty \in \mathsf{E}^{2,1}(\mathbf{P}^\infty)$ *determine a map of motivic ring cohomology theories*

$$1-u\mathsf{c}_\infty \colon \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}^{*,*}(-) \longrightarrow \mathsf{E}^{*,*}(-) \tag{5}$$

which sends β *to* u.

Proof. We note the zero-section $S \to \mathbf{G}_{\mathfrak{m}}$ induces the unit map $\mathbf{1} \to \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}$ and the cohomology class u restricts to zero over the base scheme. It follows that $1-u\mathsf{c}_\infty$ restricts to the unit in $\mathsf{E}^{0,0}$ represented by $\mathbf{1} \to \mathsf{E}$. The map is therefore unital.

To show the map is multiplicative it suffices, by Lemma 6.5, to note that the following two classes in $\mathsf{E}^{0,0}(\mathsf{B}\mathbf{G}_{\mathfrak{m}} \times \mathsf{B}\mathbf{G}_{\mathfrak{m}})$ coincide: The first is the exterior product $(1-u\mathsf{x})(1-u\mathsf{y})$ of the two classes given by the map in question, and the second the class $1-u(\mathsf{x}+\mathsf{y}-u\mathsf{x}\mathsf{y})$ obtained by composing the given map with the multiplication on $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$.

The negative of β is determined by the composite map

$$\mathbf{1}^{2,1} \longrightarrow \mathbf{1} \vee \mathbf{1}^{2,1} \cong \Sigma^\infty \mathbf{P}^1_+ \longrightarrow \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}.$$

In E-cohomology, it sends a power series in $\mathsf{E}^{0,0}(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+})$ of homogeneous degree $(0,0)$ to its coefficient in $\mathsf{x} = \mathsf{c}_\infty$. Hence $1 - u\mathsf{c}_\infty$ maps to $-u$ and the second claim follows. $\square$

Lemma 3.6: *The map (5) induces an orientation preserving map*

$$\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]^{*,*}(-) \longrightarrow \mathsf{E}^{*,*}(-). \tag{6}$$

Proof. This follows from Lemma 3.5 since $\mathsf{c}_\infty = u^{-1}\big(1 - (1 - u\mathsf{c}_\infty)\big)$ and the orientation on $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]$ is given by $\beta^{-1}(1 - \xi_\infty)$. $\square$

Lemma 3.7: *The map (6) is the unique orientation preserving map.*

Proof. Suppose $\phi\colon \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]^{*,*}(-) \to \mathsf{E}^{*,*}(-)$ is orientation preserving. It suffices to show the composite map

$$\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}^{*,*}(-) \longrightarrow \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]^{*,*}(-) \xrightarrow{\phi} \mathsf{E}^{*,*}(-)$$

equals $1 - u\mathsf{c}_\infty$, i.e. ϕ sends $1 - \xi_\infty$ to $1 - u\mathsf{c}_\infty$. To wit, ϕ maps β to u and $\beta^{-1}(1 - \xi_\infty)$ to $u^{-1}(1 - u\mathsf{c}_\infty)$. Applying $1 - \beta \cdot \_$ implies the claim. $\square$

Remark 3.8: *Adopting the same argument in topology implies Snaith's theorem because the Conner-Floyd isomorphism relating complex cobordism* MU *and unitary topological* K*-theory* KU *shows the latter represents the universal multiplicative complex oriented cohomology theory.*

4 Main proof

In this section we prove Theorem 1.1. We may assume the base scheme is the integers since the motivic spectra in question are preserved under base change. For integers p, q, define $\pi_{p,q}\mathsf{E} \equiv \mathbf{SH}(S^{p,q}, \mathsf{E})$.

Proof. (of Theorem 1.1) Since KGL and Bott inverted $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ are motivic cellular spectra it suffices according to [4, Corollary 7.2] to show there is a naturally induced isomorphism

$$\pi_{p,q}(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}] \longrightarrow \mathsf{KGL}). \tag{7}$$

We claim the map (7) is a retract of

$$\mathsf{MGL}_{p,q}(\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}] \longrightarrow \mathsf{KGL}). \tag{8}$$

In effect, recall that KGL and Bott inverted $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ acquire MGL-modules structures which combined with the unit map of MGL and the multiplication on KGL induce

$$\mathsf{KGL} \longrightarrow \mathsf{MGL} \wedge \mathsf{KGL} \longrightarrow \Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}] \wedge \mathsf{KGL} \longrightarrow \mathsf{KGL} \wedge \mathsf{KGL} \longrightarrow \mathsf{KGL},$$

and likewise for $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+}[\beta^{-1}]$. Thus it suffices to prove that (8) is an isomorphism.

