Circular meshes, cyclide splines and geometric algebra

Rimas Krasauskas

Vilnius University, Lithuania

Algebraic geometry in the sciences, Oslo, January 13, 2011
Outline

1. Introduction
   - Circular meshes in architectural design/engineering
   - Cyclides in computer aided geometric design (CAGD)

2. Generalized cyclide splines
   - Prototype example
   - Cyclide splines from framed quasi-circular meshes
   - More examples

3. Using quaternions and geometric algebra
   - Bézier parametrizations with quaternionic weights
   - Geometric algebra and conformal model

4. Conclusions
Outline

1. Introduction
   - Circular meshes in architectural design/engineering
   - Cyclides in computer aided geometric design (CAGD)

2. Generalized cyclide splines
   - Prototype example
   - Cyclide splines from framed quasi-circular meshes
   - More examples

3. Using quaternions and geometric algebra
   - Bézier parametrizations with quaternionic weights
   - Geometric algebra and conformal model

4. Conclusions
Outline

1. Introduction
   - Circular meshes in architectural design/engineering
   - Cyclides in computer aided geometric design (CAGD)

2. Generalized cyclide splines
   - Prototype example
   - Cyclide splines from framed quasi-circular meshes
   - More examples

3. Using quaternions and geometric algebra
   - Bézier parametrizations with quaternionic weights
   - Geometric algebra and conformal model

4. Conclusions
Outline

1. Introduction
   - Circular meshes in architectural design/engineering
   - Cyclides in computer aided geometric design (CAGD)

2. Generalized cyclide splines
   - Prototype example
   - Cyclide splines from framed quasi-circular meshes
   - More examples

3. Using quaternions and geometric algebra
   - Bézier parametrizations with quaternionic weights
   - Geometric algebra and conformal model

4. Conclusions
Introduction

- Circular meshes in architectural design/engineering
Freeform structures in architecture

Freeform geometries are becoming increasingly popular in contemporary architecture:

Vilnius Guggenheim Hermitage Museum is a proposed art museum in Vilnius, Lithuania. Author: a British-Iraqi architect Zaha Hadid, 2008. The museum was scheduled to open in 2011...
Freeform structures in architecture

Freeform geometries are becoming increasingly popular in contemporary architecture:

Vilnius Guggenheim Hermitage Museum is a proposed art museum in Vilnius, Lithuania. Author: a British-Iraqi architect Zaha Hadid, 2008. The museum was scheduled to open in 2011...
Planar quad (PQ) meshes possess a number of important advantages over triangular meshes:

- a smaller number of edges, i.e. supporting beams following the edges – less steel and less cost
- a lower node complexity – important for manufacturing
Planar quad (PQ) meshes possess a number of important advantages over triangular meshes:

- a smaller number of edges, i.e. supporting beams following the edges – less steel and less cost
- a lower node complexity – important for manufacturing
Planar quad (PQ) meshes possess a number of important advantages over triangular meshes:

- a smaller number of edges, i.e. supporting beams following the edges – less steel and less cost
- a lower node complexity – important for manufacturing
PQ meshes with offsets

Frequently two layers of an actual construction are needed:

**Definition**

PQ mesh is a **conical mesh** if for all vertices the four incident face planes are tangent to a common oriented cone of revolution.

**Definition**

PQ mesh is a **circular mesh** if for all faces the four incident vertices are on a circle.
Conical and circular meshes

**Theorem**

A simply connected PQ mesh possesses a face-offsett mesh if and only if it is a conical mesh.


**Theorem**

A simply connected PQ mesh possesses a vertex-offsett mesh if and only if it is a circular mesh.

Duality between conical and circular meshes

[picture by Stefan Sechelmann]
Approximate conical meshes

This conical mesh in front was obtained by a combination of Catmull-Clark subdivision and conical optimization from the control mesh behind.

Introduction

Cyclides in computer aided geometric design (CAGD)
Discovery of cyclides


Later studied by Maxwell (1868) and Cayley (1878).

