Non-abelian Brill-Noether Loci
and the Lagrangian Grassmannian $\text{LG}(3,6)$

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This paper is an elaboration of the notes for my talk at the Fano conference and reports on joint work with Atanas Iliev. Full details can be found in [2]. It is a pleasure to thank Alberto Conte, Marina Marchisio and Alberto Collino for a wonderful conference.

1. Inspiration.

In 1987 Mukai showed his famous linear section theorem:

**Theorem.** (Mukai [5],[6]) A general canonical curve $C$ of genus $6 \leq g \leq 9$ (resp. a general canonical curve of genus 10 on a $K3$–surface) is a complete intersection in a homogeneous space.

<table>
<thead>
<tr>
<th>genus</th>
<th>$C$ is a linear section of $M^\text{dim}^M_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$C = \mathbb{P}^5 \cap Q \cap \text{G}(2,5)$</td>
</tr>
<tr>
<td>7</td>
<td>$C = \mathbb{P}^6 \cap \text{OG}(5,10)$</td>
</tr>
<tr>
<td>8</td>
<td>$C = \mathbb{P}^7 \cap \text{G}(2,6)$</td>
</tr>
<tr>
<td>9</td>
<td>$C = \mathbb{P}^8 \cap \text{LG}(3,6)$</td>
</tr>
<tr>
<td>10</td>
<td>$C = \mathbb{P}^9 \cap \text{GG}_2$</td>
</tr>
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</table>

The following conditions are sufficient for the genericity in the assumption. When $g = 6$ that $C$ has no $g^1_3$ or $g^2_3$, when $g = 7$ that $C$ has no $g^1_4$, when $g = 8$ that $C$ has no $g^2_4$ and when $g = 9$ that $C$ has no $g^3_4$.

I am switching here from my notation in the talk to Mukai’s notation, so $\text{OG}(5,10)$ is the 10-dimensional orthogonal Grassmannian of isotropic 5–spaces belonging to one of the two families of such in a 8-dimensional nonsingular quadric, embedded with Spinor coordinates. $\text{G}(2,6)$ is of course the ordinary Grassmannian of rank 2 subspaces of $C^6$ embedded with Plucker coordinates. The variety $\text{LG}(3,6)$ is the Lagrangian or symplectic Grassmannian of Lagrangian 3–spaces for a nondegenerate 2–form on $C^6$, embedded by the Plucker coordinates of $\text{G}(3,6)$. The variety $\text{GG}_2 \subset \text{G}(5,7)$ is the variety of isotropic 5–spaces for a nondegenerate skew-symmetric 4-linear form on a 7-dimensional vector space. It has its name from the fact that it is homogeneous for an algebraic group of type $G_2$. Good references for the first three of these varieties is [3]. In Mukai’s terminology these varieties are called key varieties. With the letter $M$, I suggest that in this context Mukai varieties have become a common name.

Similar to the linear section theorem for curves there are linear section theorems for $K3$ surfaces and Fano 3–folds:

**Theorem.** (Mukai [5]) A $K3$ surface $S$ with $\text{Pic}(S) = \langle C \rangle$, where $C$ has genus $6 \leq g \leq 10$ is a complete intersection in a Mukai variety $M_g$.

A smooth Fano threefold $X$ of index 1 and $\text{Pic}(X) = \langle -K \rangle$, where $-K^3 = 2g - 2$ and $6 \leq g \leq 10$ is a complete intersection in a Mukai variety $M_g$.  

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2. Questions and main result.

Our main result is a relation between two Mukai varieties. It gives an answer in case $g = 9$ to the following questions:

Given a linear section $X \subseteq M_g$ with one node, and let $X_0$ be the projection of $X$ from its node.

