SYZYGIES OF GRADED MODULES AND GRADED BETTI NUMBERS

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Abstract. This is a set of lecture notes used for a course in seven lectures on *Syzygies of graded modules and graded Betti numbers* given at the Summer School at the *Sophus Lie Conference Center* in Nordfjordeid, June 15-19, 2009.

The aim is to present the new development in the theory of graded Betti numbers that has come out from the work by David Eisenbud and Frank-Olaf Schreyer [8, 9] on a set of conjectures made by myself and Jonas Söderberg in 2006 [3]. The basic idea is to look at the rational convex cone spanned by all Betti diagrams and in this way classify the possible graded Betti numbers of modules up to scalars. This was indeed acheived when Eisenbud and Schreyer proved our conjectures, but even better, they discovered an analogous dual picture in terms of cohomology tables of vector bundles.

Disclaimer: These notes have been written between the lectures and may contain lots of misprints and inaccurate statements. For the full story, please look at the original articles and books mentioned in the bibliography, which is by the way far from complete.

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Contents

1. Minimal free resolutions and Betti numbers of graded modules

This first lecture gives some background on minimal free resolutions of graded modules. A very good reference for this material is the book Cohen-Macaulay rings by Bruns and Herzog [4].

1.1. **Graded Modules over the polynomial ring and Hilbert functions.** Let S be a polynomial ring $S = k[x_1, x_2, \ldots, x_n]$ over a field k. We say that S has the standard grading if $\deg x_i = 1$, for $i = 1, 2, \ldots, n$, and we can write

$$S = \bigoplus_{d\in\mathbb{N}} S_d = \bigoplus_{d\in\mathbb{N}} \mathrm{Sym}^d(V)$$

where $V = S_1$ is the k-linear space spanned by the variables.

A graded S-module is now a module $M = \bigoplus_{d\in\mathbb{Z}} M_d$ such that the module structure maps satisfy

$$S_d \times M_e \longrightarrow M_{d+e}, \qquad \forall d \in \mathbb{N}, \forall e \in \mathbb{Z}.$$

Definition 1.1 (Hilbert function/series)**.** Let M be a finitely generated graded S-module. The *Hilbert function* of M is

$$H(M) = (\ldots, h_{-2}, h_{-1}, h_0, h_1, h_2, \ldots), \qquad h_d = \dim_k M_d, \quad d \in \mathbb{Z}.$$

The *Hilbert series* of M is the generating function

$$H(M, t) = \sum_{d\in\mathbb{Z}} h_d t^d.$$

Example 1.1. The Hilbert series of the polynomial ring itself is given by

$$H(S, t) = \sum_{d\in\mathbb{N}} \binom{n-1+d}{d} t^d = \frac{1}{(1-t)^n}.$$

An easy way to see this is by taking the multigraded Hilbert series, which is a sum over all monomials in the variables $t_1, t_2, \ldots, t_n$, since $H(S)_\alpha = 1$ for all $\alpha \in \mathbb{N}^n$. On the other hand, this sum factorizes into

$$(1 + t_1 + t_2^2 + \ldots)(1 + t_2 + t_2^2 + \ldots)\cdots(1 + t_n + t_n^2 + \ldots) = \frac{1}{(1-t_1)(1-t_2)\cdots(1-t_n)}$$

and specializing $t_1 = t_2 = \cdots = t_n = t$, we get the standard graded Hilbert series of S.

1.2. **Presentations, syzygies and free resolutions.** By definition, an S-module M is finitely generated if there is a finitely generated free S-module, F, and a surjective homomorphism

$$F \longrightarrow M \longrightarrow 0.$$

The kernel of this homomorphism is again a finitely generated module, N, and we have a short exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

and a finite presentation

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where F_0 maps to the *generators* of M, and F_0 maps to the *relations* between the generators.

Such a relation is called a *syzygy*, i.e., if $m_1, m_2, \ldots, m_\ell$ are generators of M, a syzygy is a relation

$$a_1 m_1 + a_2 m_2 + \cdots + a_\ell m_\ell = 0, \qquad a_1, a_2, \ldots, a_\ell \in S.$$

Example 1.2. Let $I = (x^2, xy, y^2) \subseteq S = k[x, y]$. Then I is an S-module generated by the $m_1 = x^2, m_2 = xy, m_3 = y^2$ and we have relations

$$y \cdot x^2 - x \cdot xy, \qquad y \cdot xy - x \cdot y^2.$$

We write these relations as columns of a matrix

$$\begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}$$

and we note that

$$\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix} \begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix} = 0$$

In fact, all the syzygies of (x^2, xy, y^2) can be expressed in terms of these two basic syzygies in the sense that we can write any syzygy as

$$f(x,y)y \cdot x^2 + (g(x,y)y - f(x,y)x) \cdot xy + (-g(x,y)x) \cdot y^2 = 0$$

for some polynomials $f(x,y)$, $g(x,y)$ and $h(x,y)$. Thus the syzygies of (x^2, xy, y^2) is a free submodule of S^3 generated by the two columns of the matrix above.

Theorem 1.1 (Hilbert's Syzygy Theorem)**.** *Let M be a finitely generated module over the polynomial ring $S = k[x_1, x_2, \ldots, x_n]$. Then there is a free resolution*

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_n \longleftarrow 0,$$

where $F_0, F_1, \ldots, F_n$ are finitely generated free S-modules.

1.3. **Minimal free resolutions and Betti numbers.** In the case of graded modules, we can choose all the generators of various syzygy modules to be homogeneous and we can define the generators of the free modules in such a way that all the maps are of degree zero.

Furthermore, if we in each step choose a minimal generating set for the syzygy modules, we get a minimal free resolution of M. In this way, we can write

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$$

for some natural numbers $\beta_{i,j}$. These numbers form a set of invariants of M as a graded S-module and we can also obtain them as the homological invariants

$$\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^S(M, k)_j$$

where k is seen as the graded S-module $k \cong S/\mathfrak{m} = S/(x_1, x_2, \ldots, x_n)$. The graded Betti numbers can be arranged into a *Betti diagram*

$$\begin{array}{r|ccccc} & \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\ \hline \vdots & \vdots & \vdots & \vdots & & \vdots \\ -1: & \beta_{0,-1} & \beta_{1,0} & \beta_{2,1} & \cdots & \beta_{n,n-1} \\ 0: & \beta_{0,0} & \beta_{1,2} & \beta_{2,2} & \cdots & \beta_{n,n} \\ 1: & \beta_{0,1} & \beta_{1,3} & \beta_{2,3} & \cdots & \beta_{n,n+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \end{array}$$

Example 1.3. The Betti diagram of the module $M = S/(x^2, xy, y^2)$ is given by

$$\begin{array}{r|ccc} & 1 & 3 & 2 \\ \hline 0: & 1 & . & . \\ 1: & . & 3 & 2 \end{array}$$

Proposition 1.1. *The Hilbert series and the graded Betti numbers are related by*

$$H(M,t) = \frac{1}{(1-t)^n} \sum_{i=0}^{n} \sum_{j\in\mathbb{Z}} (-1)^i \beta_{i,j} t^j.$$

The $\mathrm{Tor}^S(M,N)$-modules are in general defined by a projective resolution of M or N. Since tensoring by M is in general not exact as a functor on the category of graded S-modules, we can compute the homology of the complex

$$\mathbf{P}_\bullet \otimes N$$

if $\mathbf{P}_\bullet$ is a projective resolution of M as a graded S-module, or by the homology of

$$M \otimes \mathbf{Q}_\bullet$$

where $\mathbf{Q}_\bullet$ is a projective resolution of N.