Plate V. Maxwell’s real image *stereoscope* (1867), showing a stereogram of Steiner’s surface (Number 272).
Discovery of cyclides


Later studied by Maxwell (1868) and Cayley (1878).

Plate V. Maxwell's real image stereoscope (1867), showing a stereogram of Steiner’s surface (Number 272).
Dupin cyclides

An alternative viewpoint to this discovery:

 [...] the cyclide surface was first explored by Victorian geometers. It is often called the Dupin cyclide after a French mathematician who published some of its properties.

Definition

Dupin cyclides have several equivalent definitions:

- any inversion of any standard torus
- a surface with curvature lines all circles
- an envelope of a family of spheres touching 3 given spheres
- a canal surface in two different ways
- the focal surface degenerate to a couple of conics
Dupin cyclides

An alternative viewpoint to this discovery:

\[
... \text{the cyclide surface was first explored by Victorian geometers. It is often called the Dupin cyclide after a French mathematician who published some of its properties.}
\]

Definition

Dupin cyclides have several equivalent definitions:

- any inversion of any standard torus
- a surface with curvature lines all circles
- an envelope of a family of spheres touching 3 given spheres
- a canal surface in two different ways
- the focal surface degenerate to a couple of conics
Different cases of Dupin cyclides are inversions of a standard torus:
Principal patches of cyclides

Interest in cyclides revived in the 1980’s. It was motivated by research in CAGD by R. Martin (1982), who considered principal patches of cyclides, i.e. bounded by circles which are curvature lines.

Later studied by Chandru, Dutta, Hoffmann, Pratt and others.
Principal patches of cyclides

Interest in cyclides revived in the 1980’s. It was motivated by research in CAGD by R. Martin (1982), who considered *principal patches* of cyclides, i.e. bounded by circles which are curvature lines. Later studied by Chandru, Dutta, Hoffmann, Pratt and others.
Composing cyclide patches

The idea of smoothly composing principal Dupin cyclide patches:

Dutta, Martin and Pratt (1993) made a conclusion:

*Cyclides appear marginally insufficient with respect to the degrees of freedom of shape control...*
Composing cyclide patches

The idea of smoothly composing principal Dupin cyclide patches:

Dutta, Martin and Pratt (1993) made a conclusion:

Cyclides appear marginally insufficient with respect to the degrees of freedom of shape control...
Approximate methods


Computer Graphics Group, HKU: Wenping Wang, Peng Bo,...
Generalized cyclide splines

- Problem formulation
Problem formulation

Cyclide splines based on regular circular meshes have restrictions:

- they cannot represent a surface of genus other than 0 or 1,
- a cyclide either have no umbilic points, or is a sphere.

Problem

*Find a natural generalization of circular meshes and associated cyclide splines, that*

- *can represent any topology,*
- *can have isolated umbilic points.*
Generalized cyclide splines

Prototype example
Filling a $D_3$-symmetric hexagonal hole

Step 0
Filling a $D_3$-symmetric hexagonal hole

Step 1
Filling a $D_3$-symmetric hexagonal hole

Step 2
Filling a $D_3$-symmetric hexagonal hole

Step 3
Circular mesh extraction

The resulting circular quad mesh contains a planar hexagon in the center.
A **framed mesh** is a mesh with 4-valence vertices and two families of vectors: normals \( \{ N_v \} \) indexed by vertices \( v \) and tangents \( \{ T_{v,e} \} \) indexed by vertex–edge pairs \( (v, e) \), \( v \in e \), subject to the following conditions:

1. for every corner of a face defined by a vertex \( v \) and two incident edges \( e, e' \) a triple of vectors \( (N_v, T_{v,e}, T_{v,e'}) \) is an orthonormal frame;
2. any two such frames associated with adjacent corners of the same face are symmetric w.r.t. to their common edge.
Spline construction

Starting from a framed quasi-circular mesh we construct a spline surface $S$ in two steps:

1. For every edge $e$ its endpoints $v$, $v'$ are connected by a unique circular/linear segment $C(e)$ with inward tangents $T_{v,e}$, $T_{v',e}$.

