Dissimilarity vectors of metric trees and the tropical Grassmannian

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Reminder: tropical varieties

\[ T = \mathbb{R} \cup \{-\infty\} \]

\[ a \oplus b = \max\{a, b\} \]
\[ a \otimes b = a + b. \]

\[ X = \{x_1, \ldots, x_n\}, \quad f(X) = \sum C_{\vec{m}} \vec{x}^{\vec{m}} \in \mathbb{C}[X] \]

\[ T(f) = \bigoplus_{C_{\vec{m}} \neq 0} (\bigotimes_{i=1}^n m_i x_i) = \max\{\ldots, \sum_{i=1}^n m_i x_i, \ldots\} \]

\[ tr(f) = \{\vec{p} \in \mathbb{R}^{|X|} | T(f) \text{ has two maxima at } \vec{p}\} \]

\[ I \subset \mathbb{C}[X], \quad tr(I) = \bigcap_{f \in I} tr(f) \]
weighted (phylogenetic) trees

Tree $\mathcal{T}$ with $n$ leaves labeled $\{1, \ldots, n\}$

$P_{\mathcal{T}} = \{w : \text{Edge}(\mathcal{T}) \rightarrow \mathbb{R} | \geq 0 \text{ on internal edges}\}$
The space of trees $P(n)$

For $\psi : T' \to T$ a map of $n$-trees there is a map $\psi^* : P_T \to P_{T'}$.

![Figure 2:](image)

We define $P(n) = \bigsqcup_{|\text{Leaf}(T)|=n} P_T / \sim$

This space was studied by Billera, Holmes and Vogtman.
Dissimilarity vectors

For $1 < m < n$, and $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ define

$$d_{i_1, \ldots, i_m}(T, w) = \sum_{e \in C(i_1, \ldots, i_m)} w(e).$$

Figure 3: $d_{134}(T, w) = 14.26$
Dissimilarity vectors

Definition:
- We call \( d^m(\mathcal{T}, w) = (\ldots d_{i_1, \ldots, i_m}(\mathcal{T}, w) \ldots) \) the \( m \)-dissimilarity vector of \((\mathcal{T}, w)\).
- We call \( d^m : P(n) \to \mathbb{R}^{n \choose m} \) the \( m \)-dissimilarity map.

\[
d^3(\bullet) = (8.5, 7, 9.5, 8)
\]
2-dissimilarity vectors

The map $d^2 : P(n) \rightarrow \mathbb{R}^{\binom{n}{2}}$ is 1 − 1.

Figure 4: $d_{34} = 13.35$
Recall $I_{2,n} = \{ z_{ij}z_{kl} - z_{ik}z_{jl} + z_{il}z_{jk} | 1 \leq i < j < k < l \leq n \} \subset \mathbb{C}[\ldots z_{ij} \ldots | i < j]$, the $(2,n)$ Plücker ideal. 

$\mathbb{C}[z_{ij}]/I_{2,n} = \mathbb{C}[Gr_2(\mathbb{C}^n)]$

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**Theorem** [Speyer, Sturmfels]:
The image of $d^2$ coincides with $tr(I_{2,n})$.

Plücker relations form a tropical basis of $I_{2,n}$, so the $d^2(T) \in \mathbb{R}^{n \choose 2}$ satisfy

$$\max\{ d_{ij} + d_{kl}, d_{ik} + d_{jl}, d_{il} + d_{jk} \}.$$
Conjecture of Cools, Pachter, Speyer

Conjecture [Cools, Pachter, Speyer]:

Theorem [Giraldo, M]:

\[ d^m(P(n)) \subset tr(I_{m,n}) \subset \mathbb{R}^\binom{n}{m} \]

This implies \( d^m(T, w) \) satisfies \( T(f) \) for all \( f \in I_{m,n} \).

Plücker relations no longer make a tropical basis when \( m > 2 \).