This means that the Betti numbers, $\beta_{i,j}$ could also be computed via a minimal free resolution of k as a graded k-module. This is obtained via the Koszul complex:

$$0 \longleftarrow k \longleftarrow S \longleftarrow \bigwedge^1 S^n(-1) \longleftarrow \bigwedge^2 S^n(-2) \longleftarrow \cdots \longleftarrow \bigwedge^n S^n(-n) \longleftarrow 0.$$

where $d_i : \bigwedge^i S^n(-i) \longrightarrow \bigwedge^{i-1} S^n(-i+1)$ is the S-linear map given by

$$d_i(y_1 \wedge y_2 \wedge \cdots \wedge y_i) = \sum_{j=1}^{i} (-1)^{j+1} y_j y_1 \wedge y_2 \wedge \cdots \check{y}_j \cdots \wedge y_i$$

where $y_1, y_2, \ldots, y_i$ are elements of degree 0 in S^n.

1.4. **Cohen-Macaulay modules.**

Definition 1.2 (projective dimension)**.** When M is an S-module the *projective dimension* of M, $\operatorname{pd} M$, is the length of a minimal free resolution of M.

Definition 1.3 (M-regular sequence, depth)**.** The *depth*, $\operatorname{depth} M$, of an S-module M is the maximal length of an *M-regular sequence* in S, i.e., a sequence $a_1, a_2, \ldots, a_{\operatorname{depth} M} \in S$ such that a_i is a non-zero divisor on $M/(a_1, a_2, \ldots, a_{i-1})M$ and $M/(a_1, a_2, \ldots, a_i)M \neq 0$, for $i = 1, 2, \ldots, \operatorname{depth} M$.

The depth and the projective dimension is related by the Auslander-Buchsbaum thereorem which is this setting reads as follows:

Theorem 1.2 (Auslander-Buchsbaum)**.** *If M is a module over the polynomial ring S, we have that*

$$\dim S = \operatorname{depth}_S M + \operatorname{pd}_S M.$$

Definition 1.4 (Socle)**.** The socle of an S-module is the submodule

$$\operatorname{Soc}(M) = \{a \in M | x_i a = 0, i = 1, 2, \ldots, n\} = \operatorname{ann}_M(\mathfrak{m})$$

where $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ is the homogeneous maximal ideal of S.

We can use the Koszul complex to see that there is a relation between the socle of M and the nth syzygy module, F_n. In fact, when tensoring the koszul complex by M, we get that the last map

$$M(-n) \longrightarrow M^n(-n+1)$$

is given by $a \mapsto (x_1 a, x_2 a, \cdots, x_n a)$ and the kernel is therefore given by the socle of M, shifted n degrees. Hence we have

$$\operatorname{Tor}_n^S(M, k) \cong \operatorname{Soc}(M)(-n)$$

as k-linear spaces.

Having this, we can interpret the Auslander-Buchsbaum theorem in the case of a polynomial ring as stating that M has depth zero if and only if the socle of M is non-trivial, i.e.,

$$\operatorname{depth} M = 0 \qquad \Longleftrightarrow \qquad \operatorname{Soc}(M) \neq 0.$$

Definition 1.5 (Cohen-Macaulay)**.** M is *Cohen-Macaulay* if $\operatorname{depth} M = \dim M$.

Because of the Auslander-Buchsbaum theorem, we can rephrase this condition as $\operatorname{codim} M = \operatorname{pd} M$, where $\operatorname{codim} M = \dim S - \dim M$.

1.5. **Structure theorems.** There are a few structure theorems for resolutions of algebras low codimension. The first one is that Cohen-Macaulay algebras of codimension two have special form of resolutions:

Theorem 1.3 (Hilbert-Burch)**.** *If $A = S/I$ is a Cohen-Macaulay algebra of codimension two, the minimal free resolution can be written as*

$$0 \longleftarrow A \longleftarrow S \longleftarrow S^{m+1} \xleftarrow{\Phi_2} S^m \longleftarrow 0$$

and the ideal I is generated by the maximal minors of Φ_2.

Example 1.4. The matrix

$$\Phi_2 = \begin{pmatrix} x & y \\ y & z \\ z & w \end{pmatrix}$$

gives the ideal $I = (xz - y^2, xw - zy, yw - z^2)$.

Definition 1.6 (Gorenstein)**.** An algebra $A = S/I$ is *Gorenstein* if it is Cohen-Macaulay as an S-module and the Cohen-Macaulay type is one, i.e., the last non-zero syzygy module is cyclic.

Theorem 1.4 (Buchsbaum-Eisenbud [5])**.** *If $A = S/I$ is a Gorenstein algebra of codimension three, the minimal free resolution can be written as*

$$0 \longleftarrow A \longleftarrow S \longleftarrow S^{2m+1} \xleftarrow{\Phi_2} S^{2m+1} \longleftarrow S \longleftarrow 0$$

where the matrix for Φ_2 is skew-symmetric and the ideal I is generated by the maximal Pfaffians of Φ_2.

Example 1.5.

$$\Phi_2 = \begin{pmatrix} 0 & x & y & z & 0 \\ -x & 0 & z & 0 & x \\ -y & -z & 0 & x & 0 \\ -z & 0 & -x & 0 & y \\ 0 & -x & 0 & -y & 0 \end{pmatrix}$$

gives $I = (x^2 + z^2, -xy, xy - xz, y^2, x^2 + yz)$.

2. Maximal Betti numbers and cancellations

2.1. **Macaulay's theorem.**

Theorem 2.1 (Macaulay [16])**.** *If I is a homogeneous ideal in $S = k[x_1, x_2, \ldots, x_n]$, there there is a monomial ideal I' with $H(S/I) = H(S/I')$. Furthermore, if $H(S/I) = (h_0, h_1, h_2, \ldots)$, we have*

$$h_{d+1} \le h_d^{\langle d \rangle} = ((h_d)_{(d)})^{+1}_{+1}, \qquad d \ge 0.$$

The idea behind this is that the Hilbert function of any ideal grows faster thatn the ideal generated by the first monomials in the lexicographic order. In the (degree) lexicographic order, we first compare the exponents of x_1 and if they are equal we continue to x_2, and so on. This means that we can phrase it as

$$x_1^{a_1} x_1^{a_2} \cdots x_1^{a_n} <_{lex} x_1^{b_1} x_1^{b_2} \cdots x_1^{b_n} \iff \exists j : a_i = b_i, i < j, a_j < b_j$$

if the monomials have the same degree.

The Macaulay function is defined using *binomial expansion*. For an integer d we can write any natural number uniquely as

$$h = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}, \qquad k_d > k_{d-1} > \cdots > k_1 \geq 0$$

Now,

$$h^{\langle d \rangle} = (h_{(d)})^{+1}_{+1} = h = \binom{k_d + 1}{d+1} + \binom{k_{d-1}+1}{d} + \cdots + \binom{k_1+1}{2}.$$

Definition 2.1 (lex-segment ideal)**.** A monomial ideal I is a *lex-segment ideal* if $m \in I_d$, $m' \in S_d$ and $m <_{lex} m'$ implies that $m' \in I_d$. That is I_d is spanned by the lex-largest monomials for each degree d.