2. For every face $f$ the associated circular/linear boundary $\bigcup_{e \subset f} C(e)$ is filled by $n$-sided surface $S(f)$ with the normals $N_v$ at the corner vertices $v$ according to three cases:
   - $n = 2, 3$: $S(f)$ is a unique spherical patch;
   - $n = 4$: $S(f)$ is a unique principal cyclide patch;
   - $n > 4$: $S(f)$ is a special $n$-sided multipatch having a subdivision-like layout: shrinking concentric rings composed of principal cyclide patches.

This limit procedure cannot be avoided in general: there are no isolated umbilic points on cyclides.
Details on hole filling

Here: $p_i, c_{i,i+1}$ are corner vertices and boundary circles. Choose pairs of points $p_{i,i+1}^L, p_{i,i+1}^R \in c_{i,i+1}$, such that $c_{i,i+1}, p_{i-1,i}^R, p_{i+1,i+2}^L$ are on the same sphere.

Define new circles $c_i = p_{i-1,i}^R \land p_i \land p_{i,i+1}^R$.

Any point $q_1 \in c_1$ defines unique $q_2 \in c_2$ and so on...
Details on hole filling

Here: $p_i, c_{i,i+1}$ are corner vertices and boundary circles. Choose pairs of points $p_{i,i+1}^L, p_{i,i+1}^R \in c_{i,i+1}$, such that $c_{i,i+1}, p_{i-1,i}^R, p_{i+1,i+2}^L$ are on the same sphere. Define new circles $c_i = p_{i-1,i}^R \wedge p_i \wedge p_{i,i+1}^R$. Any point $q_1 \in c_1$ defines unique $q_2 \in c_2$ and so on...
Branching blend of cylinders

Blending with four non-symmetric pentagonal patches:
Using quaternions
Circular arcs in $\mathbb{R}^3$

Let $p_0$ and $p_1$ are two endpoints of a circular arc in $\mathbb{R}^3$, and let $f$ be some interior point on it. Then the arc can be rationally parametrized by the quaternionic formula (here $a/b = ab^{-1}$)

$$C(t) = \frac{p_0 w_0 (1 - t) + p_1 w_1 t}{w_0 (1 - t) + w_1 t} \in \mathbb{R}^3,$$

where ‘weights’ are $w_0 = (f - p_0)^{-1}$, $w_1 = (p_1 - f)^{-1}$, and $f = C(\frac{1}{2})$ is called a Farin point.

Alternatively, weights can be defined via a tangent vector $v_0$ at $p_0$:

$$w_0 = 1, \quad w_1 = (p_1 - p_0)^{-1} v_0.$$
Circular splines

A smooth circular spline going through a sequence of points \( p_0, \ldots, p_{n-1} \) is uniquely defined by a tangent vector \( v_0 \) at the point \( p_0 \). Then other unit tangent vectors are defined by recurrence:

\[
v_{i+1} = (p_{i+1} - p_i)^{-1} v_i (p_{i+1} - p_i),
\]

i.e. its direction is an opposite to the reflected vector w.r.t. the edge \( p_{i+1} - p_i \).

Suppose the number of points is even \( n = 2k \), and the spline curve is closed, i.e. \( p_{2k} = p_0 \).
**Multiratio condition**

**Definition**

A *multi-ratio* of points $p_0, \ldots, p_{2k-1}$

$$mr(p_0, \ldots, p_{2k-1}) = (p_0 - p_1)(p_1 - p_2)^{-1} \cdots (p_{2k-1} - p_0)^{-1}$$

For example:

- $k = 0$: $mr(p_0, p_1) = -1$, when $p_0 \neq p_1$.
- $k = 2$: $mr(p_0, p_1, p_2, p_3)$ coincides with the classical cross-ratio, which is real if and only if these four points are on a circle.