**Q1:** Is $X_0$ a linear section of $M_{g-1}$?

**Q2:** What is the maximal dimension of the sections $X$ for which the answer is yes?

**Answer to Q1:** Yes, when $X$ is a canonical curve by Mukai’s theorem since $X_0$ is a general enough canonical curve.

**Answer to Q2:** A first guess is $\dim M_g = 1$, when $X$ is a nodal hyperplane section. But notice that $X_0$ would contain a smooth quadric as exceptional divisor, so the maximal dimension of smooth quadrics in $M_g$ gives an upper bound. In the cases $7 \leq g \leq 9$ the upper bound in terms of smooth quadrics is:

<table>
<thead>
<tr>
<th>genus</th>
<th>maximal dimension of smooth quadric in $S_{g-1}$</th>
<th>upper bound for Q2</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>quadric 4 – fold</td>
<td>codimension 5</td>
</tr>
<tr>
<td>8</td>
<td>quadric 6 – fold</td>
<td>tangent hyperplane</td>
</tr>
<tr>
<td>9</td>
<td>quadric 4 – fold</td>
<td>tangent hyperplane</td>
</tr>
</tbody>
</table>

The result I want to discuss concerns the case $g = 9$:

**Main theorem** ([2]). If $X$ is a nodal hyperplane section of $\text{LG}(3, 6)$, then the projection $X_u$ from the node $u$ is a linear section of $\text{G}(2, 6)$.

3. Application to Brill–Noether loci

The main theorem has applications to moduli of stable rank 2 vector bundles on linear sections of $\text{LG}(3, 6)$, much in the spirit of Mukai’s work (cf. [7], [8]).

Let $Y$ be a smooth linear section of $\text{LG}(3, 6)$, and let $X$ be a nodal hyperplane section that contains $Y$, i.e.,

$$ Y \subseteq X \subseteq \text{LG}(3, 6) $$

then the projection from the node of $X$ restricts to an embedding

$$ i_X : Y \to G = \text{G}(2, 6). $$

Let $U_G$ be the dual of the universal rank 2 subbundle on $G$. Then

$$ E_X = i_X^* U_G $$

is a rank 2 vector bundle on $Y$ with $\det E_X = H$, the hyperplane divisor. The zero scheme $Z_X$ of a general section of $E_X$ is a smooth subvariety of codimension 2, degree 6. By adjunction the canonical divisor on $Z$ is

$$ (K_Y + H) z_X. $$
When $Y$ is a curve, $Z_X$ is empty and the important property of $E_X$ is

$$h^0(Y, E_X) = 6.$$  

In fact, the vector bundle $E_X$ is stable, so the map

$$X \mapsto [E_X]$$

maps the space of nodal hyperplane sections containing $Y$ into the moduli space

$$M_Y(2, H, [Z_X])$$

of stable rank 2 vector bundles on $Y$, with

$$c_1(E_X) = H \quad c_2(E_X) = [Z_X].$$

The nodal hyperplane sections are parameterized by the dual variety of $\text{LG}(3, 6)$. This dual variety is a quartic hypersurface $F^*$ singular in codimension 3, with a singular locus of degree 21.

**Corollary 1.** Let $Y$ be a general linear curve section of $\text{LG}(3, 6)$, and let

$$F(Y) = \{ X \mid Y \subset X \subset \text{LG}(3, 6) \quad X \text{ is a nodal hyperplane section} \}.$$  

Then $F(Y)$ is an irreducible connected component of $M_Y(2, K, 6)$, the moduli of stable rank 2 vector bundles $E$ on $Y$ with $\det E = K_Y$ and $h^0(Y, E) \geq 6$.

Furthermore, there is a compactification $\overline{F(Y)}$ of $F(Y)$, where $\overline{F(Y)}$ is a 21-nodal quartic 3-fold. The nodes form $\overline{F(Y)} - F(Y)$ and correspond to semistable bundles, more precisely to vector bundles $E$ on $Y$ that decomposes into the sum of two line bundles $L$ and $L'$ of degree 8 and $h^0(Y, L) = h^0(Y, L') = 3$.

**Remark** The general stable rank 2 vector bundle $E$ on $Y$ with $\det E = K_Y$ has $h^0(Y, E) = 0$, so $M_Y(2, K, 6)$, is called a **Non-abelian Brill Noether locus** inside the moduli space $M_Y(2, K)$, of stable rank 2 vector bundle $E$ on $Y$ with $\det E = K_Y$. Ordinary Brill Noether loci are defined similarly for line bundles.