A lex-segment in degree d will generate a lex-segment in degree $d+1$ and the Macaulay function comes from finding the location in the lexicographic order of the first monomial which is not in the ideal. If I is a lex-segment ideal and $H(S/I)_d = h$, the binomials in the expansion of h correspond to lengths of segment among the monomials outside I.

Example 2.1. Consider the 4-binomial expansion of 20. We have

$$20 = \binom{6}{4} + \binom{4}{3} + \binom{1}{1}$$

which corresponds to the monomials

$$(x_2, x_3, x_4)^4 <_{lex} x_1(x_3, x_4)^3 <_{lex} x_1x_2(x_4)^2$$

which are the 20 smallest monomials of degree four in $k[x_1, x_2, x_3, x_4]$. When looking in the next degree, we get

$$(x_2, x_3, x_4)^5 <_{lex} x_1(x_3, x_4)^4 <_{lex} x_1x_2(x_4)^3$$

as the smallest monomials of degree five.

2.2. **Lexicographic modules and the Bigatti-Hulett-Pardue theorem.** When going from monomial ideals to modules, we need to incorporate the module generators in the lexicographic order. We do so by giving the module generators higher priority than the variables. This means that if $e_1, e_2, \ldots, e_r$ are generators of $F = S^r$, we say that

$$e_1\mathcal{M}_d >_{lex} e_2\mathcal{M}_d >_{lex} \cdots >_{lex} e_r\mathcal{M}_d,$$

where $\mathcal{M}_d$ is the set of monomials of degree d in S.

Theorem 2.2 (Bigatti [1], Hulett [14], Pardue [17])**.** *Among all graded modules, M, with the same Hilbert function, generated in degree zero, the lex-segment module has the largest Betti numbers, i.e.,*

$$\beta_{i,j}(M) \leq \beta_{i,j}(M_{lex}), \qquad \forall i, j,$$

where M_{lex} is the lex-segment module with the same Hilbert function as M.

2.3. **Eliahou-Kervaire resolutions.** The upper bound given by the Bigatti-Hulett-Pardue theorem can be obtained using the Eliahou-Kervaire theorem.

Definition 2.2 (strongly stable monomial idels)**.** We say that a monomial ideal $I \subseteq S$ is *strongly stable* if

$$m \in I \text{ and } x_i|m \Longrightarrow \frac{x_j}{x_i}m \in I, \forall j < i.$$

Lex-segment ideals are examples of strongly stable monomial ideals. Eliahou-Kervaire determined the graded Betti numbers of any strongly stable monomial ideal.

Theorem 2.3 (Eliahou-Kervaire [10])**.** *Let I be a strongly stable monomial ideal with minimal generators $\mathcal{G} = \bigcup_{d\in\mathcal{N}} \mathcal{G}_d$. Then*

$$\beta_{i,i+d-1}(S/I) = \sum_{m\in\mathcal{G}_d} \binom{v(m)}{i-1}, \qquad \textit{for } i = 1, 2, \ldots, n \textit{ and } d \in \mathbb{N},$$

where $v(d) = \max\{j|x_j$ divides $m\}$.

Example 2.2. Let $H = (1, 3, 3, 1)$. Then the lexicographic ideal corresponding to H is generated by x_1^2, x_1x_2, x_1x_3 in degree two, $x_2^3, x_2^2x_3, x_2x_3^2$ in degree three and x_3^4 in degree four. Hence the maximal Betti numbers for H is given by

	1	7	10	4
0 :	1	.	.	.
1 :	.	3	3	1
2 :	.	3	5	2
3 :	.	1	2	1

where the first row can be seen as

$$(3, 3, 1) = (1, 0, 0) + (1, 1, 0) + (1, 2, 1)$$

the second as

$$(3, 5, 2) = (1, 1, 0) + (1, 2, 1) + (1, 2, 1).$$

and the last as

$$(1, 2, 1) = (1, 2, 1).$$

2.4. **Generic initial ideals and adjacent cancellations.** In characteristic zero, strongly stable monomial ideals are the same as Borel fixed ideals, i.e., monomial ideals that are stable under the action of the Borel group of upper triangular changes of coordinates.

Definition 2.3 (initial ideal, generic initial ideal)**.** Given any ideal I, and any monomial order, $<$, satisfying $m' < m' \Rightarrow m \cdot m' < m \cdot m''$, for all monomials m, m', m'', we let $in_<(I)$ denote the ideal generated by all the initial monomials of forms in I. This ideal is called the *initial ideal* of I with respect to $<$.

Furthermore, let $gin_<(I)$ denote the initial ideal of $\Phi(I)$, where $\Phi : S \longrightarrow S$ is a generic change of coordinates on S. This is the *generic initial ideal* of I with respect to $<$.

When $<$ is the reverse lexicographic order, $gin_{revlex}(I)$ is a strongly stable ideal. We can also extend these notions to monomial submodules of free modules, and we have the following theorem.

Theorem 2.4 (Peeva)**.** *The Betti numbers of I are bounded by the Betti numbers of the generic initial ideal, i.e.,*

$$\beta_{i,j}(S/I) \leq \beta_{i,j}(S/gin(I)), \qquad \text{for } i = 0, 1, \dots, n \text{ and } j \in \mathbb{Z}.$$

Furthermore, they differ by a sequence of adjacent cancellations.

Definition 2.4 (adjacent cancellation)**.** In a Betti diagram, an *adjacent cancellation* means lowering two adjacent Betti numbers by one.

Example 2.3. Let $I = (x_1^2, x_2^2, x_3^2) \subseteq S = k[x_1, x_2, x_3]$. The generic initial ideal is $gin(I) = (x_1^2, x_1x_2, x_2^2, x_1x_3^2, x_2x_3^2, x_3^4)$ which is a strongly stable ideal with Betti numbers given by

	1	6	8	3
0 :	1	.	.	.
1 :	.	3	2	.
2 :	.	2	4	2
3 :	.	1	2	1

and since I is a complete intersection ideal, we can get the Betti numbers of I by

	1	3	3	1
0 :	1	.	.	.
1 :	.	3	.	.
2 :	.	.	3	.
3 :	.	.	.	1

and the difference between those are given by five consequtive cancellations.

2.5. **Exercises.**

Exercise 2.1. Fix the Hilbert function $H = (1, 3, 3, 1)$. Find as many different Betti diagrams as possible for this Hilbert function given that $\beta_0 = 1$.

Exercise 2.2. Compute the Betti numbers of $k[x, y]/(x^2, xy, y^2)$ by means of the Koszul complex.

3. Betti numbers up to scaling and the cone of Betti diagrams

We will refer to the set of graded Betti numbers of a module as a Betti diagram, which can be thought of as an element in $\bigoplus_{j\in\mathbb{Z}}\mathbb{N}^{n+1}\subseteq\bigoplus_{j\in\mathbb{Z}}\mathbb{Q}^{n+1}$. If we look at all possible Betti diagrams of finitely generated graded modules we get a semigroup, since

$$\beta_{i,j}(M\oplus N)=\beta_{i,j}(M)+\beta_{i,j}(N),$$

when M and N are graded S-modules. It seems very hard to get a good picture of what this semigroup is like, but we will see that it will be possible to describe the rational convex cone that it spans.

