SINGULARITIES OF RESTRICTION VARIETIES IN $OG(k, n)$

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Abstract. Restriction varieties in the orthogonal Grassmannian are subvarieties of $OG(k, n)$ defined by rank conditions given by a flag that is not necessarily isotropic with respect to the relevant symmetric bilinear form. In particular, Schubert varieties of Type B and D are examples of restriction varieties. In this paper, we introduce a resolution of singularities for restriction varieties in $OG(k, n)$, and give a description of their singular locus by studying components of the exceptional locus of the resolution.

CONTENTS

1. Introduction 1
2. Singularities of Schubert Varieties in $G(k, n)$ 3
3. Preliminaries on Restriction Varieties 5
4. The Resolution of Singularities 8
5. The Exceptional Locus 15
6. The Algorithm and Examples 28
References 31

1. Introduction

In this paper, we present a resolution of singularities $\pi$ for restriction varieties in $OG(k, n)$. We also give a method for the description of the singular locus, and show that it is equal to the image of the exceptional locus of $\pi$ in most cases.

Let $Q$ be a non-degenerate symmetric bilinear form on a vector space $W$ of dimension $n$ over the complex numbers. Let $k_1 < \cdots < k_h$ be positive integers such that $2k_h \leq n$. Let $F(k_1, \ldots, k_h; n)$ be the ordinary flag variety, and let $OF(k_1, \ldots, k_h; n)$ be the orthogonal partial flag variety parameterizing subspaces $W_1 \subseteq \cdots \subseteq W_h$ of $W$ isotropic with respect to $Q$, where $W_i$ has dimension $k_i$. A restriction variety is the intersection of $OF(k_1, \ldots, k_h; n)$ with a Schubert variety in $F(k_1, \ldots, k_h; n)$ defined by a flag satisfying certain tangency conditions with respect to $Q$. Orthogonal Schubert varieties are examples of restriction varieties when the flag is isotropic.

Restriction varieties have found applications in the restriction problem in cohomology. The inclusion $i : OF(k_1, \ldots, k_h; n) \hookrightarrow F(k_1, \ldots, k_h; n)$ induces $i^* : H^*(F(k_1, \ldots, k_h; n)) \rightarrow H^*(OF(k_1, \ldots, k_h; n))$, and given a Schubert class $\tau$ in $H^*(F(k_1, \ldots, k_h; n))$, we would like to express $i^*(\tau)$ as a non-negative linear combination of the Schubert classes in $H^*(OF(k_1, \ldots, k_h; n))$. The rule used to compute the cohomology class of a restriction variety solves this problem [2]. Similarly, symplectic restriction varieties can be used to solve the same problem for the inclusion $i : SF(k_1, \ldots, k_h; n) \hookrightarrow F(k_1, \ldots, k_h; n)$, see [4, 5]. There are also applications to the rigidity problem. Restriction varieties give explicit deformations of Schubert varieties in certain instances, and hence show that the corresponding classes are not rigid. This paper studies the singularities of restriction varieties in $OG(k, n)$. We introduce a resolution of singularities, and study its exceptional locus. This resolution is inspired by the Bott-Samelson/Zelevinsky resolution for Schubert varieties, but has a more intricate construction reflecting the richer geometry of restriction varieties. We also

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describe the singular locus explicitly, and give a criterion for when it is equal to the image of the exceptional locus of \( \pi \).

Let \( F_Q \) be the quadratic polynomial associated to \( Q \). A \( k \)-plane \( \Lambda \) is isotropic with respect to \( Q \) if and only if its projectivization is contained in the quadric hypersurface defined by \( F_Q \). The orthogonal Grassmannian \( OG(k, n) \) parameterizes \( k \)-dimensional subspaces of \( W \) that are isotropic with respect to \( Q \). Equivalently, this is the Fano variety of \( (k-1) \)-planes contained in a quadric hypersurface in \( PW \).

Restriction varieties in the orthogonal Grassmannian are subvarieties of \( OG(k, n) \) that parameterize isotropic subspaces of \( W \) with respect to a flag that is not necessarily isotropic. Let \( Q^i_d \) be a quadratic form of corank \( r \) obtained by restricting \( Q \) to a vector space of dimension \( d \). Let \( L_e \) denote an \( e \)-dimensional subspace that is isotropic with respect to \( Q \). A restriction variety \( V \) in \( OG(k, n) \) is defined in terms of a sequence

\[
L_{n_1} \subseteq \ldots \subseteq L_{n_s} \subseteq Q^{r_k-e}_d \subseteq \ldots \subseteq Q^r_d.
\]

\( V \) parameterizes \( k \)-dimensional isotropic linear spaces that intersect \( L_{n_j} \) in a subspace of dimension \( j \) for all \( 1 \leq j \leq s \) and \( Q^i_d \) in a subspace of dimension \( k-i+1 \) for all \( 1 \leq i \leq k-s \). There are two important conditions we impose on these sequences: The first is that we want the isotropic linear spaces \( L_{n_j} \) and the singular loci of sub-quadrics \( Q^i_d \) to be in the most special position. This is ensured by the conditions

\[
\begin{align*}
&Q^{r_i-1} \subseteq Q^i \subseteq Q^r, \\
&\dim \left( L_{n_j} \cap Q^i \right) = \text{min}(n_j, r_i) \text{ for every } 1 \leq j \leq s \text{ and } 1 \leq i \leq k-s.
\end{align*}
\]

In accordance with this positioning we require the \( k \)-planes parameterized by \( V \) to intersect \( Q^i_d \) in a certain way. Let \( x_i \) be the number of isotropic linear spaces \( L_{n_j} \) of the sequence contained in \( Q^i_d \). We require the \((k-i+1)\)-dimensional subspace contained in \( Q^i_d \) to intersect \( Q^i \) in a subspace of dimension \( x_i \).

Secondly, we require the sub-quadrics to be irreducible. This is reflected in the condition

\[
\begin{align*}
&x_k \geq d_k - 3.
\end{align*}
\]

Informally we can think of restriction varieties as subvarieties of \( OG(k, n) \) that interpolate between Schubert varieties in \( OG(k, n) \), which is associated to maximal tangency conditions for the linear sections of the quadric hypersurface \( Q \), and restrictions of general Schubert varieties in \( G(k, n) \) to \( OG(k, n) \) which is associated to minimal tangency conditions.

The main results of this paper are the following:

1. *Theorem 4.7 gives a resolution of singularities \( \pi \) for restriction varieties.*

The resolution of singularities we introduce is inspired by the Bott-Samelson/Zelevinsky resolution for Schubert varieties. In order to resolve singularities, we construct a resolution that makes use of maximal dimensional isotropic linear subspaces at each step of the sequence. The resolution is constructed using a tower of Grassmannian and orthogonal Grassmannian bundles. We show that images of the components of the exceptional locus of \( \pi \) which have codimension larger than 1 are in the singular locus. We study the tangent space of a restriction variety at a point for the images of the remaining components, and get a complete description of the singular locus.

2. *Corollary 5.27 describes the singular locus of a restriction variety in \( OG(k, n) \).*

We give a method for finding the singular locus of a restriction variety in \( OG(k, n) \) that is based on our study of the exceptional locus of \( \pi \). In particular, this method presents an alternative to the method of describing the singular locus of a Schubert variety of Type B or D by checking for smoothness at each orbit.

3. *Given a restriction variety \( V \) in \( OG(k, n) \), Algorithm 6.1 gives the singular locus of \( V \).*

The organization of this paper is as follows: In Section 2, we review the well-known results on the singularities of Schubert varieties in \( G(k, n) \). In Section 3, we review the necessary background and the definition of restriction varieties. In Section 4, we define the resolution of singularities, and study its exceptional locus. In Section 5, we present the algorithm for the singular locus and conclude with some examples.
2. Singularities of Schubert Varieties in $G(k,n)$

In this section, we introduce a language for Schubert varieties in the Grassmanian that will generalize to restriction varieties in a straight-forward way. This section not only serves as a reminder of some classical results on Schubert varieties, but also underlines some ideas used in the following sections. We refer the reader to [1] for an extensive exposition on the singularities of Schubert varieties.

In our definition, we use sequences whose steps correspond to rank conditions giving the Schubert variety. Let $W$ be an $n$-dimensional vector space over the complex numbers, and consider $G(k,W) = G(k,n)$, the Grassmannian of $k$-planes on $W$. We define a Schubert variety $\Sigma$ in $G(k,n)$ in terms of a fixed complete flag, that is, a nested sequence of subspaces

$$0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$$

with $\dim W_i = i$. Consider a subsequence $W_\bullet$ of length $k$:

$$W_{n_1} \subseteq \cdots \subseteq W_{n_k}.$$ 

The Schubert variety $\Sigma$ associated to $W_\bullet$ is defined as the closure of the locus

$$\Sigma(W_\bullet)^0 = \{ \Lambda \in G(k,n) \mid \dim (\Lambda \cap W_{n_i}) = i \text{ for all } 1 \leq i \leq k \}.$$ 

If there are steps in $W_\bullet$ with consecutively increasing dimensions, certain conditions are implied by the others, and the number of rank conditions necessary to define the Schubert variety is less than the number of steps in the sequence. In order to define Schubert varieties in a concise way, we introduce partitions.

**Definition 2.1.** Given a sequence of increasing positive integers $n_1, \ldots, n_k$, let $n_{a_1}, \ldots, n_{a_t}$ be the subsequence such that $n_{a_g} + 1 \neq n_{a_{g+1}}$, and let

$$\alpha_g = \left\{ \text{the indices } n_i \text{ that occur in } W_\bullet \mid n_i \leq n_{a_g}, \ a_g - i = n_{a_g} - n_i \right\} \text{ for all } 1 \leq g \leq t.$$ 

Then the data $(n_{a_1}, \ldots, n_{a_t})$ is defined to be the partition associated to the sequence $n_1, \ldots, n_k$.

In other words, $n_{a_g}$ is the largest dimensional integer in each group of consecutive integers, and $\alpha_g$ counts the integers in that group. Note that we have $a_g = \sum_{i=1}^g \alpha_i$ and $a_t = k$. The Schubert variety $\Sigma$ in $G(k,n)$ associated to the partition $(n_{a_1}, \ldots, n_{a_t})$ is defined as the closure of the locus

$$\Sigma^0 = \left\{ \Lambda \in G(k,n) \mid \dim \left( \Lambda \cap W_{n_{a_l}} \right) = a_l \text{ for all } 1 \leq l \leq t \right\}.$$ 

Being homogeneous under the action of $GL(n)$, the open cell $\Sigma^0$ is smooth.

**Example 2.2.** Let $\Sigma$ be the Schubert variety in $G(7,17)$ given by the sequence

$$W_2 \subseteq W_6 \subseteq W_7 \subseteq W_{11} \subseteq W_{12} \subseteq W_{13} \subseteq W_{15}.$$ 

This variety is defined as the closure of the locus

$$\Sigma^0 = \{ \Lambda \in G(7,17) \mid \dim(\Lambda \cap W_2) = 1, \ \dim(\Lambda \cap W_7) = 3, \ \dim(\Lambda \cap W_{13}) = 6, \ \dim(\Lambda \cap W_{15}) = 7 \}.$$ 

The partition associated to this Schubert variety is $(2^1, 7^2, 13^3, 15^1)$.

The following proposition recalls the dimension of a Schubert variety using the sequence and the partition notations.

**Proposition 2.3.** The dimension of a Schubert variety $\Sigma$ in $G(k,n)$ associated to the sequence $W_\bullet : W_{n_1} \subseteq \cdots \subseteq W_{n_k}$ or the partition $(n_{a_1}, \ldots, n_{a_t})$ is given by

$$\dim \Sigma = \sum_{i=1}^k (n_i - i) = \sum_{l=1}^t \alpha_l (n_{a_l} - a_l).$$
Schubert varieties in the Grassmannian admit a natural resolution $\pi : \Sigma \to \Sigma$ such that the image of the exceptional locus of $\pi$ is equal to the singular locus of $\Sigma$. Let $\Sigma$ be given by the partition $(n_{a_1}, \ldots, n_{a_t})$ and let $\Sigma$ be the Schubert variety in the flag variety $F(a_1, \ldots, a_t; n)$ defined by

$$
\Sigma = \left\{(T^1, \ldots, T^t) \in F(a_1, \ldots, a_t; n) \mid T^l \subseteq W_{n_{a_l}} \text{ for all } 1 \leq l \leq t\right\}.
$$

Since $\Sigma$ is an iterated tower of Grassmannians, it is smooth and irreducible. The natural projection $\pi : F(a_1, \ldots, a_t; n) \to G(k, n)$ given by $(T^1, \ldots, T^t) \to T^l$ maps $\Sigma$ onto $\Sigma$ and the map is injective over the smooth open cell $\Sigma^0$. The inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda \in \Sigma^0$ is determined uniquely as

$$
T^l = \Lambda \cap W_{n_{a_l}}, \quad 1 \leq l \leq t.
$$

By Zariski’s Main Theorem, $\pi$ is an isomorphism over $\Sigma^0$ and hence a resolution of singularities of $\Sigma$.

The map has positive dimensional fibers over the locus of $k$-planes $\Lambda$ with the property that $\dim(\Lambda \cap W_{n_{a_l}}) > a_l$ for some $1 \leq l \leq t - 1$. Let $\Sigma_{s_l}$ be the closure of the locus

$$
\Sigma^0_{s_l} = \left\{ \Lambda \in G(k, n) \mid \dim \left( \Lambda \cap W_{n_{a_l}} \right) = a_l + 1, \ \dim \left( \Lambda \cap W_{n_{a_g}} \right) = a_g \text{ for all } 1 \leq g \leq t, \ i \neq l \right\}
$$

for some $1 \leq l \leq t - 1$.