Let $S = S_{2g-2}$ be a $K3$ surface with Picard group generated by a line bundle $\mathcal{O}_S(h)$ of odd genus $g$, and consider the moduli space $M_6(2, h, s)$, of stable rank 2 vector bundles $E$ on $S$ with $c_1(E) = h$ and $\chi(S, E) = \frac{1}{2}h^2 + 4 - c_2(E) = s + 2$. Mukai proves (cf. [4] and [7, section 10]) that any component of $M_6(2, h, s)$, is a nonsingular symplectic variety of dimension $2(g - 2s)$. In particular if $s = n = (g - 1)/2$, then $M_6(2, h, s)$, is a $K3$ surface. For a more recent proof of irreducibility see [1].

**Corollary 2.** For the general linear surface section

$$Y \subset \text{LG}(3, 6)$$

the moduli space

$$M_Y(2, H, 4)$$

is isomorphic to the quartic surface

$$F(Y) = \{ X \mid Y \subset X \subset \text{LG}(3, 6), \quad X \text{ is a nodal hyperplane section} \}.$$  

Notice that the Mukai vector of $E$ is $(r, h, s) = (2, H, 4)$, in particular $(r, s) = 2$, so in this case there is no universal vector bundle.
Corollary 3. For the general linear (Fano) 3-fold section \( Y \subset \Sigma \) the quartic curve
\[
F(Y) = \{X | \ Y \subset X \subset \mathbf{LG}(3, 6), \ X \text{ is a nodal hyperplane section}\}
\]
is an irreducible component of \( M_Y(2; -K, [Z_X]) \), where \([Z_X]\) is the class of a sextic elliptic curve on \( X \).

Corollary 4. For the general linear (Fano) 4-fold section \( Y \subset \mathbf{LG}(3, 6) \)
\[
F(Y) = \{X | \ Y \subset X \subset \mathbf{LG}(3, 6), \ X \text{ is a nodal hyperplane section}\}
\]
is 4 points in \( M_Y(2; H, [Z_X]) \), where \([Z_X]\) is the class of a sextic Del Pezzo surface on \( X \).

Corollary 5. The general complete intersection \( Y = H_1 \cap H_2 \cap Q \cap \mathbf{LG}(3, 6) \) is a Calabi Yau 3-fold and
\[
F(Y) = \{X | \ Y \subset X \subset \mathbf{LG}(3, 6), \ X \text{ is a nodal hyperplane section}\}
\]
is 4 points in \( M_Y(2; H, [Z_X]) \), where \([Z_X]\) is the class of a canonical curve of genus 7 on \( X \).

4. Strategy for the proof of the main theorem

Let \( X \) be a nodal hyperplane section of \( \mathbf{LG}(3, 6) \), and let \( \tilde{X} \to X \) be the blowup of \( X \) at the node \( u \) with exceptional divisor \( Q \), then we want to construct a rank 2 vector bundle \( E_X \) on \( \tilde{X} \) with
\[
h^0(\tilde{X}, E_X) = 6 \quad \text{and} \quad \det(E_X) = \mathcal{O}_X(H - Q)
\]
To construct the rank 2 vector bundle, let us first look at a simpler example.

Example: Let
\[
u \in Z = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^7
\]
Then the projection \( Z_u \) of \( Z \) from \( u \) is a linear section of the Grassmannian \( \mathbf{G}(2, 5) \)

Let \( \tilde{X} \to X \) be the blowup centered at \( u \) with exceptional divisor \( E \). Consider
\[
F_X = L_1 \oplus L_2 \oplus L_3
\]
where \( L_i \) is the pullback of \( \mathcal{O}_{\mathbf{P}^1}(1) \) from the \( i \)-th factor.

Let
\[
s_u \in H^0(X, F_X) \quad \text{s.t.} \quad V(s_u) = u,
\]
and the corresponding section
\[
s' \in H^0(X, F_{\tilde{X}}(-E)) \quad \text{s.t.} \quad V(s') = 0.
\]
The exterior multiplication with \( s' \) defines a surjective map
\[
\wedge s' : \wedge^2 F_{\tilde{X}}(-E) \to \wedge^3 F_{\tilde{X}}(-2E)
\]
that fits into an exact sequence
\[
0 \to F_u \to \wedge^2 F_{\tilde{X}}(-E) \to \wedge^3 F_{\tilde{X}}(-2E) \to 0,
\]
where \( F_u \) is a rank 2 vector bundle on \( \tilde{X} \).
It is now easy to check that

\[ h^0(\tilde{X}, F_u) = 5 \quad \text{and} \quad \det(F_u) = \mathcal{O}_{\tilde{X}}(H - E), \]

and that the global sections of \( \wedge^2 F_u \) defines embedding

\[ Z_u \to G(2, 5). \]