There are two ways of describing a convex rational cone, either by its extremal rays, or by the supporting hyperplanes of its facets. We will start from the point of extremal rays.

3.1. **The Herzog-Kühl equations and pure resolutions.** We can recover the Hilbert series from the Betti numbers by the additivity of the Hilbert series. In fact,

$$H(M,t)=\sum_{\substack{i=0\\ j\in\mathbb{Z}}}^{n}(-1)^i\beta_{i,j}(M)H(S(-j),t)=\frac{1}{(1-t)^n}\sum_{\substack{i=0\\ j\in\mathbb{Z}}}^{n}(-1)^i\beta_{i,j}(M)t^j.$$

If M is Cohen-Macaulay of codimension s, we have that

$$H(M,t)=\frac{\sum_{j\in\mathbb{Z}}h_jt^j}{(1-t)^{n-s}},$$

which means that $(1-t)^s$ divides the polynomial

$$S(M,t)=\sum_{\substack{i=0\\ j\in\mathbb{Z}}}^{s}(-1)^i\beta_{i,j}(M)t^j.$$

This condition gives rise to s linearly independent linear equations known as the *Herzog-Kühl equations* [12]. We can phrase them as

$$S^{(j)}(M,1)=0,\qquad j=0,1,\dots,s-1.$$

or as

$$\sum_{\substack{i=0\\ j\in\mathbb{Z}}}^{s}(-1)^i\beta_{i,j}(M)j^d=0,\qquad d=0,1,2,\dots,s-1.$$

If the resolution of M is *pure*, which means that there is only one non-zero Betti number in each homological degree, we get s linearly independent equations in $s+1$ indeterminates $\beta_0,\beta_1,\dots,\beta_s$, and the solutions are given by

$$\beta_i=\lambda\frac{(-1)^i}{\prod_{j=0}^{s}(d_j-d_i)},\qquad\lambda\in\mathbb{Q}.$$

It is not clear to start with that there will be Cohen-Macaulay modules with pure resolutions for any degree sequence, $d_0,d_1,\dots,d_s$, but this is now a theorem by Eisenbud, Fløystad, Schreyer and Weyman [7].

In any case, we will now look at these diagrams formally and define the diagram

$$\pi(d_0, d_1, \ldots, d_s)$$

to be the one given by the Betti numbers above when $\lambda = 1$.

Example 3.1. When $(d_0, d_1, d_2) = (0, 2, 3)$, we get

$$\pi(0,2,3) = \boxed{\begin{array}{ccc} \frac{1}{2\cdot 3} & - & - \\ - & \frac{1}{2\cdot 1} & \frac{1}{3\cdot 1} \end{array}}$$

This is not a Betti diagram of a module, but $6\pi(0,2,3) = \beta(k[x,y,z]/(x^2,xy,y^2))$ as we have seen before.

3.2. **Multiplicity Conjecture.** In relation to the Multiplicity Conjecture by Herzog, Huneke and Srinivasan [13], it is natural to also consider the normalized diagrams $\bar{\pi}(d_1, d_2, \ldots, d_s)$, when $d_0 = 0$, given by $\lambda = d_1 d_2 \cdots d_s$.

Huneke and Miller [15] had computed $e(S/I)$, when S/I had a pure resolution as

$$e(S/I) = \frac{d_1 d_2 \cdots d_s}{s!}.$$

Recall that the multiplicity can be defined as the $(n-s)!$ times the leading coefficient of the Hilbert polynomial of a graded module of codimension $s < n$. (When $s = n$, we have $e(M) = H(M, 1)$.) This lead to the following conjecture:

Conjecture 3.1 (Herzog, Huneke, Srinivasan [13])**.** *For a graded Cohen-Macaulay algebra S/I*

$$\frac{\underline{d}_1 \underline{d}_2 \cdots \underline{d}_s}{s!} \le e(S/I) \le \frac{\overline{d}_1 \overline{d}_2 \cdots \overline{d}_s}{s!}.$$

where $\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_s$ and $\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_s$ are the lower and upper shifts in the resolution, with equality on either side if and only if $\underline{d}_i = \overline{d}_i$, for $i = 1, 2, \ldots, s$.

3.3. **The simplicial structure.** The cone spanned by the pure diagrams reveals a nice combinatorial structure. In fact, the degree sequences can be partially ordered by

$$(d_0, d_2, \ldots, d_s) > (d_0', d_1', \ldots, d_t') \quad \Longleftrightarrow \quad d_i > d_i', i = 0, 1, 2, \ldots, t, \quad s < t.$$

Proposition 3.1. *The pure diagrams in any maximal chain form a basis for the solutions to the Herzog-Kühl equations.*

A maximal chain corresponds to a numbering of the positions of the Betti diagram which is increasing to the left and downwards.

Example 3.2. If we restrict our attention to Betti diagrams like

$$\begin{array}{ccc} * & * & * \\ * & * & * \end{array}$$

we have the following maximal chains:

$$\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 6 & 5 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 6 & 5 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 5 & 2 & 1 \\ \hline 6 & 4 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 6 & 5 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 6 & 4 & 2 \\ \hline \end{array}$$

where the third corresponds to the chain

$$\boxed{\begin{matrix} * & * & * \\ - & - & - \end{matrix}} < \boxed{\begin{matrix} * & * & - \\ - & - & * \end{matrix}} < \boxed{\begin{matrix} * & - & - \\ - & * & * \end{matrix}} < \boxed{\begin{matrix} * & - \\ - & * \end{matrix}} < \boxed{\begin{matrix} * \\ - \end{matrix}} < \boxed{\begin{matrix} - \\ * \end{matrix}}$$

Proposition 3.2. *The simplicial cones spanned by the chains of pure diagrams form a simplicial fan.*

3.4. **Greedy algorithm.** If the simplicial fan equals the cone spanned by the pure diagrams, we get a greedy algorithm for finding a unique expansion of any element in the cone as a sum of pure diagrams in a chain. The idea is to substract as much as possible of the pure diagram that corresponds to the lowest shifts in the diagram you are given.

Example 3.3. The coordinate ring of two skew lines in projective space has the Betti diagram

$$\begin{matrix} 1 & - & - & - \\ - & 4 & 4 & 1 \end{matrix}$$

and we start by subtracting as much as possible of the smallest diagram

$$\pi(0,2,3,4) = \begin{matrix} \frac{1}{2\cdot 3\cdot 4} & - & - & - \\ - & \frac{1}{2\cdot 1\cdot 2} & \frac{1}{3\cdot 1\cdot 1} & \frac{1}{4\cdot 2\cdot 1} \end{matrix}$$

We can subtract 8 copies of this without getting negative entries and what remains will be

$$\begin{matrix} \frac{2}{3} & - & - \\ - & 2 & \frac{4}{3} \end{matrix}$$

which is four times the pure diagram

$$\pi(0,2,3) = \begin{matrix} \frac{1}{2\cdot 3} & - & - \\ - & \frac{1}{2\cdot 1} & \frac{1}{3\cdot 1} \end{matrix}$$

4. Facet equations and inequalities

The proof by Eisenbud and Schreyer that every Betti diagram of a Cohen-Macaulay module is a sum of pure diagrams uses the other description of the cone by means of the hyperplanes supporting the maximal dimensional faces of the cone, that is the *boundary facets* of the cone. Once they found these inequalities they could recognize a connection to cohomology tables of coherent vector bundles on $\mathbb{P}^{n-1}$ which allowed them to prove the necessary inequalites, but also to prove the corresponding statement for cohomology tables.

