Irregular elliptic surfaces of degree 12
in the projective fourspace

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1 Introduction

The purpose of this paper is to give two different constructions of irregular surfaces of degree 12 in $\mathbb{P}^4$. These surfaces are new and of interest in the classification of smooth non-general type surfaces in $\mathbb{P}^4$. This classification problem is motivated by the theorem of Ellingsrud and Peskine [9] which says that the degree of smooth non-general type surfaces in $\mathbb{P}^4$ is bounded. Moreover the new family is one of a few known families of irregular surfaces in $\mathbb{P}^4$. The other known irregular smooth surfaces in $\mathbb{P}^4$ are the elliptic quintic scrolls, the elliptic conic bundles [2], the bieplicit surfaces of degree 10 [16] and degree 15 [4], the minimal abelian surfaces of degree 10 [11] and the non-minimal abelian surfaces of degree 15 (cf. [3]) up to pull-backs of these families by suitable finite maps from $\mathbb{P}^4$ to $\mathbb{P}^4$ itself.

The first author came across the irregular elliptic surfaces when studying a stable rank three vector bundle $E$ on $\mathbb{P}^4$ with Chern classes $(c_1, c_2, c_3) = (5, 12, 12)$ (see [1] for the explicit construction of this bundle). The dependency locus of two sections of $E$ is a smooth surface of the desired type.

Our first construction uses monads. The basic idea of monads is to represent a given coherent sheaf as a cohomology sheaf of a complex of simpler vector bundles. A useful way to construct monads is Horrocks’ technique of killing cohomology. As we will see in § 2.2, for the ideal sheaf of a smooth surface $X$ in $\mathbb{P}^4$ this technique is closely related to the graded finite length modules $H^i \mathcal{I}_X = \bigoplus_{m \in \mathbb{Z}} H^i(\mathbb{P}^4, \mathcal{I}_X(m))$, $i = 1, 2$, over the polynomial ring $\mathbb{C}[x_0, \ldots, x_4]$ called the Hartshorne-Rao modules of $X$. To construct an irregular elliptic surface, we construct the ideal sheaf via its Hartshorne-Rao modules.

The second construction uses liaison, and reduces the construction of an irregular surface as above to the construction of a simpler locally complete intersection surface. More precisely we show that the elliptic irregular surface of the first construction is linked (5, 5) to a reducible surface of degree 13. This reducible surface consists of a singular rational cubic scroll surface in a hyperplane of $\mathbb{P}^4$ and a smooth general type surface of degree 10 and sectional genus 10 such that they intersect in three disjoint (singular) conics on the cubic surface. We then show that these reducible surfaces can be constructed directly, in a slightly more general form. In fact we construct a cubic Del Pezzo surface $X_0$ and a smooth general type surface $T$ of degree 10 and sectional genus 10 such that $T \cap X_0$ is the disjoint union of three smooth conics and show that the general surface linked (5, 5) to $T \cup X_0$ is a smooth irregular elliptic surface. The family of irregular elliptic surfaces constructed via liaison includes the family of irregular elliptic surfaces constructed via monad as a special case.
Acknowledgment. We thank David Eisenbud, Lucian Bădescu, Sorin Popescu and Alfio Ragusa for organizing the Nato advanced workshop in Erice, September 2001. Our collaboration started in the stimulating atmosphere of that meeting. The first author would like to thank Wolfiram Decker for many stimulating conversations. The first author is also grateful to the DAAD for its financial support.

2 Preliminaries

In this part we recall basic results on smooth surfaces in $\mathbb{P}^4$ needed for the two methods of construction that we shall use.

2.1 Notation and the basic results

If not otherwise mentioned, $X$ denotes a smooth surface in $\mathbb{P}^4$ and

- $H = H_X$ its hyperplane class;
- $K = K_X$ its canonical divisor;
- $d = d(X) = H^2$ its degree,
- $\pi = \pi(X)$ its sectional genus (the genus of a general hyperplane section);
- $\chi = \chi(\mathcal{O}_X) = p_g - q + 1$ the Euler-Poincaré characteristic of its structure sheaf.

The numerical invariants of $X$ satisfy the double point formula

$$d^2 - 10d - 5H \cdot K - 2K^2 + 12\chi = 0$$

(see, for example [10]). For a curve on $X$ the arithmetic genus can be computed by the adjunction formula

$$2p_d(C) - 2 = C^2 + C \cdot K.$$  

The double point formula and the adjunction formula express the self-intersection number $K^2$ in terms of $d$, $\pi$ and $\chi$:

$$K^2 = \frac{d^2 - 5d - 10\pi + 12\chi + 10}{2}. \quad (2.1)$$

For curves $C$ and $D$ on $X$ the adjunction formula yields the following addition rule for the arithmetic genus

$$p_a(C \cup D) = p_a(C) + p_a(D) + C \cdot D - 1. \quad (2.2)$$

2.2 Monads

To construct a smooth surface $X$ in $\mathbb{P}^4$, we construct its ideal sheaf $\mathcal{I}_X$. Our construction of $\mathcal{I}_X$ (more generally coherent sheaves on $\mathbb{P}^n$) uses monads. Indeed, monads represent a given coherent sheaf in terms of simpler vector bundles, such as direct sum of line bundles, and homomorphisms between these simpler bundles:
Definition 2.1. A monad for a coherent sheaf $\mathcal{G}$ on $\mathbb{P}^n$ is a complex

$$\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$$

(2.3)

of vector bundles on $\mathbb{P}^n$ with $\alpha$ an injective map and $\beta$ a surjective map such that the cohomology $\text{Ker}(\beta)/\text{Im}(\alpha)$ is the coherent sheaf $\mathcal{G}$. The display of the monad (2.3) is the following commutative diagram:

$$\begin{array}{cccccc}
0 & & 0 & & & \\
\downarrow & & \downarrow & & & \\
0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C} & \longrightarrow & \mathcal{C} & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & & & & & 
\end{array}$$

(2.4)

where $\mathcal{K} = \text{Ker}(\beta)$ and $\mathcal{Q} = \text{Coker}(\alpha)$.

To find a monad for our ideal sheaf we use Horrocks’ technique of killing cohomology. First we recall the definition of Horrocks monad:

Definition 2.2. A monad

$$\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$$

for a coherent sheaf $\mathcal{G}$ on $\mathbb{P}^n$ is a Horrocks monad if

(a) $\mathcal{A}$ and $\mathcal{C}$ are direct sums of line bundles;

(b) $\mathcal{B}$ satisfies

$$H^i_\mathbb{P}^n \mathcal{B} \simeq \begin{cases} 
0 & i = 1, n - 1 \\
H^i_\mathbb{P}^n \mathcal{G} & 1 < i < n - 1.
\end{cases}$$

We say that this monad is minimal if

(c) $\mathcal{C}$ has no direct summand which is the image of a line subbundle of $\mathcal{B}$;

(d) there is no direct summand of $\mathcal{A}$ which is isomorphic to a direct summand of $\mathcal{B}$.

