CANONICAL CURVES AND VARIETIES OF SUMS OF POWERS OF CUBIC POLYNOMIALS

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ABSTRACT. In this note we show that the apolar cubic forms associated to codimension 2 linear sections of canonical curves of genus $g \geq 11$ are special with respect to their presentation as sums of cubes.

1. Introduction

In a graded Artinian Gorenstein ring $A$ with socle degree $d$, multiplication defines (up to scalar) a homogeneous form $f$ of degree $d$, called the socle degree generator, dual polynomial or apolar polynomial of $A$. Codimension 2 linear sections of a canonical curve of genus $g$ define Artinian Gorenstein quotients of the homogeneous coordinate ring of the curve. These quotients have socle degree 3 and therefore define (up to scalar) cubic forms in $g-2$ variables. A dimension count shows that a general cubic form is not obtained this way when $g \geq 8$. While a general cubic form in $g-2$ variables cannot be written as a sum of less than $\frac{4}{3}(g-1)$ cubes, our main result says that the cubic forms apolar to a general codimension 2 linear section of a general canonical curve of genus $g \geq 11$ can be written as a sum of $2g-4$ cubes.

Our methods give results concerning the variety of different powersum presentations. In particular we obtain partial results for genus $g=9$ (cf. 3). Results for $g \leq 6$ are classical, while $g=7$ and $g=8$ was treated in [7] and [6].

Powersum presentations of forms from a more algebraic viewpoint have been studied extensively in [5].

We work throughout over the complex numbers $\mathbb{C}$.

1.1. Powersum presentations. Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous form of degree $d$, then $f$ can be written as a sum of powers of linear forms

$$f = l_1^d + \cdots + l_s^d$$

for $s$ sufficiently large. Indeed, if we identify the map $l \mapsto l^d$ with the $d^\text{th}$ Veronese embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^{Nd}$, where $N_d = \binom{n+d}{n} - 1$, this amounts to say that the image spans $\mathbb{P}^{Nd}$. Fixing $(d,n)$, the minimal number $s$ of summands needed varies with $f$, of course. A simple dimension count shows that

$$s \geq \left\lceil \frac{1}{n+1} \left( \frac{n+d}{n} \right) \right\rceil$$

for a general $f$. With a few exceptions equality holds by a result of Alexander and Hirschowitz [1] combined with Terracini’s Lemma (cf. [4]).

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Theorem 1.1. (Alexander, Hirschowitz) A general form $f$ of degree $d$ in $n+1$ variables is a sum of $\left[\frac{n+1}{d+1}\right]^{n+1}$ powers of linear forms, unless

- $d = 2$, where $s = n + 1$ instead of $\left[\frac{n+2}{2}\right]$, or
- $d = 4$ and $n = 2, 3, 4$, where $s = 6, 10, 15$ instead of $5, 9, 14$ respectively, or
- $d = 3$ and $n = 4$, where $s = 8$ instead of $7$.

Let $F = Z(f) \subset \mathbb{P}^n$ be the hypersurface defined by $f$. For a linear form $l$ we denote by $L$ the point in $\mathbb{P}^n$ of the hyperplane $Z(l) \subset \mathbb{P}^n$. Then we define, as in [7], the variety of sums of powers as the closure

$$VSP(F, s) = \left\{ (l_1, \ldots, l_s) \in \text{Hilb}_{s}(\mathbb{P}^n) : \exists \lambda_i \in \mathbb{C} : f = \lambda_1 l_1 + \cdots + \lambda_s l_s \right\}$$

of the set of powersums presenting $f$ in the Hilbert scheme (cf. [7]). Notice that taking $d$th roots of the $\lambda_i$, we can put them into the forms $l_i$. We study these varieties of sums of powers using apolarity.

1.2. Apolarity. (cf. [7]). Consider $R = \mathbb{C}[x_0, \ldots, x_n]$ and $T = \mathbb{C}[\partial_0, \ldots, \partial_n]$. $T$ acts on $R$ by differentiation:

$$\partial^\alpha \cdot x^\beta = \alpha! \binom{\beta}{\alpha} x^{\beta - \alpha}$$

if $\beta \geq \alpha$ and 0 otherwise. Here $\alpha$ and $\beta$ are multi-indices, $\binom{\beta}{\alpha} = \prod \binom{\beta_i}{\alpha_i}$ and so on. One can interchange the role of $R$ and $T$ by defining

$$x^\beta \cdot \partial^\alpha = \beta! \binom{\beta}{\alpha} \partial^{\beta - \alpha}.$$