Back to \( LG(3, 6) \):

Let \( V = \mathbb{C}^6 \) be a 6-dimensional complex vector space, and let \( \alpha \in \wedge^2 V^* \) be a non-degenerate 2-form. Then

\[ L_\alpha : V \to V^* \quad v \mapsto \alpha(v, -) \]

is a correlation which induces isomorphisms

\[ L_\alpha : \wedge^3 V \to \wedge^3 V^*, \]

and

\[ L_\alpha : V(14) \to V(14)^*, \]

where

\[ V(14) := \{ w \in \wedge^3 V | \alpha(w) = 0 \} \subset \wedge^3 V \]

and

\[ V(14)^* := L_\alpha(V(14)) = \{ \omega \in \wedge^3 V^* | \omega \wedge \alpha = 0 \} \subset \wedge^3 V^*. \]

The Lagrangian Grassmannian \( LG = LG(3, 6) \) is

\[ \dim LG(3, 6) = 6 \quad \text{and} \quad LG(3, 6) = \mathbb{P}(V(14)) \cap G(3, V) \subset \mathbb{P}(\wedge^3 V). \]

We set \( LG^* = L_\alpha(LG) \subset \mathbb{P}(V(14)^*) \).

Consider a Lagrangian subspace \( U \subset V \) and its orthogonal subspace \( U^\perp \subset V^* \). In the universal sequence on \( G \)

\[ 0 \to U \to V \otimes \mathcal{O}_G \to Q \to 0, \]

the restriction of the quotient bundle to \( LG(3, 6) \) is naturally isomorphic to \( U^* \), induced by \( L_\alpha : U \to U^\perp \).

So the universal exact sequence on \( LG \) is

\[ 0 \to U \to V \otimes \mathcal{O}_{LG} \to U^* \to 0. \]

In particular the tangent bundle \( TLG \) becomes the subbundle of

\[ \text{Hom}(U, U^*) = U^* \otimes U^* \]

consisting of symmetric tensors, i.e.

\[ TLG = \text{Sym}^2 U^*. \]

Let \( U \) be a Lagrangian subspace, let \( u = [U] \in LG \), and let \( u^* = L_\alpha(u) \). Then \( L_\alpha \) defines a natural isomorphism

\[ TLG_u \to TLG^*_u. \]
To a point in the tangentspace of $LG^*$ at $u^*$ we may therefore associate a quadratic form:

$$\omega \in T_{LG^*}u^* \mapsto q_\omega \in \text{Sym}^2 U^*.$$  

The hyperplane $H_\omega$ in $P(V(14))$ defined by $\omega$ is tangent at $u$, so the hyperplane section of $H_\omega \cap LG$ is singular at $u$. The hyperplane section is nodal if and only if $q_\omega$ has maximal rank (3).

Fix $\omega \in \text{Sym}^2 U^*$, let $X$ be the corresponding nodal hyperplane section

$$X = H_\omega \cap LG$$

with node at $u$, and let

$$\hat{X} \to X$$

be the blowup of $X$ at the node $u$ with exceptional divisor $Q$. We want to construct a rank 2 vector bundle $E_X$ on $\hat{X}$ with

$$h^0(\hat{X}, E_X) = 6 \quad \text{and} \quad \det(E_X) = \mathcal{O}_X(H - Q).$$

We restrict and pull back $\land^2 U^*$ to $\hat{X}$, and consider the map defined by exterior multiplication by a form $x \in U^\perp \subset V^*$:

$$m_x : \land^2 U^*(-Q) \to \land^3 U^*(-2Q) \cong \mathcal{O}_{\hat{X}}(H - 2Q),$$

where $U^\perp = L_\alpha(U) \subset V^*$ is the Lagrangian subspace represented by $u^*$. The kernel

$$E'_x := \ker m_x$$

is our first candidate for a rank 2 vector bundle on $\hat{X}$.

