ORBIT CLASSIFICATION FOR THE REPRESENTATIONS
ASSOCIATED TO GRADED LIE ALGEBRAS

JERZY WEYMAN

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INTRODUCTION

The irreducible representations of the reductive groups with finitely many orbits were classified by Kac in [K80]. They were divided into classes II, III and IV. All of these classes are related to gradings of the root systems, and to the corresponding \( \theta \) groups.

The representations of type II are parametrized by a pair \( (X_n, \alpha_k) \) where \( X_n \) is a Dynkin diagram with a distinguished node \( x \in X_n \). This data defines a grading

\[
\mathfrak{g} = \bigoplus_{i=-s}^s \mathfrak{g}_i
\]

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of a simple algebra $\mathfrak{g}$ of type $X_n$ such that the Cartan subalgebra $\mathfrak{h}$ is contained in $\mathfrak{g}_0$ and the root space $\mathfrak{g}_\beta$ is contained in $\mathfrak{g}_i$ where $i$ is the coefficient of the simple root $\alpha$ corresponding to the node $x$ in the expression for $\beta$ as a linear combination of simple roots. The representation corresponding to $(X_n, x)$ is the $\mathfrak{g}_1$ with the action of the group $G_0 \times \mathbb{C}^*$ where $G_0$ is the adjoint group corresponding to $\mathfrak{g}_0$ and $\mathbb{C}^*$ is the copy of $\mathbb{C}^*$ that occurs in maximal torus of $G$ (the adjoint group corresponding to $\mathfrak{g}$) but not in maximal torus of $G_0$.

The orbit closures for the representations of type II were described in two ways by Vinberg in [V75], [V87]. The first description states that the orbits are the irreducible components of the intersections of the nilpotent orbits in $\mathfrak{g}$ with the graded piece $\mathfrak{g}_1$. In the second paper Vinberg gave a more precise description in terms of some graded subalgebras of the graded algebra $\mathfrak{g} = \oplus \mathfrak{g}_i$. In these notes we give the introduction to these methods. We use mainly the examples of triple tensor products and third exterior powers.

The notes are organized as follows. In section 1 we discuss root systems and simple Lie algebras. Section 2 introduces representations related to gradings on simple Lie algebras related to a choice of a simple root. Section 3 we state Kac’s theorem and interpret special cases of triple tensor products and third exterior powers. In section 4 we expose the Vinberg method from [V87] that classifies orbits in representations with finitely many orbits. In section 5 we work out some examples. In the following sections we give information about singularities and defining ideals of the orbit closures. We briefly describe the geometric method of calculating syzygies and give examples of its use. We give some comments on how one should be able to calculate resolutions of all the coordinate rings of orbit closures.

§1. SIMPLE LIE ALGEBRAS AND ROOT SYSTEMS

We recall the CARTAN classification of simple Lie algebra and corresponding root systems.

For a simple Lie algebra $\mathfrak{g}$ we have the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Phi$ is the associated root system.

The simple Lie algebra $\mathfrak{g}$ has also a symmetric bilinear non-degenerate form

$$(,) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$$

given by the formula $(X,Y) = Tr(ad(X) ad(Y))$.

**Definition 1.1.** Let $V$ be an Euclidean space over $\mathbb{R}$ with a scalar product $(,)$. A root system $\Phi$ is by definition a finite collection of vectors in $V$ such that

a) If $\alpha \in \Phi$ then the only multiples of $\alpha$ in $\Phi$ are $\pm \alpha$, 
b) The values of 
\[ \langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \]
are in \( \mathbb{Z} \) for \( \alpha, \beta \) in \( \Phi \).

c) If \( \alpha \in \Phi \) and \( \sigma_\alpha \) is a reflection in the hyperplane orthogonal to \( \alpha \) then \( \sigma_\alpha(\Phi) = \Phi \).

d) \( \Phi \) spans \( V \).

If \( \Phi_1 \) and \( \Phi_2 \) are two root systems in the Euclidean spaces \( V_1, V_2 \), then \( \Phi_1 \cup \Phi_2 \)
is a root system in the orthogonal direct sum \( V_1 \oplus V_2 \). A root system is \textit{irreducible} if it cannot be decomposed into a direct sum of two root systems.

In every root systems one can choose a basis \( \Delta = \{\alpha_1, \ldots, \alpha_l\} \) of simple roots which have the following property. The root system \( \Phi \) decomposes

\[ \Phi = \Phi_+ \cup \Phi_- \]

where positive roots can be written as nonnegative linear combinations of vectors in the basis, and negative roots, as non-positive linear combinations.

\textbf{Example 1.2.} The root system if type \( A_n \).

\[ V = \{ (a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1} \mid a_1 + \ldots + a_{n+1} = 0 \} \]

\[ \Phi = \{ \pm (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n+1 \} \]

\[ \Delta = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n \} \]

\text{Corresponding Lie algebra is} \( \mathfrak{sl}_{n+1} \).

\textbf{Example 1.3.} The root system if type \( B_n \).

\[ V = \mathbb{R}^n \]

\[ \Phi = \{ \pm (\epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq n, \pm \epsilon_i, 1 \leq i \leq n \} \]

\[ \Delta = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n \} \]

\text{Corresponding Lie algebra is} \( \mathfrak{so}_{2n+1} \).

\textbf{Example 1.4.} The root system if type \( C_n \).

\[ V = \mathbb{R}^n \]

\[ \Phi = \{ \pm (\epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq n, \pm 2 \epsilon_i, 1 \leq i \leq n \} \]

\[ \Delta = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, 2 \epsilon_n \} \]

\text{Corresponding Lie algebra is} \( \mathfrak{sp}_{2n} \).
Example 1.5. The root system if type $D_n$.

\[ V = \mathbb{R}^n \]

\[ \Phi = \{ \pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq n \}. \]

\[ \Delta = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n - \epsilon_{n-1} + \epsilon_n \}. \]

Corresponding Lie algebra is $\mathfrak{so}_{2n}$.

Example 1.6. The root system if type $E_6$.

\[ V = \mathbb{R}^n \]

\[ \Phi = \{ \pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq 5, \]

\[ \pm \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8), \text{with even number of minus signs} \} \]

\[ \Delta = \{ \frac{1}{2}(\epsilon_1 - \epsilon_2 - \ldots - \epsilon_7 + \epsilon_8), \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \epsilon_5, \epsilon_5 + \epsilon_6 \}. \]

To every root system $\Phi$ we associate its Dynkin diagram which is a graph whose nodes are the simple roots and the nodes $\alpha_i$ and $\alpha_j$ are joined by $\langle \alpha_i, \alpha_j \rangle$ edges.

Finally we mention that the basic representations of the Lie algebra corresponding to $\Phi$ (or the corresponding simply connected group) are the fundamental representations. They correspond to fundamental weights $\omega_i$ defined via

\[ \omega_i(\alpha_j) = \delta_{i,j} \]

for all simple roots $\alpha_j$, $1 \leq j \leq \ell$.

The fundamental representations $V(\omega_i)$ are basic in that all others can be obtained from them as summands of tensor products. For classical groups most of them have very concrete descriptions.

Example 1.7.

\[ g = \mathfrak{sl}_{n+1}, V(\omega_i) = \bigwedge^i \mathbb{C}^{n+1}. \]

Example 1.8.

\[ g = \mathfrak{so}_{2n+1}, V(\omega_i) = \bigwedge^i \mathbb{C}^{2n+1}, 1 \leq i \leq n-1, V(\omega_n) = \text{spin}(2n+1). \]
Example 1.9.

\[ g = \mathfrak{sp}_{2n}, V(\omega_i) = \bigwedge^i \mathbb{C}^{2n} / \bigwedge^{i-2} \mathbb{C}^{2n}, 1 \leq i \leq n. \]

Example 1.10.

\[ g = \mathfrak{so}_{2n}, V(\omega_i) = \bigwedge^i \mathbb{C}^{2n}, 1 \leq i \leq n - 2, \]
\[ V(\omega_{n-1}) = \text{spin}_+ (2n), V(\omega_n) = \text{spin}_- (2n). \]

We will not need the description of spinor and half-spinor representations here.

To a representation \( V \) of a reductive group \( G \) (like a triple tensor product or an exterior power) we can associate its diagram by choosing a black node (corresponding to a representation) and the white nodes corresponding to a Dynkin diagram of the group \( G \). Let the Dynkin diagram of \( G \) have several connected components \( \Phi_i \). Then the representation \( V \) is the tensor product \( \otimes_i \lambda^{(i)} \) where \( \lambda^{(i)} = \sum_j \lambda_j^{(i)} \omega_j^{(i)}. \)

The diagram of \( V \) is obtained from \( \Phi \) by joining the black node with the vertex \( \omega_j^{(i)} \) with \( \lambda_j^{(i)} \) edges. The main point is that (forgetting a few exceptions that will be listed) a representation \( V \) has finitely many \( G \)-orbits if and only if the diagram of \( V \) is a Dynkin diagram.

Exercises for section 1.