This action defines a perfect pairing between forms of degree $d$ and homogeneous differential operators of order $d$. In particular, $R_1$ and $T_1$ are natural dual vector spaces. Therefore the projective spaces with coordinate ring $R$ and $T$ respectively are natural dual to each other, we denote them by $\mathbb{P}^n$ and $\mathbb{P}^n$. A point $a = (a_0, \ldots, a_n) \in \mathbb{P}^n$ defines a form $l_a = \sum a_i x_i \in R_1$, and for a form $D \in T_1$

$$D \cdot l_a = e! \binom{d}{e} D(a) x_a^{d-e},$$

when $e \leq d$. In particular

$$(*) \quad D \cdot l_a^0 = 0 \iff D(a) = 0$$

if $e \leq d$. More generally we say that homogeneous forms $f \in R$ and $D \in T$ are apolar if $f \cdot D = D \cdot f = 0$ (According to Salmon (1885) [8] the term was coined by Reye).

Apolarity allows us to associate an Artinian Gorenstein graded quotient ring of $T$ to a form: For $f \in R$ a homogeneous form of degree $d$ and $F = Z(f) \subset \mathbb{P}^n$ define

$$F^\perp = f^\perp = \{ D \in T : D \cdot f = 0 \}$$

and

$$A^F = T/F^\perp.$$

The socle degree of $A^F$ is $d$, since

$$D \cdot (D \cdot f) = 0 \forall D \in T_1 \iff D \cdot f = 0 \text{ or } D \in T_1.$$  

In particular the socle of $A^F$ is 1-dimensional, and $A^F$ is Gorenstein. It is called the apolar Artinian Gorenstein ring of $F$. Conversely for a graded Gorenstein ring $A = T/I$ with socle degree $d$, multiplication in $A$ induces a linear form $f : \text{Sym}_d(T_1) \to \mathbb{C}$
which can be identified with a homogeneous polynomial \( f \in R \) of degree \( d \). This proves:

**Lemma 1.2.** (Macaulay, [2]) The map \( F \mapsto A^F \) is a bijection between hypersurfaces \( F = \mathcal{Z}(f) \subset \mathbb{P}^n \) of degree \( d \) and graded Artinian Gorenstein quotient rings \( A = T/I \) of \( T \) with socle degree \( d \).

Let \( X \subset \mathbb{P}^{n+m+1} \) be a \( m \)-dimensional arithmetic Gorenstein variety. Let \( S(X) \) be the homogeneous coordinate ring of \( X \), and let \( h_1, \ldots, h_{m+1} \) be general linear forms and set \( L = \mathcal{Z}(h_1, \ldots, h_{m+1}) \). Then by definition \( S(X)/(h_1, \ldots, h_{m+1}) \) is Artinian Gorenstein, i.e. by Macaulays result the apolar Artinian Gorenstein ring of a \((n-1)\)-dimensional hypersurface \( F_L \) of degree \( d \), the socle degree of the ring, \( L \) is a linear space of dimension \( n \) and by apolarity \( F_L = \mathcal{Z}(f_L) \) is a hypersurface in the dual space to \( L \). We say that \( F_L \) is apolar to the (empty) linear section \( L \cap X \).

Hence, there is a rational map

\[
\alpha_X : \mathbb{G}(n+1, m+n+2) \to H_{n,d}
\]

Where \( H_{n,d} \) is the space of \((n-1)\)-dimensional hypersurfaces of degree \( d \) modulo the action of \( \text{PGL}(n+1, k) \).

A canonical curve \( C \subset \mathbb{P}^{g(C)-1} \) is arithmetic Gorenstein, i.e. the homogeneous coordinate ring \( S(C) \) is Gorenstein. Let \( h_1, h_2 \in S(C) \) be two general linear forms, then the quotient \( S(C)/(h_1, h_2) \) is Artinian Gorenstein with values of the Hilbert function: \( 1, g-2, g-2, 1 \). Its socle degree is therefore 3. Thus we obtain a map

\[
\alpha_C : \mathbb{G}(g(C)-2, g(C)) \to H_{g(C)-3,3}
\]

to the space of cubic hypersurfaces of dimension \( g(C)-4 \). We shall study the image of this map. In particular we shall study the variety of sums of powers of the cubic hypersurfaces in this image.