$E'_x$ is a rank 2 vector bundle if the map $m_x$ is surjective. In fact the multiplication by $x$ is surjective wherever $x$ is nonzero, i.e. outside a locus of codimension 3. Thus we have an exact sequence

$$0 \to E'_x \to \land^2 U^*(-Q) \to \mathcal{O}_{\hat{X}}(H - 2Q) \to N_x \to 0,$$

where the cokernel sheaf $N_x$ has support in codimension 3. The kernel sheaf $E'_x$ is therefore torsion free and of rank 2 outside this locus.

The double dual

$$E_X = E'_x^{**}$$

is a rank 2 vector bundle on $\hat{X}$. By the defining exact sequence it has determinant $\mathcal{O}_{\hat{X}}(H - Q)$, but the problem is to get $h^0(\hat{X}, E_X) = 6$.

$$H^0(\hat{X}, E_X) = H^0(\hat{X}, E'_x)$$

so the theorem follows from

**Lemma.** $h^0(\hat{X}, E'_x) = 6$ if and only if $q_\omega(L_\alpha^{-1}(x)) = 0$.

We consider the map on global section defined by the multiplication by $x$:

$$m_x : \land^2 U^*(-Q) \to \mathcal{O}_{\hat{X}}(H - 2Q).$$

Decompose

$$V = U \oplus U_1, \quad V^* = U^\perp \oplus U_1^\perp$$
with $U_1$ also Lagrangian. Consider the corresponding decompositions of $\wedge^2 V^*$ and $\wedge^3 V^*$, then

$$V^*(14) \cap \wedge^2 U^\perp \otimes U_1^\perp + \wedge^3 U^\perp \cong \text{Sym}^2 U^* \oplus \wedge^3 U^\perp.$$  

Since $Q$ is the exceptional divisor over the point $u = [U]$, we get a natural surjection

$$\text{Sym}^2 U^* \oplus \wedge^3 U^\perp \to H^0(\mathcal{O}_X(H - 2Q)).$$

The kernel is generated by $\omega \in \text{Sym}^2 U^*$, so $h^0(X, \mathcal{O}_X(H - 2Q)) = 6$.

Similarly there is a surjection

$$U^\perp \otimes U_1^\perp + \wedge^3 U^\perp \to H^0(\wedge^2 U^*(-Q)),$$

with 12-dimensional domain, whose kernel is generated by $\alpha$ so

$$h^0(\wedge^2 U^*(-Q)) = 11.$$

The subspace

$$U_\alpha = \{ \eta = \alpha \wedge y + \beta \wedge x | \eta \wedge \alpha = 0 \} \subset V^*(14) \cap \wedge^2 U^\perp \otimes U_1^\perp + \wedge^3 U^\perp$$

has dimension 6. The image of the map $m_\alpha$ on global sections is nothing but the projection of $U_\alpha$ from the form $\omega$. This all fits in the following commutative diagram of vector spaces:

\[
\begin{array}{c}
\begin{array}{c}
\langle \alpha \rangle \\
\downarrow \\
\langle \omega \rangle \\
\downarrow \\
U^\perp \otimes U_1^\perp + \wedge^3 U^\perp \\
\downarrow \\
0 \\
\end{array}
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
H^0(E_\alpha) \\
H^0(\wedge^2 U^*(-Q)) \\
H^0(\mathcal{O}_X(H - 2Q)) \\
0 \\
\end{array}
\end{array}
\]

Therefore $m_\alpha$ is not surjective on global sections if and only if $\omega$ is an element of $U_\alpha$.

It is now a straightforward computation in local coordinates to show that:

$$\omega \in U_\alpha \text{ if and only if } q_\omega(L_\alpha^{-1}(x)) = 0.$$  

Q.E.D.

**Remark** The vector bundle $E_X$ is independent of which $x \in U^\perp$ we choose, as long as $q_\omega(L_\alpha^{-1}(x)) = 0$. Furthermore the fact that $X$ was a nodal hyperplane section played a role in the proof only to produce enough global sections for $E_X$.

**Problem:** Prove an analogue of the main theorem in the cases $g = 7, 8$.

**References**