For any vector bundle on $\mathbb{P}^n$, $n \geq 2$, there is a Horrocks minimal monad (see, for example [5]). We include a proof of the following special case for lack of a suitable reference.

Proposition 2.3. There is a Horrocks minimal monad for the ideal sheaf $\mathcal{I}_X$ of any 2-codimensional locally Cohen-Macaulay closed subscheme $X$ of $\mathbb{P}^n$. 

3
Proof. Every minimal set of generators for $H^k_{0}\mathcal{J}_X$ gives an epimorphism

$$L_0 \to H^1_0\mathcal{J}_X \to 0,$$

where $L_0$ is a free $S = \mathbb{C}[x_0, \ldots, x_n]$-module. This epimorphism defines an extension

$$0 \to \mathcal{J}_X \to \Omega \to \widetilde{L}_0 \to 0$$

(2.6)

of $\mathcal{J}_X$ by $\widetilde{L}_0$, because the given epimorphism (2.5) can be regarded as an element of $H^1(\mathcal{J}_X \otimes \widetilde{L}_0) \simeq \text{Ext}^1(\widetilde{L}_0, \mathcal{J}_X)$. Taking cohomology (2.6) gives $H^1_0\Omega = 0$ because of (2.5). Given a minimal set of generators for the $S$-module $H^0_0\omega_X(n + 1)$, we have an epimorphism

$$P_0 \to H^0_0\omega_X(n + 1) \to 0,$$

where $P_0$ is a free $S$-module. Note that $\omega_X(n + 1) = \mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}) \simeq \mathcal{E}xt^1(\mathcal{J}_X, \mathcal{O})$. So $H^0(\omega_X(n + 1) \otimes P_0) \simeq H^0(\mathcal{E}xt^1(\mathcal{J}_X, \mathcal{O}) \otimes P_0) \simeq \mathcal{E}xt^1(\mathcal{J}_X, \widetilde{P}_0)$, and hence we obtain an extension

$$0 \to \widetilde{P}_0 \to \mathcal{K} \to \mathcal{J}_X \to 0.$$  (2.7)

By construction it is clear that $H^k\mathcal{K}^\vee = 0$, and hence $H^{k-1}_0\mathcal{K} \simeq \mathcal{E}xt^{k+1}_0(H^k_0\mathcal{K}^\vee, S) = 0$. The exact sequence (2.6) induces

$$\cdots \to \mathcal{E}xt^1(\widetilde{L}_0, \widetilde{P}_0) \to \mathcal{E}xt^1(\mathcal{O}, \widetilde{P}_0) \to \mathcal{E}xt^1(\mathcal{J}_X, \widetilde{P}_0) \to \mathcal{E}xt^2(\widetilde{L}_0, \widetilde{P}_0) \to \cdots.$$  

Since $\mathcal{E}xt^i(\widetilde{L}_0, \widetilde{P}_0) = 0$ for $i \geq 1$, we have an isomorphism

$$\mathcal{E}xt^1(\mathcal{O}, \widetilde{P}_0) \cong \mathcal{E}xt^1(\mathcal{J}_X, \widetilde{P}_0).$$

Let us denote by

$$0 \to \widetilde{P}_0 \to \mathcal{B} \to \Omega \to 0$$

the extension corresponding to (2.7). This extension can be used to complete the exact sequences (2.6) and (2.7) to the display

\[
\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{P}_0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{J}_X & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{P}_0 & \overset{\alpha}{\longrightarrow} & \mathcal{B} & \longrightarrow & \Omega & \longrightarrow & 0 \\
\quad & & \quad & & \quad & & \quad & & \\
0 & & \quad & & \quad & & \quad & & \\
\end{array}
\]

(2.8)

of a monad

$$\widetilde{P}_0 \overset{\alpha}{\longrightarrow} \mathcal{B} \overset{\beta}{\longrightarrow} \widetilde{L}_0$$

(2.9)

\[
\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & \quad & & \quad & & \quad & & \\
\end{array}
\]

4
for $J_X$. From the display (2.8) it follows that $B$ satisfies (b), because both the modules $H^4_4\Omega$ and $H^{n-1}_4\mathcal{K}$ vanish. So the monad (2.9) is a Horrocks monad.

Finally we show the minimality of (2.9). It is obvious that $l_0$ (resp. $\bar{F}_0$) has a direct summand which is the image of a line subbundle of $\mathbb{B}$ (resp. $\mathbb{B}^c$) if and only if $\Omega$ (resp. $\mathcal{K}$) has a direct summand which is the image of a line subbundle of $\mathbb{B}$ (resp. $\mathbb{B}^c$). By construction, however, $\Omega$ and $\mathcal{K}$ do not have such a direct summand, which completes the proof. 

**Remark 2.4.** For the ideal sheaf $J_X$ of a smooth surface $X$ in $\mathbb{P}^4$ the Horrocks monad is closely related to the minimal free resolutions of the Hartshorne-Rao modules. Suppose that $H^2_4J_X$ does not vanish. Let us denote the minimal free resolution of $H^2_4J_X$ by

$$0 \leftarrow H^2_4J_X \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow F_4 \leftarrow F_5 \leftarrow 0.$$ 

Then $B$ has the same intermediate cohomology modules as the second sheafified syzygy module of $H^2_4J_X$. Hence $H^0_2B$ is a direct sum of $B = \text{Ker}(f_1)$ and a free $S$-module (see [7] for the proof). Frequently, $B$ is isomorphic to $\bar{B}$, and the minimal free resolution of $H^2_4J_X$ decomposes as

![Diagram showing the minimal free resolution of $H^2_4J_X$]

including the minimal free presentation of $H^4_4J_X$:

$$0 \leftarrow H^4_4J_X \leftarrow L_0 \leftarrow L_1.$$ 

So to construct a smooth surface $X$ in $\mathbb{P}^4$, we construct the Hartshorne-Rao modules $H^4_4J_X$ and $H^2_4J_X$ first.

In this case the bundle $\mathcal{K}$ can be described explicitly. So to construct the ideal sheaf $J_X$, we use the determinantal construction following [7], namely we establish that the determinantal locus of the general morphism $\psi \in \text{Hom}(\bar{F}_0', \mathcal{K})$ is indeed a smooth surface.