**1.3. Variety of apolar subschemes.** Let \( F = \mathcal{Z}(f) \subset \mathbb{P}^n \) denote a hypersurface of degree \( d \). We call a subscheme \( \Gamma \subset \mathbb{P}^n \) apolar to \( F \) if the homogeneous ideal \( I_{\Gamma} \subset T^1 \subset T \).

**Apolarity Lemma 1.3.** Let \( l_1, \ldots, l_s \) be linear forms in \( R \), and let \( L_i \in \mathbb{P}^n \) be the corresponding points in the dual space. Then \( f = \lambda_1 l_1^0 + \cdots + \lambda_s l_s^0 \) for some \( \lambda_i \in \mathbb{C}^* \) if and only if \( \Gamma = \{L_1, \ldots, L_s\} \subset \mathbb{P}^n \) is apolar to \( F = \mathcal{Z}(f) \).

**Proof.** Assume \( f = \lambda_1 l_1^0 + \cdots + \lambda_s l_s^0 \). If \( g \in I_{\Gamma} \), then \( g \cdot l_i^0 = 0 \) for all \( i \) by (*), so by linearity \( g \in I^1 \). Therefore \( \Gamma \) is apolar to \( F \).

For the converse, assume that \( I_{\Gamma} \subset F^1 \). Then we have surjective maps between the corresponding homogeneous coordinate rings

\[
T \to A_{\Gamma} = T/I_{\Gamma} \to A^F.
\]

Consider the dual inclusions of the degree \( d \) part of these rings:

\[
\text{Hom}(A_{\Gamma}^d, C) \to \text{Hom}(A_{\Gamma}^d, C) \to \text{Hom}(T_d, C).
\]

\( D \mapsto D \cdot f \) generates the first of these spaces, while the second is spanned by the forms \( D \mapsto D \cdot l_i^d \), thus \( f \) lies in the span of the \( l_i^d \). \( \square \)

This is the crucial lemma in the study of powersum presentations of \( f \). Furthermore it allows us to define a variety of apolar subschemes to \( f \), which naturally extends our definition of the variety of sums of powers.

\[
VPS(F, s) = \{ \Gamma \in Hilb_{s}(\mathbb{P}^n) \mid I_{\Gamma} \subset F^1 \},
\]
where $\text{Hilb}_s(\mathbb{P}^n)$ is the Hilbert scheme of length $s$ subschemes of $\mathbb{P}^n$. Clearly $\text{VSP}(F, s)$ is the closure of the set parametrizing smooth subschemes in $\text{VPS}(F, s)$. In general they do not coincide.

2. Apolar varieties of singular sections

2.1. Apolar varieties. Let $X \subset \mathbb{P}^{n+m+1}$ be a reduced and irreducible $m$-dimensional nondegenerate variety of degree $d \geq 3$ and codimension $n+1 \geq 2$. Let $p \in X$ be a general smooth point. Let $C_pX$ be the cone over $X$ with vertex at $p$. Since $p$ is a smooth point, the degree of the cone $C_pX$ is $d-1$, while the dimension is $m+1$. Clearly $X \subset C_pX$.

We apply this construction to describe powersum presentations of hypersurfaces in the image of the map $\alpha_X$ in 1.2. Let again $X \subset \mathbb{P}^{n+m+1}$ be a $m$-dimensional arithmetic Gorenstein variety of degree $d$. Fix a general $n$-dimensional linear subspace $L \subset \mathbb{P}^{n+m+1}$, in particular we fix the hypersurface $F_L$ in the image of $\alpha_X$. Let $p$ be a smooth point on $X$, then the intersection $C_pX \cap L$ is clearly nonempty, and if it is proper it is 0-dimensional of degree $d-1$. We may assume that this intersection is proper and smooth for a general $L$ or general $p$, so we get an apolar subscheme of degree $d-1$ to $F_L$, i.e. a point in $\text{VSP}(F_L, d-1)$. We have shown:

**Proposition 2.1.** Let $X \subset \mathbb{P}^{n+m+1}$ be a $m$-dimensional arithmetic Gorenstein variety of degree $d$, and let $L \subset \mathbb{P}^{n+m+1}$ be a $n$-dimensional linear subspace such that $L \cap X = \emptyset$. Let $F_L$ be the associated apolar hypersurface. Then there is a rational map $X \dashrightarrow \text{VSP}(F_L, d-1)$ defined by $p \mapsto C_pX \cap L$.

**Problem 2.2.** When is this map a morphism? When can $F_L$ and $X$ be recovered from the image of this map?

We may improve slightly on the degree of the apolar subschemes by considering cones on special linear sections of $X$.