There exist different H-space structures on $\mathbf{Z} \times \mathsf{BU}$ and $\mathbf{Z} \times \mathsf{B}\mathbf{GL}$ corresponding to the tensor product and direct sum operations on vector bundles. In order to distinguish between the two, we write $\mathbf{Z} \times \mathsf{BU}^{\otimes}$ and $\mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes}$ for the multiplicative (respectively $\mathbf{Z} \times \mathsf{BU}^{\oplus}$ and $\mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\oplus}$ for the additive) H-space structures induced by the tensor products (respectively direct sums). Note that $\mathsf{B}\mathbf{G}_{\mathfrak{m}} \to \{1\} \times \mathsf{B}\mathbf{GL}$ induces a map of motivic ring spectra $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathfrak{m}+} \to \Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes}$. Next we express the MGL-homology of $\mathsf{B}\mathbf{G}_{\mathfrak{m}}$ and $\mathbf{Z} \times \mathsf{B}\mathbf{GL}$ in terms of the coefficient ring $\mathsf{MGL}_{*,*}$ of the algebraic cobordism spectrum. As for the MU-homology of $\mathbf{CP}^\infty$ and $\mathbf{Z} \times \mathsf{BU}$, which corresponds to complex points when the base scheme is $\mathbf{C}$, the basis elements β_i are obtained using the Kronecker pairing with x^i where $\mathsf{MGL}^{*,*}(\mathsf{B}\mathbf{G}_{\mathfrak{m}}) = \mathsf{MGL}^{*,*}[[\mathsf{x}]]$. We will use $\underline{0} \in \{0\} \times \mathsf{B}\mathbf{GL}$ to view $\mathbf{Z} \times \mathsf{B}\mathbf{GL}$ as a pointed motivic space.

Proposition 4.1: *The following hold for the additive H-space structure on $\mathbf{Z} \times \mathsf{B}\mathbf{GL}$.*

1. $\mathsf{MGL}_{*,*}(\mathsf{B}\mathbf{G}_{\mathfrak{m}})$ *is a free* $\mathsf{MGL}_{*,*}$*-module with generators β_i for $i \geq 0$.*
2. $\mathsf{MGL}_{*,*}(\mathbf{Z} \times \mathsf{B}\mathbf{GL})$ *is a polynomial ring over* $\mathsf{MGL}_{*,*}$ *in β_0, β_0^{-1}, β_i for $i \geq 1$.*
3. *There is an isomorphism of* $\mathsf{MGL}_{*,*}$*-modules*

$$\widetilde{\mathsf{MGL}}_{*,*}(\mathbf{Z} \times \mathsf{B}\mathbf{GL}) = \mathsf{MGL}_{*,*}[\beta_0^{\pm 1}, \beta_1, \beta_2, \cdots]/(\mathsf{MGL}_{*,*} \cdot 1).$$

Let $\phi\colon \mathsf{MU}_* \to \mathsf{MGL}_{*,*}$ be the map from the Lazard ring classifying the formal group law over the coefficient ring $\mathsf{MGL}_{*,*}$ which sends MU_{2n} to $\mathsf{MGL}_{2n,n}$. Using the bijection between the generators, we shall view $\mathsf{MGL}_{*,*}(\mathsf{B}\mathbf{G}_{\mathfrak{m}})$ and $\mathsf{MGL}_{*,*}(\mathbf{Z} \times \mathsf{B}\mathbf{GL})$ as the base change of the MU_*-modules $\mathsf{MU}_*(\mathbf{CP}^\infty)$ and $\mathsf{MU}_*(\mathbf{Z} \times \mathsf{BU})$ with respect to ϕ.