The exceptional locus of $\pi$ consists of the union of the inverse images of $\Sigma_{s_l}$ for all $1 \leq l \leq t - 1$. Let us study the codimension of the components of the exceptional locus of $\pi$. Over each $\Sigma_{s_l}$, the inverse image $\Sigma_{s_l}$ is irreducible of codimension

$$
\text{codim} \left( \pi^{-1}(\Sigma_{s_l}) \right) = \text{codim} (\Sigma_{s_l}) - \text{dim} \left( \pi^{-1}(\Lambda) \right)
$$

for a general $\Lambda \in \Sigma_{s_l}$. By Proposition 2.3 we have

$$
\text{codim} (\Sigma_{s_l}) = \alpha_l(n_{a_l} - a_l) + \alpha_{l+1}(n_{a_{l+1}} - a_{l+1}) - (\alpha_l + 1)(n_{a_l} - a_l - 1) - (\alpha_{l+1} - 1)(n_{a_{l+1}} - a_{l+1}) = n_{a_{l+1}} - n_{a_l} - (a_{l+1} - a_l) + a_l + 1.
$$

On the other hand, for a general $\Lambda \in \Sigma_{s_l}$ we have

$$
\pi^{-1}(\Lambda) = \left\{ (T^1, \ldots, T^t) \mid T^g = \Lambda \cap W_{n_{a_g}} \text{ for all } 1 \leq g \leq t, \ g \neq l \right\}
$$

and $T^{l-1} \subseteq T^l \subseteq \Lambda \cap W_{n_{a_l}}$.

So, for an element of $\pi^{-1}(\Lambda)$, the coordinate $T^l$ is the only one that is not determined uniquely and it can be parameterized by $G(a_l - a_{l-1}, a_l + 1 - a_{l-1})$. This Grassmannian has dimension $a_l - a_{l-1} = \alpha_l$. Therefore we have

$$
\text{codim} \left( \pi^{-1}(\Sigma_{s_l}) \right) = n_{a_{l+1}} - n_{a_l} - (a_{l+1} - a_l) + 1 \geq 2
$$

since $n_{a_{l+1}} - n_{a_l} \geq a_{l+1} - a_l + 1$.

This shows that a component of the exceptional locus of $\pi$ has codimension larger than 1. This observation with the following lemma determines the singular locus of a Schubert variety.

**Lemma 2.4.** ([3], Lemma 2.3) Let $f : X \to Y$ be a birational morphism from a smooth, projective variety $X$ onto a normal projective variety $Y$. Assume that $f$ is an isomorphism in codimension one. Then $p \in Y$ is a singular point if and only if $f^{-1}(p)$ is positive dimensional.

**Corollary 2.5.** The image of the exceptional locus of the resolution of singularities $\pi : \Sigma \to \Sigma$ is equal to the singular locus of $\Sigma$.

**Example 2.6.** For the Schubert variety given by the partition $(2^1, 7^2, 13^3, 15^1)$, the variety $\Sigma$ is defined as

$$
\Sigma = \{(T^1, T^2, T^3, T^4) \in F(1, 3, 6, 7; 17) \mid T^1 \subseteq W_2, \ T^2 \subseteq W_7, \ T^3 \subseteq W_{13}, \ T^4 \subseteq W_{15}\}.
$$
The projection $\pi : (T^1, T^2, T^3, T^4) \mapsto T^4$ maps $\bar{\Sigma}$ onto $\Sigma$. The exceptional locus consists of the union of the inverse images of the closures of the following loci:

$$\Sigma^0_{s_1} = \{ \Lambda \in G(7,17) \mid \dim(\Lambda \cap W_2) = 2, \ \dim(\Lambda \cap W_7) = 3, \ \dim(\Lambda \cap W_{13}) = 6, \ \dim(\Lambda \cap W_{15}) = 7 \}.$$ 

$$\Sigma^0_{s_2} = \{ \Lambda \in G(7,17) \mid \dim(\Lambda \cap W_2) = 1, \ \dim(\Lambda \cap W_7) = 4, \ \dim(\Lambda \cap W_{13}) = 6, \ \dim(\Lambda \cap W_{15}) = 7 \}.$$ 

$$\Sigma^0_{s_3} = \{ \Lambda \in G(7,17) \mid \dim(\Lambda \cap W_2) = 1, \ \dim(\Lambda \cap W_7) = 3, \ \dim(\Lambda \cap W_{13}) = 7 \}.$$ 

Consequently the singular locus of the Schubert variety $\Sigma$ is given by

$$\Sigma^{sing} = \Sigma_{s_1} \cup \Sigma_{s_2} \cup \Sigma_{s_3}.$$ 

Remark 2.7. The subvarieties $\Sigma_{s_i}$ of the Schubert variety $\Sigma$ correspond to the hooks in the Young diagram of $\Sigma$.

3. Preliminaries on Restriction Varieties

In this section, we review the definition of restriction varieties and their basic properties. Restriction varieties in $OG(k,n)$ parameterize isotropic $k$-planes that intersect elements of a fixed flag in specified dimensions. The flag does not need to be isotropic but there are some conditions imposed by basic facts about quadrics. Additionally, there are some conditions we impose to ensure that the isotropic linear spaces and the singular loci of quadrics are in the most special position. We review these conditions, and refer the reader to [2] for a detailed discussion.

Let $W$ be an $n$-dimensional vector space and let $Q$ a non-degenerate symmetric bilinear form on $W$. We recall the following basic facts about quadrics:

- **The corank bound.** Let $Q^r_{d_2} \subset Q^r_{d_1}$ be two linear sections of $Q$ such that the singular locus of $Q^r_{d_1}$ is contained in the singular locus of $Q^r_{d_2}$. Then $d_2 + r_2 \leq d_1 + r_1$.

- **The linear space bound.** The largest dimensional isotropic linear space with respect to a quadratic form $Q^r_d$ has dimension $\left\lfloor \frac{d+r}{2} \right\rfloor$. A linear space of dimension $j$ intersects the singular locus of $Q^r_d$ in a subspace of dimension at least $\max(0, j - \left\lfloor \frac{d+r}{2} \right\rfloor)$.

- **Irreducibility.** A sub-quadric $Q^r_{d-2}$ of $Q$ is reducible and equal to the union of two linear spaces of (vector space) dimension $d - 1$ meeting along a linear space of dimension $d - 2$. If $n = 2k$, then the linear spaces constituting $Q^k_{k+1}$ belong to two distinct connected components.

- **The variation of tangent spaces.** Let a quadric $Q^r_d$ be singular along a codimension $j$ linear subspace $M$ of a linear space $L$. Then the image of the Gauss map of $Q^r_d$ restricted to the smooth points of $L$ has dimension at most $j - 1$. In other words, the tangent spaces to $Q^r_d$ along the smooth points of $L$ vary at most in a $(j - 1)$-dimensional family.

Let $F_Q$ denote the quadratic polynomial associated to $Q$. Let $L_{n_j}$ be an isotropic linear space of vector space dimension $n_j$. If $2n_j = n$, we denote isotropic linear spaces in different connected components as $L_{n_j}$ and $L'_{n_j}$. Let $Q^r_{d_1}$ denote a sub-quadric of corank $r_i$ cut out by a $d_i$-dimensional linear section of $Q$ and denote this linear space by $Q^r_{d_i}$. Let $F_Q$ denote the restriction of $F$ to $Q^r_{d_i}$ so that $Q^r_{d_i}$ is given by the zero locus of $F_Q$. We denote the singular locus of $Q^r_{d_i}$ by $Q^r_{d_i,sing}$. We use the same notation for projectivizations contained in $\mathbb{P}W$. For convenience let $r_0 = 0$ and $d_0 = n$.

We use sequences of the form

$$L_{n_1} \subseteq \ldots \subseteq L_{n_s} \subseteq Q^r_{d_k-s} \subseteq \ldots \subseteq Q^r_{d_s}$$

consisting of isotropic linear spaces $L_{n_j}$ and sub-quadrics $Q^r_{d_i}$ of $Q$ to define restriction varieties. The restriction variety $V$ defined via this sequence parameterizes $k$-dimensional isotropic linear spaces that intersect $L_{n_j}$ in a subspace of dimension $j$ for all $1 \leq j \leq s$ and $Q^r_{d_i}$ in a subspace of dimension $k - i + 1$ for all $1 \leq i \leq k - s$. Let $x_i$ be the number of isotropic linear spaces $L_{n_j}$ of the sequence contained in $Q^r_{d_i,sing}$. We
require the \((k - i + 1)\)-dimensional subspace \(\Lambda \cap Q_{d_i}^r\) of a \(k\)-plane \(\Lambda\) contained in \(Q_{d_i}^r\) to intersect \(Q_{d_i}^{r,s,\text{sing}}\) in a subspace of dimension \(x_i\).

**Definition 3.1.** A sequence of linear spaces and quadrics \((L_s, Q_s)\) associated to \(OG(k, n)\) is an admissible sequence if the following conditions are satisfied.

\[
\begin{align*}
(1) \quad & 2n_s \leq d_{k-s} + r_{k-s}. \\
(2) \quad & 2(k - i + 1) \leq r_i + d_i \text{ for every } 1 \leq i \leq k - s. \\
(3) \quad & r_{i+1} + d_{i+1} \leq r_i + d_i \leq n \text{ for every } 1 \leq i \leq k - s. \\
(4) \quad & Q_{d_i}^{r_i,\text{sing}} \subseteq Q_{d_i}^{r,s,\text{sing}} \text{ for every } 1 \leq i \leq k - s. \\
(5) \quad & \dim \left( L_{n_j} \cap Q_{d_i}^{r_i,\text{sing}} \right) = \min(n_j, r_i) \text{ for every } 1 \leq j \leq s \text{ and } 1 \leq i \leq k - s. \\
(6) \quad & \text{For every } 1 \leq i \leq k - s \text{ either } r_i = r_i = x_1 \text{ or } r_i - r_i \geq l - i - 1 \text{ for every } l > i. \text{ Furthermore, if } r_i = r_{i-1} > x_1 \text{ for some } l, \text{ then } d_i - d_{i+1} = r_{i+1} - r_i \text{ for all } i \geq l \text{ and } d_{i-1} - d_i = 1. \\
(7) \quad & r_{k-s} \leq d_{k-s} - 3. \\
(8) \quad & \text{For every } 1 \leq i \leq k - s, \\
& x_i \geq k - i + 1 - \frac{d_i - r_i}{2}. \\
(9) \quad & \text{For any } 1 \leq j \leq s, \text{ there does not exist } 1 \leq i \leq k - s \text{ such that } n_j - r_i = 1.
\end{align*}
\]

**Remark 3.2.** Conditions (1), (2) and (3) follow from the corank bound. In conditions (4) and (5), we require the isotropic linear spaces and the singular loci of sub-quadrics to be in the most special position. This gives a motivation for counting the sub-quadrics \(Q_{d_i}^r\) from the right; the singular loci form a nested sequence of subspaces \(Q_{d_i}^{r_i,\text{sing}} \subseteq \ldots \subseteq Q_{d_{k-s}}^{r_{k-s},\text{sing}}\). Condition (6) is a technical condition that puts a restriction on the singular loci of the sub-quadrics in the sequence; it disallows a sudden gap between \(Q_{d_i}^{r_i,\text{sing}}\). Condition (7) reflects the irreducibility property of quadrics. Condition (8) is a result of the linear space bound and the special positioning of the isotropic linear spaces and the singular loci of sub-quadrics. Finally, condition (9) follows from the variation of tangent spaces property of quadrics.

**Definition 3.3.** Let \((L_s, Q_s)\) be an admissible sequence for \(OG(k, n)\). A restriction variety \(V(L_s, Q_s)\) is the subvariety of \(OG(k, n)\) defined as the closure of

\[
V^0(L_s, Q_s) = \{ \Lambda \in OG(k, n) \mid \dim(\Lambda \cap L_{n_j}) = j, \ 1 \leq j \leq s, \\
\dim(\Lambda \cap Q_{d_i}^r) = k - i + 1, \ \dim(\Lambda \cap Q_{d_i}^{r,s,\text{sing}}) = x_i, \ 1 \leq i \leq k - s \}.
\]

**Example 3.4.** Schubert varieties in \(OG(k, n)\) are restriction varieties defined via a sequence satisfying \(d_i + r_i = n\) for all \(1 \leq i \leq k - s\), that is, when the quadrics in the sequence are as singular as possible. The restriction of a general Schubert variety in \(G(k, n)\) to \(OG(k, n)\) is also a restriction variety associated to a sequence with \(s = 0\) and \(r_i = 0\) for all \(1 \leq i \leq k - s\). Hence, restriction varieties interpolate between the restrictions of Schubert varieties in \(G(k, n)\) to \(OG(k, n)\) and Schubert varieties in \(OG(k, n)\).

When the inequality \(x_i \geq k - i + 1 - \frac{d_i - r_i}{2}\) is an equality for an index \(i\), then the \(\frac{d_i + r_i}{2}\)-dimensional linear spaces in \(Q_{d_i}^r\) form two irreducible components.

**Example 3.5.** \(V\) defined by

\[
Q_3^0 \subseteq Q_4^0
\]

in \(OG(2, 5)\) parameterizes lines on a smooth quadric surface \(Q_3^0\) in \(\mathbb{P}^3\) and consists of two irreducible components.

The \((k - i + 1)\)-dimensional subspaces contained in \(Q_{d_i}^r\) may be distinguished by their parity of the dimension of their intersection with linear spaces in each of these components.
Definition 3.6. Let \((L_\bullet, Q_\bullet)\) be an admissible sequence. An index \(1 \leq i \leq k - s\) such that
\[
x_i = k - i + 1 - \frac{d_i - r_i}{2}
\]
is called a special index. For each special index, a marking \(m_\bullet\) of \((L_\bullet, Q_\bullet)\) designates one of the irreducible components of \(\frac{d_i + r_i}{2}\)-dimensional linear spaces of \(Q_i^r\) as even and the other one as odd, such that
- If \(d_i + r_i = d_{i_2} + r_{i_2}\) for two special indices \(i_1 < i_2\) and the component containing a linear space \(\Gamma\) is designated even for \(i_2\), then the component containing \(\Gamma\) is designated even for \(i_1\) as well; and
- If \(2n_s = d_i + r_i\) for a special index \(i\), then the component to which \(L_{n_s}\) belongs is assigned the parity of \(s\); and
- If \(n = 2k\), \(m_\bullet\) assigns the component containing \(L_k\) the parity that characterizes the component \(OG(k, 2k)\).