1. Let \( F \) be an orthogonal space of dimension \( 2n+1 \) with a non-degenerate quadratic form \( (,). \) Choose a hyperbolic basis \( \{e_1, \ldots, e_n, \bar{e}_n, \ldots, \bar{e}_1\} \). The orthogonal Lie algebra \( \mathfrak{so}_{2n+1} \) consists of endomorphisms \( \varphi \) of \( F \) such that
\[ (\varphi(u), v) + (u, \varphi(v)) = 0 \quad \forall u, v \in F. \]

Find the root decomposition of \( \mathfrak{so}(2n+1) \) and identify it with the set \( \bigwedge^2 F. \)

2. Let \( F \) be an symplectic space of dimension \( 2n \) with a non-degenerate skew-symmetric form \( (,). \) Choose a hyperbolic basis \( \{e_1, \ldots, e_n, \bar{e}_n, \ldots, \bar{e}_1\} \). The symplectic Lie algebra \( \mathfrak{sp}_{2n} \) consists of endomorphisms \( \varphi \) of \( F \) such that
\[ (\varphi(u), v) + (u, \varphi(v)) = 0 \quad \forall u, v \in F. \]

Find the root decomposition of \( \mathfrak{sp}(2n) \) and identify it with the set \( S_2 F. \)

3. Let \( F \) be an orthogonal space of dimension \( 2n \) with a non-degenerate quadratic form \( (,). \) Choose a hyperbolic basis \( \{e_1, \ldots, e_n, \bar{e}_n, \ldots, \bar{e}_1\} \). The orthogonal Lie algebra \( \mathfrak{so}_{2n} \) consists of endomorphisms \( \varphi \) of \( F \) such that
\[ (\varphi(u), v) + (u, \varphi(v)) = 0 \quad \forall u, v \in F. \]

Find the root decomposition of \( \mathfrak{so}(2n) \) and identify it with the set \( \bigwedge^2 F. \).
Hint. The weight of $e_i$ is $\epsilon_i$, the weight of $\bar{e}_i$ is $-\epsilon_i$. The Cartan subalgebra $\mathfrak{h}$ is the subspace of vectors of weight zero in $\mathfrak{g}$.

§2. THE REPRESENTATIONS OF TYPE II AND $\theta$ GROUPS

Let $X_n$ be a Dynkin diagram and let $\mathfrak{g}$ be the corresponding simple Lie algebra. Let us distinguish a node $x \in X_n$. Let $\alpha$ be a corresponding simple root in the root system $\Phi$ corresponding to $X_n$. The choice of $\alpha$ determines a $\mathbb{Z}$-grading on $\Phi$ by letting the degree of a root $\beta$ be equal to the coefficient of $\alpha$ when we write $\beta$ as a linear combination of simple roots. On the level of Lie algebras this corresponds to a $\mathbb{Z}$-grading

$$\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i.$$ 

We define the group $G_0 := (G,G) \times \mathbb{C}^*$ where $(G,G)$ is a connected semisimple group with the Dynkin diagram $X_n \setminus x$. A representation of type II is the representation of $G_0$ on $\mathfrak{g}_1$.

By construction the representations of type II correspond to the Dynkin diagrams $X_n$ with distinguished nodes.

Denoting by $l$ the Levi factor $\mathfrak{g}_0$ we have

$$l = l' \oplus j(l)$$

where $l'$ denotes the Lie algebra associated to $X_n$ with the omitted node $x$.

Let us look at some interesting examples of these gradings. We look at all possible grading of Lie algebra of type $E_6$.

Notice that in each case the bracket

$$\mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$$

has to be $G_0$-equivariant, so it is determined up to a non-zero scalar. So in each case we get a model of Lie algebra of type $E_6$ in terms of classical Lie algebras and their representations. In each case we exhibit only $\mathfrak{g}_i$ for $i \geq 0$ because we always have

$$\mathfrak{g}_{-i} = \mathfrak{g}_i^*.$$ 

In each case we exhibit the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{g}_1$.

• $k = 1, 6$. The representation in question is $V = V(\omega_4, D_5)$ is a half-spinor representation for the group $G = \text{Spin}(10)$. Here $\omega_4 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. $\dim(X) = 16$. The weights of $X$ are vectors in 5 dimensional space, with coordinates equal to $\pm \frac{1}{2}$, with even number of negative coordinates.
We label the weight vectors in $g_1$ by $[I]$ where $I$ is the subset of $\{1, 2, 3, 4, 5\}$ of even cardinality where the sign of the component is negative.

The graded Lie algebra of type $E_6$ is

$$g(E_6) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1$$

with $g_0 = \mathbb{C} \oplus \mathfrak{so}(10)$.

The invariant scalar product $(\cdot, \cdot)$ on $g$ restricted to $g_1$ is given by the formula

$$([I], [J]) = 2 - \frac{1}{2} \#([I \setminus J] \cup [J \setminus I]).$$

Thus possible scalar products are only $2, 1, 0$. So the possible root systems we can get are $A_1 \times A_1$ and $A_1$. Indeed, the triple product $A_1 \times A_1 \times A_1$ is not possible because there are no three subsets $I, J, K$ with cardinalities of three symmetric differences being $4$. This means we get

* $k = 2$. $V = \wedge^3 F$, $F = \mathbb{C}^6$, $G = GL(F)$.

The graded Lie algebra of type $E_6$ is

$$g(E_6) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$$

with $g_0 = \mathbb{C} \oplus \mathfrak{sl}(6)$, $g_1 = \wedge^3 \mathbb{C}^6$, $g_2 = \wedge^6 \mathbb{C}^6$.

The weights of $g_1$ are $\epsilon_i + \epsilon_j + \epsilon_k$ for $1 \leq i < j < k \leq 6$. We label the corresponding weight vector by $[I]$ where $I$ is a cardinality $3$ subset of $\{1, 2, 3, 4, 5, 6\}$.

The invariant scalar product on $g$ restricted to $g_1$ is

$$([I], [J]) = \delta - 1$$

where $\delta = \#(I \cap J)$.

* $k = 3, 5$. $V = E \otimes \wedge^2 F$, $E = \mathbb{C}^2$, $F = \mathbb{C}^5$, $G = SL(E) \times SL(F) \times \mathbb{C}^*$

The graded Lie algebra of type $E_6$ is

$$g(E_6) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$$

with $g_0 = \mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(5)$, $g_1 = \mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^5$, $g_2 = \wedge^2 \mathbb{C}^2 \otimes \wedge^4 \mathbb{C}^5$.

Let $\{e_1, e_2\}$ be a basis of $E$, $\{f_1, \ldots, f_5\}$ be a basis of $F$. We denote the tensor $e_a \otimes f_i \wedge f_j$ by $[a; ij]$. The invariant scalar product on $g$ restricted to $g_1$ is

$$([a; ij], [b; kl]) = \delta - 1$$

where $\delta = \#(\{a\} \cap \{b\}) + \#(\{i, j\} \cap \{k, l\})$. 

$\bullet k = 4$. $V = E \otimes F \otimes H$, $E = \mathbb{C}^2$, $F = H = \mathbb{C}^3$, $G = SL(E) \times SL(F) \times SL(H) \times \mathbb{C}^*$.

The graded Lie algebra of type $E_6$ is

$$g(E_6) = g_{-3} \oplus g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \oplus g_3$$

with $g_0 = \mathbb{C} \oplus sl(2) \oplus sl(3) \oplus sl(3)$, $g_1 = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, $g_2 = \bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^3$, $g_3 = S_2 \mathbb{C}^2 \otimes \bigwedge^3 \mathbb{C}^3 \otimes \bigwedge^3 \mathbb{C}^3$.

Let $\{e_1, e_2\}$ be a basis of $E$, and $\{f_1, f_2, f_3\}$, $\{h_1, h_2, h_3\}$ bases of $F$, $H$ respectively. We label $e_a \otimes f_i \otimes h_u$ by $[a; i; u]$.

The invariant scalar product on $g$ restricted to $g_1$ is

$$([a; i; u] \mid [b; j; v]) = \delta_1$$

where $\delta = \#(\{a\} \cap \{b\}) + \#(\{i\} \cap \{j\}) + \#(\{u\} \cap \{v\})$.

It is worthwhile exhibiting explicitly the center $z(l)$. Denote the $\alpha_1, \ldots, \alpha_n$ the simple roots, and let $x = i_0$. Define the elements $x_i \in h$ by setting $\alpha_j(x_i) = \delta_{i,j}$.

Then

1.1. Proposition.

a) We have $z(l) = \mathbb{C}x_{i_0}$.

b) For a representation $V$ of $g$ we get the weights of the restriction $V|_V$ by applying the induced epimorphism $h^* \rightarrow h(l)^*$ whose kernel is $x_{i_0}$, and then expressing $\epsilon_{i_0}$ by means of other $\epsilon_i$ by modifying the weight by a multiple of $x_{i_0}$.

c) Consider the irreducible representation $V_\omega$ of $g$ with the highest weight $\omega = \sum_i m_i \omega_i$. Let us describe the restriction $V_i$. We want to label every irreducible representation in this restriction by the labeling of the Dynkin diagram $X_n$ with the marked node $i_0$. This is done as follows. We use b) to determine the weights of $(V_\omega)|_V$ and identify the highest weights. This will give us the markings on all the nodes except $i_0$. To determine the marking at the node $i_0$ we need to describe the action of $z(l)$. We have

$$\omega(x_{i_0}) = \sum_i m_i \omega_i(x_{i_0}) = \sum_i m_i (\sum_j m^i_j \alpha_j)x_{i_0} = \sum_i m_i m^i_{i_0},$$

where we write

$$\omega_i = \sum_j m^i_j \alpha_j.$$ 

The numbers $m^i_j$ can be read from the Bourbaki tables.

Vinberg in [V75] gave a description of the $G_0$-orbits in the representations of type $II$ in terms of conjugacy classes of nilpotent elements in $g$. Let $e \in g_1$ be a
nilpotent element in \( g \). Consider the irreducible components of the intersection of the conjugacy class of \( e \) in \( g \)

\[ C(e) \cap g_1 = C_1(e) \cup \ldots \cup C_n(e). \]

The sets \( C_i(e) \) are clearly \( G_0 \)-stable. Vinberg’s result shows that these are precisely the \( G_0 \)-orbits in \( g_1 \).

**1.2 Theorem.** The \( G_0 \)-orbits of the action of \( G_0 \) on \( g_1 \) are the components \( C_i(e) \), for all choices of the conjugacy classes \( C(e) \) and all \( i, 1 \leq i \leq n(e) \).

