Real monoid surfaces

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Outline

Introduction

Monoid surfaces

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The first part of this work is joint with M. Løberg and P. H. Johansen, and complements the work of Rohn (1884) and Takahashi–Watanabe–Higuchi (1982); the last part is due to Johansen. The figures are made using SURF.
Monoid surfaces

Consider a surface $X = Z(F) \subset \mathbb{P}^3$ of degree $d$, such that the point $O : = (1 : 0 : 0 : 0)$ is a point of multiplicity $d − 1$. It is “almost” a cone!
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Then

$$F(x_0, x_1, x_2, x_3) = x_0 f_{d-1}(x_1, x_2, x_3) + f_d(x_1, x_2, x_3),$$

where $f_{d-1}$ and $f_d$ are homogeneous polynomials of degrees $d - 1$ and $d$. 
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where $f_{d-1}$ and $f_d$ are homogeneous polynomials of degrees $d - 1$ and $d$.

The (projective) tangent cone to $X$ at $O$ is the plane curve $Z(f_{d-1}) \subset \mathbb{P}^2$.

The curve $Z(f_d) \subset \mathbb{P}^2$ is the intersection of $X$ with the plane at infinity $Z(x_0)$. 
Example. The surface $X \subset \mathbb{P}^3$ defined by

$$F = x_0(x_1x_2^2 + x_3^3) + x_4^4$$

is a quartic monoid. Its singular points are $O$ and $(0 : 0 : 1 : 0)$. 
The natural parameterization

The natural parameterization of the monoid $X$ is the map

$$\theta_F : \mathbb{P}^2 \to X \subset \mathbb{P}^3$$

given by

$$\theta_F(a) = (-fd(a) : fd_{d-1}(a)a_1 : fd_{d-1}(a)a_2 : fd_{d-1}(a)a_3),$$

for $a = (a_1 : a_2 : a_3) \in \mathbb{P}^2 \setminus Z(fd_{d-1}(a), fd(a))$. 
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For every $a = (a_1 : a_2 : a_3) \in \mathbb{P}^2$, the line

$$L_a := \{(s : ta_1 : ta_2 : ta_3) | (s : t) \in \mathbb{P}^1\}$$

intersects $X = Z(F)$ with multiplicity at least $d - 1$ in $O$. 
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If $f_{d-1}(a) \neq 0$ or $f_d(a) \neq 0$, then the line $L_a$ also intersects $X$ in the point

$$\theta_F(a) = (-f_d(a) : f_{d-1}(a)a_1 : f_{d-1}(a)a_2 : f_{d-1}(a)a_3).$$
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For each base point $b \in Z(f_{d-1}, f_d)$, the line $L_b$ is contained in the monoid surface. Conversely, every line of type $L_b$ contained in the monoid surface corresponds to a base point $b$. 
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If $P \in X$ is a singular point on the monoid $X$, then the line $L$ through $P$ and $O$ has intersection multiplicity at least $d - 1 + 2 = d + 1$ with $X$. Hence, by Bezout’s theorem, $L$ is contained in $X$. 
Lemma

(i) All singular points of $X$ are on lines $L_b$, where $b \in \mathbb{Z}(f_{d-1}, f_d)$ is a base point.

(ii) Both $Z(f_{d-1})$ and $Z(f_d)$ are singular in a point $b \in \mathbb{P}^2$ if and only if all points on $L_b$ are singular on $X$.

(iii) If not all points on $L_b$ are singular, then at most one point other than $O$ on $L_b$ is singular.
If $Z(f_{d-1})$ and $Z(f_d)$ have no common singular points, then each line $L_b$ contains at most one singular point in addition to $O$. 
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Hence in this case the surface has only finitely many singular points.
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The singular point on $L_b$ is of type $A_{m-1}$, where $m$ is the intersection multiplicity of $Z(f_{d-1})$ and $Z(f_d)$ at $b$.