A marked restriction variety \(V(L_\bullet, Q_\bullet, m_\bullet)\) is the Zariski closure of the subvariety of \(V^0(L_\bullet, Q_\bullet)\) parameterizing \(k\)-dimensional isotropic subspaces \(W\), where, for each special index \(i\), \(W\) intersects subspaces of dimension \(\frac{d_i + r_i}{2}\) of \(Q_i^r\) designated even (respectively, odd) by \(m_\bullet\) in a subspace of even (respectively, odd) dimension.

Partitions can be used to define restriction varieties using only the conditions that are not automatically satisfied as a result of others.

Definition 3.7. Given a restriction variety \(V\) in \(OG(k, n)\) defined by the admissible sequence
\[
L_1 \subseteq \ldots \subseteq L_n \subseteq Q_{d_{k-s}}^r \subseteq \ldots \subseteq Q_1^r,
\]
let \((n_{a_1}, \ldots, n_{a_t})\) be the partition for \(n_1, \ldots, n_s\), and let \((d_{b_1}, \ldots, d_{b_u})\) be the partition for \(d_{k-s}, \ldots, d_1\). Then the data
\[
(n_{a_1}^\alpha, \ldots, n_{a_t}^\alpha), (d_{b_1}^\beta, \ldots, d_{b_u}^\beta), (r_1, \ldots, r_{k-s})
\]
is defined to be the partitions associated to \(V\).

Remark 3.8. For a group of sub-quadrics whose dimensions are consecutive integers, the coranks are not necessarily consecutive. In other words, a partition for \(r_1, \ldots, r_{k-s}\) may contain more than \(u\) entries. However, by condition (6) in the Definition 3.1, the value of \(x_i\) is fixed for each group of sub-quadrics with consecutive dimensions.

Remark 3.9. We have \(a_g = \sum_{i=1}^g \alpha_i + k - b_h + 1 = s + \sum_{i=1}^h \beta_i\) for every \(1 \leq g \leq t\) and \(1 \leq h \leq u\). The restriction variety defined by the partitions \((n_{a_1}^\alpha, \ldots, n_{a_t}^\alpha), (d_{b_1}^\beta, \ldots, d_{b_u}^\beta), (r_1, \ldots, r_{k-s})\) parameterizes \(k\)-dimensional isotropic linear spaces that satisfy
\[
\dim \left( \Lambda \cap L_{n_g} \right) = a_g, \quad \dim \left( \Lambda \cap Q_{d_{b_h}}^r \right) = k - b_h + 1, \quad \text{and} \quad \dim \left( \Lambda \cap Q_{d_{b_h}}^{r_h, \text{sing}} \right) = x_b,
\]
where \(1 \leq g \leq t\) and \(1 \leq h \leq u\).

Example 3.10. To the sequence \(L_2 \subseteq L_3 \subseteq L_6 \subseteq L_7 \subseteq L_8 \subseteq Q_{16}^1 \subseteq Q_{17}^1 \subseteq Q_{18}^3\) we associate the partitions \((3^2, 8^3), (12^2, 18), (13, 17, 18)\).

We recall the definition of a restriction variety in the next proposition using both sequence and partition notations.

Proposition 3.11. ([2], Prop 4.16) The marked restriction variety \(V(L_\bullet, Q_\bullet, m_\bullet)\) associated to a marked admissible sequence is an irreducible variety of dimension
\[
\dim(V(L_\bullet, Q_\bullet, m_\bullet)) = \sum_{j=1}^s (n_j - j) + \sum_{i=1}^{k-s} (d_i + x_i - 2(k - i + 1))
\]

\[
= \sum_{g=1}^t a_g (n_{a_g} - a_g) + \sum_{h=1}^u \beta_h \left( d_{b_h} + x_{b_h} - 2(k - b_h + 1) + \frac{\beta_h - 1}{2} \right)
\]

Note that this expression does not depend on the marking \(m_\bullet\). The restriction variety \(V(L_\bullet, Q_\bullet)\) has an irreducible component for every marking \(m_\bullet\) and every irreducible component of \(V(L_\bullet, Q_\bullet)\) has this dimension.
Consider the variety defined by
\[
Q_{11}^{4, \text{sing}} \cap T^1 \subseteq O \subseteq L_7 \cap \widetilde{V} \quad \cap \quad T^2 \subseteq Z \subseteq Q_1^{11}
\]

**Example 3.12.** The restriction variety \([L_6 \subseteq L_7 \subseteq L_8]\) is the Grassmannian \(G(3, 8)\) which parameterizes planes contained in a projective space of dimension 7. It is given by \((8^3), (), ()\) in terms of partitions, and has dimension \(a_1(n_{a_1} - a_1) = 3(8 - 3) = 15\).

**Example 3.13.** The restriction variety \([Q_{11}^{11} \subseteq Q_{12}^{11} \subseteq Q_{13}^{11}]\) is the Fano variety of planes contained in a quadric 11-fold in \(\mathbb{P}^{12}\) singular along a line. In terms of partitions this is given by \(((), (13^3)), (2, 3, 4)\) and has dimension \(\beta_1(d_{b_1} + x_{b_1} - 2(3) + \frac{b_1 - 1}{2}) = (13 + 0 - 6 + 1) = 24\).

**Example 3.14.** The restriction variety \([L_2 \subseteq L_3 \subseteq Q_{17}^{11} \subseteq Q_{18}^{11}]\) parameterizes 3-dimensional projective linear spaces that are contained in a quadric hypersurface in \(\mathbb{P}^{17}\) of corank 6 and that intersect a plane contained in the singular locus of the quadric along a line. In terms of partitions this variety is given by \((3^2), (18^2), (6, 7)\) and has dimension \(\alpha_1(n_{a_1} - a_1) + \beta_1(d_{b_1} + x_{b_1} - 2(5) + \frac{b_1 - 1}{2}) = (2(3 - 2) + 2(18 + 2 - 8 + \frac{1}{2})) = 27\).

4. **The Resolution of Singularities**

In this section, we present a resolution of singularities for restriction varieties in \(OG(k, n)\). We first illustrate the resolution on a few examples and then introduce the general definition.

**Example 4.1.** Let \(V\) be the restriction variety in \(OG(1, n)\) defined by the one-step sequence \(Q_{11}^{4}\), where \(n \geq 15\). This variety is a singular quadric contained in a 10-projective dimensional linear space whose singular locus is isomorphic to \(\mathbb{P}^3\). Consider the flag variety \(\widetilde{V}\) defined by
\[
\widetilde{V} = \{(T, Z) \in OF(1, 5; n) \mid Q_{11}^{4, \text{sing}} \subseteq Z \subseteq Q_{11}^{4}\} \subseteq OG(1, n) \times OG(5, n).
\]
The second projection map \(\pi_2 : (T, Z) \mapsto Z\) maps \(\widetilde{V}\) onto \(\{Z \in OG(5, n) \mid Q_{11}^{4, \text{sing}} \subseteq Z \subseteq Q_{11}^{4}\}\) which is isomorphic to \(OG(1, 7)\). Over such \(Z\), the map has fibers \(G(1, 5)\) of dimension 4 so \(\widetilde{V}\) is irreducible of dimension 9. The first projection map \(\pi_1 : (T, Z) \mapsto T\) maps \(\widetilde{V}\) onto \(V\) where the inverse image is determined uniquely over the smooth locus of \(V\). By Zariski’s theorem, \(\pi_1 : \widetilde{V} \to V\) is a resolution of singularities for \(V\) where the image of the exceptional locus gives the singularities of \(V\). Note that, in this case \(\pi_1\) is the blowup of the quadric \(V = Q_{11}^{4}\) along its singular locus.

**Example 4.2.** Let \(V = [L_7 \subseteq Q_{11}^{4}]\) contained in \(OG(2, n)\), where \(n \geq 15\). The variety \(V\) parameterizes the lines in a singular quadric intersecting a fixed linear space that contains the singular locus of the quadric. Consider the variety defined by
\[
\widetilde{V} = \{(T, T^2, O, Z) \mid T^1 \subseteq T^2, \quad Q_{11}^{4, \text{sing}} \subseteq O \subseteq Z, \quad T^1 \subseteq O \subseteq L_7 \text{ and } T^2 \subseteq Z \subseteq Q_{11}^{4}\}
\]
where \(\dim T^1 = j, \dim O = 5\) and \(\dim Z = 6\). The properties defining the variety \(\widetilde{V}\) can be visualized by the diagram in Figure 1: Consider the following forgetful maps:
\[
(T^1, T^2, O, Z) \mapsto (T^1, O, Z) \mapsto (T^1, O) \mapsto (O).
\]
We show \(\widetilde{V}\) is an iterated tower of \(G(l, n)\) and \(OG(l, n)\) bundles via these maps. The linear space \(O\) satisfies \(Q_{11}^{4, \text{sing}} \subseteq O \subseteq L_7\) and hence can be parameterized by \(G(5 - 4, 7 - 4) = G(1, 3)\). For fixed \(O\), the linear space
Example 4.3. Let \( V = \left[ L_5 \subseteq Q_{10}^7 \subseteq Q_{20}^2 \right] \) contained in \( OG(3,n) \), where \( n \geq 22 \). For this restriction variety we consider \( \tilde{V} \) defined by

\[
\tilde{V} = \{ (T^1, T^2, T^3, O^1, O^2, Z^1, Z^2) \mid Q_{20}^{2,\text{sing}} \subseteq O^1 \subseteq O^2 \subseteq Z^2, \ Q_{10}^{7,\text{sing}} \subseteq Z^1, \\
T^1 \subseteq O^1 \subseteq L_5, \ T^2 \subseteq O^2 \subseteq Q_{10}^7 \text{ and } T^3 \subseteq Z^2 \subseteq Q_{20}^2 \}
\]

where \( \dim T^j = j, \ \dim O^1 = 3, \ \dim O^2 = 4, \ \dim Z^1 = 8 \) and \( \dim Z^2 = 5 \). The corresponding diagram is given in Figure 2. We consider the following forgetful maps:

\[
(T^1, T^2, T^3, O^1, O^2, Z^1, Z^2) \rightarrow (T^1, T^2, O^1, O^2, Z^1, Z^2) \rightarrow (T^1, T^2, O^1, O^2, Z^1)
\]

\[
\rightarrow (T^1, O^1, O^2, Z^1) \rightarrow (T^1, O^1, Z^1) \rightarrow (T^1, O^1) \rightarrow (O^1).
\]

The linear space \( O^1 \) is parameterized by \( G(1,3) \) and for fixed \( O \), \( T^1 \) is parameterized by \( G(1,3) \). The linear space \( Z^1 \) is parameterized by \( OG(1,3) \). For fixed \( Z^1 \), \( O^2 \) satisfies \( O^1 \subseteq O^2 \subseteq Z^1 \) and hence can be parameterized by \( G(1,5) \). Then \( T^2 \) is parameterized by \( G(1,3) \). In the last row, as \( O^2 \subseteq Z^2 \subseteq Q_{20}^2 \), \( Z^2 \) is parameterized by \( OG(1,14) \). Then \( T^3 \) is parameterized by \( G(1,3) \). Thus \( \tilde{V} \) is a tower of the discussed \( G(1,3), G(1,3), OG(1,3), G(1,5), G(1,3), OG(1,14) \) and \( G(1,3) \) bundles. Thus \( \tilde{V} \) is an irreducible smooth variety of dimension 25. The third projection map

\[
\pi : (T^1, T^2, T^3, O^1, O^2, Z^1, Z^2) \rightarrow T^3
\]

gives the resolution of singularities in this example.

Example 4.4. Let us consider the restriction variety in \( OG(10,70) \) given by the sequence

\[
L_2 \subseteq L_6 \subseteq L_{13} \subseteq L_{14} \subseteq L_{19} \subseteq Q_{10}^7 \subseteq Q_{10}^{11} \subseteq Q_{15}^{11} \subseteq Q_{16}^7 \subseteq Q_{50}^3.
\]

In this case \( \tilde{V} \) satisfies the diagram in Figure 3. The dimensions of the \( T \), \( Z \) and \( O \)'s are noted as subscripts. The variety \( \tilde{V} \) is a tower of \( G(k,n) \) and \( OG(k,n) \) bundles via 25 successive forgetful maps in this case. Starting with an element of \( \tilde{V} \), the forgetful maps trail each row from left to right going from the bottom row to the top row.
Figure 3. $\tilde{V}$ for \[L_2 \subseteq L_6 \subseteq L_{13} \subseteq L_{14} \subseteq L_{19} \subseteq Q_{30}^{17} \subseteq Q_{40}^{11} \subseteq Q_{46}^{5} \subseteq Q_{50}^{3} \]

\[
\begin{array}{c}
T_1^1 \subseteq L_2 \\
\cap \\
\cap \\
\cap \\
\cap \\
Q_{r,s}^{3.sing} \subseteq Q_{46}^{7.sing} \subseteq Q_{40}^{11.sing} \subseteq Q_{30}^{17.sing} \\
T_2^2 \subseteq O_{4,na_1}^4 \subseteq L_6 \\
T_3^3 \subseteq O_{6,na_2}^4 \subseteq O_{9,na_2}^3 \subseteq O_{13}^{2,na_2} \subseteq L_{14} \\
T_4^4 \subseteq O_{6,na_3}^4 \subseteq O_{10}^{3,na_3} \subseteq O_{14}^{2,na_3} \subseteq O_{18}^{1,na_3} \subseteq L_{19} \\
T_5^5 \subseteq O_{8,rb_1}^4 \subseteq O_{11}^{3,rb_1} \subseteq O_{15}^{2,rb_1} \subseteq Z_{19}^{1} \subseteq Q_{30}^{17} \\
T_6^6 \subseteq O_{9,rb_2}^4 \subseteq O_{12}^{3,rb_2} \subseteq Z_{16}^{2} \subseteq Q_{40}^{10} \\
T_7^7 \subseteq O_{11}^{4,rb_3} \subseteq Z_{14}^{3} \subseteq Q_{46}^{7} \\
T_8^8 \subseteq Z_{12}^{3} \subseteq Q_{50}^{3} \\
T_9^9 \subseteq Z_{14}^{3} \subseteq Q_{46}^{7} \\
T_10^9 \subseteq Z_{12}^{3} \subseteq Q_{50}^{3}
\end{array}
\]

Figure 4. $\tilde{V}$ for \[L_7 \subseteq Q_{3}^{5} \subseteq Q_{10}^{4} \]

\[
Q_{10}^{4, sing} \\
T_1^1 \subseteq O_5 \subseteq L_7 \\
T_2^2 \subseteq Z_7 \subseteq Q_{10}^{4}
\]

Example 4.5. As a final example, let us illustrate $\tilde{V}$ for the restriction variety $V = [L_7 \subseteq Q_{3}^{5} \subseteq Q_{10}^{4}]$ contained in $OG(3, n)$, where $n \geq 14$, with a given marking $m_\bullet$ for the special index 1. The variety $\tilde{V}$ satisfies the diagram in Figure 4. By considering the forgetful maps

\[(T^1, T^2, O, Z) \mapsto (T^1, O, Z) \mapsto (T^1, O) \mapsto (O),\]

we obtain that $\tilde{V}$ is a tower of $G(1, 3), G(1, 5), OG(2, 4)$ and $G(1, 6)$. Here $Z_7$, which satisfies $O_5 \subseteq Z_7 \subseteq Q_{10}^{4}$, is parameterized by $OG(2, 4)$, and the component that contains $Z$ is the component determined by the marking $m_\bullet$ of $V$.