Theorem 1.2 makes the connection between the orbits in \( g_1 \) and the nilpotent orbits in \( g \). This means we will be needing the classification of nilpotent orbits in simple Lie algebras. This was obtained by Bala and Carter in the papers [BC76a], [BC76b]. A good account of this theory is the book [CM93]. Here we recall that the nilpotent orbit of an element \( e \) in a simple Lie algebra \( g \) is characterized by the smallest Levi subalgebra \( l \) containing \( e \). One must be careful because sometimes \( l \) is equal to \( g \). If the element \( e \) is a principal element in \( l \), then this orbit is denoted by the Dynkin diagram of \( l \) (but there might be different ways in which the root system \( R(l) \) sits as a subroot system of \( R(g) \)).

There are, however, the nonprincipal nilpotent orbits that are not contained in a smaller reductive Lie algebra \( l \). These are called the distinguished nilpotent orbits and are described in sections 8.2, 8.3, 8.4 of [CM93]. They are characterized by their associated parabolic subgroups (as their Dynkin characteristics are eve, see section 8 in [CM93]). Let us remark that for Lie algebras of classical types, for type \( A_n \) the only distinguished nilpotent orbits are the principal ones, and for types \( B_n, C_n, D_n \) these are orbits corresponding to the partitions with different parts. For exceptional Lie algebras the distinguished orbits can be read off the tables in section 8.4 of [CM93].

This theorem is not easy to use because it is not very explicit. In section 4 we describe more precise method from second Vinberg paper [V87].

**Exercises for section 2.** 1. Identify the groups \( G_0 \) and the representations \( g_1 \) for the gradings of \( A_n, B_n, C_n, D_n, E_7 \) and \( E_8 \) related to simple roots. 2. For \((E_7, \alpha_2)\) and \((E_8, \alpha_2)\) find the models of corresponding Lie algebras.

§3. Kac’s classification of representations with finitely many orbits

The representations of reductive groups with finitely many orbits were classified by Kac in [K80] and [DK85]. The result is
Theorem 3.1. Let $H$ be a reductive group operating on an irreducible representation with finitely many orbits. Then the pair $(H, V)$ is on the following list

a) $H = G_0, V = g_1$ where $g = \oplus_{i \in \mathbb{Z}} g_1$ is a grading on a simple Lie algebra $g$ related to a simple root $\alpha_k$.

b) $H = SL(2) \times SL(3) \times SL(n) \times \mathbb{C}^*$ operating on $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$, $n \geq 6$.

c) $H = SL(2) \times Spin(7), V = \mathbb{C}^2 \otimes V(\omega_3)$.

In fact Kac classifies a more general class of representations, so-called visible representations.

The nullcone $N(V)$ of a representation $V$ of $G$ is the set of $v \in V$ for which $f(v) = 0$ for every $G$-invariant function $f \in \text{Sym}(V^*)$.

Definition 3.2. The nullcone $N(V)$ of a representation $V$ of $G$ is the set of $v \in V$ for which $f(v) = 0$ for every $G$-invariant function $f \in \text{Sym}(V^*)$. A representation $V$ of $G$ is visible if there are finitely many $G$-orbits in the null-cone $N(V)$ of $V$.

A very important class of visible representations is related to finite order automorphisms of the root systems classified by Kac.

Let $\tilde{X}_n$ be an extended Dynkin graph corresponding to the Dynkin graph $X_n$. Let $x_k$ be a node in $\tilde{X}_n$. Let $a_k$ be the coefficient of the isotropic root of the graph $\tilde{X}_n$ at the vertex $x_k$. Then there is a grading on $g$ by $\mathbb{Z}/a_k\mathbb{Z}$ where $g_0$ is a subalgebra corresponding to $\tilde{X}_n \setminus \alpha_k$ and $g_1$ is given by the same recipe as for the cases in previous section.

Alternative description os that the Kac-Moody Lie algebra $\hat{g}$ has a grading

$$\hat{g} = \oplus_{i \in \mathbb{Z}} \hat{g}_i$$

and the components $\hat{g}_i$ are periodic with period $a_k$.

The visible representations $g_1$ have analogous properties to the adjoint representations. Thy have a Cartan subspace $h$ which is the slice representation. The Weyl group $W := N(h)/Z(h)$ acts on $h$ as a reflection group generated by pseudoreflections. Vinberg proves the analogue of Chevalley Theorem. The natural map

$$\text{Sym}(g_1^*)^G \to \text{Sym}(h^*)^W$$

is an isomorphism. Both rings are polynomial rings. Vinberg proves only existence of $h$ and identifies $W$ using Sheppard-Todd table and the knowledge of degrees of invariants. It would be very interesting to find the direct and explicit proofs of these results.

Example 3.1. Let us consider the case $(\tilde{E}_6, \alpha_4)$. The order of $\theta$ is 3. We have

$$g_0 = sl_3 \times sl_3 \times sl_3, g_1 = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3, g_2 = \bigwedge^2 \mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^3.$$
Exercises for section 3.

1. Determine the graded components $g_k$ for the case $(\tilde{E}_7, \alpha_4)$. Answer:

$$g_0 = \mathfrak{sl}_2 \times \mathfrak{sl}_4 \times \mathfrak{sl}_4, g_1 = C^2 \otimes C^4 \otimes C^4,$$

$$g_2 = \bigwedge^2 C^2 \otimes \bigwedge^2 C^4 \otimes \bigwedge^2 C^4, g_3 = S_2, C^2 \otimes \bigwedge^3 C^4 \otimes \bigwedge^3 C^4.$$

2. Do the same for the case $(\tilde{E}_7, \alpha_5)$.

§4. The Vinberg method for classifying orbits.

All Lie algebras $g$ we will consider will be Lie algebras of some algebraic group $G$. Let $(X_n, \alpha_k)$ be one of the representations on our list. It defines the grading $g = \bigoplus_{i \in \mathbb{Z}} g_i$ where $g_i$ is the span of the roots which, written as a combination of simple roots, have $\alpha_k$ with coefficient $i$. The component $g_0$ contains in addition a Cartan subalgebra. $G_0$ denotes the connected component of the subgroup of $G$ corresponding to $g_0$. In the sequel $Z(x)$ denotes the centralizer of an element $x \in G$. $Z_0(x)$ is $Z(x) \cap G_0$. The gothic letters, $\mathfrak{z}$, $\mathfrak{z}_0$ denote corresponding Lie algebras. Similarly, $N(x)$ denotes the normalizer of an element $x \in G$. $N_0(x)$ is $N(x) \cap G_0$.

We denote $R(g)$ the set of roots of a reductive Lie algebra $g$, and $\Pi(g)$ denotes the set of simple roots.

In order to state Vinberg Theorem we need some definitions.

We will be dealing with the graded Lie subalgebras $s = \bigoplus_{i \in \mathbb{Z}} s_i$.

**Definition 4.1.** A graded Lie subalgebra $s$ of $g$ is regular if it is normalized by a maximal torus in $g_0$. A reductive graded Lie algebra $s$ of $g$ is complete if it is not a proper graded Lie subalgebra of any regular reductive $\mathbb{Z}$-graded Lie algebra of the same rank.

**Definition 4.2.** A $\mathbb{Z}$-graded Lie algebra $g$ is locally flat if one of the following equivalent conditions is satisfied, for $e$ a point in general position in $g_1$:

1. The subgroup $Z_0(e)$ is finite,
2. $\mathfrak{z}_0(e) = 0$,
3. $\dim g_0 = \dim g_1$.

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a graded reductive Lie algebra. Let us fix a nonzero nilpotent element $e \in \mathfrak{g}_1$. Let us choose some maximal torus $H$ in $N_0(e)$. Its Lie algebra $\mathfrak{h}$ is the accompanying torus of the element $e$. We denote $\varphi$ the character of the torus $H$ defined by the condition

$$[u, e] = \varphi(u)e$$
for $u \in \mathfrak{h}$. Consider the graded Lie subalgebra $\mathfrak{g}(\mathfrak{h}, \varphi)$ of $\mathfrak{g}$ defined as follows

$$\mathfrak{g}(\mathfrak{h}, \varphi) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\mathfrak{h}, \varphi)_i$$

where

$$\mathfrak{g}(\mathfrak{h}, \varphi)_i = \{ x \in \mathfrak{g}_{ia} \mid [u, x] = i \varphi(u) \forall u \in H \}.$$

**Definition 4.3.** The support $\mathfrak{s}$ of the nilpotent element $e \in \mathfrak{g}_a$ is the commutant of $\mathfrak{g}(\mathfrak{h}, \varphi)$ considered as a $\mathbb{Z}$ graded Lie algebra.

Clearly $e \in \mathfrak{s}_1$.

We are ready to state the main theorem of [V87].

**Theorem 4.4. (Vinberg).** The supports of nilpotent elements of the space $\mathfrak{g}_i$ are exactly the complete regular locally flat semisimple $\mathbb{Z}$ graded subalgebras of the algebra $\mathfrak{g}$. The nilpotent element $e$ can be recovered from the support subalgebra $\mathfrak{s}$ as the generic element in $\mathfrak{s}_1$.

It follows from the theorem that the nilpotent $e$ is defined uniquely (up to conjugation by an element of $G_0$) by its support. This means it is enough to classify the regular semisimple $\mathbb{Z}$-graded subalgebras $\mathfrak{s}$ of $\mathfrak{g}$.

Let us choose the maximal torus $t$ of $\mathfrak{g}_0$. The $\mathbb{Z}$-graded subalgebra $\mathfrak{s}$ is standard if it is normalized by $t$, i.e. if for all $i \in \mathbb{Z}$ we have

$$[t, s_i] \subset s_i.$$

Vinberg also proves that every $\mathbb{Z}$ graded subalgebra $\mathfrak{s}$ is conjugated to a standard subalgebra by an element of $G_0$. Moreover, he shows that if two standard $\mathbb{Z}$-graded subalgebras are conjugated by an element of $G_0$, then they are conjugated by an element of $N_0(t)$.

