Problem Sheets

— The Do-It-Yourself Lecture —

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Contents

1 – Gröbner Bases and Border Bases

Problem 1.1. Show that, if an ideal $I \subset P$ has an $\mathcal{O}$-border basis, it is uniquely determined.

Problem 1.2. Find an $\mathcal{O}$-border basis such that $\mathcal{O}$ is not of the form $\mathcal{O}_\sigma(I)$ for any term ordering σ.

(*Hint:* Consider $\mathcal{O} = \{1, x, y, x^2, y^2\}$.)

Problem 1.3. Find an $\mathcal{O}$-border basis G and an infinite sequence of proper reduction steps $f_1 \xrightarrow{G} f_2 \xrightarrow{G} f_3 \xrightarrow{G} \cdots$.

Problem 1.4. Consider the order ideal $\mathcal{O} = \{1, x, y, xy\}$.

(a) Write down the *generic $\mathcal{O}$-border prebasis*, i.e. the $\mathcal{O}$-border prebasis having 16 generic coefficients c_{ij}.

(b) Using the border basis version of Buchberger's criterion, find 12 equations in those 16 indeterminates which have to be satisfied in order for the prebasis to be an $\mathcal{O}$-border basis.

2 – Multiplication Matrices and PoSSo

Problem 2.1. (a) For the order ideal $\mathcal{O} = \{1, x, y, xy\}$, find the multiplication matrices corresponding to the generic $\mathcal{O}$-border prebasis.

(b) Write down the equations which have to be satisfied for $\mathcal{A}_x$ and $\mathcal{A}_y$ to commute. Compare them to the equations you found in Problem 1.4.

Problem 2.2. Verify the claims in Example 4.3 of the second lecture.

Problem 2.3. Consider the order ideal $\mathcal{O} = \{1, x, y, x^2, y^2\}$ and the ideal

$$I \;=\; \langle x^2 + xy - \tfrac{1}{2}\, y^2 - x - \tfrac{1}{2}\, y,\; y^3 - y,\; xy^2 - xy \rangle$$

in $\mathbb{Q}[x, y]$.

(a) Show that I has an $\mathcal{O}$-border basis. Compute it!

(b) Determine the multiplication matrices and their eigenvalues.

(c) Find the zeros of I using the first and the second method for eigenvalue solving.

3 – Points, Designs and Fractions

Problem 3.1. Consider the full factorial design $\mathbb{D} = \{-1, 0, 1\}^2 \subset \mathbb{Q}^3$ of level $(3, 3)$ and the order ideal $\mathcal{O} = \{1, x, y\}$.

(a) Write down the **generic multiplication matrices**, i.e. the formal multiplication matrices for the $\mathcal{O}$-border prebasis having generic coefficients.

(b) Using the **Fraction-Algorithm**, find the ideal defining all fractions of $\mathbb{D}$ which identify $\mathcal{O}$.

(c) Show that there are 18 such fractions. Find them!

Problem 3.2. Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$ having border $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$, and let $\mathbb{X} = \{p_1, \ldots, p_\mu\} \subset \mathbb{Q}^n$ with $p_i = \log(t_i)$. Prove that the vanishing ideal of $\mathbb{X}$ satisfies $I_{\mathbb{X}} = \langle \Delta_\pi(b_1), \ldots, \Delta_\pi(b_\nu) \rangle$ for $\pi = \{0, 1, 2, \ldots,\}^n$.

Problem 3.3. Find all possible Hilbert functions of 5 points in $\mathbb{P}^2(\mathbb{Q})$. Characterise each of them by a geometrical property of the point set.

4 – Approximate Exercises

Problem 4.1 In $\mathbb{R}^2$, let $\mathbb{X}$ be the set of (approximate) points $\mathbb{X} = \{(0.01, 0.01), (0.49, 0), (0.51, 0), (0, 0.99)\}$. We use the threshold numbers $\varepsilon = 0.05$ and $\tau = 0.001$ and compute an approximate border basis using the AVI Algorithm.

(a) Show that in degree $d = 1$ the singular values are $s_1 = 2.12$, $s_2 = 0.9$, and $s_3 = 0.35$. No singular value truncation is needed.
Hint: The command `Dec(...)` may come in handy here.

(b) Show that in degree $d = 2$ the singular values are $s_1 = 2.22$, $s_2 = 1.2$, $s_3 = 0.4$, and $s_4 = 0.006$. Perform the singular value truncation and show that an ONB of $\operatorname{apker}(\mathcal{A}, \varepsilon)$ is given by the rows of

$$\mathcal{B}^{\rm tr} = \begin{pmatrix} 0.89 & 0 & -0.03 & -0.45 & 0.02 & 0 \\ -0.02 & -0.46 & -0.62 & 0.02 & 0.62 & 0 \\ 0.01 & -0.88 & 0.32 & -0.01 & -0.32 & 0 \end{pmatrix}$$

(c) Now apply the rest of the AVI Algorithm and show that the set $G = \{x^2 - 0.51x,\ xy,\ y^2 - 0.02x - y + 0.01\}$ is an approximate border basis for $\mathcal{O} = \{1, x, y\}$.

(d) Find an exact border basis close to G. Which set of points does it represent? Interpret the result.

Problem 4.2. Verify the results of Examples 6.1 and 6.2!