Let us fix terminology before giving the definition. In the following we say a sequence $A = [A_1 \subseteq \ldots \subseteq A_k]$ is contained in a sequence $B = [B_1 \subseteq \ldots \subseteq B_k]$ if $A_i \subseteq B_i$ for all $1 \leq i \leq k$. We will denote by $A$ both the sequence $[A_1 \subseteq \ldots \subseteq A_k]$ and the ordered set $(A_1, \ldots, A_k)$.

Let $V(L_\bullet, Q_\bullet)$ be a restriction variety defined by the sequence

\[L_{a_1} \subseteq \ldots \subseteq L_{a_n} \subseteq Q_{db_{k-s}}^{r_{k-s}} \subseteq \ldots \subseteq Q_{d_1}^{r_1} \]

or equivalently, by the partitions $(a_{a_1}, \ldots, a_{a_n}), (d_{b_1}, \ldots, d_{b_s}), (r_1, \ldots, r_{k-s})$. For each $Q_{db_{k-s}}^{r_{k-s}}$, let $V(Q_{db_{k-s}}^{r_{k-s}})$ be the subsequence consisting of isotropic linear subspaces $L_{ma_s}$ and sub-quadratics $Q_{d_{b_{s}}}$ that strictly contain
Proof. Consider the successive forgetful maps omitting one coordinate of possible types of rows in a diagram: in each row, starting at the bottom row and going up. The proof of this proposition is based on constructing to \( V \).

Proposition 4.6. Let \( L_1 \leq \ldots \leq Q_{d_{bh}}^{r_{bh} - 1} \) obtained by omitting the last \( \beta_u \) sub-quadrics from the defining sequence will have a crucial role in the following definition.

Define:
\[
V(L_\bullet, L_\bullet) := \left\{ (T^1, \ldots, T^{t+u}, Z^1, \ldots, Z^u, O(Q^{r_{bh}}_{d_{bh}^1}), \ldots, O(Q^{r_{bh}}_{d_{bh}^u})) \mid Q^{r_{bh}}_{d_{bh}^i} \subseteq O(Q^{r_{bh}}_{d_{bh}^i}) \subseteq Z^h \subseteq Q^{r_{bh}}_{d_{bh}}, \right.
\]
\[
O^{h,n_{ag}} \subseteq L_{n_{ag}} \text{ for all } L_{n_{ag}} \text{ in } V(Q^{r_{bh}}_{d_{bh}}),
\]
\[
O^{h,n_{ag}} \subseteq O^{h+1,n_{ag}} \text{ for all } L_{n_{ag}} \text{ that lies in both } V(Q^{r_{bh}}_{d_{bh}}) \text{ and } V(Q^{r_{bh}+1}_{d_{bh}+1}),
\]
\[
O^{h,r_{bh}} \subseteq Q^{r_{bh}}_{d_{bh}} \text{ for all } Q^{r_{bh}}_{d_{bh}} \text{ in } V(Q^{r_{bh}}_{d_{bh}}),
\]
\[
O^{h,r_{bh}} \subseteq O^{h+1,r_{bh}} \text{ for all } Q^{r_{bh}}_{d_{bh}} \text{ that lies in both } V(Q^{r_{bh}}_{d_{bh}}) \text{ and } V(Q^{r_{bh}+1}_{d_{bh}+1}),
\]
\[
T^1 \subseteq \ldots \subseteq T^{t+u} \text{ for all } 1 \leq g \leq t \text{ and } 1 \leq h \leq u
\]
\[
(T^1, \ldots, T^{t+u-1}) \subseteq \left[ L_1 \subseteq \ldots \subseteq Q^{r_{bh}+1}_{d_{bh}-1} \right] \text{ and } T^{t+u} \subseteq Z^u
\]
where \( T^g = a_g, \dim T^{t+h} = k - b_h + 1, \dim Z^h = r_{bh} + (k - b_h + 1) - x_{bh}, \dim O^{h,n_{ag}} = r_{bh} + a_g - x_{bh}, \) and \( \dim O^{h,r_{bh}} = r_{bh} + (k - b_h + 1) - x_{bh} \) for all \( 1 \leq g \leq t \) and \( 1 \leq h \leq u \).

Drawing a diagram, as in the examples above, provides a tidier framework and gives the intuition behind this construction. Let \( L_{n_{ag}} \subseteq \ldots \subseteq L_{n_{ag}} \) be the isotropic linear subspaces in the defining sequence contained in \( Q^{r_{bh},sing} \), thus contained in all other \( Q^{r_{bh},sing} \), \( 1 \leq h \leq u \). The defining properties of \( \tilde{V} \) are visualized in the diagram in Figure 5. Here, the linear spaces \( O^{h,\bullet} \) that lie in the column of \( Q^{r_{bh}}_{d_{bh}} \) form the sequence \( O(Q^{r_{bh}}_{d_{bh}}) \) in the definition of \( \tilde{V} \) above.

Let \( \tilde{V}(L_\bullet, Q_\bullet, m_\bullet) \) be the variety obtained by considering \( \tilde{V}(L_\bullet, Q_\bullet) \) with the marking for each special index inherited from \( V(L_\bullet, Q_\bullet, m_\bullet) \). There is a natural projection from \( \tilde{V}(L_\bullet, Q_\bullet, m_\bullet) \) to \( V(L_\bullet, Q_\bullet, m_\bullet) \) given by
\[
\pi : (T^1, \ldots, T^{t+u}, Z^1, \ldots, Z^u, O(Q^{r_{bh}^1}_{d_{bh}^1}), \ldots, O(Q^{r_{bh}^u}_{d_{bh}^u})) \mapsto T^{t+u}.
\]

Proposition 4.6. Let \( V(L_\bullet, Q_\bullet, m_\bullet) \) be a marked restriction variety. The variety \( \tilde{V}(L_\bullet, Q_\bullet, m_\bullet) \) associated to \( V(L_\bullet, Q_\bullet, m_\bullet) \) is a smooth irreducible variety of the same dimension as \( V(L_\bullet, Q_\bullet, m_\bullet) \).

Proof. Consider the successive forgetful maps omitting one coordinate of \( \tilde{V} \) at a time, going from left to right in each row, starting at the bottom row and going up. The proof of this proposition is based on constructing a tower of \( G(l,n) \) and \( OG(l,n) \) bundles via these forgetful maps. In the following, we study the following possible types of rows in a diagram:

1. For \( L_{n_{ag}} \subseteq Q^{r_{bh}}_{d_{bh}} \), we have \( T^{g-1} \subseteq T^g \subseteq L_{n_{ag}} \). Hence \( T^g \) is parameterized by \( G(a_g - a_{g-1}, n_{ag} - a_{g-1}) \) which has dimension \( (a_g - a_{g-1})(n_{ag} - a_{g}) = a_g(n_{ag} - a_g) \).

2. Suppose for \( L_{n_{ag}} \), the sub-quadrics whose singular loci lie between \( L_{n_{ag}} \) and \( L_{n_{ag} - 1} \) are \( Q^{r_{bh}+c} \), \ldots, \( Q^{r_{bh}} \) for some number \( c \), that is,
\[
Q^{r_{bh}}_{d_{bh}^i,sing} \subseteq \ldots \subseteq Q^{r_{bh}+c+1,sing}_{d_{bh}^i} \subseteq L_{n_{ag} - 1} \subseteq Q^{r_{bh}+c,sing}_{d_{bh}^i} \subseteq \ldots \subseteq Q^{r_{bh},sing}_{d_{bh}^i} \subseteq L_{n_{ag}}.
\]
Figure 5. Definition of $\tilde{V}$

\[
\begin{array}{c}
T_1 \subseteq L_{n_{a_1}} \\
\vdots \\
T_\omega \subseteq L_{n_\omega} \\
\quad \quad Q_{db_\omega, sing} \subseteq Q_{db_\omega-1, sing} \subseteq \ldots \subseteq Q_{db_2, sing} \subseteq Q_{db_1, sing} \\
T_{\omega+1} \subseteq O^{n_{\omega+1}}_{a_1} \\
\vdots \\
T_t \subseteq O^{n_{at}}_{a_1} \\
\quad \quad T_{t+1} \subseteq O^{\cdot, r_{b_1}}_{\cdot} \subseteq O^{\cdot-1, r_{b_1}}_{\cdot} \subseteq \ldots \subseteq O^{3, r_{b_1}}_{\cdot} \subseteq O^{2, r_{b_1}}_{\cdot} \subseteq Z^1 \subseteq Q^{r_{b_1}}_{db_1} \\
\quad \quad T_{t+2} \subseteq O^{\cdot, r_{b_2}}_{\cdot} \subseteq O^{\cdot-1, r_{b_2}}_{\cdot} \subseteq \ldots \subseteq O^{3, r_{b_2}}_{\cdot} \subseteq Z^2 \subseteq Q^{r_{b_2}}_{db_2} \\
\quad \quad T_{t+3} \subseteq \ldots \subseteq Z^3 \\
\vdots \\
\quad \quad T_{t+u-1} \subseteq O^{\cdot, r_{b_{u-1}}} \subseteq Z^{u-1} \\
T_{t+u} \subseteq Z^u
\end{array}
\]

Figure 6. Row of Type 2 in $\tilde{V}$

\[
\begin{array}{c}
Q^{r_{b_\omega, sing}}_{db_\omega} \subseteq \ldots \subseteq Q^{r_{b_{\omega+c+1}, sing}}_{db_{\omega+c+1}} \subseteq Q^{r_{b_{\omega+c}, sing}}_{db_{\omega+c}} \subseteq \ldots \subseteq Q^{r_{b_1, sing}}_{db_1} \\
\vdots \\
T_{y-1} \subseteq O^{n_{ag-1}}_{\cdot} \subseteq \ldots \subseteq O^{\eta+c+1, n_{ag-1}}_{\cdot} \\
\vdots \\
T_y \subseteq O^{n_{ag}}_{\cdot} \subseteq \ldots \subseteq O^{\eta+c+1, n_{ag}}_{\cdot} \subseteq O^{\eta+c, n_{ag}}_{\cdot} \subseteq \ldots \subseteq O^{n_{ag}}_{\cdot} \subseteq L_{n_{ag}} \\
\vdots \\
\vdots
\end{array}
\]

Note that $x_{b_\omega} = \ldots = x_{b_{\omega+c}} = a_{g-1}$ in this setting. The row consisting of $T_y$, $O^{n_{ag}}_{\cdot}$, $L_{n_{ag}}$ satisfies the diagram in Figure 6.