We shall start to look at the problem of finding the equations for the boundary facets.

Proposition 4.1. *A boundary facet of the cone corresponds to removing from a maximal chain an element which cannot be replaced by any other element.*

Proposition 4.2. *There are four kinds of boundary facets:*

(1) *Remove* π_1 *in* $\pi_0 < \pi_1 < \pi_2$ *where* π_0 *and* π_2 *differ only in one position.*
(2) *Remove* π_1 *in* $\pi_0 < \pi_1 < \pi_2$ *where* π_0 *and* π_2 *differ in two adjacent positions and the degrees there differ by one.*
(3) *Remove* π_1 *in* $\pi_0 < \pi_1 < \pi_2$ *where* π_0 *and* π_2 *differ in length by two.*

Example 4.1. The first kind correspond to

$$\begin{array}{ccc} * & * & - \\ - & - & - \\ - & - & * \end{array} < \begin{array}{ccc} * & - & - \\ - & * & - \\ - & - & * \end{array} < \begin{array}{ccc} * & - & - \\ - & - & - \\ - & * & * \end{array},$$

the second type to

$$\begin{array}{cc} * & * \\ - & - \end{array} < \begin{array}{cc} * & - \\ - & * \end{array} < \begin{array}{cc} - & - \\ * & * \end{array}$$

and the third kind to

$$\begin{array}{ccc} * & * & - \\ - & - & * \end{array} < \begin{array}{cc} * & * \\ - & - \end{array} < \begin{array}{c} * \\ - \end{array}.$$

4.1. **Facet equations.** We can easily give the equation for a supporting hyperplane of a facet in terms of the basis that corresponds to maximal chain from which we remove one element to get the facet. The equation is simply that the coordinate of the element removed should be zero. Now we would like to express this in the natural basis, given by the entries of the Betti diagram.

We can do this step by step, starting with a maximal chain including the short chain $\pi_0 < \pi_1 < \pi_2$, where we would like to compute the coordinate of π_1. We know that the coordinate will be zero for all the diagrams in the chain starting at π_2. We can then work backwards to solve for the contributions from positions in the Betti diagram below π_2. We know that the π_1-coordinate for π_1 should be one. It is easiest to see what happens by an example.

Example 4.2. Suppose we want to compute the coordinate of $\pi(0,4,6)$ in a chain containing the short chain $\pi(0,4,5) < \pi(0,4,6) < \pi(0,5,6)$. We know that the coefficient will be zero in degrees above or equal to the degrees of $\pi(0,5,6)$.

$$\begin{array}{llll} 0: & 0 & \square & \square \\ 1: & 0 & \square & \square \\ 2: & 0 & \square & \square \\ 3: & 0 & \square & \square \\ 4: & 0 & 0 & 0 \\ 5: & 0 & 0 & 0 \end{array}$$

Since the diagram for $(0,4,6)$ is given by

$$\pi(0,4,6) = \begin{array}{llll} 0: & \frac{1}{4\cdot 6} & - & - \\ 1: & - & - & - \\ 2: & - & - & - \\ 3: & - & \frac{1}{4\cdot 2} & - \\ 4: & - & - & \frac{1}{6\cdot 2} \end{array}$$

we can deduce that the coefficient in position $(1,4)$ should be 8. The next element of the chain is

$$\pi(0,4,5) = \begin{array}{llll} 0: & \frac{1}{4\cdot 5} & - & - \\ 1: & - & - & - \\ 2: & - & - & - \\ 3: & - & \frac{1}{4\cdot 1} & \frac{1}{5\cdot 1} \end{array}$$

and we need the coefficient in position $(2,5)$ to be -10. We have now filled in two positions and have

$$\begin{array}{llll} 0: & 0 & \square & \square \\ 1: & 0 & \square & \square \\ 2: & 0 & \square & \square \\ 3: & 0 & 8 & -10 \\ 4: & 0 & 0 & 0 \\ 5: & 0 & 0 & 0 \end{array}$$

We can continue in the same way, using the chain

$$\pi(0,4,5) > \pi(0,3,5) > \pi(0,3,4) > \pi(0,2,4) > \pi(0,2,3) > \pi(0,1,3) > \pi(0,1,2)$$

and successively solve for one coefficient at a time. We will get

$$\begin{array}{llll} 0: & 0 & 2 & -4 \\ 1: & 0 & 4 & -6 \\ 2: & 0 & 6 & -8 \\ 3: & 0 & 8 & -10 \\ 4: & 0 & 0 & 0 \\ 5: & 0 & 0 & 0 \end{array}$$

We can at this point certainly guess the formula for the entries in this particular case and prove that the coefficient will be $(-1)^{i+1}2j$, in positions (i,j) below $\pi_2 = \pi(0,5,6)$.

In general, we can also guess what the general formula will be and we will be able to prove that it is indeed correct, just using the Herzog-Kühl equations. For example we get the following:

Proposition 4.3. *The coefficient of* $\pi_1 = \pi(d_0, d_1, \dots, d_p)$ *when a Betti diagram* β *in is expanded in a basis containing* $\pi_0 < \pi_1 < \pi_2$ *is given by*

$$\sum_{i=0}^{n} \sum_{d \le d_i(\pi_0)} (-1)^i (d_\ell - d_k) \prod_{\substack{j=0 \\ j \notin \{k,\ell\}}}^{p} (d_j - d)\beta_{i,d},$$

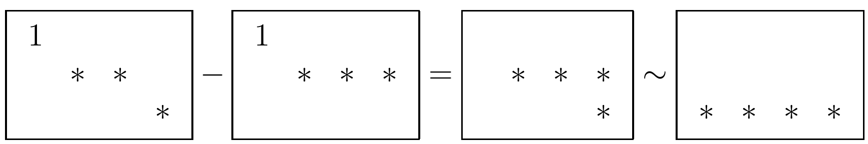
when π_1 *differs from* π_2 *in column* k *and from* π_0 *in column* $\ell \neq k$*.*

Observe that this formula does not depend of the rest of the maximal chain containg the small chain $\pi_0 < \pi_1 < \pi_2$ it is proved by computing it for $\beta = \pi_1$, where the result is one. By construction, it is zero for all pure diagrams $\pi \geq \pi_2$ and by the Herzog-Kühl equations, which are valid for all pure diagrams $\pi \leq \pi_0$, it is zero.

Proposition 4.4. *The Hilbert series is strictly increasing along chains of normalized pure diagrams,*

$$\overline{\pi}(d_1, d_2, \dots, d_p) = d_1 d_2 \cdots d_p \pi(d_0, d_1, \dots, d_p).$$

We can see this from the following picture which computes the difference between two normalized diagrams:



The last two diagrams have the same Hilbert function, and since it satisfies the Herzog-Kühl equation and has one positive entry, it has a positive Hilbert series.