To construct $H^4_4J_X$ and $H^2_4J_X$, we need information on the dimensions $h^0j_X(m)$ in some range of twists. This information can be obtained from the Riemann-Roch theorem:

**Theorem 2.5 (Riemann-Roch).** Let $X$ be a smooth surface in $\mathbb{P}^4$ of degree $d$, sectional genus $\pi$, geometric genus $p_g$ and irregularity $q$. Then

$$\chi(J_X(m)) = \chi(O(m)) - \left(\frac{m+1}{2}\right)d + m(\pi - 1) - 1 + q - p_g.$$ 

Since surfaces on quadric and cubic hypersurfaces are completely classified (cf. [12]), we may assume that

$$h^0j_X(2) = h^0j_X(3) = 0.$$ 

We observe that some of the cohomology groups vanish:
Proposition 2.6 ([7]). Let $X$ be a smooth non-general type surface in $\mathbb{P}^4$ which is not lying on any cubic hypersurface. Then we have the following table for the dimensions $h^i\mathcal{J}_X(j)$:

\[
\begin{array}{cccccc}
 & & & & & 4 \\
 & & & & & 3 \\
 & & & & & 2 \\
 & & & & & 1 \\
 \hline
0 & 0 & 0 & 0 & 0 & 4 \\
N + 1 & p_g & 0 & 0 & 0 & 3 \\
0 & q & s & h^2\mathcal{J}_X(2) & h^2\mathcal{J}_X(3) & 2 \\
0 & 0 & 0 & h^2\mathcal{J}_X(2) & h^2\mathcal{J}_X(3) & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where $N = \pi - q + p_g - 1$ and $s = \pi - d + 3 + q - p_g$.

In the sequel we represent a zero in a cohomology table by an empty box.

2.3 Liaison

We recall the definition and some basic results from [15].

Definition 2.7. Let $X$ and $X'$ be surfaces in $\mathbb{P}^4$ with no irreducible component in common. Two surfaces $X$ and $X'$ are said to be linked $(m, n)$ if there exist hypersurfaces $V$ and $V'$ of degree $m$ and $n$ respectively such that $V \cap V' = X \cup X'$.

There are two standard sequences of linkage, namely

\[0 \to \mathcal{O}_X(K) \to \mathcal{O}_{X \cup X'}(m + n - 5) \to \mathcal{O}_{X'}(m + n - 5) \to 0\]

and

\[0 \to \mathcal{O}_X(K) \to \mathcal{O}_X(m + n - 5) \to \mathcal{O}_{X \cap X'}(m + n - 5) \to 0.\]

For the construction we will use the following theorem:

Proposition 2.8 ([15]). Let $X$ be a locally complete intersection surface in $\mathbb{P}^4$. If $X$ is scheme-theoretically cut out by hypersurfaces of degree $d$, then $X$ is linked to a smooth surface $X'$ in the complete intersection of two hypersurfaces of degree $d$.

3 A monad construction

Let $\mathbb{P}^4$ be the 4-dimensional projective space over $\mathbb{C}$ with homogeneous coordinate ring $S = \mathbb{C}[x_0, \ldots, x_4]$. We construct an example of a smooth irregular proper elliptic surface in $\mathbb{P}^4$ using Horrocks' technique of killing cohomology.

Theorem 3.1. There exists a smooth, minimal, proper elliptic surface in $\mathbb{P}^4$ with $d = 12$, $\pi = 13$, $p_g = 3$ and $q = 1$. 
Proof. For a smooth surface $X$ in $\mathbb{P}^4$ with the given invariants, formula (2.1) leads to $K^2 = 0$. By the Riemann-Roch formula and Proposition 2.6, the table

<table>
<thead>
<tr>
<th>$i$</th>
<th>15</th>
<th>3</th>
<th></th>
<th></th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>$a$</td>
<td>$b$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$a$</td>
<td>$b+4$</td>
<td>1</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

reflects the relations between the dimensions $h^i\mathcal{J}_X(j)$ in the range $-1 \leq j \leq 3$ of twists. A simple example of a finite length graded $S$-module with Hilbert function $(1, 2, \cdots)$ is the module $M$ with minimal free presentation:

$$0 \leftarrow M \leftarrow S \leftarrow f_0 \leftarrow 3S(-1) \oplus 2S(-2),$$

where $f_0 = (x_0, x_1, x_2, x_3, x_4)$. The minimal free resolution of $M$ is of the following form:

$$0 \leftarrow M \leftarrow S \leftarrow 3S(-1) \oplus 3S(-2) \oplus S(-3) \oplus 2S(-2) \oplus 6S(-3) \oplus 6S(-4) \oplus 2S(-5) \oplus S(-4) \oplus 3S(-5) \oplus 3S(-6) \leftarrow S(-7) \leftarrow 0.$$

Suppose that $H^2_{et}X = M$. Then $a = 1$ and $b = 0$. Let $B$ be the second syzygy module of $M$. We will suppose that $\tilde{B}$ is the middle term of the Horrocks minimal monad for $\mathcal{J}_X$:

$$\tilde{F}_0 \to \tilde{B} \to \tilde{F}_0.$$

With suitably chosen bases, $B$ is generated by the column of the matrix

$$f_2 = \begin{pmatrix}
  x_2 & x_3^2 & 0 & 0 & x_4^2 & 0 & 0 & 0 & 0 & 0 \\
-x_1 & 0 & x_3^2 & 0 & x_4^2 & 0 & 0 & 0 & 0 & 0 \\
x_0 & 0 & 0 & 0 & x_4^2 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_1 & -x_2 & 0 & 0 & 0 & 0 & x_4^2 & 0 & 0 \\
0 & x_0 & 0 & -x_2 & 0 & 0 & 0 & 0 & x_4^2 & 0 \\
0 & 0 & x_0 & x_1 & 0 & 0 & 0 & 0 & 0 & x_4^2 \\
0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & 0 & 0 & x_4^2 \\
0 & 0 & 0 & 0 & 0 & x_0 & 0 & -x_2 & 0 & -x_4^2 \\
0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & 0 & 0 & x_4^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & 0 & x_4^2
\end{pmatrix}.$$

For simplicity we denote by $F_2$ and $F_3$ the target and source of $f_2$ respectively. From the minimal free resolution of $M$ it follows that $\tilde{B}$ has the following Beilinson cohomology table:

<table>
<thead>
<tr>
<th>$j$</th>
<th>$-1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<td>3</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>
Suppose that \( H^3\mathcal{J}_X \) is generated in the first non-zero twist, that is, monogeneous. Then \( \tilde{L}_0 = \mathcal{O}(-2) \). Since \( p_g = h^3\mathcal{J}_X = 3 \), the map \( 3S \to H^0_0\mathcal{J}_X \) forms part of the minimal generating set of \( H^0_0\mathcal{J}_X \). From the cohomology table of \( \tilde{B} \),
\[
h^4\tilde{P}^0_0(-1) = h^4\mathcal{J}_X(-1) + h^4\tilde{B}(-1) = 15 + 1 = 16.
\]
So \( \tilde{P}^0_0 \) must contain \( \mathcal{O}(-4) \) as a direct summand. Since \( \text{rank}(\tilde{B}) = 6 \), we can deduce that \( \tilde{P}^0_0 \simeq 3\mathcal{O}(-5) \oplus \mathcal{O}(-4) \). So the Horrocks monad for \( \mathcal{J}_X \) is of type
\[
3\mathcal{O}(-5) \oplus 0(-4) \to \tilde{B} \to \mathcal{O}(-2)
\]
and we have the display of the monad:
\[
\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
0 & 3\mathcal{O}(-5) \oplus 0(-4) & \mathcal{K} & \mathcal{J}_X & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 3\mathcal{O}(-5) \oplus 0(-4) & \mathcal{B} & \mathcal{O} & 0.
\end{array}
\]
(3.1)

Next we look for an appropriate finite length graded module \( N \) with Hilbert function \((1, 4, \cdots)\) which is monogeneous. Recall that \( P^0_0 \) contains \( S(-4) \) as a direct summand. So to construct such a module, we have to find a map \( \phi : F_2 \to S(-2) \) such that the source of the syzygy matrix of \( f_2 \circ \phi : F_3 \to S(-2) \) includes \( S(-4) \) as a direct summand. For instance, the map given by the matrix
\[
\phi = (1, 0, 0, 0, -x_0, 0, 0, x_1, 0) : F_2 \to S(-2)
\]
satisfies the desired condition. One shows by straightforward calculations that the cokernel of \( f_2 \circ \phi \) is of finite length and that \( N \) has the minimal free resolution of type
\[
0 \to N \to S(-2) \overset{\mathcal{O}(-3)}{\to} \begin{array}{c}
5S(-4) \\
\oplus \\
7S(-5) \\
\oplus \\
2S(-6) \\
\oplus \\
7S(-6) \\
\oplus \\
11S(-7) \\
\oplus \\
4S(-8) \\
\oplus \\
3S(-8) \\
\oplus \\
5S(-9) \\
\oplus \\
2S(-10) \to 0.
\end{array}
\]
In this case \( \mathcal{K} \) has syzygies of type
\[
0 \to \mathcal{K} \to 8\mathcal{O}(-5) \overset{2\mathcal{O}(-6)}{\to} \begin{array}{c}
4\mathcal{O}(-6) \\
\oplus \\
100(-7) \\
\oplus \\
4\mathcal{O}(-8) \\
\oplus \\
3\mathcal{O}(-8) \\
\oplus \\
5\mathcal{O}(-9) \\
\oplus \\
2\mathcal{O}(-10) \to 0.
\end{array}
\]
The ideal sheaf $J_X$ will be obtained via an exact sequence

$$0 \to 3\mathcal{O}(-5) \oplus \mathcal{O}(-4) \to \mathcal{K} \to J_X \to 0.$$ 

However $X$ will be obtained as the determinantal locus of the general map between the vector bundles $3\mathcal{O}(-5)$ and $\mathcal{E}$, where $\mathcal{E}$ is the cokernel of a non-zero map $\mathcal{O}(-4) \to \mathcal{K}$. Indeed, the map

$$\psi = t(0,0,0,1,0,1,0,0,0) : S(-4) \to F_3$$

forms part of the minimal generating set of $\text{Syz}(f_2 \circ \phi)$ and a straightforward calculation shows that the cokernel of $(\psi \circ f_2)^\vee$ is of finite length. This means that for a non-zero element of $\text{Hom}(\mathcal{O}(-4), \mathcal{K})$ the cokernel $\mathcal{E}$ is a rank four vector bundle. Clearly the minimal free resolution of $\mathcal{E}$ is of the form

$$
\begin{array}{cccccc}
8\mathcal{O}(-5) & 2\mathcal{O}(-6) & \\
0 & \mathcal{E} & \oplus & \oplus & \\
& 4\mathcal{O}(-6) & \left\langle 10\mathcal{O}(-7) \right\rangle & \left\langle 4\mathcal{O}(-8) \right\rangle & \\
& & \oplus & \oplus & \\
& & 3\mathcal{O}(-8) & 5\mathcal{O}(-9) & \left\langle 2\mathcal{O}(-10) \right\rangle & 0
\end{array}
$$

By using Macaulay 2, we can check that the determinantal locus $V(f)$ of the general map $f \in \text{Hom}(3\mathcal{O}(-5), \mathcal{E})$ is a smooth surface. The invariants are computed from the minimal free resolution of $J_X$. Since $J_X$ has the presentation

$$0 \to 3\mathcal{O}(-5) \to \mathcal{E} \to J_X \to 0,$$ 

the minimal free resolution of $J_X$ can be computed from that of $\mathcal{E}$ and has the shape

$$
\begin{array}{cccccc}
5\mathcal{O}(-5) & 2\mathcal{O}(-6) & \\
0 & J_X & \oplus & \oplus & \\
& 4\mathcal{O}(-6) & \left\langle 10\mathcal{O}(-7) \right\rangle & \left\langle 4\mathcal{O}(-8) \right\rangle & \\
& & \oplus & \oplus & \\
& & 3\mathcal{O}(-8) & 5\mathcal{O}(-9) & \left\langle 2\mathcal{O}(-10) \right\rangle & 0
\end{array}
$$

Computing the Hilbert Polynomial of $X$ from this resolution, we conclude that $X$ has degree $d = 12$, sectional genus $\pi = 13$, and Euler characteristic $\chi = 3$. From (2.1) we now get $K^2 = 0$. Dualizing (3.2), we obtain

$$0 \to \mathcal{O}(-5) \to \mathcal{E}^\vee \to 3\mathcal{O} \to \omega_X \to 0.$$ 