The maximal number of singular points that a monoid surface of degree $d$ can have is $\frac{d(d-1)}{2} + 1$. 
Real surfaces and real singularities

In the case that the singular point is real, it is of type $A_{m-1}^-$. 
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In the case that the singular point is *real*, it is of type $A_{m-1}^-$. The two real versions of the $A_2$ singularity:
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\[ A_2^+: x^2 + y^2 - z^3 = 0 \]
\[ A_2^-: x^2 - y^2 - z^3 = 0 \]
Real monoid with max number of singularities

To construct a monoid with the maximal number of real singularities, it is sufficient to construct two affine real curves in the $xy$-plane defined by equations $f_{d-1}$ and $f_d$ of degrees $d-1$ and $d$ such that the curves intersect in $d(d-1)/2$ points with multiplicity 2. Assume $d-1$ is odd. Set

$$f_{d-1} = \varepsilon - \prod_{i=1}^{d-1} \left( x \sin \left( \frac{2i\pi}{d-1} \right) + y \cos \left( \frac{2i\pi}{d-1} \right) + 1 \right).$$
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For $\varepsilon > 0$ sufficiently small there exist at least $\frac{d}{2}$ radii $r > 0$, one for each (positive real) root of the univariate polynomial $f_{d-1}|x=0$, such that the circle $x^2 + y^2 - r^2$ intersects $f_{d-1}$ in $d - 1$ points with multiplicity 2. Let $f_d$ be a product of such circles. The homogenizations of $f_{d-1}$ and $f_d$ define a monoid surface with $1 + \frac{1}{2}d(d - 1)$ singularities.
The curves $f_{d-1}$ for $d - 1 = 3, 5$ and corresponding circles.
**Real monoid with max $A_m$-singularity**

The maximal Milnor number of a singularity other than $O$ is $d(d - 1) - 1$. The following example shows that this bound can be achieved on a *real* monoid surface:
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**Example.** The surface $X \subset \mathbb{P}^3$ defined by $F = x_0(x_1x_2^{d-2} + x_3^{d-1}) + x_1^d$ has precisely two singular points. The point $O$ is a singularity of multiplicity 3 with Milnor number $\mu = (d^2 - 3d + 1)(d - 2)$, while the point $(0 : 0 : 1 : 0)$ is an $A_d(d(d - 1) - 1)$ singularity.
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For $d = 4$:  

![Monomial Surface](image)
Quartic monoid surfaces

Theorem
On a quartic monoid surface, all singularities other than the monoid point $O$ can occur as given in the following table. Moreover, all possibilities are realizable on real quartic monoids with a real monoid point, and with the additional singularities being real and of type $A^-$. 
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(In the table, the first column gives the type of the tangent cone: nonsingular cubic, nodal cubic, cuspidal cubic, . . . , triple line.)
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The third column refers to the possible intersections of the curves $Z(f_3)$ and $Z(f_4)$; their total number of intersections is 12.)
<table>
<thead>
<tr>
<th>Case</th>
<th>Triple point</th>
<th>Invariants and constraints</th>
<th>Other singularities</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$P_8 = T_{3,3,3}$</td>
<td>$m = 2, \ldots, 12$</td>
<td>$A_{m_i-1}, \sum m_i = 12$</td>
</tr>
<tr>
<td>1</td>
<td>$T_{3,3,4}$</td>
<td>$m = 2, 3$</td>
<td>$A_{m_i-1}, \sum m_i = 12$</td>
</tr>
<tr>
<td></td>
<td>$T_{3,3,3+m}$</td>
<td>$r_0 = \max(j_0, k_0), r_1 = \max(j_1, k_1)$, $j_0 &gt; 0 \leftrightarrow k_0 &gt; 0$, $\min(j_0, k_0) \leq 1$, $j_1 &gt; 0 \leftrightarrow k_1 &gt; 0$, $\min(j_1, k_1) \leq 1$</td>
<td>$A_{m_i-1}, \sum m_i = 12$</td>
</tr>
<tr>
<td></td>
<td>$Q_{10}$</td>
<td>$j_0 \leq 8, k_0 \leq 4$, $\min(j_0, k_0) \leq 2$, $j_0 &gt; 0 \leftrightarrow k_0 &gt; 0$, $j_1 &gt; 0 \leftrightarrow k_0 &gt; 1$</td>
<td>$A_{m_i-1}, \sum m_i = 4 - k_0$</td>
</tr>
<tr>
<td>4</td>
<td>$S$ series</td>
<td>$m_1 + l_1 \leq 4$, $k_2 + m_2 \leq 4$, $k_3 + l_3 \leq 4$, $k_2 &gt; 0 \leftrightarrow k_3 &gt; 0$, $l_1 &gt; 0 \leftrightarrow l_3 &gt; 0$, $m_1 &gt; 0 \leftrightarrow m_2 &gt; 0$, $\min(k_2, k_3) \leq 1$, $\min(l_1, l_3) \leq 1$, $\min(m_1, m_2) \leq 1$, $j_k = \max(k_2, k_3)$, $j_l = \max(l_1, l_3)$, $j_m = \max(m_1, m_2)$</td>
<td>$A_{m_i-1}, \sum m_i = 4 - m_1 - l_1$, $A_{m_i-1}, \sum m_i = 4 - k_2 - m_2$, $A_{m_i-1}, \sum m_i = 4 - k_3 - l_3$</td>
</tr>
<tr>
<td>5</td>
<td>$T_{4+j_k,4+j_l,4+jm}$</td>
<td>$j_1 &gt; 0 \leftrightarrow j_2 &gt; 0 \leftrightarrow j_3 &gt; 0$, at most one of $j_1$, $j_2$, $j_3 &gt; 1$, $j_1, j_2, j_3 \leq 4$</td>
<td>$A_{m_i-1}, \sum m_i = 4 - j_1$, $A_{m_i-1}, \sum m_i = 4 - j_2$, $A_{m_i-1}, \sum m_i = 4 - j_3$</td>
</tr>
<tr>
<td>6</td>
<td>$U$ series</td>
<td>$j_0 \geq 0 \leftrightarrow k_0 \geq 0$, $\min(j_0, k_0) \leq 1$, $j_0 \leq 4$, $k_0 \leq 4$</td>
<td>$A_{m_i-1}, \sum m_i = 4 - j_0$, None</td>
</tr>
<tr>
<td>7</td>
<td>$V$ series</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$V'$ series</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