This leads to the following generalization of the Multiplicity conjecture, once we know that any Betti diagram can be written as a sum of pure diagrams:

Theorem 4.1. *If* M *is a graded* S*-module generated in degree zero of codimension* s *and projective dimension* p*, we have that*

$$H(\overline{\pi}(\underline{d}_1, \underline{d}_2, \dots, \underline{d}_p), t) \leq \frac{H(M,t)}{\beta_0(M)} \leq H(\overline{\pi}(\overline{d}_1, \overline{d}_2, \dots, \overline{d}_p), t),$$

where $\underline{d}_1, \underline{d}_2, \dots, \underline{d}_p$ *and* $\overline{d}_1, \overline{d}_2, \dots, \overline{d}_s$ *are the lower and upper shifts in a minimal free resolution of* M*. (The inequalities are termwise.)*

Example 4.3. We expanded the Betti diagram of the coordinate ring of two skew lines in $\mathbb{P}^3$ and the normalized Betti diagrams occuring were

$$\begin{array}{cccc} 1 & - & - & - \\ - & 6 & 8 & 3 \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 & - & - \\ - & 3 & 2 \end{array}$$

and the inequality of Hilbert series says that

$$\frac{1+3t}{(1-t)} \leq \frac{1+2t-t^2}{(1-t)^2} \leq \frac{1+2t}{(1-t)^2}$$

or more explicitly

$$1+4t+4t^2+4t^3+\cdots \leq 1+4t+6t^2+8t^3+\cdots \leq 1+4t+7t^2+10t^3+\cdots$$

5. Resolutions over the exterior algebra and cohomology of coherent sheaves

Resolutions over the exterior algebra shares some properties with resolutions over the polynomial algebra. The resolutions are not finite anymore, but since the exterior algebra E is injective as well as projective as a module over itself, we can extend any exact sequence of free modules into a doubly infinite sequence

$$\cdots \leftarrow F_{-2} \leftarrow F_{-1} \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$$

There is a correspondence between such *Tate resolutions* over the exterior algebra and the cohomology of coherent sheaves on $\mathbb{P}^{n-1}$ via the Bernstein-Gel'fand-Gel'fand (BGG) correspondence, which has been explored by Eisenbud, Fløystad and Schreyer [6]

Example 5.1. We look at the cohomology table of the structure sheaf on $\mathbb{P}^1$, $\mathcal{O}_{\mathbb{P}^1}$:

$$\begin{array}{r|ccccccccc} 1: & \cdots & 3 & 2 & 1 & - & - & - & - & \cdots \\ 0: & \cdots & - & - & - & 1 & 2 & 3 & 4 & \cdots \\ \hline & \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \end{array}$$

This can also be seen as the Betti diagram of the Tate resolution over the exterior algebra with maps

$$\begin{pmatrix} x & 0 \\ y & x \\ 0 & y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x & y \end{pmatrix}, \quad \begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}, \quad \begin{pmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \end{pmatrix}, \ldots$$

In fact, the maps on the lower line corresponds to the multiplication maps

$$S_i \longrightarrow S_{i+1}$$

on the polynomial ring, when using the basis $(x^i, x^{i-1}y, \ldots, y^i)$ for S_i.

We define the cohomology table of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{n-1}$ to be the array

$$\gamma_{i,j}(\mathcal{F}) = \dim_k H^i(\mathbb{P}^{n-1}, \mathcal{F}(j)), \qquad i = 0, 1, \ldots, n-1, j \in \mathbb{Z}.$$

Recall that the coherent sheaf is given by the module $M = \bigoplus_{j \geq j_0} H^0(\mathbb{P}^{n-1}, \mathcal{F}(j))$, for large j_0. If we take the multiplication maps $M_j \longrightarrow M_{j+1}$ and consider them in the exterior algebra, we get the maps in the Tate resolution.

If $\mathcal{F}$ is a vector bundle, the Tate resolution is eventually linear in both directions. For coherent sheaves in general, it is linear to the right, i.e., in the projective direction.

5.1. **Pure Tate resolutions and supernatural cohomology.**

Definition 5.1. A vector bundle supported on a linear subspace of $\mathbb{P}^{n-1}$ is said to have *supernatural cohomology*[1] if the corresponding Tate resolution is a pure resolution.

[1] This notion was introduced by Eisenbud and Schreyer [8] as a strengthening of *natural cohomology*

We can use the analogues of the Herzog-Kühl equations to determine the cohomology tables of supernatural vector bundles, or equivalently, of pure Tate resolutions. If we cut the resolution at any point, and view the rest as the minimal free resolution of the cokernel, M, of the map $F_m \leftarrow F_{m+1}$, we get

$$H(M,t) = (1+t)^n \sum_{i\geq m} \sum_{j\in\mathbb{Z}} (-1)^i \beta_{i,j}(M) t^j.$$

Since $H(M,t)$ is just a polynomial, we have that the function

$$d \mapsto \sum_{j\in\mathbb{Z}} (-1)^j \beta_{d,j}(M)$$

is polynomial for large d. Since we can cut arbitrarily far to the left, we get that it is in fact polynomial of degree less than n for all d. This polynomial is the Hilbert polynomial

$$\chi(\mathcal{F}(d)) = \sum_{i=0}^{n-1} (-1)^i H^i(\mathbb{P}^{n-1}, \mathcal{F}(d)).$$

If $\mathcal{F}$ has supernatural cohomology, the Hilbert polynomial has zeroes at exactly the degrees where there is no non-zero Betti number in the pure diagram. If $z_1 > z_2 > \cdots > z_s$ are these zeroes, we get

$$\chi(\mathcal{F}(d)) = \lambda(d - z_1)(d - z_2)\cdots(d - z_s).$$

The number of zeroes is one less than the number of lines, so the number of zeroes equals the degree of the Hilbert polynomial, which shows that the roots are simple.

Example 5.2. The pure diagram that corresponds to the roots $0 > -1 > -3$ is given by

3 :	168	90	20	12	–	–	–	–	–
2 :	–	–	–	–	2	–	–	–	–
1 :	–	–	–	–	–	–	–	–	–
0 :	–	–	–	–	–	8	30	72	140
	−4	−3	−2	−1	0	1	2	3	4

As in the case of Betti diagrams of modules over S, there is a cone of cohomology tables of vector bundles, and it has a similar simplicial structure.

Theorem 5.1 (Eisenbud-Schreyer '08)**.** *The cohomology table of a vector bundle on* $\mathbb{P}^{n-1}$ *can by uniquely expressed as a sum of supernatural cohomology tables in a chain.*

The proof of their theorems both rely on the inequalities given by the facets of the cones, and in both cases, they come from a pairing:

$$\langle \beta, \gamma \rangle = \sum_{\{i,j,k \mid j \leq i\}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k},$$

which can also be described in terms of the Euler characteristic:

$$\langle F_\bullet, E^\bullet \rangle = \sum_{j\in\mathbb{Z}} \chi(F_{\geq j} \otimes H^j(E^\bullet))$$

where $F_\bullet$ is a free resolution with Betti diagram β and $E^\bullet$ is a free complex with cohomology γ.

