ON THE RANK OF A SYMMETRIC FORM

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Abstract. We give a lower bound for the degree of a finite apolar subscheme of a symmetric form \( F \), in terms of the degrees of the generators of the annihilator ideal \( F^\perp \). In the special case, when \( F \) is a monomial \( x_0^{d_0} \cdot x_2^{d_2} \cdots x_n^{d_n} \) with \( d_0 \leq d_1 \leq \ldots \leq d_{n-1} \leq d_n \), we deduce that the minimal length of an apolar subscheme of \( F \) is \( (d_0 + 1) \cdot \ldots \cdot (d_n - 1 + 1) \), and if \( d_0 = \ldots = d_n \), then this minimal length coincides with the rank of \( F \).

Let \( F \in T = K[x_0, \ldots, x_n] \) be a homogeneous form and let \( S = K[y_0, \ldots, y_n] \) be the ring of commuting differential operators acting on \( T \). The action is called apolarity, and defines \( S \) as a natural coordinate ring on the projective space \( \mathbb{P}(T_1) \) of 1-dimensional subspaces of \( T_1 \). The annihilator of \( F \) is an ideal \( F^\perp \subset S \). A finite subscheme \( \Gamma \subset \mathbb{P}(T_1) \) is apolar to \( F \) if the homogeneous ideal \( I_{\Gamma} \subset S \) is contained in \( F^\perp \).

We define the cactus rank \( \text{cr}(F) \) as

\[
\text{cr}(F) = \min\{\deg \Gamma | \Gamma \subset \mathbb{P}(T_1), \dim \Gamma = 0, I_{\Gamma} \subset F^\perp\},
\]

the smoothable rank \( \text{sr}(F) \) as

\[
\text{sr}(F) = \min\{\deg \Gamma | \Gamma \subset \mathbb{P}(T_1) \text{ smoothable}, \dim \Gamma = 0, I_{\Gamma} \subset F^\perp\}
\]

and the rank \( r(F) \) as

\[
r(F) = \min\{\deg \Gamma | \Gamma \subset \mathbb{P}(T_1) \text{ smooth}, \dim \Gamma = 0, I_{\Gamma} \subset F^\perp\}.
\]

Clearly \( \text{cr}(F) \leq \text{sr}(F) \leq r(F) \). We shall give lower bounds for these ranks in terms of the generators of the ideal \( F^\perp \). The related notion of border rank, \( \text{br}(F) \), is defined as the minimal \( k \) such that \( [F] \) lies in the Zariski closure of the set of forms of rank \( k \) in \( \mathbb{P}(T_{\text{deg}F}) \). In general \( \text{br}(F) \leq \text{sr}(F) \), and strict inequality occurs, so our lower bounds for \( \text{sr}(F) \) do not apply unconditionally to \( \text{br}(F) \). Notice also that cactus rank coincides with the notion of scheme length as defined by Iarrobino [Iarrobino 1995, Definition 4D]. Applications of these notions of rank to powersum decompositions of symmetric forms and to equations of secant varieties, see [Ranestad, Schreyer 2000], [Landsberg, Teitler 2010] and [Buczynska, Buczyński 2011], the latter inspired our use of the name cactus rank.

We define the degree of \( F^\perp \) to be the length of the quotient algebra \( S_F = S/F^\perp \).

**Proposition 1.** If the ideal of \( F^\perp \) is generated in degree \( d \) and \( \Gamma \subset \mathbb{P}(T_1) \) is a finite apolar subscheme to \( F \), then

\[
\deg \Gamma \geq \frac{1}{d} \deg F^\perp.
\]
Proof. Taking cones, we may assume that $F^\perp$ and $I_Y$ define subschemes $X$ and $Y$ of pure dimension $r$ and $r + 1$ in $\mathbb{P}^N$. Furthermore $\deg Y = \deg \Gamma$ and $\deg X = \deg F^\perp$. The apolarity condition says that $I_X \supset I_Y$, i.e. that $X \subset Y$ as schemes. Now, take an element $g$ in $I_X$ that does not contain any component of $Y$. Then the hypersurface $G = \{g = 0\}$ has proper intersection with $Y$ and contains $X$. Therefore, by Bezout,
\[ \deg G \cdot \deg Y \geq \deg X. \]
The proposition follows by taking $g$ of degree $d$. \hfill \Box

Corollary 1. If the ideal of $F^\perp$ is generated in degree $d$, then the cactus rank
\[ \text{cr}(F) \geq \frac{1}{d} \deg F^\perp. \]

Corollary 2. If $F$ is a monomial, $F = x_0^{d_0} \cdot x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$ with $d_0 \leq d_1 \leq \ldots \leq d_n$, then the cactus rank and the smoothable rank coincide and equals
\[ \text{cr}(F) = \text{sr}(F) = (d_0 + 1) \cdot (d_1 + 1) \cdot \ldots \cdot (d_n - 1 + 1). \]
If furthermore $d_n = d_0 = d$, i.e. $F = (x_0 \cdot x_1 \cdot \ldots \cdot x_n)^d$, then $r(F) = cr(F) = sr(F) = (d+1)^n$.

Proof. When $F = x_0^{d_0} \cdot x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$, then $F^\perp$ is the complete intersection generated by the forms
\[ y_0^{d_0+1}, y_1^{d_1+1}, \ldots, y_n^{d_n+1}. \]
So it is generated in degree $d_n + 1$, while $F^\perp$ has degree
\[ (d_0 + 1) \cdot (d_1 + 1) \cdot \ldots \cdot (d_n + 1). \]
The formula for the cactus rank follows, since the first $n$ generators define a finite apolar subscheme of degree $(d_0 + 1) \cdot \ldots \cdot (d_n - 1 + 1)$. Now, any complete intersection is smoothable, so the smoothable rank equals the cactus rank for $F$. If $d_0 = d_n = d$, then the forms of degree $d + 1$ in $F^\perp$ has no basepoints so, by Bertini, $n$ general forms in $F^\perp$ of degree $(d + 1)$ define a smooth finite subscheme of degree $(d + 1)^n$ in $\mathbb{P}(T_1)$.
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References


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