In order to compare Bott inverted $\mathsf{BG_m}$ and KGL we recall the diagram

$$\Sigma^\infty \mathsf{BG}_{\mathfrak{m}+} \xrightarrow{\beta} \Sigma^{-2,-1}\Sigma^\infty \mathsf{BG}_{\mathfrak{m}+} \xrightarrow{\Sigma^{-2,-1}\beta} \cdots \tag{9}$$

defining the former, and note that the homotopy colimit of the diagram

$$\Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes} \xrightarrow{i_0} \Sigma^{-2,-1}\Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes} \xrightarrow{i_0} \cdots \tag{10}$$

maps by a weak equivalence to KGL, where i_0 is the negative reduced class of $\mathbf{P}^1 \to \{0\} \times \mathsf{B}\mathbf{GL} \to \mathbf{Z} \times \mathsf{B}\mathbf{GL}$. Moreover, the natural map $\Sigma^\infty \mathsf{BG}_{\mathfrak{m}+} \to \Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}$ induced by $\mathsf{BG}_\mathfrak{m} \to \{1\} \times \mathsf{B}\mathbf{GL}$ sends β to i_1, the negative reduced class of $\mathbf{P}^1 \to \{1\} \times \mathsf{B}\mathbf{GL} \to \mathbf{Z} \times \mathsf{B}\mathbf{GL}$. In order to get an induced map from (9) we form the following diagram, where the horizontal and vertical maps are induced by i_1 and i_0 respectively:

$$\begin{array}{ccccc}
\Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes} & \xrightarrow{i_1} & \Sigma^{-2,-1}\Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes} & \xrightarrow{\Sigma^{-2,-1}i_1} & \cdots \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle \Sigma^{-2,-1}i_0} & & \\
\Sigma^{-2,-1}\Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes} & \xrightarrow{\Sigma^{-2,-1}i_1} & \Sigma^{-4,-2}\Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes} & \xrightarrow{\Sigma^{-4,-2}i_1} & \cdots \\
\downarrow{\scriptstyle \Sigma^{-2,-1}i_0} & & \downarrow{\scriptstyle \Sigma^{-4,-2}i_0} & & \\
\vdots & & \vdots & &
\end{array} \tag{11}$$

The pointed map $(\mathbf{Z} \times \mathsf{B}\mathbf{GL}_+, +) \to (\mathbf{Z} \times \mathsf{B}\mathbf{GL}, \underline{0})$ induces a map of motivic ring spectra with respect to the H-space structure obtained from the tensor product. Using the composite map

$$\Sigma^\infty \mathsf{BG}_{\mathfrak{m}+} \longrightarrow \Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}_+ \longrightarrow \Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL},$$

it follows there is a commutative diagram:

$$\begin{array}{ccc}
\Sigma^\infty \mathsf{BG}_{\mathfrak{m}+} & \xrightarrow{\beta} & \Sigma^{-2,-1}\Sigma^\infty \mathsf{BG}_{\mathfrak{m}+} \\
\downarrow & & \downarrow \\
\Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes} & \xrightarrow{i_1} & \Sigma^{-2,-1}\Sigma^\infty \mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes}
\end{array}$$

This shows there is a natural map from (9) to (11), and applying MGL-homology yields (8). In the remainder of the proof we compare the above with MU-homology and the topological analogs of (9), (10) and (11).

First, consider the diagram with naturally induced horizontal maps:

$$\begin{array}{ccc} \mathsf{MU}_*(\mathbf{CP}^\infty) & \longrightarrow & \mathsf{MU}_{*+2}(\mathbf{CP}^\infty) \\ \downarrow & & \downarrow \\ \mathsf{MGL}_{*,*}(\mathrm{B}\mathbf{G}_\mathfrak{m}) & \longrightarrow & \mathsf{MGL}_{*+2,*+1}(\mathrm{B}\mathbf{G}_\mathfrak{m}) \end{array} \tag{12}$$

The vertical maps in (12) are defined using $\phi\colon \mathsf{MU}_* \to \mathsf{MGL}_{*,*}$ and the bijection between basis elements over the coefficient rings noted in Proposition 4.1. Compatibility of the formal group laws with respect to ϕ and duality between multiplication in homology and comultiplication in cohomology - giving rise to the formal group laws - implies (12) commutes.

Second, by using the same recipe as for (12), we obtain the diagram:

$$\begin{array}{ccc} \mathsf{MU}_*(\mathbf{CP}^\infty) & \longrightarrow & \widetilde{\mathsf{MU}}_*(\mathbf{Z}\times \mathsf{BU}^\otimes) \\ \downarrow & & \downarrow \\ \mathsf{MGL}_{*,*}(\mathrm{B}\mathbf{G}_\mathfrak{m}) & \longrightarrow & \widetilde{\mathsf{MGL}}_{*,*}(\mathbf{Z}\times \mathrm{B}\mathbf{GL}^\otimes) \end{array} \tag{13}$$

Here $\mathsf{MGL}_{*,*}(\mathrm{B}\mathbf{G}_\mathfrak{m}) \to \widetilde{\mathsf{MGL}}_{*,*}(\mathbf{Z}\times \mathrm{B}\mathbf{GL}^\otimes)$ sends β_i to $\beta_0\beta_i$ since it is defined using i_1 and similarly for the upper horizontal map. It follows that (13) commutes.