This allows to give a combinatorial method for classifying regular semisimple $\mathbb{Z}$-graded subalgebras of $\mathfrak{g}$.

Let $\mathfrak{s}$ be a standard semisimple $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$. The subalgebra $\mathfrak{s}$ defines the degree map $\text{deg} : R(\mathfrak{s}) \to \mathbb{Z}$. For a standard $\mathbb{Z}$-graded subalgebra $\mathfrak{s}$ we also get the map

$$f : R(\mathfrak{s}) \to R(\mathfrak{g}).$$

The map $f$ has to be additive, i.e. it satisfies

$$f(\alpha + \beta) = f(\alpha) + f(\beta) \quad \forall \alpha, \beta \in R(\mathfrak{s}),$$

$$f(-\alpha) = -f(\alpha) \quad \forall \alpha \in R(\mathfrak{s}).$$

Moreover we have
Proposition 4.5. The map $f$ satisfies the following properties:

a) $\frac{(f(\alpha), f(\beta))}{(f(\alpha), f(\alpha))} = \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ for all $\alpha, \beta \in \mathcal{R}(\mathfrak{s})$.

b) $f(\alpha) - f(\beta) \notin R(\mathfrak{g})$ for all $\alpha, \beta \in \Pi(\mathfrak{s})$.

c) $\deg f(\alpha) = \deg \alpha$ for all $\alpha \in \Pi(\mathfrak{s})$.

Conversely, every map satisfying a), b), c) defines a standard regular $\mathbb{Z}$-graded subalgebra $\mathfrak{s}$ of $\mathfrak{g}$.

Remark 4.6. The subalgebra $\mathfrak{s}$ corresponding to the map $f$ is complete if and only if there exists an element $w$ in the Weyl group $W$ of $\mathfrak{g}$ such that $w f(\Pi(\mathfrak{s})) \subset \Pi(\mathfrak{g})$ (see [V87], p.25).

Proposition 4.5 means that in order to classify the nilpotent elements $e \in \mathfrak{g}_1$ we need to classify the possible maps $f$ corresponding to its support, i.e. the corresponding complete regular $\mathbb{Z}$-graded subalgebra $\mathfrak{s}$. Since we are interested in the nilpotents $e \in \mathfrak{g}_1$, we need to classify the maps $f$ for which $\deg f(\alpha) \in \{0, 1\}$ for every $\alpha \in \Pi(\mathfrak{s})$.

One should also mention the connection with the Bala-Carter classification. The Bala-Carter characteristic of a nilpotent element is given by the type of the algebra $\mathfrak{s}$. So to find the components of a given nilpotent orbit with Bala-Carter characteristic $\mathfrak{s}$ it is enough to see in how many ways the Lie algebra $\mathfrak{s}$ can be embedded as a $\mathbb{Z}$-graded Lie algebra into $\mathfrak{g}$.

One can determine the orbits explicitly using the following strategy.

Proposition 4.7. The following procedure allows to find the components of the intersection of a nilpotent orbit in $\mathfrak{g}$ with $\mathfrak{g}_1$.

1. Establish the restriction of the invariant scalar product $(,)$ on $\mathfrak{g}$ to $\mathfrak{g}_1$.
2. Fix a nilpotent element $e \in \mathfrak{g}$ with Bala-Carter characteristic $\mathfrak{s}$.
3. Find in how many ways $\mathfrak{s}$ embeds into $\mathfrak{g}$ as a standard $\mathbb{Z}$-graded Lie subalgebra by exhibiting corresponding map $f$ as in Proposition 4.5.
4. For a characteristic $\mathfrak{s}$ such that $e$ is a principal nilpotent in $\mathfrak{s}$ the map $f$ sends all simple roots of $\mathfrak{s}$ to weight vectors in $\mathfrak{g}_1$. Thus it is enough to classify the subsets of weights vectors in $\mathfrak{g}_1$ for which the pattern of scalar products is the same as the one for simple roots of $\mathfrak{s}$.
5. For non-principal orbits one needs to make a more detailed analysis, but still one reduces to finding sets of weight vectors in $\mathfrak{g}_1$ with certain patterns of scalar products.

Exercises for section 4. Classify the subalgebras $\mathfrak{s}$ of type $A_3 + 2A_1$ and $A_2 + 2A_1$ in bigwedge^3C^8 (the case $(E_8, \alpha_2)$). Prove that we do not have any cases of type $A_5$ and $D_4$. 


Example 5.1. Consider the diagram $X_n = A_n$. Let $x$ be the note corresponding to $\alpha_m := \epsilon_m - \epsilon_{m+1}$. We have $\mathfrak{g} = sl_{n+1}$, $G_0 = SL(m, \mathbb{C}) \times SL(n-m+1, \mathbb{C}) \times \mathbb{C}^*$. The $\mathbb{Z}$ grading is

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_0 = \mathbb{C} \oplus sl(m) \oplus sl(n-m+1)$, $\mathfrak{g}_1 = Hom_{\mathbb{C}}(\mathbb{C}^{n+1-m}, \mathbb{C}^m)$.

The weight vector $e_i \otimes f_j$ corresponds to the root $\epsilon_i - \epsilon_{m+j}$. Let us denote it by the label $[i,j]$. The invariant scalar product $(,)$ on $\mathfrak{g}$ restricted to $\mathfrak{g}_1$ is

$$([[i_1,j_1],[i_2,j_2]]) = \delta$$

where $\delta = \#(\{i_1\} \cap \{i_2\}) + \#(\{j_1\} \cap \{j_2\})$. This means the scalar products can take only values $2, 1, 0$. So the only root systems whose simple roots can be embedded into $\mathfrak{g}_1$ are $(A_1)^r$ for $1 \leq r \leq \min(m,n)$. The corresponding set of weights has to be orthogonal, so (up to permutation of indices which means conjugation) we have one $r$-tuple $\{[1,1], \ldots, [r,r]\}$. The representative is the tensor $e_1 \otimes f_1 + \ldots + e_r \otimes f_r$.

The corresponding orbit is the set of tensors of rank $r$.

Consider a matrix of rank $r$ in $\mathfrak{g}_1$. After changing the bases in $\mathbb{C}^m$ and in $\mathbb{C}^{n+1-m}$ we can assume that the corresponding nilpotent $e(r) = \sum_{j=1}^r E_{j,m+j}$. The intersection of the orbit of $e(r)$ with $\mathfrak{g}_1$ is a determinantal variety of $m \times (n+1-m)$ matrices of rank $\leq r$.

Let’s go to triple tensor product cases.

Example 5.2. Consider the case $(E_6, \alpha_4)$.

$$\mathfrak{g}(E_6) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with $\mathfrak{g}_0 = \mathbb{C} \oplus sl(2) \oplus sl(3) \oplus sl(3)$, $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, $\mathfrak{g}_2 = \bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^3$, $\mathfrak{g}_3 = S_{2,1} \mathbb{C}^2 \otimes \bigwedge^3 \mathbb{C}^3 \otimes \bigwedge^3 \mathbb{C}^3$. 
Example 5.3. Consider the case \((E_7, \alpha_4)\). The graded Lie algebra of type \(E_7\) is

\[
\mathfrak{g}(E_7) = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \\
\]

with \(\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(3) \oplus \mathfrak{sl}(4)\), \(\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4\), \(\mathfrak{g}_2 = \bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^4\), \(\mathfrak{g}_3 = S_{2,1} \mathbb{C}^2 \otimes \bigwedge^3 \mathbb{C}^3 \otimes \bigwedge^3 \mathbb{C}^4\), \(\mathfrak{g}_4 = S_{2,2,1,1} \mathbb{C}^3 \otimes \bigwedge^4 \mathbb{C}^4\).

Let \(\{e_1, e_2\}\) be a basis of \(E\), and \(\{f_1, f_2, f_3\}\), \(\{h_1, h_2, h_3, h_4\}\) bases of \(F\), \(H\) respectively. We label \(e_a \otimes f_i \otimes h_u\) by \([a; i; u]\).

The invariant scalar product on \(\mathfrak{g}\) restricted to \(\mathfrak{g}_1\) is

\[
([a; i; u], [b; j; v]) = \delta - 1
\]

where \(\delta = \#(\{a\} \cap \{b\}) + \#(\{i\} \cap \{j\}) + \#(\{u\} \cap \{v\})\).

There are six \(H\)-nondegenerate orbits. They can be described by observing that the castling transform establishes a bijection between \(H\)-nondegenerate orbits and \(H'\)-nondegenerate orbits for the \(2 \times 3 \times 2\) matrices corresponding to representation \(E \otimes F \otimes H'\). The six orbits in the representation \(E \otimes F \otimes H'\) are: generic, hyperdeterminant hypersurface and four \(F\)-degenerate orbits, coming from \(2 \times 2 \times 2\) matrices: generic, hyperdeterminant and two determinantal varieties. Combining this knowledge with the case \((E_6, 4)\) we get 23 orbits in our representation.
Example 5.4. Consider the case \((E_8, \alpha_4)\).

The graded Lie algebra of type \(E_8\) is

\[ \mathfrak{g}(E_8) = \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_5 \oplus \mathfrak{g}_6 \]

with \( \mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(3) \oplus \mathfrak{sl}(5) \), \( \mathfrak{g}_1 = \Lambda^2 \mathbb{C}^2 \oplus \Lambda^2 \mathbb{C}^3 \oplus \Lambda^2 \mathbb{C}^5 \), \( \mathfrak{g}_2 = \Lambda^3 \mathbb{C}^3 \oplus \Lambda^3 \mathbb{C}^5 \), \( \mathfrak{g}_3 = S_{2,1} \mathbb{C}^2 \oplus \Lambda^4 \mathbb{C}^3 \oplus \Lambda^4 \mathbb{C}^5 \), \( \mathfrak{g}_4 = S_{2,2} \mathbb{C}^2 \oplus S_{2,1,1} \mathbb{C}^3 \oplus \Lambda^4 \mathbb{C}^5 \), \( \mathfrak{g}_5 = S_{3,2} \mathbb{C}^2 \oplus S_{2,2,1} \mathbb{C}^3 \oplus \Lambda^5 \mathbb{C}^5 \), \( \mathfrak{g}_6 = S_{3,3} \mathbb{C}^2 \oplus S_{2,2,2} \mathbb{C}^3 \oplus S_{2,1,1} \mathbb{C}^5 \).