Finding a tropical basis is difficult
Tropical theory: Lifting I

Valuations, \( v : K \rightarrow \mathbb{T} \)

\[
v(a + b) \leq \max\{v(a), v(b)\}
\]

\[
v(ab) = v(a) + v(b)
\]

\[
v(0) = -\infty, v(C) = 0, C \in \mathbb{C}^\times
\]

Theorem:

Let \( I \subset \mathbb{C}[X], \mathbb{C} \subset K \) algebraically closed, complete with respect to \( v : K \rightarrow \mathbb{T} \).

The map \( v : V(K \otimes_{\mathbb{C}} I) \rightarrow \mathbb{T}^{|X|} \), given by \( \vec{p} \rightarrow (\ldots v(p_i) \ldots) \) has image equal to \( \text{tr}(I) \).
Strategy:

Take $K$ to be the Puiseaux series, carefully construct a matrix $M(\mathcal{T}, w) \in K^{n \times m}$.

Tropicalize the $m \times m$-minors of $M(\mathcal{T}, w)$ to get $d^m(\mathcal{T}, w)$.

By tropical lifting, this is a point in $tr(I_{m,n})$. 
Tropical theory: Lifting II

For $A$ an algebra over $\mathbb{C}$, let $\mathcal{V}_\mathbb{T}(A)$ be the set of valuations of $A$ into $\mathbb{T}$ over $\mathbb{C}$.

**Theorem:**
For any presentation

$$0 \rightarrow I \rightarrow \mathbb{C}[X] \rightarrow A \rightarrow 0$$

there is a surjective map $\pi_X : \mathcal{V}_\mathbb{T}(A) \rightarrow tr(I)$ given by $\pi_X(v) = (\ldots v(x_i) \ldots)$.

Instead of the ideal, we consider the algebra

$$P_{m,n} = \mathbb{C}[\ldots z_{i_1,\ldots,i_m} \ldots]/I_{m,n}.$$  

**Strategy:** For $(\mathcal{T}, w)$, build a valuation of $P_{m,n}$ which evaluates $z_{i_1,\ldots,i_m}$ to $d_{i_1,\ldots,i_m}(\mathcal{T}, w)$. 
Irreducible representations $V(\eta)$ correspond to dominant weights $\eta \in \Delta$, a chosen Weyl chamber.

\begin{align*}
V(\omega_1) &= C^m \\
V(\omega_1^*) &= \bigwedge^{m-1}(C^m) \\
V(r\omega_1^*) &= Sym^r(\bigwedge^{m-1}(C^m))
\end{align*}

- We say $\eta < \lambda$ if $\lambda - \eta$ is a sum of positive roots.

- There is a dual Weyl Chamber $\Delta^*$, elements are linear functions on $\Delta$.

- An element of $\Delta^*$ gives an ordering on $\Delta$ which is compatible with $\prec$. 
By the first fundamental theorem of invariant theory,

\[ P_{m,n} = \bigoplus_{\vec{r} \in \mathbb{Z}_n^* \geq 0} (V(r_1\omega_1^*) \otimes \ldots \otimes V(r_n\omega_1^*))^{SL_m(\mathbb{C})} \]

\[(V(r_1\omega_1^*) \otimes \ldots \otimes V(r_n\omega_1^*))^{SL_m(\mathbb{C})} = \text{Hom}(V(r_1\omega_1), V(r_2\omega_1^*) \otimes \ldots \otimes V(r_n\omega_1^*)) \]

We call this space \( W_{\vec{r}} \), so \( P_{m,n} = \bigoplus_{\vec{r}} W_{\vec{r}} \), and \( W_{\vec{r}} W_{\vec{s}} \subset W_{\vec{r}+\vec{s}} \).