So $p_g = 3$ and $X$ is irregular with $g = 1$. Furthermore $\omega_X$ is globally generated by its own three sections, and thus these sections define the canonical map

$$\Phi = \Phi_{[K]} : X \to \mathbb{P}^2.$$

Then we have the Stein factorization of $\Phi$:

$$\xymatrix{ X \ar[rr]^(0.55){\Phi} & & C \ar[r] & \mathbb{P}^2, \\
& & B \ar[lu]_p &}
$$

where $p$ has connected fibers. Let $F$ be a fiber of $p$. Then $F \cdot (F + K) = 2K^2 = 0$. So it follows from the adjunction formula that $p_h(F) = 1$, which implies that the smooth fibers of $p$ are elliptic curves. Therefore $X$ is an elliptic surface. Finally, since $K^2 = 0$, the surface is minimal, which completes the proof. \qed
Remark 3.2. The surface constructed above may also be obtained as the dependency locus of two general sections of a rank three vector bundle on $\mathbb{P}^4$. In fact, in the notation of the above proof, define a homomorphism by

$$\psi' = \iota(0, x_1, 0, 0, x_0, 0, 0, 0, 0, 1) : S(-5) \to F_3.$$ 

Then $\psi'$ forms part of the minimal set of generators for $\text{Syz}(f_2 \circ \phi)$. So $\psi'$ induces a map $g : \mathcal{O}(-5) \to \mathcal{E}$. One can show by a straightforward calculation that $g$ is an injective bundle map. The cokernel $\mathcal{I}$ of $g$ is a rank three vector bundle on $\mathbb{P}^4$, and thus $X$ can be regarded as the dependency locus of two general sections of $\mathcal{I}$.

4 A liaison construction

We first motivate this second construction. Let $X$ be the irregular elliptic surface in $\mathbb{P}^4$ constructed in the previous section. With Macaulay 2 we can check that the quintics containing $X$ intersect in

$$V(\mathcal{H}^0(J_X(S))) = X \cup X_0,$$

where $X_0$ is a singular rational cubic scroll surface in a hyperplane $H_0$ of $\mathbb{P}^4$. The scroll $X_0$ may be obtained by projecting a smooth rational cubic scroll $S$ in $\mathbb{P}^4$ from a point off $S$. The scroll $S$ contains a unique directrix line. Let $L$ be the image of this line on $X_0$.

We can check a few more facts with Macaulay 2. The surface $X$ does not intersect the line $L$. On the other hand the hyperplane section $H_0 \cap X$, denoted by $C_0$, is equal to $X_0 \cap X$, and hence deg$(C_0) = 12$ and $\pi(C_0) = 13$. Let $\tilde{C}_0$ and $\tilde{L}$ be the inverse images of $C_0$ and $L$ respectively on $S$. Then $\tilde{C}_0 \equiv 3(H_5 + \tilde{L})$, since $\tilde{C}_0$ does not intersect $\tilde{L}$. Now, $C_0 = X \cap X_0 = X \cap H_0$, so $X \cup X_0$ is locally a complete intersection cut out by quintics. Therefore, by Proposition 2.8, $X \cup X_0$ can be linked $(5, 5)$ to a smooth general type surface $T$ of degree $d = 10$, $\pi = 10$ and $\chi = 4$. This implies that $X$ is reobtained as the residual surface of $X_0 \cup T$ in the complete intersection of two quintics in $\mathcal{H}^0(J_{X_0 \cup T}(5))$. Let $C = X_0 \cap (X \cup T)$. Compare the second standard exact sequence

$$0 \to \mathcal{O}_{X_0}(K_{X_0}) \to \mathcal{O}_{X_0}(5) \to \mathcal{O}_C(5) \to 0$$

with the exact sequence

$$0 \to \mathcal{O}_{X_0}(-C) \to \mathcal{O}_{X_0} \to \mathcal{O}_C \to 0.$$ 

Then we can deduce that $C \equiv 5H_{X_0} - K_{X_0} \equiv 6H_{X_0}$. On $S$ we get $\tilde{C} \equiv 6H_S$. Let $C_1 := X_0 \cap T$ and let $\tilde{C}_1$ be its inverse image on $S$. Then $\tilde{C}_1 \equiv 3(H_S - \tilde{L})$, and hence $\tilde{C}_1$, is the disjoint union of six members of the ruling on $S$. The singular locus of $X_0$ is a line, so $C$ has six double points along this line. Three of these appear already on $C_0$ since $p(C_0) = 10 = p(C_0) - 3$ and are precisely the intersection of $C_0$ with the singular locus. The other three points are therefore intersections of pairs of rulings in $C_1$. Thus $C_1$ consists of three disjoint singular conic sections that each lie in a plane that contains the line $L$.

We recall some facts about smooth general type surfaces $T$ of degree $d = 10$, $\pi = 10$ and $\chi = 4$ from [14, Proposition 4.18]. First of all, any such surface $T$ can be linked $(4, 4)$ to the union of a degenerate cubic scroll $U_0$ formed by three planes and a Del Pezzo surface $U_1$ of degree 3 such that each plane in $U_0$ intersect the hyperplane of $U_1$ along the same line $L$ on $U_1$. The line $L$ is the unique 6-secant line to $T$. Every hyperplane through $L$ intersect $T$.
in a section which is contained in a cubic surface. Furthermore, the surface $T$ is embedded by the linear system

$$[2K - A_1 - A_2 - A_3]$$

in $\mathbb{P}^4$, where $A_1$, $A_2$ and $A_3$ are pairwise disjoint $(-2)$-curves embedded as conics that each intersect $L$ in two points.

We now want to choose an appropriate surface $T$ together with a cubic surface $X_0$ such that their union is linked $(5,5)$ to a smooth irregular minimal elliptic surface. On such a surface $T$ the three conics in $C_1 = X_0 \cap T$ all have a common secant line which coincides with $L$, so we assume that $C_1 = A_1 \cup A_2 \cup A_3$.

In particular we want $C_1$ to be contained in a hyperplane. Let $D$ be the residual curve $H_T - C_1$. Then $C_1 \cdot D = C_1 \cdot (H_T - C_1) = 6 + 6 = 12$ and $p_a(D) = 1$ by (2.2). This means that $D$ is an elliptic curve of degree 4. So to construct an irregular elliptic surface $X$, we find a smooth general type surface $T$ in $\mathbb{P}^4$ with $d = 10$, $\pi = 10$ and $\chi = 4$ and a hyperplane $H_0$ of $\mathbb{P}^4$ such that

- $H_0$ contains the 6-secant $L$ to $T$;
- $H_0 \cap T = C_1 \cup D$, where $C_1$ is the disjoint union of three smooth $(-2)$-conics and $D$ is an elliptic curve of degree 4 such that $D \cap L = \emptyset$ and $C_1 \cap D$ consists of 12 points.