Third, by employing the H-space structures corresponding to the tensor product of vector bundles, Proposition 4.1 and $\phi\colon \mathsf{MU}_* \to \mathsf{MGL}_{*,*}$ as above, we obtain the diagram:

$$\begin{array}{ccc} \widetilde{\mathsf{MU}}_*(\mathbf{Z}\times \mathsf{BU}^\otimes) & \longrightarrow & \widetilde{\mathsf{MU}}_{*+2}(\mathbf{Z}\times \mathsf{BU}^\otimes) \\ \downarrow & & \downarrow \\ \widetilde{\mathsf{MGL}}_{*,*}(\mathbf{Z}\times \mathrm{B}\mathbf{GL}^\otimes) & \longrightarrow & \widetilde{\mathsf{MGL}}_{*+2,*+1}(\mathbf{Z}\times \mathrm{B}\mathbf{GL}^\otimes) \end{array} \tag{14}$$

In what follows we show the corresponding diagram in unreduced homology commutes by proving an explicit formula for multiplication with the image of i_1 in MU-homology, i.e. for $\beta_0\beta_1 \in \mathsf{MU}_*(\mathbf{Z}\times \mathsf{BU})$. A verbatim argument shows the same formula holds in $\mathsf{MGL}_{2*,*}(\mathbf{Z}\times \mathrm{B}\mathbf{GL})$. This suffices to conclude (14) commutes. We denote by $\cdot$ and $\star$ the two multiplications in MU-homology arising from the H-space structures $\mathbf{Z}\times \mathsf{BU}^\oplus$ and $\mathbf{Z}\times \mathsf{BU}^\otimes$ respectively.

Denote by $\triangle^{(n)}_{\mathbf{CP}^\infty}\colon \mathbf{CP}^\infty \to (\mathbf{CP}^\infty)^n$ and $\triangle^{(n)}_{\mathbf{Z}\times\mathsf{BU}}\colon \mathbf{Z}\times\mathsf{BU} \to (\mathbf{Z}\times\mathsf{BU})^n$ the n-fold diagonal maps. We use the same notations for the induced maps in MU-homology. Recall there is an isomorphism $\mathsf{MU}_*\big((\mathbf{Z}\times\mathsf{BU})^n\big) = \mathsf{MU}_*(\mathbf{Z}\times\mathsf{BU})\otimes_{\mathsf{MU}_*}\cdots\otimes_{\mathsf{MU}_*}\mathsf{MU}_*(\mathbf{Z}\times\mathsf{BU})$ with n copies of $\mathsf{MU}_*(\mathbf{Z}\times\mathsf{BU})$. Thus, if $a \in \mathsf{MU}_*(\mathbf{Z}\times\mathsf{BU})$, $\triangle^{(n)}_{\mathbf{Z}\times\mathsf{BU}}(a) = \sum_{i\in I} a_i^{(1)}\otimes\cdots\otimes a_i^{(n)}$ for some indexing set I and $a_i^{(k)} \in \mathsf{MU}_*(\mathbf{Z}\times\mathsf{BU})$. Our aim is to compute the products $\beta_0\beta_1\star(\beta_0^m\beta_{i_1}\cdots\beta_{i_n})$ for $m\in\mathbf{Z}$, $n\geq 0$, $i_1,\ldots,i_n\geq 1$. If $b_1,\ldots,b_n \in \mathsf{MU}_*(\mathbf{Z}\times\mathsf{BU})$, then

$$a\star(b_1\cdots b_n) = \sum_{i\in I}(a_i^{(1)}\star b_1)\cdots(a_i^{(n)}\star b_n). \tag{15}$$

Clearly the n-fold product of

$$\Sigma^\infty\mathbf{CP}^\infty_+ \longrightarrow \Sigma^\infty\mathbf{Z}\times\mathsf{BU}_+ \tag{16}$$

induced by $\mathbf{CP}^\infty \to \{1\}\times\mathsf{BU} \subset \mathbf{Z}\times\mathsf{BU}$ sends $\triangle^{(n)}_{\mathbf{CP}^\infty}(\beta)$ to $\triangle^{(n)}_{\mathbf{Z}\times\mathsf{BU}}(\beta_0\beta_1)$. Now since the diagonal in homology is the dual of the multiplication in cohomology we have

$$\triangle^{(n)}_{\mathbf{CP}^\infty}(\beta) = \sum_{i=1}^n 1\otimes\cdots\otimes\beta\otimes\cdots\otimes 1,$$

where β is in the ith tensor factor. Together with (15) this implies

$$\beta_0\beta_1\star(b_1\cdots b_n) = \sum_{i=1}^n b_1\cdots(\beta_0\beta_1\star b_i)\cdots b_n.$$