We start by choosing $O^{n_{ag}}_{\cdot}$. The linear space $O^{n_{ag}}_{\cdot}$ satisfying $Q^{r_{b_{\omega, sing}}}_{db_\omega} \subseteq O^{n_{ag}}_{\cdot} \subseteq L^{r_{b_{\omega}}}_{n_{ag}}$ is parameterized by the Grassmannian $G(r_{b_\omega} + a_g - x_{b_\omega} - r_{b_\omega}, n_{ag} - r_{b_\omega})$. In a similar fashion, the parameterization of $T_y, O^{n_{ag}}_{\cdot}, \ldots, O^{n_{ag}}_{\cdot}$ are given by Grassmannians whose dimensions add up to $\alpha_g(n_{ag} - a_g)$ as in Table 1.
Table 1. Dimensions for a row of Type 2 in $\tilde{V}$

<table>
<thead>
<tr>
<th>Coordinates of $\tilde{V}$ in the $g$-th row:</th>
<th>Dimensions of the corresponding Grassmannian:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{\mu}^{r_{\mu},\text{sing}} \subseteq Q_{\mu}^{r_{\mu+1},\text{sing}} \subseteq \cdots \subseteq Q_{\mu}^{r_{\mu+c},\text{sing}} \subseteq O^{\eta_{x_{\mu}}}$</td>
<td>$(a_g - x_{b_{\mu}})(n_{a_{\mu}} - a_g - (r_{b_{\mu}} - x_{b_{\mu}}))$</td>
</tr>
<tr>
<td>$Q_{d_{\mu}}^{r_{\mu}} \subseteq Q_{d_{\mu+1}}^{r_{\mu+1}} \subseteq \cdots \subseteq Q_{d_{\mu+c}}^{r_{\mu+c}} \subseteq O^{\eta_{x_{\mu}}}$</td>
<td>$(a_g - x_{b_{\mu}})((r_{b_{\mu}} - x_{b_{\mu}}) - (r_{b_{\mu+1}} - x_{b_{\mu+1}}))$</td>
</tr>
<tr>
<td>$O^{\eta_{x_{\mu}}+c,n_{a_{\mu}}} \subseteq Q^{\eta_{x_{\mu}}+c-1,n_{a_{\mu}}}$</td>
<td>$(a_g - x_{b_{\mu}})((r_{b_{\mu+c}} - x_{b_{\mu+c}}) - (r_{b_{\mu+c-1}} - x_{b_{\mu+c-1}}))$</td>
</tr>
<tr>
<td>$O^{\eta_{x_{\mu}}+c+1,n_{a_{\mu}}-1} \subseteq O^{\eta_{x_{\mu}}+c+1,n_{a_{\mu}}}$</td>
<td>$(a_g - a_{\mu-1})((r_{b_{\mu-1}} - x_{b_{\mu-1}}) - (r_{b_{\mu}} - x_{b_{\mu}}))$</td>
</tr>
<tr>
<td>$O^{u,n_{a_{\mu}}-1} \subseteq O^{u,n_{a_{\mu}}} \subseteq O^{u-1,n_{a_{\mu}}}$</td>
<td>$(a_g - a_{\mu-1})(r_{b_{\mu}} - x_{b_{\mu}})$</td>
</tr>
<tr>
<td>$T^{\mu-1} \subseteq T^\mu \subseteq O^{u,n_{a_{\mu}}}$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

Figure 7. Row of Type 3 in $\tilde{V}$

(3) Consider the row that corresponds to $Q_{\mu}^{r_{\mu}}$. Depending on $r_{b_{\mu}}$, there are two possibilities for the diagram. If $r_{b_{\mu}} \geq n_{a_{\mu}}$, then $Z^{1}$ is determined by $Q_{\mu}^{r_{\mu},\text{sing}} \subseteq Z^{1} \subseteq Q_{\mu}^{r_{\mu}}$. Explicitly, suppose $L_{n_{a_{\mu}}}$ is positioned as $Q_{d_{\mu+c}}^{r_{\mu+c},\text{sing}} \subseteq L_{n_{a_{\mu}}} \subseteq Q_{d_{\mu+c}}^{r_{\mu},\text{sing}} \subseteq \cdots \subseteq Q_{d_{\mu+1}}^{r_{\mu+1},\text{sing}}$ for some number $c$. Note that $x_{b_{\mu}} = \cdots = x_{b_{\mu}} = t$ in this setting. The diagram is as in Figure 7. We start by choosing $Z^{1}$. The linear space $Z^{1}$ satisfies $Q_{d_{\mu+c}}^{r_{\mu+c},\text{sing}} \subseteq Z^{1} \subseteq Q_{d_{\mu+1}}^{r_{\mu+1}}$ and dim $Z^{1} = r_{b_{\mu}} + (k-b_{\mu})_1 - x_{b_{\mu}} = r_{b_{\mu}} + \frac{1}{k}$. Hence $Z^{1}$ can be parameterized by $OG(\beta_{1}, d_{b_{\mu}}, -r_{b_{\mu}})$. Note that $OG(\beta_{1}, d_{b_{\mu}}, -r_{b_{\mu}})$ is irreducible as $d_{b_{\mu}}, -r_{b_{\mu}} - 2\beta_{1} \geq 3$. The linear spaces $T^{\mu+1}, O^{u,r_{\mu}}, \ldots, O^{2,r_{\mu}}$ can be parameterized by Grassmannians whose dimensions add up to $\beta_{1}(d_{b_{\mu}} + x_{b_{\mu}} - 2(k-b_{\mu} + 1) - \frac{\beta_{1} - 1}{k})$, see Table 2. Note that dim $OG(k, n) = k(n-2k+\frac{k-1}{2})$ (see [2] for a proof).

(4) As another case for the row that corresponds to $Q_{\mu}^{r_{\mu}}$, if $r_{b_{\mu}} < n_{a_{\mu}}$, then $Z^{1}$ is determined by $O^{1,n_{a_{\mu}}} \subseteq Z^{1} \subseteq Q_{d_{\mu+1}}^{r_{\mu}}$. The linear space $Z^{1}$ has to be contained in the orthogonal complement of $O^{1,n_{a_{\mu}}}$, so $Z^{1} \subseteq Q_{d_{\mu+1}}^{r_{\mu}+1, \langle a_{\mu}, -x_{\mu} \rangle}$. Hence $Z^{1}$ can be parameterized by $OG(\beta_{1}, d_{b_{\mu}} - r_{b_{\mu}} - 2(a_{\mu} - x_{b_{\mu}})).$ If $OG(\beta_{1}, d_{b_{\mu}} - r_{b_{\mu}} - 2(a_{\mu} - x_{b_{\mu}}))$ has two components, then $Z^{1}$ belongs to the component determined by the marking $m_{\bullet}$. The parameterizations of $T^{\mu+1}, O^{u,r_{\mu}}, \ldots, O^{2,r_{\mu}}$ are similar to the previous case, the total dimension is $\beta_{1}(d_{b_{\mu}} + x_{b_{\mu}} - 2(k-b_{\mu} + 1) - \frac{\beta_{1} - 1}{k})$ as before. The diagram and the parameterizations in this case are as in Figure 8 and Table 3.

(5) Finally, the $(t+h)$-th row for some $h \geq 2$ is similar to the case above. The parameterizations are given by a tower of Grassmannians contained in an orthogonal Grassmannian and the total
The variety $\widetilde{V}$ is smooth as it is an iterated tower of the ordinary and the orthogonal Grassmannian bundles observed above. The inverse image $\pi^{-1}(\Lambda)$ of a point $\Lambda$ in $V$ is irreducible by the same observations, hence $\widetilde{V}$ is irreducible for a marked restriction variety. Furthermore, combining the results from each row of
Table 4. Dimensions for a row of Type 5 in $\tilde{V}$

<table>
<thead>
<tr>
<th>Coordinates of $\tilde{V}$ in the $(t + h)$-th row:</th>
<th>Dimension of the corresponding Grassmannian:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O^{h,r_{bh} - 1} \subseteq Z^h \subseteq Q^{r_{bh}}$</td>
<td>$\beta_h(d_{bh} + x_{bh} - 2(k - b_h + 1) - (r_{bh} - x_{bh}) + \frac{\beta_h - 1}{2})$</td>
</tr>
<tr>
<td>$O^{h+1,r_{bh} - 1} \subseteq O^{h+1,r_{bh}} \subseteq Z^h$</td>
<td>$(b_{h-1} - b_h)(r_{bh} - x_{bh}) - (r_{bh+1} - x_{bh+1}))$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$T^{t+h-1} \subseteq T^{t+h} \subseteq O^{u,r_{bh}}$</td>
<td>$(b_{h-1} - b_h)(r_{bh} - x_{bh})$</td>
</tr>
</tbody>
</table>

the diagram, $\dim \tilde{V}$ is given by

$$\dim \tilde{V} = \sum_{g=1}^{t} \alpha_g (n_{ag} - a_g) + \sum_{h=1}^{u} \beta_h \left( d_{bh} + x_{bh} - 2(k - b_h + 1) - \frac{\beta_h - 1}{2} \right) = \dim V$$

which concludes the proof. □

Over $V^0(L_\bullet, Q_\bullet)$, the inverse image of a point $\pi^{-1}(\Lambda)$ is determined uniquely by

- $T^g = \Lambda \cap L_{n_{ag}}$
- $T^{t+h} = \Lambda \cap Q^{r_{bh}}_{dh}$
- $O^{h,r_{ag}} = Q^{r_{ag}}_{d_{bh}} \cap \Lambda$, $O^{h,n_{ag}} = Q^{r_{ag}}_{d_{bh}} \cap \Lambda \cap L_{r_{ag}}$
- $Z^h = Q^{r_{bh}}_{d_{bh}} \cap \Lambda \cap Q^{r_{bh}} \cap \Lambda \cap L_{r_{ag}}$ for all $1 \leq g \leq t$, $1 \leq h \leq u$.

$V^0(L_\bullet, Q_\bullet)$ is in the smooth locus of $V(L_\bullet, Q_\bullet)$ since it is homogeneous under the action of $SO(n)$. Then, Zariski’s main theorem shows that $\pi$ is an isomorphism over $V^0(L_\bullet, Q_\bullet)$. Therefore we have

**Theorem 4.7.** The map $\pi : \tilde{V}(L_\bullet, Q_\bullet) \to V(L_\bullet, Q_\bullet)$ is a resolution of singularities.

5. THE EXCEPTIONAL LOCUS

We now study the exceptional locus of $\pi$. More specifically, we are interested in the codimension of the components of the exceptional locus.

Corresponding to the three types of conditions in Definition 3.3, namely,

$$\dim(\Lambda \cap Q^{r_{ag}}_{d_{ag}}) = x_i \ , \ \dim(\Lambda \cap L_{r_{ag}}) = j \ , \ \text{and} \ \dim(\Lambda \cap Q^{r_{ag}}_{d_i}) = k - i + 1 \ ,$$

we consider three types of loci $\Sigma$ where $\pi$ has positive dimensional fibers. The image of the exceptional locus of $\pi$ is equal to the union of the following $\Sigma$’s

**I:** $\Sigma_{r_{bh}}$: The Zariski closure of the subvariety of $V(L_\bullet, Q_\bullet, m_\bullet)$ parameterizing $k$-dimensional isotropic subspaces $\Lambda$ such that $\dim(\Lambda \cap Q^{r_{bh}}_{d_{bh}}) = x_{bh} + 1$ for some $1 \leq h \leq u$.

**II:** $\Sigma_{n_{ag}}$: The Zariski closure of the subvariety of $V(L_\bullet, Q_\bullet, m_\bullet)$ parameterizing $k$-dimensional isotropic subspaces $\Lambda$ such that $\dim(\Lambda \cap L_{n_{ag}}) = a_g + 1$ for some $1 \leq g \leq t$, or $\dim(\Lambda \cap L_{r_{ag}}) = s + 2$ in a certain case that is discussed in Remark 5.1.

**III:** $\Sigma_{d_{bh}}$: The Zariski closure of the subvariety of $V(L_\bullet, Q_\bullet, m_\bullet)$ parameterizing $k$-dimensional isotropic subspaces $\Lambda$ such that $\dim(\Lambda \cap Q^{r_{bh}}_{d_{bh}}) = k - b_h + 2$ for some $1 \leq h \leq u - 1$.

**Remark 5.1.** Note that these loci do not exist for every restriction variety. There are natural restrictions for their existence resulting from the properties of quadrics. The locus $\Sigma_{n_{ag}}$, for some $1 \leq g \leq t$, exists only if $n_{ag} > a_g$, and the locus $\Sigma_{d_{bh}}$, for some $1 \leq h \leq u - 1$, exists only if $u > 1$ and $d_{bh} - r_{bh} - 2\beta_1 \geq 3$ (requirement for the irreducibility of the quadric that arises in the sequence of $\Sigma_{d_{bh}}$). Similarly, the locus $\Sigma_{r_{bh}}$, for some $1 \leq h \leq u$, exists only if $r_{bh} > x_{bh}$.
Furthermore, different components of $\text{OG}(m, 2m)$ must be kept in mind in a certain case. Suppose $b_1$ is a special index (that is, $x_{b_1} = k - b_1 + 1 - b_2 - r_{a_2}$ as in Definition 3.6) and $2n_s = r_{b_1} + d_{b_1}$. The linear space $L_{n_s}$ belongs to one of the components of the Fano variety of maximal dimensional linear spaces contained in $Q_{d_{b_1}}^{r_{b_1}}$. For a general $k$-plane $\Lambda$, the $n_s$-dimensional linear subspace $\Lambda \cap Q_{d_{b_1}}^{r_{b_1}}$ lies in the other component. Note that two linear spaces in $\text{OG}(m, 2m)$ belong to the same component if and only if their intersection is equal to $m \mod 2$. Therefore, no $\Lambda$ in $V$ satisfies $\dim(\Lambda \cap L_{n_s}) = s + 1$, but there may be elements with $\dim(\Lambda \cap L_{n_s}) = s + 2$.

**Example 5.2.** The locus $\Sigma_{r_{b_1}}$ does not make sense for the restriction variety given by $\left[Q_8^0 \subseteq Q_9^0\right]$ since $Q_9^{0, \text{sing}}$ is empty. Similarly the locus $\Sigma_{r_{b_1}}$ does not exist for the restriction variety given by $\left[L_1 \subseteq Q_7^4\right]$ since $x_1 = 1$ and it is not possible to intersect $Q_7^{1, \text{sing}}$ in a higher dimension.

**Example 5.3.** The loci $\Sigma_{n_s}$ do not exist for the restriction variety given by $\left[L_1 \subseteq L_7 \subseteq L_8\right]$; lines contained in $L_8$ containing $L_1$ cannot intersect $L_8$ or $L_1$ in higher dimensions. Similarly, $\Sigma_{d_{b_1}}$ does not exist for the restriction variety given by $\left[Q_7^3 \subseteq Q_8^1\right]$.

**Example 5.4.** Let $V = \left[L_2 \subseteq Q_4^4\right]$, the variety of lines contained in a smooth quadric surface intersecting a fixed line on the surface. This (marked) restriction variety is one of the components of the lines on the quadric surface. The locus $\Sigma_{n_s}$, given by $\left[L_1 \subseteq L_2\right] = L_2$, lies in the other component, and hence is not contained in $V$. It is easy to see $V$ is smooth in this example as it is isomorphic to $\mathbb{P}^1$.

**Example 5.5.** Let $V$ be the restriction variety in $\text{OG}(4, 8)$ given by $\left[L_1 \subseteq L_3 \subseteq L_4 \subseteq Q_4^4\right]$. A general element $\Lambda$ of $V$ satisfies $\dim(\Lambda \cap L_4) = 3$, therefore $L_4$ and $\Lambda$ lie in different components of $\text{OG}(4, 8)$. This shows that the restriction variety given by the sequence $\left[L_1 \subseteq L_2 \subseteq L_3 \subseteq L_4\right] = L_4$ is not in the image of the exceptional locus of $\pi$ in this case.