Since the hyperplane $H_0$ of $C_1 \cup D$ contains $L$, the hyperplane section lies in a unique cubic surface, which we denote by $U_1$. On the other hand the curve $C_1$ lies in the cubic surface defined by the three planes of the conics in $C_1$. The general element of $\mathbb{H}^0_{U_1,H_0}(3)$ therefore defines a smooth cubic surface $X_0$, which does not coincide with $U_1$. We will show that the surface linked $(5,5)$ to the reducible surface $T \cup X_0$ is a smooth irregular minimal elliptic surface. But first we make the explicit construction.

**Construction:** Let $H_0 = V(h_0)$ be a hyperplane off $\mathbb{P}^4$ and $L = V(h_0, h_1, h_2)$ a line in $\mathbb{P}^4$. Then three quadric minors $Q_1, Q_2$ and $Q_3$ of a $2 \times 3$ matrix whose entries are quadratic combinations of $h_0, h_1$ and $h_2$, define a degenerate cubic scroll $U_0$. By construction, $U_0$ then consists of three planes through the line $L$. Next, we define a cubic Del Pezzo surface $U_1$ in $H_0$ given by a general cubic $f$ in $\mathbb{H}^3_{U_1,H_0}(3)$. An elliptic curve $D$ on $U_1$, which does not meet $L$ can be obtained as the general member of the linear system $[H_{U_1} + L]$. Explicitly, we consider a plane conic $A \in [H_{U_1} - L]$ on $U_1$ and take a quadric $q \in \mathbb{H}^0_{A,H_0}(2)$. Then $D = B - A$, where $B = V(h_0, q)$.

The homogeneous ideal $I_D$ of $D$ can be written as

$$I_D = (h_0, q_1, q_2),$$

where $q_1, q_2 \in \mathbb{H}^0_{U_1,H_0}(2)$ such that $f \in (q_1, q_2)$. Consider the double structure $D_2 = V(h_0^2, q_1, q_2) \subset \mathbb{P}^4$. Then $D_2$ is not contained in $U_1$. Multiplication by $h_0$ defines the following exact sequence:

$$0 \longrightarrow \mathcal{J}_{U_0 \cup U_0}(3) \xrightarrow{h_0} \mathcal{J}_{U_0 \cup U_0}(4) \longrightarrow \mathcal{J}_{U_0 \cup U_0 \cup D_2}(4) \longrightarrow \mathcal{J}_{U_0 \cup U_0 \cup D_2 \cup H_0}(4) \longrightarrow 0.$$

Consider the exact sequence

$$0 \longrightarrow \mathcal{J}_{U_0}(2) \xrightarrow{h_0} \mathcal{J}_{U_0 \cup D}(3) \longrightarrow \mathcal{J}_{U_0 \cup D \cup H_0}(3) \longrightarrow 0.$$

The curve $(U_0 \cup D) \cap H_0$ is the union of the first order neighborhood on $L$ and the elliptic curve $D$ of degree 4 with $D \cap L = \emptyset$, so we have $\mathbb{H}^0_{U_0 \cup D \cup H_0}(3) = 0$. We therefore conclude that $\mathbb{H}^0_{U_0 \cup D}(3) = \mathbb{H}^0_{U_0}(2) = 3$. Furthermore, $\mathbb{H}^0_{U_0 \cup U_0 \cup D \cup H_0}(4) \simeq \mathbb{H}^0_{L,H_0}(1)$, and hence $\mathbb{H}^0_{U_0 \cup U_0 \cup D_2}(4) \geq 5$.

On the other hand, the following example tells us that $\mathbb{H}^0_{U_0 \cup U_0 \cup D_2}(4) \geq 5$.
Example 4.1. Let \( S = \mathbb{C}[x_0, \ldots, x_4] \) be the homogeneous coordinate ring of \( \mathbb{P}^4 \). We define the homogeneous ideals \( I_{H_0}, I_L, I_{U_0}, I_{U_1} \) and \( I_D \) by

\[
\begin{align*}
I_{H_0} &= (h_0) = (x_1); \\
I_L &= (h_0, h_1, h_2) = (x_0, x_1, x_4); \\
I_{U_0} &= (Q_1, Q_2, Q_3) = ((x_1 - x_4)(x_4 - x_0), (x_4 - x_0)(x_1 + x_0), (x_1 + x_0)(x_1 - x_4)); \\
I_{U_1} &= (h_0, f) = (x_1, x_4x_0 - x_2x_3)x_0 + (x_2^2 + x_3^2 + x_4^2)x_4; \\
I_D &= (h_0, q_1, q_2) = (x_1, x_4x_0 - x_2x_3, x^2 + x_3^2 + x_4^2).
\end{align*}
\]

Then the following quartics form part of the minimal generating set of \( I_{U_1} \cup (H_0 \cup D_2) \):

\[
\begin{align*}
g_1 &= (x_1^2 - x_4^2)h_0^2; \\
g_2 &= (-x_1^2 + x_3^2 + Q_3)h_0^2; \\
g_3 &= h_0^2Q_2 - (x_0x_2 + x_1x_4)Q_1 + (-q_1 + x_1x_4 - x_3^2)Q_2 + q_2Q_3 + x_4^2; \\
g_4 &= (x_1^2 - x_3^2 - Q_2 - Q_3)h_0^2; \\
g_5 &= q_2h_0^2 + (-x_1x_4 + x_3^2)Q_2 + (-x_2x_3 + x_4^2)Q_3 - x_4^2.
\end{align*}
\]

The sequence of global sections

\[
0 \rightarrow H^0(I_{U_1} \cup D(3)) \xrightarrow{j_0} H^0(I_{U_1} \cup U_0 \cup D_2(4)) \rightarrow H^0(I_{U_1} \cup U_0 \cup D_2(3)) \xrightarrow{i_0} H^0(H_0 \cup H_0 \cup (4)) \rightarrow 0.
\]

is therefore exact. Let \( f_1, f_2 \) be general elements in \( H^0(I_{U_1} \cup U_0 \cup D_2(3)) \). By Bertini, the complete intersection \( V(f_1) \) and \( V(f_2) \) is singular at most along the base locus of the quartics in \( H^0(I_{U_1} \cup U_0 \cup D_2(3)) \). Therefore \( U_0 \cup U_1 \) is linked \( (4, 4) \) to a smooth surface \( T \) of degree \( 10 \) and \( \pi = 10 \) in the complete intersection \( V(f_1, f_2) \). The surface \( T \) contains \( D_2 \), and hence \( D_2 \). Notice that \( T \cap H_0 = T \cap U_1 \). We denote \( T \cap U_1 \) by \( C \) and the residual curve \( C - D \) by \( C_1 \). Since \( \deg(C) = 10 \) and \( p_a(C) = 10 \), the divisor \( C \) of \( U_1 \) can be written as

\[
C = 4H_1 - 2L.
\]