The monoids $Z(x^3 + y^3 + 5xyz - z^3(x + y))$ and $Z(x^3 + y^3 + 5xyz - z^3(x - y))$ both have a $T_{3,3,5}$ singularity.
The monoids \( Z(z^3 + xy^3 + x^3y) \) and \( Z(z^3 + xy^3 - x^3y) \) are of the same type over \( \mathbb{C} \), but are different over \( \mathbb{R} \).
**Stratification of the space of quartic monoids**

The space of quartic surfaces with a triple point $O$ has dimension 24. The space of quartic monoid surfaces (with only isolated singularities) is an open subset of this space and is contained in

$$(\mathbb{A}^{10} \setminus \{0\}) \times (\mathbb{A}^{15} \setminus \{0\})/\sim \subset \mathbb{P}^{24}.$$
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The *stratum* of a given $X = Z(F)$ is the set of quartic monoid surfaces that have the same type of tangent cone $Z(f_3)$, and the same kind of intersections between the tangent cone and the curve at infinity $Z(f_4)$. 


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Each stratum \( S \) has a (not necessarily rational) parameterization \( B_S \times G \to S \), where \( B_S \) is an open in (a hypersurface of) an affine space and \( G \) is the group of projective transformations fixing \( O \).
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For each tangent cone type, compute (use Singular) a certain matrix group, which is used to compute the components of dimension of $B_S$, and the dimension of $S$. 
<table>
<thead>
<tr>
<th>Type</th>
<th>Invariants</th>
<th>$\text{dim } S$</th>
<th>Comp</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$m_1 + \ldots + m_r = 12$</td>
<td>$12 + r$</td>
<td>?</td>
</tr>
<tr>
<td>1</td>
<td>$m = 0, m_1 + \ldots + m_r = 12, 2^{e_1}3^{e_2} := \gcd(m_1, \ldots, m_r)$</td>
<td>$11 + r$</td>
<td>$1 + e_1$</td>
</tr>
<tr>
<td></td>
<td>$m = 2, \ldots, 12, m_1 + \ldots + m_r = 12 - m$</td>
<td>$12 + r$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$m = 0, m_1 + \ldots m_r = 12, m = 2, 3, m_1 + \ldots + m_r = 12 - m$</td>
<td>$10 + r$</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>7</td>
<td>$j_0 = k_0 = 0, m_1 + \ldots + m_r = m'<em>1 + \ldots + m'</em>{r'} = 4$</td>
<td>$r + r' + 11$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$j_0, k_0 &gt; 0, m_1 + \ldots + m_r = 4 - j_0, m'<em>1 + \ldots + m'</em>{r'}4 - j_0$</td>
<td>$r + r' + r'' + 11$</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>$m_1 + \ldots + m_r = 4$</td>
<td>$r + 13$</td>
<td>1</td>
</tr>
</tbody>
</table>
References