Moreover, $\beta_0\beta_1\star\_$ is an MU_*-module map and $\beta_0\beta_1\star 1 = 0$. It follows that $\beta_0\beta_1\star\_$ is an MU_*-derivation for the additive MU_*-algebra structure on $\mathsf{MU}_*(\mathbf{Z}\times\mathsf{BU})$.

Since β_0 is the unit for the $\star$-multiplication on $\mathsf{MU}_*(\mathbf{Z}\times\mathsf{BU})$, we get

$$\beta_0\beta_1\star\beta_0^m = m\beta_0^{m-1}(\beta_0\beta_1\star\beta_0) = m\beta_0^{m-1}\beta_0\beta_1. \tag{17}$$

Now (16) sends β_i to $\beta_0\beta_i$, so the products $\beta_0\beta_1\star(\beta_0\beta_i)$ can be computed in $\mathsf{MU}_*(\mathbf{CP}^\infty)$. Thus, writing $\beta_0^m\beta_{i_1}\cdots\beta_{i_n} = \beta_0^{m-n}(\beta_0\beta_{i_1})\cdots(\beta_0\beta_{i_n})$ and combining (15) with (17),

$$\beta_0\beta_1\star(\beta_0^m\beta_{i_1}\cdots\beta_{i_n}) = \tag{18}$$

$$(m-n)\beta_0^{m-1}\beta_{i_1}\cdots\beta_{i_n} + \beta_0^{m-1}\sum_{j=1}^n \beta_{i_1}\cdots\big(\beta_0\beta_1\star(\beta_0\beta_{i_j})\big)\cdots\beta_{i_n}.$$

The same argument shows there exists a motivic analog of (18) for the action of $\beta \star \_$ on $\mathsf{MGL}_{2*,*}(\mathbf{Z} \times \mathsf{B}\mathbf{GL})$. Since $\mathsf{MU}_*(\mathbf{Z} \times \mathsf{BU}) \to \mathsf{MGL}_{*,*}(\mathbf{Z} \times \mathsf{B}\mathbf{GL})$ preserves the products $\beta_0\beta_1\star(\beta_0\beta_{i_j})$ and the additive algebra structure in (18), we conclude that (14) commutes.

Finally we show that multiplication by i_0 induces a commutative diagram:

$$\begin{tikzcd} \widetilde{\mathsf{MU}}_*(\mathbf{Z} \times \mathsf{BU}^{\otimes}) \arrow[r] \arrow[d] & \widetilde{\mathsf{MU}}_{*+2}(\mathbf{Z} \times \mathsf{BU}^{\otimes}) \arrow[d] \\ \widetilde{\mathsf{MGL}}_{*,*}(\mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes}) \arrow[r] & \widetilde{\mathsf{MGL}}_{*+2,*+1}(\mathbf{Z} \times \mathsf{B}\mathbf{GL}^{\otimes}) \end{tikzcd} \tag{19}$$

In MU-homology, i_0 maps to β_1, $\beta_1 \star \beta_0^m = m\beta_1$ since $1 \star \beta_0^m = 1$, and

$$\beta_1 \star (\beta_0^m \beta_{i_1} \cdots \beta_{i_n}) = \begin{cases} \beta_1 \star \beta_{i_1} & n = 1, \\ 0 & n > 1. \end{cases}$$

To find a formula for products of the form $\beta_1 \star \beta_n$ we use $\triangle^{(2)}_{\mathbf{Z}\times\mathsf{BU}}(\beta_n)$ to conclude

$$\begin{aligned} \beta_0\beta_1 \star \beta_n &= \sum_{i+j=n\, i,j>0} (\beta_0 \star \beta_i)(\beta_1 \star \beta_j) + (\beta_0 \star 1)(\beta_1 \star \beta_n) + (\beta_0 \star \beta_n)(\beta_1 \star 1) \\ &= \sum_{i+j=n,\, i,j>0} \beta_i(\beta_1 \star \beta_j) + (\beta_1 \star \beta_n), \end{aligned}$$

which allows to deduce the recursive formula

$$\beta_1 \star \beta_n = \beta_0\beta_1 \star \beta_n - \sum_{i+j=n,\, i,j>0} \beta_i(\beta_1 \star \beta_j).$$

By specialization of (18) to the product $\beta_0\beta_1 \star \beta_n$ it follows that (19) commutes.