2.2. Tangent hyperplane sections. Let $X \subset \mathbb{P}^{n+m+1}$ be a reduced and irreducible $m$-dimensional nondegenerate variety of degree $d$ and codimension $n+1 \geq 2$. We assume additionally that $X$ satisfies the following condition:

A general tangent hyperplane section of $X$ has a double point at

the point of tangency, and the projection of the tangent hyperplane section from the point of tangency is birational.

In particular, $X$ is not a scroll and $d \geq 4$. Let $p \in X$ be a general smooth point. Let $H_p$ be a general hyperplane tangent to $X$ at $p$. Since $H_p \cap X$ has multiplicity 2 at $p$ and the projection of $H_p \cap X$ from $p$ is birational, the image of the projection is $(m-1)$-dimensional of degree $d-2$. Therefore $H_p \cap X$ is contained in an $m$-dimensional cone $C_p(H_p \cap X)$ of degree $d-2$ with vertex at $p$. Similarly, if $H_p$ and $H'_p$ are two general hyperplanes tangent at $p$, then the intersection $H_p \cap H'_p \cap X$ has a singularity at $p$ of multiplicity 4, the complete intersection of two singularities of multiplicity 2. In this case we say that the codimension 2 space $H_p \cap H'_p$ is doubly tangent to $X$ at $p$. If

the projection of $H_p \cap H'_p \cap X$ from $p$ is birational,

(***)
then the image \((m - 2)\)-dimensional of degree 4 less than the degree of \(X\). Hence \(H_p \cap H'_p \cap X\) is contained in a \((m - 1)\)-dimensional cone \(C_p(H_p \cap H'_p \cap X)\) of degree \(d - 4\) with vertex at \(p\). This proves the

**Lemma 2.3.** Let \(X \subset \mathbb{P}^{n+m+1}\) be a smooth \(m\)-dimensional nondegenerate variety of degree \(d\), and assume that \(X\) satisfies condition (**). Let \(p \in X\) be a general smooth point, and let \(H_p\) be a general hyperplane tangent to \(X\) at \(p\). Then the cone \(C_p(H_p \cap X)\) is an \(m\)-dimensional variety of degree \(d - 2\) that contains \(H_p \cap X\). Assume furthermore that \(X\) satisfies condition (**), and let \(H_p\) and \(H'_p\) be two general hyperplanes tangent to \(X\) at \(p\). Then the cone \(C_p(H_p \cap H'_p \cap X)\) is a \((m - 1)\)-dimensional variety of degree \(d - 4\) that contains \(H_p \cap H'_p \cap X\).

As above we apply this lemma to describe powersum presentations of hypersurfaces in the image of the map \(\alpha_X\) in section 1.2. Let again \(X \subset \mathbb{P}^{n+m+1}\) be a \(m\)-dimensional arithmetic Gorenstein variety of degree \(d\), with \(m \geq 1\). We assume additionally that \(X\) satisfies condition (**). Fix a general \(n\)-dimensional linear subspace \(L \subset \mathbb{P}^{n+m+1}\), in particular we fix the hypersurface \(F_L\) in the image of \(\alpha_X\). If \(L \subset H_p\) where \(H_p\) is a general hyperplane tangent at \(p\), then according to 2.1 there is a \((m - 1)\)-dimensional variety \(Y \supset H_p \cap X\) of degree \(d - 2\). The intersection \(Y \cap L\) is clearly nonempty, and if it is proper it is 0-dimensional of degree \(d - 2\). We may assume that this intersection is proper for a general \(L\), so we get a point in \(VSP(F_L, d - 2)\). Let \(X \subset \mathbb{P}^{n+m+1}\) be the dual variety of \(X\), i.e. the set of hyperplanes tangent at some point \(p \in X\). Then we have set up a rational map

\[X_L \rightarrow VSP(F_L, d - 4)\]

where \(X_L = \{[H] \in X_L | H \supset L\}\). The subvariety \(X_L\) has dimension \(m - \text{codim} X\), which equals \(m - 1\) when \(X\) is nondegenerate. In particular this is the case when \(X\) is a curve.