**Example 5.6.** Let $V$ be given by $\left[L_3 \subseteq L_4 \subseteq Q_4^4 \subseteq Q_8^0\right]$. A general element $\Lambda$ of $V$ satisfies $\dim(\Lambda \cap L_4) = 2$, and hence $L_4$ lies in the same component of $\text{OG}(k, 2k)$ as $V$. Since we have $\dim(\Lambda \cap L_4) = 4 \mod 2$ for linear spaces $\Lambda$ in the same component as $L_4$, we conclude $\dim(\Lambda \cap L_4)$ must be either 2 or 4. Therefore, in this case we have $\Sigma_{n_s} = \left[L_1 \subseteq L_2 \subseteq L_3 \subseteq L_4\right]$.

In the following, we study loci $\Sigma$ that are contained in the restriction variety $V$. Over each $\Sigma$, $\pi^{-1}(\Sigma)$ is irreducible of codimension

$$\text{codim}(\pi^{-1}(\Sigma)) = \text{codim}(\Sigma) - \dim(\pi^{-1}(\Lambda))$$

for a general point $\Lambda$ in $\Sigma$. We now consider each $\Sigma$ separately, observe the sequence that defines the restriction variety $\Sigma$, and study $\text{codim}(\pi^{-1}(\Sigma))$ in each case. Our aim is to find the components of the exceptional locus with codimension greater than 1.

**Observation 5.7.** A component of the exceptional locus of $\pi$ with image of one of the types

- $\Sigma_{r_{b_h}}$ with $r_{b_h} < n_s$
- $\Sigma_{n_s}$ with $1 \leq g \leq t - 1$
- $\Sigma_{d_{b_h}}$ for all $1 \leq h \leq u - 1$

has codimension larger than 1 (by I.B, I.C, II.B and III below). A component with image of type $\Sigma_{r_{b_h}}$ with $r_{b_h} \geq n_s$ has codimension equal to 1 (by I.A and I.D below). A component with image of type $\Sigma_{n_s}$ has codimension given by $\text{codim}(\pi^{-1}(\Sigma_{n_s})) = d_{k-s} + x_{k-s} - s - n_s - 1$ which may be larger than or equal to 1.

In the following computation which results in Observation 5.7, each component of the exceptional locus is studied by dividing it into subcases.
Given the corank $r_{bh}$, we divide this case into sub-cases depending on the relation between $r_{bh}$ and the dimensions $n_s$, of the isotropic linear spaces appearing in the sequence defining $V$. The sub-cases we consider in the following are:

**I.A:** $r_{bh} > n_s$

**I.B:** $r_{bh} < n_s$ and $r_{bh} ≠ n_j$ for all $j$

**I.C:** $r_{bh} = n_j$ for some $n_j < n_s$

**I.D:** $r_{bh} = n_s$

**I.A:** Suppose $r_{bh} > n_s$. A general element of $\Sigma_{r_{bh}}$ intersects $Q_{d_{bh}}^{r_{bh} - \text{sing}}$ in one more dimension. Equivalently, this is the restriction variety associated to the sequence obtained by replacing the sub-quadratic $Q_{d_{bh}}^{r_{bh} - \text{sing}}$ with the isotropic linear space $L_{r_{bh}}$ in the fixed full flag of dimension $r_{bh}$. Note that $\Sigma_{r_{bh} - 1}$ contains $\Sigma_{r_{bh}}$, so all $\Sigma_{r_{bh}}$ with $r_{bh} > n_s$ are contained in $\Sigma_{r_{bh} - 1}$. Therefore it is sufficient to consider $\Sigma_{r_{bh} - 1}$.

**Example 5.8.** Let $V$ be the restriction variety given by the sequence $[L_3 \subseteq Q_{10}^5 \subseteq Q_{20}^5$]. The loci $\Sigma_{r_{bh}}$ and $\Sigma_{r_{bh} - 2}$ are defined as the closures of the loci:

$$\Sigma^0_{r_{bh}} := \{ \Lambda \in V \mid \dim(\Lambda \cap Q_{10}^5) = 2 \text{ with other conditions of } V^0 \text{ unchanged} \}$$

$$\Sigma^0_{r_{bh} - 2} := \{ \Lambda \in V \mid \dim(\Lambda \cap Q_{20}^5) = 2 \text{ with other conditions of } V^0 \text{ unchanged} \}$$

Since $\Sigma_{r_{bh} - 2}$ is contained in $\Sigma_{r_{bh} - 1} = [L_3 \subseteq L_7 \subseteq Q_{20}^5$], it is sufficient to consider codim($\pi^{-1}(\Sigma_{r_{bh}})$).

The introduced isotropic linear space $L_{r_{bh}}$ is contained in $Q_{d_i}^i$ for $b_1 \leq i < k - s$. Therefore, in the resulting restriction variety, the value of $x_i$ increases by one for $b_1 \leq i < k - s$. Thus we have

$$\dim(\Sigma_{r_{bh}}) = (d_{k-s} + x_{k-s} - 2(s + 1)) - ((r_{bh} - (s + 1)) - (\beta_1 - 1))$$

$$= d_{k-s} - r_{bh} - \beta_1$$

since $x_{k-s} = s$ by our assumption that $r_{bh} > n_s$.

Now we study the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{r_{bh}}$. By assumption there is no $O$ containing $Q_{d_{bh}}^{r_{bh} - \text{sing}}$ and $O$’s contained in $Q_{d_{bh}}^{r_{bh} - \text{sing}}$ are determined uniquely by $\Lambda$. We have $T_{i+1}, Q_{d_{bh}}^{r_{bh} - \text{sing}} \subseteq Z_1 \subseteq Q_{d_{bh}}$ where $\dim(T_{i+1}, Q_{d_{bh}}^{r_{bh} - \text{sing}}) = r_{bh} + (k - b_1 + 1) - (x_{bh} + 1)$ and dim $Z_1 = \dim(T_{i+1}, Q_{d_{bh}}^{r_{bh} - \text{sing}}) + 1$.

Since $Z_1$ has to lie in the orthogonal complement of $T_{i+1}, Q_{d_{bh}}^{r_{bh} - \text{sing}}$, we have $T_{i+1}, Q_{d_{bh}}^{r_{bh} - \text{sing}} \subseteq Z_1 \subseteq Q_{d_{bh}}^{r_{bh} - (k - b_1 + 1 - x_{bh})}$. Such $Z_1$ can be parameterized by $OG(1, d_{bh} - r_{bh} - 2(k - b_1 + 1 - x_{bh} - 1))$. Therefore

$$\text{codim}(\pi^{-1}(\Sigma_{r_{bh}})) = d_{k-s} - r_{bh} - \beta_1 - (d_{bh} - r_{bh} - 2(k - b_1 + 1 - x_{bh} - 1) - 2)$$

$$= d_{k-s} - d_{bh} + 2(k - b_1 + 1 - x_{bh}) - \beta_1$$

$$= 1$$

since $d_{bh} - d_{k-s} = \beta_1 - 1$ and $k - b_1 + 1 - s = \beta_1$.

**Example 5.9.** Let $V = [L_3 \subseteq Q_{10}^5 \subseteq Q_{20}^5$], then

$$\tilde{V} = \{(T^1, T^2, T^3, Z^1, Z^2, O^{2,r_{bh}}) | Q_{20}^5 \subseteq O^{2,r_{bh}} \subseteq Z^2, Q_{10}^5 \subseteq Z^1, T^1 \subseteq L_3, T_2 \subseteq O^{2,r_{bh}} \subseteq Z^1 \subseteq Q_{10}^5, T^3 \subseteq Z^2 \subseteq Q_{20}^5 \}$$

equivalently, the diagram is given in Figure 10.

The subvariety $\Sigma_{r_{bh}} = [L_3 \subseteq L_7 \subseteq Q_{20}^5] \subseteq V$ has codimension 2. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{r_{bh}}$, we have $T^3 = \Lambda, T^2 = \Lambda \cap Q_{10}^5 = \Lambda \cap L_7, T^1 = \Lambda \cap L_3$. 

17
I.B: Next we consider $\Sigma^6_{20}$. Let $L_i$ in the sequence satisfying $r_i < n_j$, then let $r_{ia} := \max\{r_i < n_j\}$, and replace $Q_{d_{ia}}^{r_{ia}}$ with $Q_{d_{ia}-(n_j-r_{ia})}^{r_{ia}}$.

**Example 5.10.** Let $V = [L_1 \subseteq Q^3_6 \subseteq Q^1_8]$, an orthogonal Schubert variety in $OG(3,9)$. The diagram in Figure 11 defines $V$.

The subvariety $\Sigma_{r_{a1}} = [L_1 \subseteq L_3 \subseteq Q^1_8]$ has codimension 2. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{r_{a1}}$, only $Z^1$ is not determined uniquely. We have $\dim Z^1 = 4$ and $Q^3_{6,\text{sing}} \subseteq Z^1 \subseteq Q^1_{20}$, from which we conclude $Z^1$ is parameterized by $OG(1,3)$. Thus $\dim(\pi^{-1}(\Lambda)) = 1$ and $\text{codim}(\pi^{-1}(\Sigma_{r_{a1}})) = 2 - 1 = 1$.

**Rule 1.** Given the defining sequence of a restriction variety, consider the modified sequence where an isotropic linear space $L_{n_j}$ is replaced with a smaller dimensional isotropic linear space. If there are subquadrats $Q_{d_{ia}}^{r_{ia}}$ in the sequence satisfying $r_i < n_j$, then let $r_{ia} := \max\{r_i < n_j\}$, and replace $Q_{d_{ia}}^{r_{ia}}$ with $Q_{d_{ia}-(n_j-r_{ia})}^{r_{ia}}$.

Figure 10. $\bar{V}$ for $[L_3 \subseteq Q^1_{10} \subseteq Q^5_{20}]$

\[
\begin{align*}
T^1 & \subseteq \{L_3 \subseteq Q^5_{20}\} \\
& \subseteq Q^7_{10,\text{sing}} \\
& \subseteq Q^7_{20,\text{sing}} \\
T^2 & \subseteq \{O^2_{r_{a1}} \subseteq Z^1 \subseteq Q^7_{10}\} \\
& \subseteq \{Z^2 \subseteq Q^5_{20}\} \\
T^3 & \subseteq \{Z^2 \subseteq Q^5_{20}\}
\end{align*}
\]

Figure 11. $\bar{V}$ for $[L_1 \subseteq Q^3_6 \subseteq Q^1_8]$

\[
\begin{align*}
T^1 & \subseteq \{L_1 \subseteq Q^3_{6,\text{sing}}\} \\
& \subseteq Q^3_{6,\text{sing}} \\
& \subseteq Q^3_{6,\text{sing}} \\
T^2 & \subseteq \{Z^1 \subseteq Q^3_6\} \\
& \subseteq \{Q^3_6\} \\
T^3 & \subseteq \{Q^3_6\}
\end{align*}
\]
In the inverse image \( G \) by \( \Sigma \) dim(\( L \)) \( \subseteq \) \( O^{1,n_{a_1}} \subseteq \) \( L_7 \) \( \cap \) \( T^1 \subseteq \) \( O^{1,n_{a_1}} \subseteq \) \( L_7 \), therefore, \( \Sigma \) dim(\( L \)) \( \subseteq \) \( Z^1 \subseteq \) \( Q_{15}^2 \).

Since \( L_{r_{_{bh}}} \) is contained in the singular locus of every sub-quadric in the group of \( Q^r_{d_{bh}} \), for each of these sub-quadrics, \( x_i \) increases by one. Hence we get
\[
\text{codim}(\Sigma_{r_{_{bh}}}) = n_{j_1} - r_{bh} + n_{j_2} - r_{bh} = \beta_h.
\]

**Example 5.11.** Let \( V = \left[ L_7 \subseteq Q_{15}^2 \subseteq Q_{25}^2 \right] \), then \( \Sigma_{r_{_{bh}}} = \left[ L_5 \subseteq Q_{15}^2 \subseteq Q_{25}^2 \right] \). Specializing a general element \( \Lambda \) of \( V \) so that it intersects \( L_5 \) increases \( x_2 \) by 1. In this example, \( \text{codim}(\Sigma_{r_{_{bh}}}) = 2 + 2 - 1 = 3 \).

Note that the linear space \( L_{n_{a_1}} \) may not be among \( L_{n_{a_2}} \), that is, the largest dimensional isotropic linear space in a group with consecutively increasing dimensions. Let \( L_{n_{a_2}} \) be the smallest \( L_{n_{a_2}} \) containing \( L_{n_{a_1}} \). In the inverse image \( \pi^{-1}(\Lambda) \) of a general point \( \Lambda \) in \( \Sigma_{r_{_{bh}}} \), all coordinates are determined uniquely except for \( O^{h,n_{a_2}} \) and \( Z^h \). We have \( \frac{Q^r_{d_{bh}}}{\text{sing}} \subseteq O^{h,n_{a_2}} \subseteq O^{h,n_{a_2}} \) thus \( O^{h,n_{a_2}} \) can be parameterized by \( G(1,n_{a_2} - (r_{bh} + a_{g_1} - x_{bh}) + 1) \). Then \( Z^h \) is determined uniquely as \( \overline{O^{h,n_{a_2}, \Lambda \cap Q^r_{d_{bh}}}} \). Thus \( \text{dim}(\pi^{-1}(\Lambda)) = n_{a_{g_2}} - (r_{bh} + a_{g_2} - x_{bh}) \) and
\[
\text{codim}(\pi^{-1}(\Sigma_{r_{_{bh}}})) = n_{j_2} - r_{bh} + n_{j_3} - r_{bh} + 1 - \beta_h - (n_{a_{g_2}} - (r_{bh} + a_{g_2} - x_{bh}))
\]
\[
= \left( (n_{j_2} - r_{bh}) - (n_{a_{g_2}} - (r_{bh} + a_{g_2} - x_{bh})) \right) + n_{j_3} - r_{bh} + 1 - \beta_h
\]
\[
\geq 2
\]
since \( (n_{j_2} - r_{bh}) - (n_{a_{g_2}} - (r_{bh} + a_{g_2} - x_{bh})) \geq 1 \) and \( n_{j_3} - r_{bh} + 1 - \beta_h \geq 1 \) by construction.