By combining the commutative diagrams (12), (13), (14) and (19) we may identify the induced map in MGL-homology (8) with

$$\left(\operatorname*{colim}_{n\in\mathbf{N}} \big(\mathsf{MU}_{*+2n}(\mathbf{CP}^{\infty})\big) \longrightarrow \operatorname*{colim}_{n,m\in\mathbf{N}\times\mathbf{N}} \big(\widetilde{\mathsf{MU}}_{*+2(n+m)}(\mathbf{Z} \times \mathsf{BU}^{\otimes})\big) \right) \otimes_{\mathsf{MU}_*} \mathsf{MGL}_{*,*}. \tag{20}$$

The map in (20) is an isomorphism by Snaith's theorem, cp. Remark 3.8. □

Remark 4.2: *The proof of Theorem 1.1 employs the algebraic cobordism spectrum. However, the argument goes through for every oriented motivic spectrum equipped with an orientation preserving map to the Bott inverted infinite projective space.*

5 Maps to KGL

The motivic spectrum KGL over the integers acquires a unique commutative monoidal structure which is compatible with the ring structure on K_0 [16]. Thus KGL is equipped with a distinguished associative, commutative and unital product for every base scheme.

Proposition 5.1: *Suppose the group $\mathsf{KGL}^{-1,0}$ is finite or divisible. Then there is a bijection between maps from the Bott inverted infinite projective space to KGL in the sense of spectra and in the sense of cohomology theories. The same holds for smash products of $\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathsf{m}+}[\beta^{-1}]$. In particular, there is a unique map of oriented motivic spectra*

$$\Sigma^\infty \mathsf{B}\mathbf{G}_{\mathsf{m}+}[\beta^{-1}] \longrightarrow \mathsf{KGL}$$

which lifts the map between the corresponding oriented cohomology theories.

Corollary 5.2: *Suppose S is the integers, a finite field or an algebraically closed field. Then there is a unique map of oriented motivic spectra from the Bott inverted infinite projective space to KGL. Thus for every base scheme there is a distinguished such map.*

Proof. (of Proposition 5.1) The multiplication on $\mathsf{B}\mathbf{G}_\mathsf{m}$ restricts to the Segre embedding $\mathbf{P}^{n-1} \times \mathbf{P}^{m-1} \to \mathbf{P}^{nm-1}$, in particular to $\mathbf{P}^{2^n-1} \times \mathbf{P}^1 \to \mathbf{P}^{2^{n+1}-1}$. Thus

$$\operatorname{hocolim} \left(\Sigma^\infty \mathbf{P}^1_+ \longrightarrow \Sigma^{-2,-1} \Sigma^\infty \mathbf{P}^3_+ \longrightarrow \Sigma^{-4,-2} \Sigma^\infty \mathbf{P}^7_+ \longrightarrow \cdots \right) \tag{21}$$

lifts the cohomology theory represented by the Bott inverted infinite projective space. Since $\mathsf{KGL}^{*,*}$ is $(2,1)$-periodic, the group $\mathsf{KGL}^{-1+2r,r}(\mathbf{P}^n)$ is a finite product of copies of $\mathsf{KGL}^{-1,0}$. We need to analyze the $\lim^1$-exact sequence of the system

$$\cdots \longrightarrow \mathsf{KGL}^{3,2}(\mathbf{P}^7) \longrightarrow \mathsf{KGL}^{1,1}(\mathbf{P}^3) \longrightarrow \mathsf{KGL}^{-1,0}(\mathbf{P}^1), \tag{22}$$

obtained by mapping the tower (21) to KGL. If $\mathsf{KGL}^{-1,0}$ is finite, then the $\lim^1$-term of (22) vanishes. Using the isomorphism $\Sigma^\infty_+ \mathbf{P}^1 = \mathbf{1} \vee \mathbf{1}^{2,1}$ the negative of multiplication by β is given by including $\Sigma^{2,1}\Sigma^\infty \mathsf{B}\mathbf{G}_\mathsf{m}$ into the source of the composite map

$$\Sigma^\infty \mathsf{B}\mathbf{G}_\mathsf{m} \vee \Sigma^{2,1}\Sigma^\infty \mathsf{B}\mathbf{G}_\mathsf{m} = \Sigma^\infty \mathsf{B}\mathbf{G}_\mathsf{m} \wedge \Sigma^\infty_+ \mathbf{P}^1 \longrightarrow \Sigma^\infty \mathsf{B}\mathbf{G}_\mathsf{m} \wedge \Sigma^\infty \mathsf{B}\mathbf{G}_\mathsf{m} \longrightarrow \Sigma^\infty \mathsf{B}\mathbf{G}_\mathsf{m}.$$