Theorem 5.2 (Eisenbud-Schreyer '08)**.** *$\langle \beta, \gamma \rangle \geq 0$, when β is the Betti diagram of a free resolution and γ comes from a free complex.*

Example 5.3. When γ is the cohomology table

2 :	⋯	15	8	3	–	–	–	–	–	⋯
1 :	⋯	–	–	–	1	–	–	–	–	⋯
0 :	⋯	–	–	–	–	3	8	15	24	⋯
	⋯	−3	−2	−1	0	1	2	3	4	⋯

we get the functional

	⋮	⋮	⋮	⋮
−3 :	15	−8	3	0
−2 :	8	−3	0	1
−1 :	3	0	−1	0
0 :	0	1	0	−1
1 :	0	0	3	−8
2 :	0	0	8	−15
3 :	0	0	15	−24
	⋮	⋮	⋮	⋮

which is non-negative on all Betti diagrams over $S = k[x, y, z]$ - even the non-minimal.

However, this does not look exactly like the inequalities that define the cone, and Eisenbud and Schreyer had to modify the pairing slightly with two integer parameters c and τ, they define a pairing

$$\langle \beta, \gamma \rangle_{c,\tau}$$

which allows them to get the boundary inequalities for the cone.

Theorem 5.3 (Eisenbud-Schreyer '08 [8])**.**

$$\langle \beta, \gamma \rangle_{c,\tau} \geq 0$$

for all Betti diagrams β of minimal free resolutions, all c and τ and all γ coming from a free complex.

Since they also have constructions of modules with pure resolutions for any degree sequence and supernatural vector bundles for all sequences of zeroes, they could prove

Theorem 5.4 (Eisenbud-Schreyer '08 [8])**.** *β is the Betti diagram of a CM module if and only if it is a sum of pure diagrams and γ is the cohomology table of a vector bundle if and only if it is a sum of supernatural tables.*

After this, their theorem has been extended to the non-Cohen-Macaulay case by Söderberg and myself [2] and recently Eisenbud and Schreyer made the following extension on the cohomology side:

Theorem 5.5 (Eisenbud-Schreyer '09 [9])**.** *The cohomology table of a coherent sheaf is a sum (possibly infinite) of a chain of supernatural tables of various dimensions.*

Example 5.4. Let $\mathcal{I}_X$ be the ideal sheaf of two skew lines in $\mathbb{P}^3$. The cohomology table of this is given by

3 :	84	56	35	20	10	4	1	–	–	–	–	–	–	–	–
2 :	16	14	12	10	8	6	4	2	–	–	–	–	–	–	–
1 :	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–
0 :	–	–	–	–	–	–	–	–	–	4	12	25	44	70	104
	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7

where the rows are shifted one step to the right for each step upwards and lowest row corresponds to the global sections $H^0(\mathcal{I}_X)$. In this way, we can see that it looks like a Betti diagram for a Tate resolution, and in fact this resolution can be computed from the ideal (xz, xw, yz, yw) in the exterior algebra on $\langle x, y, z, w\rangle$. The maps $F_0 \stackrel{\Phi_1}{\leftarrow} F_1 \stackrel{\Phi_2}{\leftarrow} F_2$ are given by the matrices

$$\Phi_1 = \begin{pmatrix} zw \\ xy \end{pmatrix} \quad \text{and} \quad \Phi_2 = \begin{pmatrix} xz & xw & yz & yw \end{pmatrix}.$$

We can now expand this diagram into a sum of pure diagrams, starting with $\pi(0, -2, -4)$ which equals

3 :	⋯	105	48	15	–	–	–	–	–	
2 :	⋯	–	–	–	3	–	–	–	–	
1 :	⋯	–	–	–	–	3	–	–	–	
0 :	⋯	–	–	–	–	–	15	48	105	⋯
	⋯	−3	−2	−1	0	1	2	3	4	⋯

we can subtract 1/15 of this diagram, and will continue by the pure diagram $\pi(0, -2, -5)$. It turns out that we can continue in this fashion and we can write the diagram of the ideal of skew lines as

$$\sum_{n=0}^{\infty} \frac{4}{(n+2)(n+3)(n+4)} \pi(0, -2, -4 - n).$$

Here we use the notation that $\pi(z_1, z_2, \ldots, z_{n-1})$ corresponds to the pure diagram with Hilbert polynomial $(d - z_1)(d - z_2) \cdots (d - z_{n-1})$.

6. Constructions of pure resolutions

The existence of modules with pure resolutions and of vector bundles with supernatural cohomology was essential in the proof of Eisenbud and Schreyer's theorems. They used them not only in order to prove the only if part, but also in producing the inequalities that give the boundary facets of the cones on either side.

The first construction of modules with pure resolution is due to Eisenbud, Fløystad and Weyman [7] and uses Schur modules. It is very beautiful, but has the drawback that it requieres the field to be of characteristic zero. Later, Eisenbud and Schreyer [8] came up with a new construction that is characteristic independent, and even works over finite fields.

We will here concentrate more on the first construction, and hence we need some preliminaries about Schur modules.

The symmetric algebra $S = k[x_1, x_2, \dots, x_n]$ can be written as

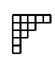
$$S = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V)$$

where V is the vector space spanned by $x_1, x_2, \dots, x_n$ and $\mathrm{Sym}^i(V)$ denotes the symmetric power of V. Since V is naturally a representation of $\mathrm{Gl}(V)$, we get that the symmetric powers are also representations of $\mathrm{Gl}(V)$ and they are examples of Schur-modules, which are irreducible representations of $\mathrm{Gl}(V)$. In fact, for each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$, there is an irreducible $\mathrm{Gl}(V)$-representation $S_\lambda(V)$. When taking the tensor product of two representations, we get a new representation, but this is usually not irreducible, even if the two factors are irreducible.

In representation theory, it is an important question how to decompose the tensror product of two given irreducible representations into a sum of irreducible representation. The first easy example is when we have the representation V itself and write

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$$

The symmetric powers are the Schur modules corresponding to the partition (n) and the exterior powers correspond to the partitions $(1, 1, \dots, 1)$. The above decomposition is a special case of the *Pieri rule*. We can represent the partions by *Young diagrams*, where the boxes in the columns represent the parts of the partition, e.g.,

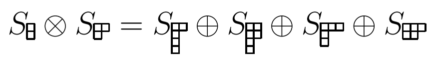
$(5, 3, 2, 1, 1) \quad \longleftrightarrow$ 

Theorem 6.1 (Pieri's rule). *If λ is any partition and we have*

$$\mathrm{Sym}^n(V) \otimes S_\lambda(V) = \bigoplus_{\mu} S_\mu$$

where the sum is taken over all μ which are obtained from λ by adding n boxes, no two in the same row of the Young diagram.

Example 6.1.



A very important fact from representation theory is that equivariant maps between irreducible representations have to be zero unless the representations are isomorphic. If they are isomorphic, there is up to scalars only one equvivariant map between them. Thus we have from the Pieri rule up to scalars only one equivariant map from one of the terms into the tensor product.

Example 6.2. The injection

$$S_{(3,2)} \longrightarrow S_{(1,1)} \otimes S_{(2,1)}$$

is uniquely determined up to scalars.