**Example 5.12.** Let \( V = \left[ L_6 \subseteq L_7 \subseteq Q_{15}^2 \right] \), then \( \tilde{V} \) is given by the diagram in Figure 12. The subvariety \( \Sigma_{r_{_{bh}}} = \left[ L_2 \subseteq L_7 \subseteq Q_{15}^2 \right] \) has codimension 7. In the inverse image \( \pi^{-1}(\Lambda) \) of a general point \( \Lambda \) in \( \Sigma_{r_{_{bh}}} \), we have \( T^3 = \Lambda, T^2 = \Lambda \cap L_7 \cap L_7 \) as above, \( Z^3 \) is determined as \( Z^3 = O^{1,n_{a_1}}, \Lambda \) so the nontrivial part is the parametrization of \( O^{1,n_{a_1}} \). We have \( Q^r_{15, \text{sing}}, \Lambda \cap L_7 \subseteq O^{1,n_{a_1}} \subseteq L_7 \) which is parameterized by \( G(1,4) \). Thus \( \text{dim}(\pi^{-1}(\Lambda)) = 3 \) and \( \text{codim}(\pi^{-1}(\Sigma_{r_{_{bh}}})) = 7 - 3 = 4 \).

**Example 5.13.** Let \( V = \left[ L_7 \subseteq Q_{15}^2 \subseteq Q_{25}^2 \right] \), then \( \tilde{V} \) is given by the diagram in Figure 13. The subvariety \( \Sigma_{r_{_{bh}}} = \left[ L_5 \subseteq L_7 \subseteq Q_{25}^2 \right] \) has codimension 3. In the inverse image \( \pi^{-1}(\Lambda) \) of a general point \( \Lambda \) in \( \Sigma_{r_{_{bh}}} \), we have \( T^3 = \Lambda, T^2 = \Lambda \cap Q_{15}^2 \subseteq \Lambda \cap Q_{15}^2, T^1 = \Lambda \cap L_7 = \Lambda \cap L_5, O^{2,n_{a_1}} = \overline{Q_{25}^2, \Lambda \cap L_7}, O^{2,r_{_{bh}}} = \overline{Q_{15, \text{sing}}, \Lambda \cap Q_{15}^2, Z^3 = \overline{Q_{25}^2, \Lambda}} \). The linear space \( O^{1,n_{a_1}} \) satisfies \( Q_{15}^r \subseteq Q_{15}^r \subseteq L_7 \) and hence can be parameterized by \( G(1,2) \). Then \( Z^1 \) is determined uniquely as \( Z^1 = O^{1,n_{a_1}}, \Lambda \cap Q_{15}^2 \). Thus \( \text{dim}(\pi^{-1}(\Lambda)) = 1 \) and \( \text{codim}(\pi^{-1}(\Sigma_{r_{_{bh}}})) = 3 - 1 = 2 \).

**I.C:** Consider \( r_{bh} \) such that there is \( L_{n_{j}} \) with \( n_{j} < s \) in the defining sequence satisfying \( n_{j} = r_{bh} \). Since there is no \( L_{n_{j}} \) in the sequence with \( n_{j} = r_{bh} + 1 \), we have \( r_{bh} = n_{a_g} \) for some \( 1 \leq g \leq t - 1 \).

In the sequence of \( \Sigma_{r_{_{bh}}} \), the isotropic linear space that appears next to \( L_{n_{a_g}} \) in the sequence of \( V \), namely \( L_{n_{a_g+1}} \), is replaced with \( L_{n_{a_g}} \). Consequently, \( L_{n_{a_g}} \) is replaced with the isotropic linear space of
one less dimension, namely \( L_{n_{ag}-1} \). Similarly, each isotropic linear space \( L_\tau \), where \( n_{ag} - \alpha_g + 1 \leq \tau \leq n_{ag} \), is replaced with \( L_{\tau-1} \). Applying Rule 1, the sub-quadric \( Q^r_{\nu} \), where \( r_{io} := \max\{r_i \leq n_{ag}+1\} \), is replaced with \( Q^{n_{ag}+1}_{d_i (n_{ag}+1-r_{io})} \). Observe that \( x_i \) increases by one for each \( i \) satisfying \( b_h + \beta_h - 1 \geq i \geq b_h \).

Comparing the dimensions of both sequences, we have

\[
\text{codim}(\Sigma_{r_{bh}}) = \alpha_g (n_{ag} - a_g) + \alpha_{g+1} (n_{ag+1} - a_{g+1})
- (\alpha_g + 1) (n_{ag} - a_g - 1) - (\alpha_{g+1} - 1) (n_{ag+1} - a_{g+1})
+ (n_{ag+1} - r_{io}) - \beta_h.
\]

In the inverse image \( \pi^{-1}(\Lambda) \) of a general point \( \Lambda \) in \( \Sigma_{r_{bh}} \), all coordinates are determined uniquely except for \( O^{h,n_{ag}+1}_{\nu} \), \( Z^h \) and the coordinates in the \( g \)-th row. We have

\[
Q^{r_{bh},-\text{sing}}_{d_{bh}}, \Lambda \cap L_{n_{ag}+1} \subseteq O^{h,n_{ag}+1}_{\nu} \subseteq L_{n_{ag}+1}
\]
thus \( O^{h,n_{ag}+1}_{\nu} \) can be parameterized by

\[
G(1, n_{ag}+1 - (r_{bh} + a_{g+1} - x_{bh}) + 1) = G(1, n_{ag}+1 - n_{ag} - \alpha_{g+1} + 1).
\]

Then \( Z^h \) is determined uniquely as \( O^{h,n_{ag}+1}_{\nu} \), \( \Lambda \cap Q^{r_{bh}}_{d_{bh}} \). On the other hand, the \( g \)-th row is determined uniquely once \( T^g \) is determined.

The linear space \( T^g \) satisfies \( T^{g-1} \subseteq T^g \subseteq \Lambda \cap L_{n_{ag}} \) and hence can be parameterized by \( G(\alpha_g, a_g + 1) \). Thus \( \text{dim}(\pi^{-1}(\Lambda)) = n_{ag}+1 - n_{ag} - \alpha_{g+1} + a_g \) and

\[
\text{codim}(\pi^{-1}(\Sigma_{r_{bh}})) = n_{ag}+1 - n_{ag} - \alpha_{g+1} + a_g + 1
\]

which is greater than 1, as \( n_{ag}+1 - n_{ag} - \beta_h \geq 1 \) by construction.

**Example 5.14.** Let \( V = [L_2 \subseteq L_4 \subseteq Q_7^2] \), an orthogonal Schubert variety in \( OG(3,9) \). The definition of \( \tilde{V} \) is given by the diagram in Figure 14.

The subvariety \( \Sigma_{r_{bh}} \) is given by the sequence \( [L_1 \subseteq L_2 \subseteq L_4] \) as \( Q_7^2 \) becomes \( L_4 \) if its corank is increased by 2. The variety \( \Sigma_{r_{bh}} \) has codimension 4. In the inverse image \( \pi^{-1}(\Lambda) \) of a general point \( \Lambda \) in \( \Sigma_{r_{bh}} \), the coordinates \( T^3 \) and \( O^1 \) are determined uniquely as \( T^3 = O^1 = \Lambda \) and \( T^2 = L_2 \). The coordinate
Figure 15. $V$ for \[ L_5 \subseteq L_{10} \subseteq Q_{19}^{6} \subseteq Q_{20}^{5} \subseteq Q_{30}^{2} \]\n
\[
\begin{array}{c}
T^1 \subseteq O^{2,n_{a_1}} \subseteq Q_{30}^{2,sing} \\
\cap \quad \cap \quad \cap \quad \cap \quad L_5 \\
T^2 \subseteq O^{2,n_{a_2}} \subseteq Q_{30}^{5,sing} \\
\cap \quad \cap \quad \cap \quad \cap \quad L_5 \\
T^3 \subseteq O^{2,n_{a_3}} \subseteq Z \subseteq Q_{20}^{5} \\
\cap \quad \cap \quad \cap \quad \cap \\
T^4 \subseteq Z^2 \subseteq Q_{30}^{2} \\
\end{array}
\]

$T^1$ satisfies $T^1 \subseteq L_2$ and is parameterized by $G(1, 2)$. The coordinate $Z^1$ satisfies $O^1 \subseteq Z^1 \subseteq Q_7^3$ and is parameterized by $OG(1, 3)$. Thus $\dim(\pi^{-1}(\Lambda)) = 2$ and $\dim(\pi^{-1}(\Sigma_{r_{a_1}})) = 2$.

**Example 5.15.** Let $V = \left[ L_5 \subseteq L_{10} \subseteq Q_{19}^{6} \subseteq Q_{20}^{5} \subseteq Q_{30}^{2} \right]$, then $V$ is given by the diagram in Figure 15. The subvariety $\Sigma_{r_{a_1}} = \left[ L_4 \subseteq L_5 \subseteq Q_{19}^{6} \subseteq Q_{20}^{5} \subseteq Q_{30}^{2} \right]$ has codimension 12. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{r_{a_1}}$, we have $T^4 = \Lambda$, $T^3 = \Lambda \cap Q_{16}^0 = \Lambda \cap Q_{20}^5$, $T^2 = \Lambda \cap L_5 = \Lambda \cap L_{10}$, $O^{2,n_{a_2}} = Q_{20}^{5,sing}$, $\Lambda \cap L_{10}$, $O^{2,n_{a_3}} = Q_{20}^{5,sing}$, $\Lambda \cap Q_{20}^{5}$, $Z^2 = Q_{30}^{2,sing}$, $\Lambda$. The linear space $O^{1,n_{a_2}}$ satisfies $Q_{20}^{5,sing}$, $\Lambda \cap L_{10} \subseteq O^{1,n_{a_2}} \subseteq L_{10}$ and hence can be parameterized by $G(1, 5)$. Then $Z^1$ is determined uniquely as $Z^1 = O^{1,n_{a_2}}, \Lambda \cap Q_{20}^{5}$. On the other hand, $T^1$ satisfies $T^1 \subseteq L_5$ and hence can be parameterized by $G(1, 2)$. Then $O^{2,n_{a_1}} = Q_{30}^{2,sing}, T^1$. Thus $\dim(\pi^{-1}(\Lambda)) = 5$ and $\dim(\pi^{-1}(\Sigma_{r_{a_1}})) = 12 - 5 = 7$.

**I.D.** Suppose $r_{a_1} = n_s$. Note that $\Sigma_{r_{a_1-1}}$ contains $\Sigma_{r_{a_1}}$, so all $\Sigma_{r_{a_1}}$ are contained in $\Sigma_{r_{a_1}}$. Therefore it is sufficient to consider $\Sigma_{r_{a_1}}$.

In the sequence of $\Sigma_{r_{a_1}}$, the sub-quadric $Q_{d_k-s}$ is replaced with $L_{n_{a_1}}$, and consequently each isotropic linear space $L_\tau$, where $n_{a_1} - \alpha_1 + 1 \leq \tau \leq n_{a_1}$, is replaced with the isotropic linear space of one less dimension, $L_{\tau-1}$. This increases the value of $x_i$ by one for $i$ satisfying $b_1 \leq i < k - s$. We have

\[
\text{codim} \Sigma_{r_{a_1}} = \alpha_1 (n_{a_1} - a_t - b_1) + \sum_{t=1}^{\beta_1} (d_{b_1} + x_{b_1} - 2(s + \beta_1) + t - 1) \\
- (\alpha_1 + 1) (n_{a_1} - a_t - 1) - \sum_{t=1}^{\beta_1-1} (d_{b_1} + x_{b_1} - 2(s + \beta_1) + t - 1) - (\beta_1 - 1) \\
= \alpha_1 + d_{b_1} - n_a - 2\beta_1 + 1
\]

Note that the resulting sequence may contradict condition (9) in Definition 3.1. The sub-quadric with maximal corank that is smaller than $r_{a_1}$, namely $Q_{d_{b_2}+\beta_2-1}$, may have corank one less than the dimension of the introduced isotropic linear space, namely, $L_{n_{a_1} - \alpha_1}$. We remedy this by replacing this sub-quadric with one with larger corank which reflects the geometry of the resulting restriction variety better. Explicitly, if $n_{a_1} - \alpha_1 = n_{b_2} + \beta_2$, we replace the sub-quadric $Q_{d_{b_2}+\beta_2-1}$ with $Q_{d_{b_2}+\beta_2+1}$. The changes in the dimension and the value of $x_i$ cancel each other, hence we get the same codimension computation.

This scenario arises in the study of other types of components of the exceptional locus. Here we give the general rule that applies whenever a sub-quadric is replaced with an isotropic linear space.

**Rule 2.** Given the defining sequence of a restriction variety, consider the modified sequence where a sub-quadric is replaced with an isotropic linear space. If $n_j - r_i = 1$ for an isotropic linear space $L_{n_j}$,
Figure 16. $\tilde{V}$ for $[L_2 \subseteq L_3 \subseteq Q^3_7]$

\[
\begin{align*}
T^1 & \subseteq L_3 \\
\cap & Q^3_{6,sing} \\
T^2 & \subseteq Z^1 \subseteq Q^3_6 \\
\end{align*}
\]

Figure 17. $\tilde{V}$ for $[L_5 \subseteq Q^5_10 \subseteq Q^2_{30}]$

\[
\begin{align*}
Q^2_{30, sing} & \subseteq Q^5_{10, sing} \\
T^1 & \subseteq O^{2,n_{a_1}} \subseteq L_5 \\
\cap & Q^5_{10} \\
T^2 & \subseteq O^{2,r_{a_1}} \subseteq Z^1 \subseteq Q^5_{10} \\
\cap & Q^2_{30} \\
T^3 & \subseteq Z^2 \subseteq Q^2_{30}
\end{align*}
\]

and a sub-quadric $Q_{d_i}^{r_i}$ in the modified sequence, then let $n_{a_{g_0}} = \min\{n_{a_g} \geq n_j\}$, and replace $Q_{d_i}^{r_i}$ with $Q_{d_i - n_{a_{g_0} - r_i}}^{r_i}$.