Roughly, we build valuations out of the representation theory data in the spaces \( W_{\vec{r}} \).
A tree $\mathcal{T}$ determines a nesting of parentheses in the tensor product.

$$\text{Hom}(X_1, X_2 \otimes (X_3 \otimes X_4))$$
Trees and tensor Products

\[ W_{\vec{r}} = Hom(V(r_1 \omega_1), V(r_2 \omega^*_1) \otimes [V(r_3 \omega^*_1) \otimes V(r_4 \omega^*_1)]) \]

\[ V(r_3 \omega^*_1) \otimes V(r_4 \omega^*_1) = \bigoplus_{\eta \in \Delta} Hom(V(\eta), V(r_3 \omega^*_1) \otimes V(r_4 \omega^*_1)) \otimes V(\eta) \]

\[ W_{\vec{r}} =\]
\[ Hom(V(r_1 \omega_1), V(r_2 \omega^*_1) \otimes [ \bigoplus_{\eta} Hom(V(\eta), V(r_3 \omega^*_1) \otimes V(r_4 \omega^*_1)) \otimes V(\eta)]) \]
\[ = \bigoplus_{\eta} [Hom(V(r_1 \omega_1), V(r_2 \omega^*_1) \otimes V(\eta))] \otimes [Hom(V(\eta), V(r_3 \omega^*_1) \otimes V(r_4 \omega^*_1))] \]
\[ = \bigoplus_{\eta} W_{\vec{r}, \eta, T} \]
Trees and tensor products

For a tree $\mathcal{T}$ with $n$ leaves, we obtain a finer grading of $P_{m,n}$ as a vector space.

$$P_{m,n} = \bigoplus W_{\vec{r},\vec{\eta},\mathcal{T}}$$

This is a filtration on the level of rings.

$$W_{\vec{r},\vec{\eta},\mathcal{T}} W_{\vec{s},\vec{\lambda},\mathcal{T}} \subset \bigoplus_{\beta < \vec{\eta} + \vec{\lambda}} W_{\vec{r} + \vec{s},\vec{\beta},\mathcal{T}}$$
Trees and valuations

To produce a valuation on $P_{m,n}$ we choose a labelling of $T$ by elements of the dual Weyl chamber, $\Delta^*$. 

$$\chi_1(\eta_1) + \chi_2(\eta_2) + \chi_3(\eta_3) + \chi_4(\eta_4)$$

$$\chi(\vec{r}, \vec{\eta}, T) = \sum_{e \in \text{Edge}(T)} \chi_e(\eta_e)$$
This defines a valuation $v_\chi : P_{m,n} \to \mathbb{T}$.

For $f_{r_1,\eta_1} + \ldots + f_{r_m,\eta_m} \in \bigoplus W_{r_i,\eta_i,\mathcal{T}}$, we define

$$v_\chi(f_{r_1,\eta_1} + \ldots + f_{r_m,\eta_m}) = \max\{\ldots \chi(\eta_i, r_i, \mathcal{T}) \ldots \}$$
Theorem [M]:
For each metric tree $(T, w)$ there is a $\Delta^*$-labelling of $T$ such that the induced valuation $v_{T,w}$ satisfies

$$v_{T,w}(z_{i_1, \ldots, i_m}) = d_{i_1, \ldots, i_m}(T, w)$$

for all Plücker generators $z_{i_1, \ldots, i_m} \in X_{m,n} \subset P_{m,n}$.

This defines a 1–1 map $\phi_m : P(n) \to \mathbb{V}_T(P_{m,n})$.
Constructing $\nu_{\mathcal{T},w}$

Let $\rho \in \Delta^*$ be the sum of the primitive co-weights, this satisfies $\rho(\omega_k) = 1$ for all primitive weights $\omega_k$.

For $e \in Edge(\mathcal{T})$, we let $\chi_e = w_e \rho$.
Constructing $v_{\mathcal{T},w}$

The weight of the space $W_{\mathcal{T},\bar{\eta},\bar{r}}$ containing a Plücker generator $z_{i_1...i_m}$ is non-zero exactly on the edges in $C(i_1...i_m)$.

Figure 5: $z_{234} \in P_{3,4}$
The valuation construction generalizes to many algebras from representation theory.

There is a similar construction for Kac-Moody algebras which produces tropical invariants of graphs (as opposed to just trees).
References


B. Giraldo: Dissimilarity vectors of trees are contained in the tropical Grassmannian, The Electronic Journal of Combinatorics 17, no. 1, 2010


Thankyou!