Definition 6.1 (Eisenbud-Fløystad-Weyman)**.** For a sequence of non-negative integers, $e_1, e_2, \dots, e_n$, there is a sequence of partitions $\lambda_0, \lambda_1, \dots, \lambda_n$ such that λ_i is obtained from λ_{i-1} by adding e_i boxes in column i and an overlap of exactly one box from the previous addition.

$$S_{\lambda_0}(V) \otimes S(-d_0) \longleftarrow S_{\lambda_1}(V) \otimes S(-d_1) \longleftarrow \cdots \longleftarrow S_{\lambda_n}(V) \otimes S(-d_n) \longleftarrow 0,$$

where the maps are defined by the Pieri rule and the degrees are given by $d_i = e_i + d_{i-1}$. Observe that S_{λ_i} is by the Pieri rule a summand in $S_{\lambda_{i-1}} \otimes S_{e_i}$ in a unique way, so there is a unique equivariant map up to scalars.

We can see from the Pieri rule alone that this will be a complex, since the same Schur module can only occur in adjacent free modules. We can also see that every Schur module that occurs in one of the free modules occurs in one of the adjacent, except for a finite number at the last step. The cokernel of the last map is an artinian module.

Example 6.3. When $(d_0, d_1, d_2, d_3) = (0, 2, 4, 5)$, we get $e_1 = 2$, $e_2 = 2$ and $e_3 = 1$. Thus we get the resolution

$$S_{(1)}(V) \otimes S \longleftarrow S_{(1,1,1)}(V) \otimes S(-3) \longleftarrow S_{(2,2,1)}(V) \otimes S(-4) \longleftarrow S_{(3,2,1)}(V) \otimes S(-5) \longleftarrow 0.$$

We can look at what the first map of this resolution is. We have to figure out how $\mathrm{Sym}^3(V)$ sits inside $\mathrm{Sym}^1(V) \otimes \mathrm{Sym}^2(V)$. We can choose the basis of each of them to be the monomials. The matrix will be given by the derivatives of the monomials in the columns by the variables:

$$\begin{pmatrix} 3x^2 & 2xy & 2xz & y^2 & yz & z^2 & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 2xy & xz & 0 & 3y^2 & 2yz & z^2 & 0 \\ 0 & 0 & x^2 & 0 & xy & 2xz & 0 & y^2 & 2yz & 3z^2 \end{pmatrix}$$

To understand why this is the case, we see that the map $S_3 \longrightarrow S_2 \otimes S_1$ given by

$$f \mapsto \frac{\partial f}{\partial x} \otimes x + \frac{\partial f}{\partial y} \otimes y + \frac{\partial f}{\partial z} \otimes z$$

behaves well under the action of $\mathrm{Gl}(V)$. More generally, this is how the maps

$$\mathrm{Sym}^{m+n}(V) \longrightarrow \mathrm{Sym}^m(V) \otimes \mathrm{Sym}^n(V)$$

look like for any positive integers m and n.

In general, there is no known formula for the matrices in these resolutions, butthere are algorithms that allows to compute them for small examples implemented in Macaulay2 by Steven Sam.

6.1. **Supernatural vector bundles.** Also for supernatural vector bundles, there is an equivariant construction, completely analogous to the construction for modules with pure resolution, but with the difference that we use the Pieri rule for tensoring with exterior powers. In fact,

$$E = \bigoplus_{i=0}^{n} \mathrm{Alt}^i(V)$$

and there is a similar Pieri rule to tensor where we add the boxes in a way that there are no two in the same column.

7. Applications and open problems

When we came up with the conjectures on how Betti diagrams can look up to scaling, we were looking at the Multiplicity conjecture by Herzog-Huneke-Srinivasan [13] which says that

$$\frac{\underline{d}_1\underline{d}_2\cdots\underline{d}_s}{s!} \le e(S/I) \le \frac{\overline{d}_1\overline{d}_2\cdots\overline{d}_s}{s!}$$

the $\underline{d}$ and $\overline{d}$ are the lower and upper shifts in a minimal free resolution of S/I. This follows easily from the theorems, since the product of the degrees is clearly an increasing function on chains degree sequences.

There is a similar inequality on the side of vector bundles, bounding the slope.

One of the other nice things that came up when looking at Hilbert functions and Betti diagrams up to scaling is a version of Macaulay's bound up to scaling. It is nice because it is linear, rather than using the irregular Macaulay functions.

Proposition 7.1 (Macaulay up to scaling). $(h_0, h_1, \ldots,)$ *is a multiple of the Hilbert function of a module generated in degree zero if and only if*

$$\frac{h_{i+1}}{s_{i+1}} \le \frac{h_i}{s_i},$$

where $s_i = \dim_k S_i$*, for* $i \ge 0$*.*

The proof of this is very simple when looking at how lex-modules look. In a similar way, we also get a simpler bound on the Betti numbers, given the Hilbert function, as

Proposition 7.2 (Bigatti-Hulett-Pardue up to scaling). *The Betti numbers of* M *are bounded by*

$$\beta_{i,j}(M) \le \left(\frac{h_{j-i+1}}{s_{j-i+1}} - \frac{h_{j-i+2}}{s_{j-i+2}}\right)\binom{j-1}{i-1}\binom{n+j-i}{j}$$

and this bound is sharp after scaling. [2]

One of the consequences of the conjectures that Söderberg proved in his thesis is a beatiful characterization of h-vectors of level modules up to scaling. A level module is a Cohen-Macaulay module which is pure in the first and the last part of the resolution, i.e., it is generated in one degree and has pure Cohen-Macaulay type.

Theorem 7.1 (Söderberg [18]). $H = (h_0, h_1, \ldots, h_d)$ *is a multiple of the Hilbert function of a level module if and only if the consequtive minors of*

$$\begin{pmatrix} 0 & h_0 & h_1 & h_2 & \cdots & h_{d-1} & h_d & 0 \\ 0 & s_0 & s_1 & s_2 & \cdots & s_{d-1} & s_d & s_{d+1} \\ s_{d+1} & s_d & s_{d-1} & s_{d-2} & \cdots & s_1 & s_0 & 0 \end{pmatrix}$$

are non-negative.

[2] Given H, there is a multiple nH such that there exists a module M with Hilbert function nH and with equality in the above inequality.

We can interpret these inequalities geometrically as a kind of convexity.

Erman has proved a version of the Buchsbaum-Eisenbud-Horrocks conjecture using this theory:

Theorem 7.2 (Erman [11]).

$$\beta_i(M) \geq \binom{s}{i}$$

if M is Cohen-Macaulay, generated in degrees ≤ 0 and with $\operatorname{reg}(M) \leq 2\underline{d}_1(M) - 2$.

7.1. Open problems.

Open problem 1. What is the lowest multiple of a given pure diagram that actually corresponds to a module?

In some special cases, all the integral points on the extremal rays are actual Betti diagrams, but the constructions known produce rather large integral multiples of the smallest integral points. Eisenbud, Fløystad and Weyman conjectured that all integral points sufficiently far out correspond to modules. This is not true for some interior rays, as Erman has proved.

Open problem 2. Do the decomposisions of Betti diagrams have any meaning in terms of actual deformations of modules?

There has been in principle no progress in this direction, but it leads to the following problem where there are some minor results:

Open problem 3. How to the parameter spaces for modules of a given Hilbert function nH behave for different integers $n \geq 1$?

Another direction where there is probably much to yet discover is the following:

Open problem 4. What are the extremal rays of the cone when the grading of S is not the standard grading?

On the side of cohomology tables, Eisenbud and Schreyer have some ideas on what should be the extremal rays some some other varieties than $\mathbb{P}^n$, but it is not clear what should be the corresponding objects on the module side.

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