**Lemma**

*A closed circular spline exists for every initial tangent $v_0$ and even number of points* $\Leftrightarrow$ *their multi-ratio is real.*
Quasi-circular meshes

**Definition**

A *Quasi-circular even* (QCE) mesh is a mesh with even sided faces and 4-valency vertices satisfying multi-ratio condition

$$mr(p_0, \ldots, p_{2k-1}) \in \mathbb{R}$$

for every even loop of edges $$(p_0, p_1), (p_1, p_2), \ldots, (p_{2k-1}, p_0)$$

This is more general definition than in:


Any QCE mesh with a fixed frame at a vertex naturally defines a framed mesh which can be used to generate a cyclide spline (as was shown above).

Similarly a QCE mesh defines a vertex-offset and a face-offset meshes.
Spherical patches

Further applications of quaternions.

Let $S$ be a spherical triangle with corner points $p_0$, $p_1$, $p_2$ bounded by three circular arcs, such that these three circles intersect in a point $p_\infty$.

Then this spherical triangle $S$ can be rationally parametrized by the quaternionic formula

$$S(s, t) = \frac{p_0 w_0 (1 - s - t) + p_1 w_1 s + p_2 w_2 t}{w_0 (1 - s - t) + w_1 s + w_2 t},$$

with weights: $w_i = (p_i - p_\infty)^{-1}$, $i = 0, 1, 2$. 

R. Krasauskas (VU, Lithuania)
Principal cyclide patches

Let \( p_0, p_1, p_2, p_3 \) be any 4 points on a circle in \( \mathbb{R}^3 \), and let \( v_1, v_2 \) be two orthonormal vectors.

Then there is a unique principal Dupin cyclide patch \( D \) with corners in these points, and bounded by circular arcs with tangent vectors \( v_1, v_2 \) at the corner \( p_0 \), which can be rationally parametrized by the quaternionic formula

\[
D(s, t) = \frac{p_0 w_0 (1 - s)(1 - t) + p_1 w_1 s(1 - t) + p_2 w_2 (1 - s)t + p_3 w_3 st}{w_0 (1 - s)(1 - t) + w_1 s(1 - t) + w_2 (1 - s)t + w_3 st},
\]

where \( w_i \) are defined using vectors \( q_{ij} = (p_i - p_j) / |p_i - p_j| : \)

\[
w_0 = 1, \quad w_1 = q_{10} v_1, \quad w_2 = q_{20} v_2, \\
w_3 = |p_2 - p_1| |p_3 - p_0|^{-1} q_{31} w_1 q_{20} w_2.
\]
Geometric algebra

All formulas involving quaternions can be written in terms of geometric algebra.

Geometric algebra (GA) = Clifford algebra + geometric content

Start from a vector space with a given inner product. The geometric product of two vectors $a$ and $b$ is defined to be associative and distributive over addition, with additional rule: $a \cdot a = a^2 \in \mathbb{R}$. Define symmetric inner and anti-symmetric exterior products:

$$a \cdot b = \frac{1}{2}(ab + ba), \quad a \wedge b = \frac{1}{2}(ab - ba).$$
Conformal model of Euclidean space $\mathbb{R}^3$

Consider homogenous coordinates in $\mathbb{R}^4$ associated with a standard basis $\{e_i\}, \; i = 1, \ldots, 5$ ($x_1 = 0$ is a plane at infinity). Then slightly change the basis:

$$e_\infty = e_4 + e_5, \quad e_0 = (-e_4 + e_5)/2.$$ 

Use a stereographic projection $\mathbb{R}^3 \rightarrow S^3 \subset \mathbb{R}^4$ to the unit sphere

$$F : x \mapsto x + \frac{1}{2}x^2 e_\infty + e_0,$$

where $x^2 = x \cdot x$ is a Euclidean inner product.
The basis: \( \{ e_i \} \), \( i = 1, \ldots, 5 \) generates an algebra that is spanned by \( 1 + 5 + 10 + 10 + 5 + 1 = 32 \) terms:

\[
1, \{ e_i \}, \{ e_i \wedge e_j \}, \{ e_i \wedge e_j \wedge e_k \}, \{ le_i \}, l,
\]

where \( l = e_1 e_2 e_3 e_4 e_5 \) is called a pseudo-scalar, \( l^2 = -1 \).