Now the multiplication map in KGL-cohomology can be read off from the formal group law, e.g. its restriction to $\mathsf{B}\mathbf{G}_\mathsf{m} \times \mathbf{P}^1$ is its image in $\mathsf{KGL}^{**}[[\mathsf{x},\mathsf{y}]]/(\mathsf{y}^2)$, and the restriction

to the summand $\Sigma^{2,1}\Sigma^\infty \mathsf{B}\mathbf{G}_\mathfrak{m}$ is given by the coefficient of y (as a power series in x). Here x and y denote the pullbacks of the orientation classes to $\mathsf{B}\mathbf{G}_\mathfrak{m} \times \mathsf{B}\mathbf{G}_\mathfrak{m}$. Thus it sends x^n in $\mathsf{KGL}^{**}(\mathbf{P}^\infty) = \mathsf{KGL}^{**}[[\mathsf{x}]]$ to the coefficient of y in $-\big(\mathsf{x}+\mathsf{y}(1-a\mathsf{x})\big)^n$, that is $-n\mathsf{x}^{n-1}(1-a\mathsf{x}) = -n\mathsf{x}^{n-1} + na\mathsf{x}^n$. The map $\mathsf{KGL}^{*,*}(\mathbf{P}^{2^{n+1}-1}) \to \mathsf{KGL}^{*-2,*-1}(\mathbf{P}^{2^n-1})$ has the same description on generators, and tensoring with $\mathsf{KGL}^{-1,0}$ identifies (22) with

$$\cdots \longrightarrow \mathbf{Z}[\mathsf{x}]/(\mathsf{x}^{2^n}) \longrightarrow \cdots \longrightarrow \mathbf{Z}[\mathsf{x}]/(\mathsf{x}^4) \longrightarrow \mathbf{Z}[\mathsf{x}]/(\mathsf{x}^2), \tag{23}$$

where the transition maps are given by $\mathsf{x}^m \longmapsto -m\mathsf{x}^{m-1} + m\mathsf{x}^m$. Now since $\mathsf{KGL}^{-1,0}$ is a divisible group, the Mittag-Leffler condition holds and hence $\lim^1 = 0$.

For a k-fold smash product the analog of (22) arise from (23) by a termwise k-fold tensor product of (23) tensored with $\mathsf{KGL}^{-1,0}$. $\square$

6 Appendix

In this appendix we collect some results on cohomology theories in motivic homotopy theory. For a more thorough discussion we refer to [14].

A cohomology theory $\mathsf{E}^{*,*}(-)$ is defined on finite objects of $\mathbf{SH}$, where the values in bidegree (p,q) are given by appropriate suspensions so that every motivic spectrum gives an example. An oriented cohomology theory is a ring cohomology theory together with compatible classes in $\mathsf{E}^{2,1}(\mathbf{P}^n)$ for $n \geq 1$ with the canonical one for $n = 1$. If F is a motivic spectrum, let $\mathsf{E}^{p,q}(\mathsf{F})$ be the group of natural transformations $\mathbf{SH}(\mathsf{F},-) \to \mathsf{E}^{p,q}(-)$ of contravariant functors from finite objects of $\mathbf{SH}$ to abelian groups. If X is a motivic space, $\mathsf{E}^{p,q}(X) \equiv \mathsf{E}^{p,q}(\Sigma^\infty X_+)$ and similarly for reduced cohomology. We note the natural map from $\mathbf{SH}\big(\mathsf{F}, \Sigma^\infty(S_s)^{\wedge(p-2q)} \wedge \Sigma^\infty(\mathbf{P}^1)^{\wedge q} \wedge \mathsf{E}\big)$ to $\mathsf{E}^{p,q}(\mathsf{F})$ is not an isomorphism in the event of nontrivial phantom maps when F is not finite. For legibility we allow a uniform notation trusting that the precise meaning will be clear from the context. Cohomology theories associated to spectra are ind-representable functors. For cellular spaces such as infinite Grassmannians and Thom spaces of universal vector bundles these systems are determined by their respective finite subspaces, so we get:

Proposition 6.1: *If* $\mathsf{E}^{*,*}(-)$ *is an oriented cohomology theory, then*

- $\mathsf{E}^{*,*}((\mathbf{P}^\infty)^n) = \mathsf{E}^{*,*}[[x_1,\ldots,x_n]]$,
- $\mathsf{E}^{*,*}(\mathsf{MGL}^{\wedge n}) = \mathsf{E}^{*,*}[[c_i^{(k)} \,|\, 0 < i, 1 \leq k \leq n]]$.

It follows that orientations on $\mathsf{E}^{*,*}(-)$ are in bijection with elements in $\mathsf{E}^{2,1}(\mathbf{P}^\infty)$ which restrict to the canonical element in $\mathsf{E}^{2,1}(\mathbf{P}^1)$

Proposition 6.2: $\mathsf{MGL}^{*,*}(-)$ *is the universal oriented cohomology theory.*

Proof. This follows as in [19, Corollaries 3.6, 3.10, Lemma 4.1, Theorem 4.3]. In outline, multiplicative maps $\Sigma^\infty \mathsf{B}\mathbf{GL}^{*,*}(-) \to \mathsf{E}^{*,*}(-)$ correspond bijectively to multiplicative power series in Chern roots and the Thom isomorphism implies the corresponding result for $\mathsf{MGL}^{*,*}(-)$. These power series correspond to normalized orientations by restricting to $\mathsf{B}\mathbf{GL}_1$ and the Thom space of the universal line bundle MGL_1 respectively. $\square$

It follows now that the coefficient ring of any oriented cohomology theory is graded commutative, cf. the proof of [8, Proposition 2.16]. Proposition 6.1 implies there is an induced formal group law on $\mathsf{E}^{*,*}$. Recall that L denotes the Lazard ring.

Lemma 6.3: *Suppose* $\mathsf{L} \to \mathsf{A}$ *is a map of evenly graded commutative rings and* $\mathsf{E}^{*,*}(-)$ *an oriented cohomology theory. If* $\mathsf{L} \to \mathsf{E}^{2*,*}$ *factors through* A*, then there is a unique multiplicative map* $\mathsf{MGL}^{*,*}(-) \otimes_{\mathsf{L}} \mathsf{A} \to \mathsf{E}^{*,*}(-)$ *such that precomposing with the natural maps from* $\mathsf{MGL}^{*,*}(-)$ *and* A *yields the unique map of oriented cohomology theories* $\mathsf{MGL}^{*,*}(-) \to \mathsf{E}^{*,*}(-)$ *and the factorization* $\mathsf{A} \to \mathsf{E}^{*,*}(-)$*.*

Proof. If $\mathsf{MGL}^{2*,*} \to \mathsf{B}$ is a map of evenly graded commutative rings, then multiplicative maps from $\mathsf{MGL}^{*,*}(-) \otimes_{\mathsf{MGL}^*} \mathsf{B}$ to $\mathsf{E}^{*,*}(-)$ are in bijection with maps $\mathsf{MGL}^{*,*}(-) \to \mathsf{E}^{*,*}(-)$ together with a factorization of the induced map on the point $\mathsf{MGL}^{2*,*} \to \mathsf{B} \to \mathsf{E}^{2*,*}$. $\square$

From the above and motivic Landweber exactness [14] we get:

Corollary 6.4: $\mathsf{MGL}^{*,*}(-) \otimes_{\mathsf{L}} \mathbf{Z}[\beta, \beta^{-1}]$ *is the universal multiplicative oriented cohomology theory on strongly dualizable objects.*

Lemma 6.5: *Let* R *be a motivic ring spectrum and* $\mathsf{E}^{*,*}(-)$ *a ring cohomology theory. Then* $\iota\colon \mathsf{R}^{*,*}(-) \to \mathsf{E}^{*,*}(-)$ *is multiplicative if and only if it is unital and the exterior product of* ι *with itself coincide with the class in* $\mathsf{E}^{0,0}(\mathsf{R} \wedge \mathsf{R})$ *given by the composite of* ι *and the multiplication map on* R*.*

Proof. A ring structure on a cohomology theory extends canonically to ind-representable functors on finite motivic spectra. Thus the claimed compatibility is the universal one. $\square$

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