So \( C_1 = 3(H_1 - L) \), which implies that \( C_1 \) is the disjoint union of three conics and the intersection \( D \cap C_1 \) is 12 points. Each conic of \( C_1 \) is contained in a unique plane, and these three planes form a cubic surface \( U_\infty \) in \( H_0 \). So we have the pencil of cubics containing \( C_1 \) spanned by \( U_1 \) and \( U_\infty \), and the general member of this pencil is smooth. Therefore we may choose a smooth cubic surface \( X_0 \) containing \( C_1 \) such that \( X_0 \neq U_1 \) and does not contain \( D_2 \). But \( D_2 \) is a quartic curve, so \( X_0 \cap D_2 \) is 12 points (distinct by a general choice of \( X_0 \)). Therefore \( X_0 \cap D = C_1 \cap D_2 \), and \( C_1 = X_0 \cap T \). In particular \( X_0 \) and \( T \) intersect transversally along \( C_1 \), and hence \( Y = T \cup X_0 \) is locally complete intersection.

Theorem 4.2. The reducible surface \( Y \) is linked \((5, 5)\) to a minimal smooth elliptic surface with \( d = 12 \), \( \pi = 13 \), \( p_g = 3 \), and \( q = 1 \).

Proof. For smoothness it suffices, by Proposition 2.8, to show that \( J_Y(5) \) is globally generated. We first estimate \( h^0(J_Y(5)) \):

Lemma 4.3. The residual exact sequence

\[
0 \rightarrow J_T(4) \xrightarrow{j_0} J_Y(5) \rightarrow J_Y \cap H_0 \cap H_0(5) \rightarrow 0
\]

remains exact on global sections.
Proof. We have already shown that the residual curve $D = C - C_1$ is an elliptic curve of degree 4. So $H^0(J_Y|_{H_0 \cap H_0}) \cong H^0(T_{D\cup X_0}H_0) \cong H^0(D,H_0)(2)$, and thus $H^0(J_Y|_{H_0 \cap H_0}) = 2$. On the other hand $H^0(Y|_{T}) = 3$ and $h^1(Y|_{T}) = 1$ by [14]. In order to prove the lemma, it is therefore enough to show that $h^0(J_Y(5)) = 5$. Let $H = V(h)$ be a hyperplane containing the 6-secant line $L$ to $T$, but not $X_0$. Then multiplication by $h$ defines the exact sequence

$$0 \longrightarrow J_Y(4) \longrightarrow J_Y(5) \longrightarrow J_Y|_{H \cap H_0}(5) \longrightarrow 0.$$ 

If $h^0(J_Y|_{H \cap H_0}) = 7$, then $h^0(J_Y(5)) = 5$, because $h^0(J_Y(4)) = 0$, $h^1(J_Y(4)) = 2$ and $h^2(J_Y(4)) = 0$. So we shall show that $h^0(J_Y|_{H \cap H_0}) = 7$.

Since $H$ contains $L$, the hyperplane section $Y \cap H$ consists of $L$, a conic $W_0$ in $X_0$ and the hyperplane section $Z = T \cap H$. For simplicity, $Z \cap L$ will be denoted by $W_1$. Then multiplication by $h_0$ defines the exact sequence

$$0 \rightarrow J_{W_1}(4) \rightarrow J_{W_0 \cup W_1}(5) \rightarrow J_{(W_0 \cup W_1) \cap P}(5) \rightarrow 0, \quad (4.2)$$

where $P = H \cap H_0$. The vector space $H^0(J_{(W_0 \cup W_1) \cap P}(5))$ is spanned by the conics in $P$ containing the 4 points obtained as $T \cap P - T \cap L$, so $h^0(J_{(W_0 \cup W_1) \cap P}(5)) = 2$.

On the other hand $Z \cdot L = 6$, so

$$p_a(W_1) = p_a(Z) + p_a(L) + Z \cdot L - 1 = 10 - 0 + 6 - 1 = 15.$$ 

By the Riemann-Roch formula,

$$\chi(CW_1(4)) = 4 \deg(W_1) + 1 - p_a(W_1) = 44 + 1 - 15 = 30,$$

and hence $\chi(CW_1(4)) = 5$, which implies that $h^0(W_1(4)) \geq 5$. But $h^0(W_1(4)) \leq h^0(J_Y(4)) \leq 5$, so $h^0(W_1(4)) = 5$ and $h^1(W_1(4)) = 0$. Thus from (4.2) we have $h^0(J_{W_1 \cup W_2}(5)) = 7$. \hfill $\square$

Lemma 4.4. The reducible surface $Y = T \cup X_0$ is scheme-theoretic cut out by 5 quintic hypersurfaces.

Proof. By Lemma 4.3 the residual sequence

$$0 \rightarrow J_T(4) \rightarrow J_Y(5) \rightarrow J_{D \cup X_0 \cap H_0}(5) \rightarrow 0$$

remains exact after taking global sections, so the intersection of the base locus in the middle with $H_0$ coincides with the base locus on the right. But $J_{D \cup X_0 \cap H_0}(5)$ is globally generated by its own two global sections, because $H^0(J_{D \cup X_0 \cap H_0}(5)) \cong H^0(D,H_0)(2)$, so the base locus of the quintics through $Y$ would be in $U_0$. As we can check in the example given in [1], p.107, there is a quintic hypersurface $V$ through $Y$ such that $V$ does not contain any plane of $U_0$. So this is the general property, and hence we may assume the existence of such a quintic hypersurface $V$. Note that $V$ intersects each plane of $U_0$ in a quartic plane curve in $T$ and the line $L$ in $X_0$. So $Y$ is scheme-theoretically cut out by 5 quintics. \hfill $\square$

Now we compute the sectional genus $\pi$, the geometric genus $p_g$ and the irregularity $q$ of $X$. From the standard liaison sequence

$$0 \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_{T \cup X_0 \cup X}(5) \rightarrow \mathcal{O}_{T \cup X_0}(5) \rightarrow 0,$$

we have the exact sequence

$$0 \rightarrow J_{T \cup X_0 \cup X}(5) \rightarrow J_{T \cup X_0}(5) \rightarrow \mathcal{O}_X(K_X) \rightarrow 0, \quad (4.3)$$