Also a *meet* product will be useful:

\[
(L_1 \vee L_2)^* = L_1^* \wedge L_2^*, \quad X^* = lX.
\]
Geometric meaning

Points: \( p, p \cdot p = 0 \).
Pairs of points: \( p_1 \wedge p_2, p_i \) – points.
Circles: \( p_1 \wedge p_2 \wedge p_3 \).
Spheres: \( p_1 \wedge p_2 \wedge p_3 \wedge p_4 \).
Intersection of two geometric objects: \( L_1 \vee L_2 \).
For example, intersection of a sphere \( s \) and a circle \( c \) with a common point \( p_1 \).

\[
q = s \vee c
\]

Then we compute a plane of symmetry of the pair \( Q := q^* \wedge e_\infty \) and extract the second point by reflecting the known point: \( p_2 = Qp_1 \).
Conclusions

We generalized
  - the notion of a regular circular quad mesh
  - the associated Dupin cyclide patchwork
to:
  - the notion of framed quasi-circular mesh involving non-quad faces
  - the associated cyclide spline surface of arbitrary topology.

It seems all constructions have a nice description in terms of geometric algebra.

Problems:
  - hole filling problem - extra shape control required
  - relations Laguerre and Lie sphere geometries.
Conclusions

We generalized

- the notion of a regular circular quad mesh
- the associated Dupin cyclide patchwork

to:

- the notion of framed quasi-circular mesh involving non-quad faces
- the associated cyclide spline surface of arbitrary topology.

It seems all constructions have a nice description in terms of geometric algebra.

Problems:

- hole filling problem - extra shape control required
- relations Laguerre and Lie sphere geometries.
Conclusions

We generalized

- the notion of a regular circular quad mesh
- the associated Dupin cyclide patchwork

to:

- the notion of framed quasi-circular mesh involving non-quad faces
- the associated cyclide spline surface of arbitrary topology.

It seems all constructions have a nice description in terms of geometric algebra.

Problems:

- hole filling problem - extra shape control required
- relations Laguerre and Lie sphere geometries.
Conclusions

We generalized

- the notion of a regular circular quad mesh
- the associated Dupin cyclide patchwork
to:

- the notion of framed quasi-circular mesh involving non-quad faces
- the associated cyclide spline surface of arbitrary topology.

It seems all constructions have a nice description in terms of geometric algebra.

Problems:

- hole filling problem - extra shape control required
- relations Laguerre and Lie sphere geometries.
Conclusions

We generalized

- the notion of a regular circular quad mesh
- the associated Dupin cyclide patchwork

to:

- the notion of framed quasi-circular mesh involving non-quad faces
- the associated cyclide spline surface of arbitrary topology.

It seems all constructions have a nice description in terms of geometric algebra.

Problems:

- hole filling problem - extra shape control required
- relations Laguerre and Lie sphere geometries.
Conclusions

We generalized
- the notion of a regular circular quad mesh
- the associated Dupin cyclide patchwork
to:
- the notion of framed quasi-circular mesh involving non-quad faces
- the associated cyclide spline surface of arbitrary topology.
It seems all constructions have a nice description in terms of geometric algebra.

Problems:
- hole filling problem - extra shape control required
- relations Laguerre and Lie sphere geometries.
Conclusions

We generalized
- the notion of a regular circular quad mesh
- the associated Dupin cyclide patchwork
to:
- the notion of framed quasi-circular mesh involving non-quad faces
- the associated cyclide spline surface of arbitrary topology.

It seems all constructions have a nice description in terms of geometric algebra.

Problems:
- hole filling problem - extra shape control required
- relations Laguerre and Lie sphere geometries.
References (including figures)

Colaborators

Severinas Zubė (Vilnius University)
Wenping Wang (The University of Hong Kong)
Pengbo Bo (The University of Hong Kong)

Acknowledgements

Kestas Karčiauskas (VU), Heidi Dahl (VU, SINTEF).

CLUCalc interactive visualization software created by C. Perwass (http://www.clucalc.info/) was used intensively for preparation of this talk.
Questions

Thank you!