Let \( \{e_1, e_2\} \) be a basis of \(E\), and \( \{f_1, f_2, f_3\}, \{h_1, h_2, h_3, h_4, h_5\} \) bases of \(F, H\) respectively. We label \( e_a \otimes f_i \otimes h_u \) by \([a; i; u]\).

The invariant scalar product on \( \mathfrak{g} \) restricted to \( \mathfrak{g}_1 \) is

\[ (\langle a; i; u \rangle, [b; j; v]) = \delta - 1 \]

where \( \delta = \#(\{a\} \cap \{b\}) + \#(\{i\} \cap \{j\}) + \#(\{u\} \cap \{v\}) \).
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Exercises for section 5. Analyze one of the smaller cases, for example $(D_5, \alpha_2)$.


In the case of $(E_6, \alpha_4), (E_7, \alpha_4), (E_8, \alpha_4)$ we are mostly interested in, there is another approach using representations of Kronecker quiver.

The Kronecker quiver is the quiver

$$Q : \begin{array}{c}
\overset{a}{\longrightarrow} \\
\downarrow{b} \\
2
\end{array}$$
A representation $V$ of $Q$ is just a pair of vector spaces $\{V(1), V(2)\}$ and a pair of linear maps. Each representation $V$ of $Q$ has dimension vector $\alpha = (\dim V(1), \dim V(2)) \in \mathbb{Z}^2$. The representations of $Q$ form an Abelian category (in fact a category of $KQ$-modules where $KQ$ is a path algebra of $Q$). By Krull-Remak-Schmidt theorem each representation decomposes uniquely (up to permutation of factors) to a direct sum of indecomposable representations.

In the case of Kronecker quiver one can list all indecomposable representations. The are as follows.

In each dimension vector $\langle n, n \rangle$ there is $P^n$ worth of representations, with the standard $A_1$ is given by $V(1) = K^n, V(2) = K^n, V(a) = 1, V(b) = J^n\lambda$ where $J^n\lambda$ is an $n \times n$ Jordan block of dimension $n$. We denote these representations by $V^n$.

Moreover, there are two discrete infinite families: $\{P^n\}$ of dimension vector $\langle n, n + 1 \rangle, n \geq 0$, and $\{Q^n\}$ of dimension vector $\langle n + 1, n \rangle, n \geq 0$, defined as follows

$$P^n(a) = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

$$P^n(b) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

The matrices of $Q^n$ are just transposes of the matrices of $P^n$. Proof. The idea is to introduce reflection functors $C^+(2)$ and $C^-(1)$, and their compositions $\tau$ and $\tau^{-1}$. They take $Q$ into itself. Thus they take indecomposables into indecomposables. $C^+(2)$ takes dimension vector $\langle m, n \rangle$ into $\langle 2m - n, m \rangle$. So, the vector $\langle p+1, p \rangle$ goes to $\langle p+2, p+1 \rangle$ and $\langle p, p+1 \rangle$ goes to $\langle p-1, p \rangle$. Respectively, $C^-(1)$ takes $\langle m, n \rangle$ into $\langle n, 2n - m \rangle$. This means $\langle q, q+1 \rangle$ goes to $\langle q+1, q+2 \rangle$ and $\langle q+1, q \rangle$ goes to $\langle q, q-1 \rangle$.

Notice that the action of both functors on dimension vectors $\langle m, n \rangle$ preserve $m - n$. This means there are no indecomposables in dimension vectors $\langle m, n \rangle$ with $|m - n| \geq 2$, as they would imply the existence of indecomposables in dimension vectors $\langle |m-n|, 0 \rangle$ or $\langle 0, |m-n| \rangle$. In the dimension vectors $\langle p+1, p \rangle$ and $\langle p, p+1 \rangle$ there is one indecomposable, as it is the case in dimension vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$. Finally, in dimension vector $\langle n, n \rangle$, assuming one of the matrices is nonsingular, we can use it to identify $V(1)$ and $V(2)$ so the description of indecomposables follows.
from Jordan canonical form. This is the case if any linear combination of \( V(a) \) and \( V(b) \) is nonsingular. If not, then there has to be subspace of dimension \( p + 1 \) in \( V(1) \) that is taken to a subspace of dimension \( p \) in \( V(2) \) by both \( V(a) \) and \( V(b) \) (apply Hilbert-Mumford criterion exercise), which means \( V \)‘s not indecomposable, as after applying reflection functors we reduce to a representation of dimension \((n, n)\) having a subrepresentation of dimension \((1, 0)\) which is injective, so it splits.

To a tensor in \( \varphi \in \mathbb{C}^2 \otimes \mathbb{C}^p \otimes \mathbb{C}^q \) we can associate the representation \( V(\varphi) \) by choosing a fixed basis \( \{e_1, e_2\} \) of \( \mathbb{C}^2 \), write \( \varphi = e_1 \otimes \varphi(a) + e_2 \otimes \varphi(b) \) and take \( V(\varphi)(a) = \varphi(a) \), \( V(\varphi)(b) = \varphi(b) \). Alternatively, the group \( GL(2) \) acts on the set of representations of \( Q \) of dimension vector \((p, q)\) by replacing \( V(a) \), \( V(b) \) by the linear combinations of these maps, and the orbits in \( \mathbb{C}^2 \otimes \mathbb{C}^p \otimes \mathbb{C}^q \) are the \( GL(2) \)-orbits of this action. In particular the \( \mathbb{P}^1 \) families become one \( GL(2) \)-orbit, and we get finitely many possibilities, as long as the number of summands of that type is \( \leq 3 \).

In this language our three tables have the following meaning.

\[(E_6, \alpha_4)\].

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We end this section with some remarks on the castling transform. This is a version of the reflection functors, suitable for the triple tensor products. The method allows to classify the cases when the triple tensor product has an open orbit. This method goes back at least to SATO–KIMURA [SK77] who classified the irreducible representations of reductive groups with an open orbit.
We will work with the triple tensor products $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$. The main point of the approach is the following observation. Assume that $p \leq q \leq r$. If $r > pq$ then any orbit in $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$ is inherited from $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^{pq}$. Moreover, if $r = pq$ then $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^{pq}$ has an open orbit.

**Proposition 6.1.** Assume that $p \leq q \leq r$, $r < pq$. Then there are open subsets in $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$ and in $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^{pq} - r$ such that there is a bijection between the orbits within these open subsets.

**Proof.** Let $U \subset \mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$ be the open set of tensors

$$t = \sum_{i=1}^r t_i \otimes e_i$$

where $t_i$ are linearly independent in $\mathbb{C}^p \otimes \mathbb{C}^q$. Notice that the $GL_p \times GL_q \times GL_r$-orbits in $U$ are in bijection with the $GL_p \times GL_q$-orbits of $r$-dimensional subspaces in $CC^p \otimes \mathbb{C}^q$. But from an $r$-dimensional subspace in $\mathbb{C}^p \otimes \mathbb{C}^q$ we can produce a $pq-r$-dimensional subspace in $(\mathbb{C}^p)^* \otimes (\mathbb{C}^q)^*$ by taking the set of linear forms vanishing on a given $r$-dimensional subspace in $\mathbb{C}^p \otimes \mathbb{C}^q$. □

The move from the triple $(p, q, r)$ to $(p, q, pq - r)$ is called CASTLING. Let us introduce the invariant

$$N(p, q, r) := p^2 - q^2 - r^2 - pqr.$$ 

Notice that the invariant $N(p, q, r)$ does not change when we apply castling.

**Theorem 6.2.** The triple tensor product $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$ has an open orbit only if $N(p, q, r) \geq 2$. This happens precisely when by a sequence of castling moves (exchanging $(p, q, r)$ to $(p, q, pq - r)$) the triple can be reduced to $(1, q, r)$ or to $(2, q, r)$. The open orbit occurs unless we are in the case linked to $(2, r, r)$ with $r \geq 4$.

**Proof.** Obviously $pqr = \dim \mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$ and $p^2 + q^2 + r^2 = \dim(GL_p \times GL_q \times GL_r)$.

We have a chance to have an open orbit only if $N(p, q, r) \geq 2$ (the number 2 comes from the fact that homoteties from three linear groups act in the same way on $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$).

Let us assume that $p \leq q \leq r$. Recall that the number $N(p, q, r)$ will not change if we replace $(p, q, r)$ by $(p, q, pq - r)$. Thus we may assume that $r \leq \frac{pq}{2}$. Now we write

$$N(p, q, r) = p(p - \frac{rq}{6}) + q(q - \frac{pr}{3}) + r(r - \frac{pq}{2}).$$

This shows that if $p \geq 3$ and $q \geq 6$ all three summands will be negative. Thus it remains to classify cases $p = 1$, $p = 2$ and $3 \leq p \leq q \leq 6$. When $p = 1$ we have

$$N(p, q, r) = 1 + q^2 + r^2 - qr = 1 + (q - r)^2 + qr \geq 2.$$
When \( p = 2 \) we have

\[
N(p, q, r) = 4 + q^2 + r^2 - 2qr = 4 + (q - r)^2 \geq 4.
\]

Detailed case by case analysis shows that in the cases \( 3 \leq p \leq q \leq r \) the value of \( N \) is never \( \geq 2 \).

In the cases with \( p = 1 \) example 5 shows that the algebra \( A_3(1, q, r) \) is finite dimensional.