Similarly, assume that \(X\) satisfies (**), and let \(L \subset H_p \cap H'_p\) where \(H_p\) and \(H'_p\) are two general hyperplanes tangent at \(p\). Then, according to 2.1, there is a \((m - 1)\)-dimensional variety

\[Y \supset H_p \cap H'_p \cap X\]

of degree \(d - 4\). The intersection \(Y \cap L\) is clearly nonempty, and if it is proper it is 0-dimensional of degree \(d - 4\). We may assume that this intersection is proper for a general \(L\), so we get a point in \(VSP(F_L, d - 4)\). Let \(Z_L \subset G(m + n, m + n + 2)\) be the set of codimension 2 subspaces doubly tangent at some point \(p \in X\). Then we have set up a rational map

\[G(m + n, m + n + 2) \supset Z_L \rightarrow VSP(F_L, d - 4)\]

where \(Z_L = \{[V] \in Z_L | V \supset L\}\). If the dual variety of \(X\) is nondegenerate, then the subvariety \(Z_L\) has dimension \(m + 2(n-1)\). The codimension in \(G(m + n, m + n + 2)\) of subspaces that contains \(L\) is \(2(m+n) - 2(m-1) = 2n+2\) so the expected dimension of \(Z_L\) is \(m - 4\).

Notice that it is essential for the dimension count that \(X\) is not a cone, i.e. that the dual variety is nondegenerate.

**Proposition 2.4.** Let \(X \subset \mathbb{P}^{n+m+1}\) be a \(m\)-dimensional arithmetic Gorenstein variety of degree \(d\), with \(m \geq 1\). Assume that \(X\) satisfies the condition (**), and has nondegenerate dual variety. Let \(L \subset \mathbb{P}^{n+m+1}\) be a general \(n\)-dimensional linear subspace, and let \(F_L = \alpha_X([L])\) be the hypersurface apolar to \(L \cap X\). Then
$VSP(F_r, d - 2) \neq \emptyset$ and of dimension at least $m - 1$. Assume furthermore that $m \geq 4$ and that $X$ also satisfies condition $(* \ast \ast)$. Then the dimension of $VSP(F_r, d - 4)$ is at least $m - 4$, when $m \geq 4$ and $L$ is contained in at least one codimension 2 linear space doubly tangent to $X$.

3. CANONICAL CURVES AND APOLAR CUBIC POLYNOMIALS

For a general cubic $n$-fold $F$, the result of Alexander and Hirschowitz (1.1) implies that $VSP(F, k) = \emptyset$, when $k < \frac{1}{2}(n+4)(n+3)$. In 1.2 we defined a map $\alpha_c$ that associates an apolar cubic $n$-fold to an empty codimension two linear section of a canonical curve $C$ of genus $g = n+4$. The following theorem shows that cubic $n$-folds in the image of this map are special with respect to the possible powersum presentations as soon as $n \geq 7$.

**Theorem 3.1.** If $F$ is a cubic $n$-fold apolar to a general codimension two linear section of a general canonical curve of genus $g = n + 4$, then $VSP(F, 2n + 4) \neq \emptyset$.

**Proof.** This is immediate from 2.4 since a canonical curve has nondegenerate dual variety and satisfies $(\ast \ast \ast)$. \qed

**Remark 3.2.** By Hurwitz' formula, the degree of the dual variety of a canonical curve is $6g - 6$, so $VSP(F, 2n + 4)$ contains at least $6n + 18$ points. We do not know whether there are more.

For $n \leq 3$, the general cubic is apolar to a section of a canonical curve. This fact can be used to describe completely the powersum presentations of the cubic form (cf. [7]).

For $n = 3, 4, 5$ the general canonical curve of genus $g = n + 4$ is a linear section of a homogeneous space of dimension at least 6 (cf. [3]). For $n = 4, 5$, these homogeneous spaces of dimension 8 (resp. 6) have nondegenerate dual varieties. For a cubic 4-fold $F$ apolar to a general canonical curve of genus 8 proposition 2.4 gives us a 4-dimensional component of $VSP(F, 10)$. It is shown in [6] that this is in fact all of $VSP(F, 10)$.

A general canonical curve of genus 9 is a linear section of the symplectic grassmannian $Sp(3)/U(3) \subset G(3, 6)$. For a cubic 5-fold $F$ apolar to a canonical curve of genus 9, which is contained in a codimension two linear section doubly tangent to $Sp(3)/U(3)$, proposition 2.4 gives us a 2-dimensional subvariety of $VSP(F, 12)$. On the other hand, for a general cubic 5-fold $F$ it follows from 1.1 that $VSP(F, 12)$ is finite.

The general canonical curve of genus 10 is not a section of a $K3$-surface (cf. [3]), so only 3.1 applies i.e. $VSP(F, 16) \neq \emptyset$, while already $VSP(F, 15) \neq \emptyset$ for a general cubic 6-fold $F$.

**References**


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