Proof Mining in Ergodic Theory and Topological Dynamics

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Introduction

“Proof mining” is the subfield of mathematical logic concerned with extracting additional information from proofs in mathematics and computer science.

This activity has its roots in the 1950s in G.Kreisel’s unwinding program. Kreisel suggested to apply proof-theoretic techniques invented to settle questions of consistency to the analysis of “interesting” proofs.

Recently, this has been coined proof mining.
Introduction

What kind of additional information may we extract?

- Quantitative: Realizers and bounds.
  
  E.g. from a proof that a certain iteration sequence in a compact metric space converges, we extract a rate of convergence.

- Qualitative: Uniformities, weakenings of premises.
  
  E.g. we establish that the convergence is independent of the starting point or that it holds for bounded metric spaces not only compact metric spaces.
Introduction

The main tools of proof mining are so-called proof interpretations, which transform a proof into an enriched proof from which the desired additional information can be read off.

Example: Gödel’s functional interpretation (and variants).

Idea: Give a computational interpretation of constants, axioms and derivation rules of a given formal system. Inductively transform a proof, starting with the axioms. Read off realizers for the existential quantifiers at the conclusion of the proof.
Introduction

Aims of proof mining:

- Study techniques to carry out proof mining.
- Prove general metatheorems that classify theorems and proofs from which information can be extracted.
- Carry out cases studies, analyse actual proofs.
**Example: Mean Ergodic Theorem**

Let \((X, \langle \cdot, \cdot \rangle)\) be a Hilbert space. We call \(T : X \to X\) nonexpansive, if \(\|Tf\| \leq \|f\|\) for all \(f \in X\).

**Mean Ergodic Theorem:** Let \((X, \langle \cdot, \cdot \rangle)\). Let \(T : X \to X\) be linear and nonexpansive and define \(A_n f := \frac{1}{n+1} \sum_{i=0}^{n+1} T^i f\). Then

\[
\forall \varepsilon > 0 \exists m \forall m \geq n(\|A_m f - A_n f\| < \varepsilon).
\]

**Computational challenge:** Find a rate of convergence.
Example: Mean Ergodic Theorem

Limits: No full rate of convergence possible.
There is a Hilbert space and a mapping $T$ such that “full rate of convergence exists” $\iff$ “Halting problem decidable”.

Classically equivalent no-counterexample version:

Mean Ergodic Theorem, version 2: Let $(X, \langle \cdot, \cdot \rangle)$. let $T : X \to X$ be linear and nonexpansive and define $A_n f := \frac{1}{n+1} \sum_{i=0}^{n+1} T^i f$.

Then

$$\forall \varepsilon > 0 \forall M : \mathbb{N} \to \mathbb{N} \exists m (\| A_{M(n)} f - A_n f \| < \varepsilon).$$
Example: Mean Ergodic Theorem

The formal system $\mathcal{A}^\omega[\mathcal{X}, \langle\cdot, \cdot\rangle]$: 

- $\mathcal{A}^\omega$ is Peano arithmetic in all finite types over $\mathbb{N}$ + dependent choice. 
- For $[\mathcal{X}, \langle\cdot, \cdot\rangle]$ add new type $\mathcal{X}$ representing the Hilbert space. 
- ... extend $\mathcal{A}^\omega$ to finite types over $\mathbb{N}$ and $\mathcal{X}$. 
- ... add new constants for inner product and vector space operations on $\mathcal{X}$. 
- ... add defining algebraic axioms for Hilbert spaces. 

The proof of the Mean Ergodic Theorem is formalizable in the theory $\mathcal{A}^\omega[\mathcal{X}, \langle\cdot, \cdot\rangle]$. 
Example: Mean Ergodic Theorem

Using Gödel functional interpretation + Howard-Bezem majorization one proves metatheorems for the extraction of bounds from proofs of $\forall \exists$-statements in $\mathcal{A}^\omega$.

Extending majorization to new type $X$ one extends these metatheorems to $\mathcal{A}^\omega[X, \langle \cdot, \cdot \rangle]$.

For the no-counterexample version of the Mean Ergodic Theorem these metatheorems predict bounds on $\exists n$

- that depend on $M$, on $\varepsilon > 0$ and on a bound $b \geq \| f \|$.
- that do not depend on the particular space $(X, \langle \cdot, \cdot \rangle)$ or the mapping $T$. 
Example: Mean Ergodic Theorem

From a standard proof of the Mean Ergodic Theorem the following bounds were extracted (Avigad, Towsner, G.):

Define:

\[ i_0 = 0, \quad n_k = \left\lfloor \frac{b}{\varepsilon^2} \sum_{l=0}^{i_k} M\left(\frac{2lb}{\varepsilon}\right) \right\rfloor \]

\[ i_k + 1 = i_k + \left\lfloor \frac{2^{15}M(n_k)^4b^4}{\varepsilon^4} \right\rfloor \]

Let \( d = \frac{512b^2}{\varepsilon^2} \), then for some \( n \leq N(b, \varepsilon, M) = \frac{2n_d b}{\varepsilon} \), we have that \( \|A_{M(n)}f - A_n f\| < \varepsilon \).