Let us look at the examples with \( p = 2 \). We will think of \( \mathbb{C}^2 \otimes \mathbb{C}^q \otimes \mathbb{C}^r \) as of the set of representations of Kronecker quiver of dimension vector \((q, r)\) this will allow us to identify the generic tensors. The generic tensors can be determined by the generic decomposition. The indecomposables occur in dimensions \((n, n)\) (one parameter families) and \((n, n + 1)\) and \((n + 1, n)\). Thus canonical decomposition of \((n, n + k)\) is \((k - v)(u + 1) + v(u + 1, u + 2)\) where \( n = uk + v \) is division with remainder. Similarly \((n + k, n) = (k - v)(u + 1, u) + v(u + 2, u + 1)\). This decomposition allows to write easily the generic tensors. □

**Exercises for section 6.** 1. Apply Hilbert-Mumford criterion to prove that if a representation \( V \) of the Kronecker quiver has the property that for each \( \alpha, \beta \) the linear combination \( \alpha V(a) + \beta V(b) \) is singular, then there exist \( p, 0 \leq p \leq n - 1 \) and a subspace \( W \) of dimension \( p + 1 \) in \( V(1) \) such that

\[
\dim (V(a)W + V(b)W) \leq p.
\]

2. Fill the rest of the entries in the table \((E_6, \alpha_4)\)

---

§7. Geometric technique.

In this section we provide a quick description of main facts related to geometric technique of calculating syzygies. We work over an algebraically closed field \( K \).

Let us consider the projective variety \( V \) of dimension \( m \). Let \( X = A_N^m \) be the affine space. The space \( X \times V \) can be viewed as a total space of trivial vector bundle \( E \) of dimension \( n \) over \( V \). Let us consider the subvariety \( Z \) in \( X \times V \) which is the total space of a subbundle \( S \) in \( E \). We denote by \( q \) the projection \( q : X \times V \rightarrow X \) and by \( q' \) the restriction of \( q \) to \( Z \). Let \( Y = q(Z) \). We get the basic diagram

\[
\begin{array}{c}
Z \subset X \times V \\
\downarrow q' \quad \downarrow q \\
Y \subset X
\end{array}
\]
The projection from $X \times V$ onto $V$ is denoted by $p$ and the quotient bundle $\mathcal{E}/\mathcal{S}$ by $\mathcal{T}$. Thus we have the exact sequence of vector bundles on $V$

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{E} \rightarrow \mathcal{T} \rightarrow 0$$

The dimensions of $\mathcal{S}$ and $\mathcal{T}$ will be denoted by $s$, $t$ respectively. The coordinate ring of $X$ will be denoted by $A$. It is a polynomial ring in $N$ variables over $\mathbb{C}$. We will identify the sheaves on $X$ with $A$-modules.

The locally free resolution of the sheaf $\mathcal{O}_Z$ as an $\mathcal{O}_{X \times V}$-module is given by the Koszul complex

$$K_\bullet(\xi) : 0 \rightarrow \bigwedge^t (p^* \xi) \rightarrow \ldots \rightarrow \bigwedge^2 (p^* \xi) \rightarrow p^*(\xi) \rightarrow \mathcal{O}_{X \times V}$$

where $\xi = T^*$. The differentials in this complex are homogeneous of degree 1 in the coordinate functions on $X$. The direct image $p_*(\mathcal{O}_Z)$ can be identified with the the sheaf of algebras $\text{Sym}(\eta)$ where $\eta = S^*$.

The idea of the geometric technique is to use the Koszul complex $K(\xi)_\bullet$ to construct for each vector bundle $\mathcal{V}$ on $V$ the free complex $F_\bullet(\mathcal{V})$ of $A$-modules with the homology supported in $Y$. In many cases the complex $F(\mathcal{O}_V)_\bullet$ gives the free resolution of the defining ideal of $Y$.

For every vector bundle $\mathcal{V}$ on $V$ we introduce the complex

$$K(\xi, \mathcal{V})_\bullet := K(\xi)_\bullet \otimes_{\mathcal{O}_{X \times V}} p^* \mathcal{V}$$

This complex is a locally free resolution of the $\mathcal{O}_{X \times V}$-module $M(\mathcal{V}) := \mathcal{O}_Z \otimes p^*\mathcal{V}$.

Now we are ready to state the basic theorem (Theorem (5.1.2) in [W]).

**Theorem 7.1.** For a vector bundle $\mathcal{V}$ on $V$ we define a free graded $A$-modules

$$F(\mathcal{V})_i = \bigoplus_{j \geq 0} H^j(V, \bigwedge^{i+j} (\xi \otimes \mathcal{V}) \otimes_k A(-i-j))$$

a) There exist minimal differentials

$$d_i(\mathcal{V}) : F(\mathcal{V})_i \rightarrow F(\mathcal{V})_{i-1}$$

of degree 0 such that $F(\mathcal{V})_\bullet$ is a complex of graded free $A$-modules with

$$H_{-i}(F(\mathcal{V})_\bullet) = \mathcal{R}^i q_* M(\mathcal{V})$$

In particular the complex $F(\mathcal{V})_\bullet$ is exact in positive degrees.
b) The sheaf \( R^i q_* M(V) \) is equal to \( H^i(Z, M(V)) \) and it can be also identified with the graded \( A \)-module \( H^i(V, \text{Sym}(\eta) \otimes V) \).

c) If \( \varphi : M(V) \to M(V')(n) \) is a morphism of graded sheaves then there exists a morphism of complexes

\[
    f_\bullet(\varphi) : F(V)_\bullet \to F(V')_\bullet(n)
\]

Its induced map \( H_{-i}(f_\bullet(\varphi)) \) can be identified with the induced map

\[
    H^i(Z, M(V)) \to H^i(Z, M(V'))(n).
\]

If \( V \) is a one dimensional trivial bundle on \( V \) then the complex \( F(V)_\bullet \) is denoted simply by \( F_\bullet \).

The next theorem gives the criterion for the complex \( F_\bullet \) to be the free resolution of the coordinate ring of \( Y \).

**Theorem 7.2.** Let us assume that the map \( q' : Z \to Y \) is a birational isomorphism. Then the following properties hold.

- a) The module \( q'_* \mathcal{O}_Z \) is the normalization of \( \mathbb{C}[Y] \).
- b) If \( R^i q'_* \mathcal{O}_Z = 0 \) for \( i > 0 \), then \( F_\bullet \) is a finite free resolution of the normalization of \( \mathbb{C}[Y] \) treated as an \( A \)-module.
- c) If \( R^i q'_* \mathcal{O}_Z = 0 \) for \( i > 0 \) and \( F_0 = H^0(V, \wedge^0 \xi) \otimes A = A \) then \( Y \) is normal and it has rational singularities.

This is Theorem (5.1.3) in [W].

The complexes \( F(V)_\bullet \) satisfy the Grothendieck type duality. Let \( \omega_V \) denote the canonical divisor on \( V \).

**Theorem 7.3.** Let \( V \) be a vector bundle on \( V \). Let us introduce the dual bundle

\[
    V^\vee = \omega_V \otimes \bigwedge^i \xi^* \otimes V^*.
\]

Then

\[
    F(V^\vee)_\bullet = F(V)_\bullet[m-t]
\]

This is Theorem (5.1.4) in [W].

In all our applications the projective variety \( V \) will be a Grassmannian. To fix the notation, let us work with the Grassmannian \( \text{Grass}(r, E) \) of subspaces of dimension \( r \) in a vector space \( F \) of dimension \( n \). Let

\[
    0 \to \mathcal{R} \to E \times \text{Grass}(r, E) \to \mathcal{Q} \to 0
\]

be a tautological sequence of the vector bundles on \( \text{Grass}(r, E) \). The vector bundle \( \xi \) will be a direct sum of the bundles of the form \( S_{\lambda_1, \ldots, \lambda_n} \mathcal{Q} \otimes S_{\mu_1, \ldots, \mu_r} \mathcal{R} \). Thus all the exterior powers of \( \xi \) will also be the direct sums of such bundles. We will apply repeatedly the following result to calculate cohomology of vector bundles \( S_{\lambda_1, \ldots, \lambda_n} \mathcal{Q} \otimes S_{\mu_1, \ldots, \mu_r} \mathcal{R} \).
**Proposition 7.4 (Bott’s algorithm).** The cohomology of the vector bundle $S_{\lambda_1, \ldots, \lambda_{n-r}} \mathcal{Q} \otimes S_{\mu_1, \ldots, \mu_r} \mathcal{R}$ on Grass$(r, E)$ is calculated as follows. We look at the weight $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_{n-r}, \mu_1, \ldots, \mu_r)$ and add to it $\rho = (n, n-1, \ldots, 1)$. Then one of two mutually exclusive cases occurs.

1. The resulting sequence $(\lambda, \mu) + \rho = (\lambda_1 + n, \ldots, \lambda_{n-r} + r, \mu_1 + r, \ldots, \mu_r + 1)$ has repetitions. In such case $H^i(\text{Grass}(r, E), S_{\lambda} \mathcal{Q} \otimes S_{\mu} \mathcal{R}) = 0$ for all $i \geq 0$.

2. The sequence $(\lambda, \mu) + \rho$ has no repetitions. Then there is a unique permutation $w \in \Sigma_n$ that makes this sequence decreasing. The sequence $\nu = w((\lambda, \mu) + \rho) - \rho$ is a non-increasing sequence. Then the only non-zero cohomology group of the sheaf $S_{\lambda} \mathcal{Q} \otimes S_{\mu} \mathcal{R}$ is the group $H^1$ where $l = l(w)$ is the length of $w$. We have $H^1(\text{Grass}(r, E), S_{\lambda} \mathcal{Q} \otimes S_{\mu} \mathcal{R}) = S_{\nu}E$.

Here $S_{\nu}E$ denotes the highest weight representation $S_{\nu}E$ of $GL(E)$ corresponding to the highest weight $\nu$ (so-called Schur module).