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Twisting by \( O(-1) \) and taking cohomology, we can compute the speciality
\[
 s = h^1 O_X(H_X) = h^1 O_X(K_X - H_X) = h^1 J_{T \cup X_0}(4) = h^1 J_T(3) = 2.
\]
The exact sequence remains exact on global sections, because \( T \cup X_0 \cup X \) is a complete intersection of two quintics. Recall, from above, that \( h^0 J_{T \cup X_0}(5) = 5 \), so we have
\[
p_g = h^0 O_X(K_X) = h^0 J_{T \cup X_0}(5) - h^0 J_{T \cup X_0 \cup X}(5) = 5 - 2 = 3.
\]
Since \( \chi(O_X) = \chi(O_X(K_X)) = \chi(J_{T \cup X_0}(5)) - \chi(J_{T \cup X_0 \cup X}(5)) = 3 \), we have \( q = 1 \). Now the speciality \( s = h^1 O_X(H_X) = 2 \) and \( h^2 O_X(H_X) = 0 \), so by Riemann Roch
\[
 H_X \cdot K_X = H_X^2 + 2\chi(O_X) - 2\chi(O_X(H_X)) = 12 + 2 \cdot 3 - 2 \cdot (5 - 2) = 12.
\]
In particular, by the adjunction formula, \( \pi = 13 \). Formula (2.1) yields \( K^2 = 0 \).

Finally we show that \( X \) is an elliptic surface. To do this, we show that \( X \) is minimal. Since \( K^2 = 0 \) and \( p_g = 3 \), it is enough to prove that \( K \) has no fixed components. Let \( C = (X \cup T) \cap X_0 \). By the standard liaison sequence
\[
 0 \to O_{X_0}(K_{X_0}) \to O_{X_0}(5) \to O_C(5) \to 0
\]
and the exact sequence
\[
 0 \to O_{X_0}(-C) \to O_{X_0} \to O_C \to 0,
\]
we can see that \( C \equiv 5H_{X_0} - K_{X_0} \equiv 6H_{X_0} \). Recall that \( C_1 = T \cap X_0 \) is the residual curve \( T \cap H_0 \). So \( C_1 \equiv 3H_{X_0} - 3L \) on \( X_0 \). Therefore \( C_0 = X \cap X_0 \equiv 3H_{X_0} + 3L \). In particular \( C_0 \) has degree 12, which implies that \( X \cap X_0 \) is equal to the hyperplane section \( H_X = X \cap H_0 \). Applying the analogous argument to \( X \) shows that \( (T \cup X_0) \cap X = 5H_X - K_X \). Thus \( X \cap T \equiv 4H_X - K_X \). This says that any quartic hypersurface containing \( T \) intersects \( X \) in \( X \cap T \) and \( K_X \). Note that there is a net of such quartics. Consider the following commutative diagram
\[
\begin{array}{ccc}
J_T(4) & \to & J_{T \cup X_0}(5) \\
\downarrow & & \downarrow \\
0 & \to & O_X(K_X) \to 0,
\end{array}
\]
The global sections of \( J_T(4) \) are mapped to \( H^0 O_X(K_X) \) injectively, because we have chosen two general quintics of \( H^0 J_{T \cup X_0}(5) \). Since \( p_g = 3 \), we have an isomorphism
\[
 H^0 J_T(4) \cong H^0 O_X(K_X).
\]
This implies that every canonical divisor of \( X \) appears in the way described above. Notice that the base locus of the quartics through \( T \) is the union of \( T \) and the three planes \( U_0 \). Therefore any base component of \( [K_X] \) would lie in \( U_0 \). As claimed in the proof of Lemma 4.4, any quintic through \( T \) which does not contain any plane of \( U_0 \) intersects each plane of \( U_0 \) in a quartic plane curve in \( T \) and the line in \( X_0 \). So if there would be a curve in \( X \cap U_0 \), then this curve would meet \( L \). On the other hand, \( C_0 \cdot L = (3H_{X_0} + 3L) \cdot L = 0 \) on \( X_0 \). This is a contradiction, and hence \( [K_X] \) has no fixed component. \( \Box \)

**Remark 4.5** Let \( X \) be an irregular elliptic surface given as above. We have shown that the canonical system \( [K_X] \) is base point free. In other words, the canonical bundle \( \omega_X \) is generated by its own three global sections, because \( p_g = 3 \). This leads to the following:
(i) The canonical system $[K_X]$ defines a morphism $\Phi : X \to \mathbb{P}^2$, and $\dim \Phi(X) = 1$, because $K_X^2 = 0$. Let $\Phi = s \circ r$ be the Stein factorization of $\Phi$. Then, as we saw in the proof of Theorem 3.1, the smooth fibers of $r$ are elliptic curves.

(ii) From the generalized Serre correspondence [13] it follows that there is a rank four vector bundle $E$ with Chern classes $(c_1, c_2, c_3, c_4) = (5, 12, 12, 0)$ and three sections of $E$ whose dependency locus is $X$. However we have not been able to prove that every irregular elliptic surface we have constructed above arises from a rank three vector bundle on $\mathbb{P}^4$ with Chern classes $(c_1, c_2, c_3) = (5, 12, 12)$ (compare Remark 3.2).

**Remark 4.6.** Let $X$ be a smooth surface in $\mathbb{P}^4$ with $d = 12$, $\pi = 13$, $p_g = 3$ and $q = 1$. Is it clear that the $X$ is as above? In this last remark we sketch an argument why this is true as soon as $h^0(\mathcal{I}_X(2)) = 1$. We conjecture, but cannot prove that this is always the case. If $h^0(\mathcal{I}_X(2)) = 1$, then $h^1(\mathcal{I}_X(2)) = 1$ by the Riemann-Roch theorem. Using Beilinson’s theorem [6] and a remark in [8], we can deduce that $h^2(\mathcal{I}_X(3)) = 0$, and thus $h^1(\mathcal{I}_X(3)) = 4$. This implies that there exists the unique hyperplane $H_0$ such that $h^0(\mathcal{I}_{X \cap H_0}(3)) = 1$. Let $X_0$ be the unique cubic surface in $H_0$ containing $X \cap H_0$. Then the residual surface $X \setminus X_0$ is locally complete intersection, because $X \cap H_0$ and $X \cap X_0$ coincide. Suppose that $X \cup X_0$ is scheme-theoretically cut out by quintics. Then the linked surface $T$ is a smooth general type surface of $d = 10$, $\pi = 10$ and $\chi = 4$. So $X$ is reobtained as the surface linked $(5, 5)$ to $T \cup X_0$, and hence $X$ is an elliptic surface as above.

**References**