§8. Singularities, defining ideals and syzygies.

The geometric method of calculating syzygies [W03] is applicable to analyze the singularities and defining ideals of orbit closures. We will look more closely at the three cases $(E_6, \alpha_4)$, $(E_7, \alpha_4)$ and $(E_8, \alpha_4)$.

Let us recall our basic notions. The coordinate ring of the triple tensor product is $A = \text{Sym}(E^* \otimes F^* \otimes G^*) = \mathbb{C}[X_{i,j,k}]$. For an orbit closure $Y = \overline{O(t)}$ we define its coordinate ring

$$\mathbb{C}[Y] = A/J, J = \{f \in A \mid f|_{\overline{O(t)}} = 0\}.$$ 

The ideal $J$ is called the defining ideal of $Y$.

In order to find the data for some orbit closure we need to find a desingularization of each orbit closure and then use the push-down of the Koszul complex to calculate the terms of a minimal free resolution. It is possible in most of the cases (and all cases for triple tensor products).

Before we start talking about orbit closures for triple tensor products let us introduce the concept of a degenerate orbit.
Definition 8.1. The $GL(E) \times GL(F) \times GL(G)$-orbit of a tensor $t \in E \otimes F \otimes G$ is $E$-degenerate (resp. $F$, $G$-degenerate) if there exists a proper subspace $E' \subset E$ (resp. $F' \subset F$, $G' \subset G$) such that $t \in E' \otimes F \otimes G$ (resp. $E \otimes F' \otimes G$, $E \otimes F \otimes G'$).

The point is that for an $E$ (resp. $F$, $G$)-degenerate orbit closure we can get an estimate on the free resolution of the coordinate ring of the orbit closure as follows. Without loss of generality we can assume that $\dim E' = \dim E - 1$.

Assume that we know the terms $F'_\bullet$ of the free resolution of the coordinate ring of the orbit closure $Y' = \overline{O(t)}$ where $O(t)$ is the closure of the orbit of a tensor $t$ treated as an element of $E' \otimes F \otimes G$.

Consider the desingularization of the subspace variety

$$Z = \{(t, R) \in E \otimes F \otimes G \times \text{Grass}(m - 1, E) \mid r \in R \otimes F \otimes G\}$$

The tautological sequence associated to the projective space is

$$0 \to \mathcal{R} \to E \times \text{Grass}(m - 1, E) \to \mathcal{Q} \to 0$$

Working over the sheaf of rings $\mathcal{B} = \text{Sym}(\mathcal{R}^* \otimes \mathcal{F}^* \otimes \mathcal{G}^*)$ we can produce the exact sequence $F'_\bullet(\mathcal{R}, \mathcal{F}, \mathcal{G})$ of $\mathcal{B}$ modules modeled after $F'_\bullet$, whose terms are $\oplus S_\alpha \mathcal{R}^* \otimes S_\beta \mathcal{F}^* \otimes S_\gamma \mathcal{G}^* \otimes \mathcal{B}(-s)$. Taking sections we get an exact complex of $A$-modules supported in the subspace variety $q(Z)$. Each of the modules $H^0(\text{Grass}(m - 1, E), S_\alpha \mathcal{R}^* \otimes S_\beta \mathcal{F}^* \otimes S_\gamma \mathcal{G}^* \otimes \mathcal{B})$ has a resolution with the terms

$$F(\alpha, \beta, \gamma)_\bullet = H^\bullet(\text{Grass}(m - 1, E), \bigwedge \xi)$$

where $\xi = \mathcal{Q}^* \otimes \mathcal{F}^* \otimes \mathcal{G}^*$. We can use the cone construction to find a (non-minimal) resolution of the coordinate ring of $O(t)$ in $E \otimes F \otimes G$.

Example 8.2. Look at the orbit $O_{10}$ in the case $(E_6, \alpha_4)$. It is clearly $G$-degenerate. In fact this is a general $G$-degenerate orbit, so we get its desingularization using the Grassmannian $\text{Grass}(2, G)$ with the tautological sequence

$$0 \to \mathcal{R} \to G \times \text{Grass}(2, G) \to \mathcal{Q} \to 0$$

and the bundles $\eta = E^* \otimes F^* \otimes \mathcal{R}^*$, $\xi = E^* \otimes F^* \otimes \mathcal{Q}^*$. the resolution $F'_\bullet$ has the terms

$$F_0 = A, F_1 = \bigwedge^3 (E^* \otimes F^*) \otimes S_{1,1,1} \mathcal{G}^* \otimes A(-3),$$

$$F_2 = \bigwedge^4 (E^* \otimes F^*) \otimes S_{2,1,1} \mathcal{G}^* \otimes A(-4),$$
\[ F_3 = \bigwedge^5 (E^* \otimes F^*) \otimes S_{3,1,1}G^* \otimes A(-5), \]
\[ F_4 = \bigwedge^6 (E^* \otimes F^*) \otimes S_{4,1,1}G^* \otimes A(-6), \]
and our complex is just the Eagon-Northcott complex associated to the 3×6 minors of a 3×6 matrix we get when we flatten our 3-dimensional matrix.

**Example 8.3.** Consider the orbit \( \mathcal{O}_8 \) in the case \((E_6, \alpha_4)\). This is a \(G\)-degenerate orbit which corresponds to an orbit of codimension one in \( E \otimes F \otimes G' \). The resolution of this orbit closure is given by an invariant of degree 6, i.e. the complex
\[ 0 \to S_{3,3}E^* \otimes S_{2,2,2}F^* \otimes S_{3,3}G^* \otimes A'(-6) \to A'. \]
We construct a complex of sheaves \( F_\bullet \) over \( B = \text{Sym}(E^* \otimes F^* \otimes R^*) \) with
\[ F_1 = S_{3,3}E^* \otimes S_{2,2,2}F^* \otimes S_{3,3}R^* \otimes B, F_0 = B. \]
Bott theorem implies both sheaves have no higher cohomology, so we get the exact sequence
\[ 0 \to H^0(\text{Grass}(2, G), F_1) \to H^0(\text{Grass}(2, G), F_0) \to \mathbb{C}[\mathcal{O}_8]. \]
Now we can find the resolutions of both modules using the geometric construction from the previous example. The resolution of \( H^0(\text{Grass}(2, G), F_0) \) was worked out in the previous example. The resolution of \( H^0(\text{Grass}(2, G), F_1) \) is (use \( V = S_{3,3}E^* \otimes S_{2,2,2}F^* \otimes S_{3,3}R^* \) and \( \xi \) from previous example).
\[ G_0 = S_{3,3}E^* \otimes S_{2,2,2}F^* \otimes S_{3,3,0}G^* \otimes A(-6), \]
\[ G_1 = S_{3,3}E^* \otimes S_{2,2,2}F^* \otimes (E^* \otimes F^*) \otimes S_{3,3,1}G^* \otimes A(-7), \]
\[ G_2 = S_{3,3}E^* \otimes S_{2,2,2}F^* \otimes \bigwedge^2 (E^* \otimes F^*) \otimes S_{3,3,2}G^* \otimes A(-8), \]
\[ G_3 = S_{3,3}E^* \otimes S_{2,2,2}F^* \otimes \bigwedge^3 (E^* \otimes F^*) \otimes S_{3,3,3}G^* \otimes A(-9), \]
\[ G_4 = S_{3,3}E^* \otimes S_{2,2,2}F^* \otimes \bigwedge^6 (E^* \otimes F^*) \otimes S_{4,4,4}G^* \otimes A(-10). \]
The mapping cone
\[ G_\bullet \to F_\bullet \]
gives a (possibly non-minimal) resolution of the coordinate ring \( \mathbb{C}[\mathcal{O}_8] \). In his case the length of our resolution is 5, i.e. equal to codimension of \( \mathcal{O}_8 \) so the orbit closure is Cohen-Macaulay.
Example 8.4. The orbit closure $\overline{O_{16}}$.
This is the hypersurface given by the $2 \times 3 \times 3$ matrices with vanishing hyperdeterminant. Its desingularization lives on $\mathbb{P}(E) \times \mathbb{P}(F) \times \mathbb{P}(G)$. We treat each projective space as the set of 2-subspaces with the tautological subbundles $\mathcal{R}_E$, $\mathcal{R}_F$ and $\mathcal{R}_G$ respectively. The bundle $\xi$ is
\[
\xi = \mathcal{Q}_E^* \otimes \mathcal{Q}_F^* \otimes G^* + E^* \otimes \mathcal{Q}_F^* \otimes \mathcal{Q}_G^* + \mathcal{Q}_E^* \otimes F^* \otimes \mathcal{Q}_G^*.
\]
The complex $\mathcal{F}(16)^\bullet$ is
\[
0 \rightarrow (4, 2; 2, 2, 2, 2, 2) \otimes A(-6) \rightarrow (2, 1; 1, 1, 1, 1) \otimes A(-3) \oplus A.
\]
The determinant of this matrix is the hyperdeterminant $\Delta$ of our matrix which has degree 12.

Example 8.5. Codimension 2 orbit closure $\overline{O_{15}}$. The orbit closure has a geometric description as the set of pencils of $3 \times 3$ matrices whose determinant is a cube of a linear form.
The bundle $\xi$ is the submodule with the weights
\[
(0, 1; 0, 0, 1; 0, 0, 1), (0, 1; 0, 1, 0; 0, 0, 1), (0, 1; 0, 0, 1; 0, 1, 0),
(0, 1; 0, 1, 0; 0, 1, 0), (0, 1; 1, 0, 0; 0, 0, 1), (0, 1; 0, 0, 1; 1, 1, 0),
(1, 0; 0, 1, 0; 0, 0, 1), (1, 0; 0, 1, 0; 0, 0, 1), (1, 0; 0, 0, 1; 0, 1, 0).
\]
This bundle can be thought of as the set of weights in the graphic form as follows.
\[
\begin{array}{cccccc}
XX & XX & X & X & O \\
XX & X & O & X & O & O \\
X & O & O & O & O & O
\end{array}
\]
Here the first matrix represents the weights with $(1, 0)$ on the first coordinate, and the second matrix represents the weights with $(0, 1)$ on the first coordinate. The symbol $X$ denotes the weight in $\eta$, the symbol $O$-the weight in $\xi$. Our incidence space $Z(15)$ has dimension $9 + 3 + 3 + 1 = 16$, so it projects on the orbit of codimension 2.

The calculation of cohomology reveals that we get a complex $\mathcal{F}(15)^\bullet$,
\[
0 \rightarrow (5, 4; 3, 3, 3; 3, 3, 3) \otimes A(-9) \rightarrow (4, 2; 2, 2, 2; 2, 2, 2) \otimes A(-6) \rightarrow A.
\]
This complex can be obtained also from the hyperdeterminant complex by looking at the kernel of the transpose of the cubic part of the complex $\mathcal{F}(16)^\bullet$. We see the complex is determinantal so it must give a resolution of the reduced ideal. This is
in fact a proof that the incidence space $Z(15)$ is a desingularization. Dividing by the regular sequence of 16 generic linear forms gives a resolution of the ring with Hilbert function

$$\frac{1 - 3t^6 + 2t^9}{(1 - t)^2} = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 4t^6 + 2t^7.$$ 

This means the degree of the orbit closure is 21.

**Example 8.6.** The codimension 3 orbit $O_{14}$. This orbit closure has a nice geometric description. It consists of pencils of $3 \times 3$ matrices of linear forms containing a matrix of rank $\leq 1$.

The bundle $\xi$ is the submodule with the weights

$$(0,1;0,1;0,0,1), (0,1;0,1;0,1;0), (0,1;0,0;0,1,0),$$

$$(0,1;0,0;0,1,0), (0,1;1,0,0,0,1), (0,1;0,1;1,0,0),$$

$$(0,1;1,0,0;0,1,0), (0,1;0,1,0;1,0,0).$$

This bundle can be thought of as the set of weights in the graphic form as follows.

```
X X X  X O O
X X X  O O O
X X X  O O O
```

Here the first matrix represents the weights with $(1,0)$ on the first coordinate, and the second matrix represents the weights with $(0,1)$ on the first coordinate. The symbol $X$ denotes the weight in $\eta$, the symbol $O$-the weight in $\xi$. Our incidence space has dimension $10 + 2 + 2 + 1 = 15$, so it projects on the orbit of codimension 3. It is clearly $O_{14}$.

The calculation of cohomology reveals that we get a complex $F(14)$.  

$$0 \to (5,1;2,2,2,2,2,2) \otimes A(-6) \to (3,1;2,1,1,2,1,1) \otimes A(-4) \to$$

$$\to (2,1;1,1,1,2,1,0) \otimes A(-3) \oplus (2,1;2,1,0;1,1,1) \otimes A(-3) \to$$

$$\to (1,1;1,1,0;1,1,0) \otimes A(-2) \oplus A.$$ 

The terms of the complex $F(14)$ can be understood by looking at the partitions on two last coordinates. They give terms of the Gulliksen-Nagard complex resolving $2 \times 2$ minors of the generic $3 \times 3$ matrix. The corresponding term on the first coordinate comes from the symmetric power on the bundle $Q^{\times}_{E}$. This means the term comes from $H^1$ on the $\mathbb{P}(E)$ coordinate (except the trivial term), hence the shift in homological degree. The orbit is not normal. One still needs to resolve the cokernel module. This is an important problem, as similar situation occurs in several other cases.
Example 8.7. The codimension 4 orbit $O_{13}$. The geometric description of this orbit can be understood from the quiver point of view. Our representation can be thought of as the set of representations of Kronecker quiver of dimension vector $(3,3)$.

$$C^3 \rightarrow C^3.$$  

Our orbit closure then is the set of representations having a subrepresentation of dimension vector $(2,1)$. The desingularization $Z(13)$ lives on $\text{Grass}(1,F) \times \text{Grass}(1,G)$ and the bundle $\xi$ is

$$\xi = E^* \otimes Q_F^* \otimes Q_G^*.$$  

Graphically we have

```
X X X X X X  
X O O X O O  
X O O X O O  
```

Our incidence space has dimension $10 + 2 + 2 = 14$, so it projects on the orbit of codimension 4. In this case one can see directly that $Z(13)$ is a desingularization of $\overline{O_{13}}$.

The calculation of cohomology reveals that we get a complex $F(13)$.

$$0 \rightarrow (4,4; 3,3,2; 3,3,2) \rightarrow (4,3; 3,2,2; 3,2,2) \rightarrow$$

$$\rightarrow (2,2; 1,1; 2,1,1) \oplus (3,1; 2,1,1; 2,1,1) \oplus (3,2; 2,2,2; 2,2,2) \rightarrow$$

$$\rightarrow (2,1; 1,1; 1,1; 2,1,0) \oplus (2,1; 1,1; 1,1; 2,1,0) \oplus (2,1; 2,1,0; 1,1,1) \oplus (3,0; 1,1,1; 1,1,1)$$

$$\rightarrow (1,1; 1,0; 1,1,0) \oplus (0,0; 0,0; 0,0; 0,0).$$

The orbit closure is obviously not normal. The complex $F(13)$ gives a minimal resolution of the normalization $N(O_{13})$. We have an exact sequence

$$0 \rightarrow \mathbb{C}[O_{13}] \rightarrow \mathbb{C}[N(O_{13})] \rightarrow C(13) \rightarrow 0.$$  

The complex $F(13)$ reveals that the $A$-module $C(13)$ has the presentation

$$(2,1; 1,1; 1,1; 1,1) \oplus (2,1; 1,1; 1,1; 2,1,0) \oplus (2,1; 2,1,0; 1,1,1) \rightarrow (1,1; 1,0; 1,1,0).$$

Indeed, the representation $(3,0; 1,1,1; 1,1,1)$ can map only into the trivial term in the complex $F(13)$. 
We can look for the module with the above presentation among twisted modules supported in smaller orbits. It turns out the right choice is the orbit closure $\overline{O_7}$. Its bundle $\xi$ has a diagram

\[
\begin{array}{cccc}
  X & O & X & O \\
  X & O & X & O \\
  O & O & O & O \\
\end{array}
\]

This bundle lives on the space Grass$(2, F) \times$ Grass$(2, G)$. Consider the twisted complex $F(\Lambda^2 E^* \otimes \Lambda^2 R^*_F \otimes \Lambda^2 R^*_{G})(7)$, Its terms are:

\[
0 \rightarrow (6, 6; 4, 4, 4, 4, 4, 4) \rightarrow (5, 4; 3, 3, 3, 4, 4, 4) \oplus (5, 4; 3, 3, 2, 3, 3, 3) \oplus (6, 3; 3, 3, 3, 3, 3) \rightarrow (4, 4; 3, 3, 2, 3, 3, 2) \oplus (4, 4; 3, 3, 2, 3, 3, 2) \oplus (4, 4; 3, 3, 2, 3, 3, 2) \oplus (5, 3; 3, 3, 2, 3, 3, 2) \rightarrow (4, 3, 3, 2, 3, 3, 1) \oplus (4, 3, 3, 2, 3, 3, 1) \oplus (5, 1; 2, 2, 2, 2, 2, 2) \rightarrow (2, 2; 2, 1, 1; 2, 1, 1) \oplus (3, 3; 2, 2, 2, 3, 3, 0) \oplus (3, 3; 3, 3, 0; 2, 2, 2) \rightarrow (2, 1; 1, 1, 1; 1, 1, 1) \oplus (2, 1; 1, 1, 1; 2, 1, 0) \oplus (2, 1; 2, 1, 0; 1, 1, 1) \rightarrow (1, 1; 1, 1, 0; 1, 1, 0).
\]

The resolution of the $A$-module $C[\overline{O_{13}}]$ can be constructed as a mapping cone of the map

\[
F(13) \rightarrow F(\Lambda^2 E \otimes \Lambda^2 Q_F \otimes \Lambda^2 Q_H)(7)
\]

covering the natural epimorphism of $A$-modules. The mapping cone is not a minimal resolution but the repeating representations might cancel out. Let us see that the pairs of representations $(2, 2; 2, 1, 1; 2, 1, 1)$ and $(3, 1; 2, 1, 1; 2, 1, 1)$ cancel out. From this we can deduce that the defining ideal of $\overline{O_{13}}$ is generated by the representations $(3, 0; 1, 1, 1; 1, 1, 1)$ in degree 3 and the representations $(3, 3; 2, 2, 2, 3, 3, 0)$ and $(3, 3; 3, 3, 0; 2, 2, 2)$ in degree 6.

Indeed, if $(3, 1; 2, 1, 1; 2, 1, 1)$ would not cancel out, it would contribute to the minimal generators of the defining ideal. However it occurs once in $S_4(E^* \otimes F^* \otimes G^*)$ so that representation is already in the ideal generated by $(3, 0; 1, 1, 1; 1, 1, 1)$. Regarding representation $(2, 2; 2, 1, 1; 2, 1, 1)$, if it would occur in the defining ideal of $\overline{O_{13}}$ the analysis of the next section will show that then the defining ideal of $\overline{O_{13}}$ would contain the defining ideal of $\overline{O_{12}}$. But this is impossible since both orbit closures have the same dimension. The homological dimension of $\mathbb{C}[\overline{O_{13}}]$ as an $A$-module equals 5 because the top of the resolution of $C(13)$ does not cancel out. So this coordinate ring is not Cohen-Macaulay. Also the non-normality locus of $\overline{O_{13}}$ equals to $\overline{O_7}$.
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[LM] Landsberg, J., Manivel, L.


DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY
360 HUNTINGTON AVENUE, BOSTON, MA 02115, USA
E-mail address: jweyman@neu.edu