We again look at the fibers of $\pi$. By assumption there is no $O$ containing $Q_{d_i}^{r_i}$ and other $O$'s are determined uniquely as there is no change in the relevant rank conditions. The only nontrivial parameterizations are observed for $Z^1$ and the coordinates in the $t$-th row. As in (I.A), we have $T^{t+1}, Q_{d_i}^{r_i, sing} \subseteq Z_1 \subseteq Q_{d_i}^{r_i}$ where $\dim(T^{t+1}, Q_{d_i}^{r_i, sing}) = r_{b_1} + (k - b_1 + 1) - (x_{b_1} + 1)$ and $\dim(Z_1) = \dim(T^{t+1}, Q_{d_i}^{r_i, sing}) + 1$. Since $Z^1$ has to lie in the orthogonal complement of $T^{t+1}, Q_{d_i}^{r_i, sing}$, we have $T^{t+1}, Q_{d_i}^{r_i, sing} \subseteq Z_1 \subseteq Q_{d_i}^{r_i} + (k - b_1 + 1 - x_{b_1} - 1)$. Such $Z_1$ can be parameterized by $OG(1, d_i - r_{b_1} - 2(k - b_1 + 1 - x_{b_1} - 1))$. On the other hand, the $t$-th row can be determined once $T^1$ is determined. The linear space $T^1$ satisfies $T^1 \subseteq L \cap L_{n_{a_t}}$ and hence can be parameterized by $G(a_t, a_t + 1)$. Thus $\dim(\pi^{-1}(\Lambda)) = d_{b_1} - r_{b_1} - 2(k - b_1 + 1 - x_{b_1} - 1) - 2 + a_t$ and we have

\[
\text{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1.
\]

Example 5.16. Let $V = \left[L_2 \subseteq L_3 \subseteq Q^3_7\right]$, an orthogonal Schubert variety in $OG(3, 9)$. The diagram in Figure 16 defines $\tilde{V}$.

The subvariety $\Sigma_{r_{b_1}} = \left[L_1 \subseteq L_2 \subseteq L_3\right]$, which consists of a single point, has codimension 4. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{r_{b_1}}$, we have $T^1 \subseteq L_3$ which is parameterized by $G(2, 3)$. Also, $\dim(Z^1) = 4$ with $Q^3_{6, sing} \subseteq Z^1 \subseteq Q^3_6$, so $Z^1$ is parameterized by $OG(1, 3)$ which has dimension 1. Thus $\dim(\pi^{-1}(\Lambda)) = 3$ and $\text{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1$.

Example 5.17. Let $V = \left[L_5 \subseteq Q^5_{10} \subseteq Q^2_{30}\right]$, then $\tilde{V}$ is given by the diagram diagram in Figure 17. The subvariety $\Sigma_{r_{b_1}} = \left[L_4 \subseteq L_5 \subseteq Q^2_{30}\right]$ has codimension 7. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{r_{b_1}}$, we have $T^3 = \Lambda$, $T^2 = \Lambda \cap L_5 = \Lambda \cap Q^5_{10}$. $Q^2_{30, sing} \subseteq \tilde{L}_{30}^2$, $\Lambda \cap Q^5_{10}$. $Z^2 = Q^2_{30, sing}$. We have $Q^2_{30, sing} \subseteq O^{2,n_{a_1}} \subseteq L_5$ which can be parameterized by $G(1, 3)$. Then the linear space $T^1$ which satisfies
Figure 18. $\tilde{V}$ for $[L_5 \subseteq Q_8^2]$

\[
\begin{array}{c}
Q_8^{2,\text{sing}} \\
T^1 \subseteq O^{1,n_{a_1}} \subseteq L_5 \\
\cap \\
T^2 \subseteq Z^1 \subseteq Q_8^5
\end{array}
\]

$T^1 \subseteq O^{2,n_{a_1}}$ can be parameterized by $G(1,3)$. On the other hand, $Z^1$ satisfies $Q_8^{5,\text{sing}} \subseteq Z^1 \subseteq Q_8^5$ and hence can be parameterized by $OG(1,5)$. Thus $\dim(\pi^{-1}(\Lambda)) = 6$ and $\operatorname{codim}(\pi^{-1}(\Sigma_{r_1})) = 1$.

II: $\Sigma_{n_{a_g}} : \dim(\Lambda \cap L_{n_{a_g}}) = a_g + 1$ for some $1 \leq g \leq t$

Depending on $n_{a_g}$, we divide this case into the following two subcases:

II.A: $g = t$

II.B: $g < t$

II.A: $\Sigma_{n_{a_g}} : \dim(\Lambda \cap L_{n_{a_g}}) = a_t + 1$ (or equivalently, $\dim(\Lambda \cap L_{r_t}) = s + 1$)

If $r_{b_1} = n_{a_t}$, then $\Sigma_{n_{a_g}}$ corresponds to $\Sigma_{r_{b_1}}$. If $r_{b_1} > n_{a_t}$ then $\Sigma_{r_{b_1}}$ contains $\Sigma_{n_{a_t}}$. So we assume $r_{b_1} < n_{a_t}$ in the following. In the sequence of $\Sigma_{n_{a_t}}$, the sub-quadric $Q_{d_{k-s}}^{t}$ is replaced with the isotropic linear space $L_{n_{a_t}}$. Consequently, each isotropic linear space $L_{\tau}$, where $n_{a_t} - \alpha_t + 1 \leq \tau \leq n_{a_t}$, is replaced with $L_{\tau-1}$. We have

\[
\operatorname{codim}(\Sigma_{r_{b_1}}) = \alpha_t (n_{a_t} - a_t) + \sum_{t=1}^{\beta_1} (d_{b_1} + x_{b_1} - 2(s + \beta_1) + t - 1)
- (\alpha_t + 1) (n_{a_t} - a_t - 1) - \sum_{t=1}^{\beta_1 - 1} (d_{b_1} + x_{b_1} - 2(s + \beta_1) + t - 1)
= \alpha_t + d_{b_1} + x_{b_1} - s - n_{a_t} - \beta_1
\]

The only nontrivial parameterizations in the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{n_{a_t}}$ are in the row of $T^t$ and once $T^t$ is fixed, the rest of the row can be determined uniquely. The linear space $T^t$ satisfies $T^t \subseteq T^t \subseteq \Lambda \cap L_{n_{a_t}}$, and hence can be parameterized by $G(\alpha_t, \alpha_t + 1)$. Thus we have

\[
\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_t}})) = d_{b_1} + x_{b_1} - s - n_{a_t} - \beta_1
\]

Since $d_{b_1} - \beta_1 + 1 = d_{k-s}$, this is equivalent to

\[
\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_t}})) = d_{k-s} + x_{k-s} - s - n_{a_t} - 1
\]

Note that $\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_t}}))$ may be $1$ or larger in this case.

Example 5.18. Let $V = [L_5 \subseteq Q_8^2]$, then $\tilde{V}$ is given by the diagram in Figure 18.

The subvariety $\Sigma_{n_{a_1}} = [L_4 \subseteq L_5]$ has codimension $2$. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{n_{a_1}}$, we have $T^2 = \Lambda$ and $Z^1 = \Lambda, Q_8^{2,\text{sing}}$. The linear space $T^1$ satisfies $T^1 \subseteq \Lambda \cap L_5$ and hence can be parameterized by $G(1,2)$. Then $O^{1,n_{a_1}}$ is determined uniquely as $O^{1,n_{a_1}} = Q_8^{2,\text{sing}}, T^1$. Thus $\dim(\pi^{-1}(\Lambda)) = 1$ and $\operatorname{codim}(\pi^{-1}(\Sigma_{r_1})) = 2 - 1 = 1$.

Example 5.19. Let $V = [L_4 \subseteq Q_8^1]$, an orthogonal Schubert variety in $OG(2,9)$. The diagram in Figure 19 gives the definition of $\tilde{V}$. 23
The subvariety $\Sigma_{n_{a_1}} = [L_4 \subseteq L_5]$ has codimension 3. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{n_{a_1}}$, we have $T^2 = \Lambda$ and $Z^1 = \Lambda, Q^{1,sing}_8$. The linear space $T^1$ satisfies $T^1 \subseteq \Lambda \cap L_4$ and hence can be parameterized by $G(1,2)$. Then $O^{1,n_{a_1}}$ is determined uniquely as $O^{1,n_{a_1}} = Q^{1,sing}_8, T^1$. Thus $\dim(\pi^{-1}(\Lambda)) = 1$ and $\codim(\pi^{-1}(\Sigma_{r_{1_1}})) = 3 - 1 = 2$.

II.B: $\Sigma_{n_{a_g}} : \dim(\Lambda \cap L_{n_{a_g}}) = a_g + 1$ for some $1 \leq g \leq t - 1$

We have already discussed in I.C the case when there is some $Q^{r_{bh}}_{d_{bh}}$ in the defining sequence with $r_{bh} = n_{a_g}$. Also, if there is $Q^{r_{bh}}_{d_{bh}}$ in the sequence with $r_{bh} > n_{a_g}$ then $\Sigma_{n_{a_g}}$ will be contained in $\Sigma_{r_{bh}}$. So it is sufficient to consider the case when $n_{a_g} > r_{bh}$ for all $1 \leq h \leq u$, equivalently, when $n_{a_g} > r_{k-s}$.

In the sequence of $\Sigma_{n_{a_g}}$, the isotropic linear space that comes after $L_{n_{a_g}}$ in the sequence of $V$, namely $L_{n_{a_g}+1}$, is replaced with $L_{n_{a_g}}$. Consequently, each isotropic linear space $L_t$, where $n_{a_g}-a_g+1 \leq t \leq n_{a_g}$, is replaced with the isotropic linear space of one less dimension, $L_{t-1}$. Rule 1 applies to the sub-quadric $Q^{r_{bh}}_{d_{bh}}$, and consequently to each $Q^{r_{bh}}_{d_{bh}}$, where $b_1 \leq i \leq k - s$; we replace $Q^{r_{bh}}_{d_{bh}}$ with $Q^{r_{bh}+(n_{a_g}-r_{k-s})}_{d_{bh}}$. This increases the value of $x_i$ by $a_g + 1$ for all $i$ with $b_1 \leq i \leq k - s$. We have

$$\text{codim}(\Sigma_{n_{a_g}}) = a_g (n_{a_g} - a_g) + a_g + 1 (n_{a_g+1} - a_g + 1)$$
$$- (a_g + 1) (n_{a_g} - a_g - 1) - (a_g + 1 - 1) (n_{a_g+1} - a_g + 1)$$
$$+ \beta_1 (n_{a_g} - r_{k-s}) - \beta_1 (a_g + 1)$$

The only nontrivial parameterizations in the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{n_{a_g}}$ are in the $q$-th row of the diagram of $\tilde{V}$ and once $T^q$ is parameterized the remaining coordinates can be determined uniquely. The linear space $T^q$ satisfies $T^q \subseteq T^q \subseteq L_{n_{a_g}}$ and hence can be parameterized by the Grassmannian $G(a_g, a_g + 1)$. Thus we have

$$\text{codim}(\pi^{-1}(\Sigma_{n_{a_g}})) = n_{a_g+1} - n_{a_g} - (a_g + 1 - a_g) + 1 + \beta_1 (n_{a_g} - a_g - r_{k-s} - 1).$$

Note that $n_{a_g+1} - n_{a_g} \geq a_g + 1 - a_g + 1$ and $n_{a_g} - a_g \geq k - s + 1$ by assumption. Therefore $\text{codim}(\pi^{-1}(\Sigma_{n_{a_g}})) \geq 2$ in this case.

**Example 5.20.** Let $V = [L_2 \subseteq L_4 \subseteq Q^9_8]$, an orthogonal Schubert variety on $OG(3,9)$. The diagram in Figure 20 gives the definition of $\tilde{V}$.

The subvariety $\Sigma_{n_{a_1}} = [L_1 \subseteq L_2 \subseteq Q^2_7]$ has codimension 3. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{n_{a_1}}$, the coordinates $T^3$ and $T^2$ are determined uniquely as $T^3 = \Lambda$ and $T^2 = \Lambda \cap L_4$. The coordinate $T^1$ satisfies $T^1 \subseteq L_2$ and hence is parameterized by $G(1,2)$. Thus $\dim(\pi^{-1}(\Lambda)) = 1$ and $\codim(\pi^{-1}(\Sigma_{r_{1_1}})) = 3 - 1 = 2$.

**Example 5.21.** Let $V = [L_5 \subseteq L_7 \subseteq Q^9_{20}]$, then $\tilde{V}$ is given by the diagram Figure 21. The subvariety $\Sigma_{n_{a_1}} = [L_4 \subseteq L_5 \subseteq Q^9_{18}]$ has codimension 3. In the inverse image $\pi^{-1}(\Lambda)$ of a general point $\Lambda$ in $\Sigma_{n_{a_1}}$,
III: \( \Sigma_{\partial_b} : \dim(\Lambda \cap Q_{\partial_b}^{r_{bh}}) = k - b_h + 2 \) for some \( 1 \leq h \leq u - 1 \)

This case is similar to the case in II.A. In the sequence of \( \Sigma_{\partial_b} \), the sub-quadric that comes after \( Q_{\partial_b}^{r_{bh}} \) in the sequence of \( V \), namely \( Q_{\partial_b}^{r_{bh} + \delta - 1} \), is replaced with \( Q_{\partial_b}^{r_{bh}} \). Consequently, each sub-quadric \( Q_{\partial_b}^{r_{bh}} \), where \( d_{bh} - \beta + 1 \leq \delta \leq d_{bh} \) and \( r_{bh} + \beta - 1 \geq \rho \geq r_{bh} \), is replaced with \( Q_{\partial_b}^{r_{bh} + 1} \).

Comparing the dimensions of the sequences, we have

\[
\text{codim}(\Sigma_{\partial_b}) = \sum_{t=1}^{\beta_h} (d_{bh} + x_{bh} - 2(k - b_h + 1) + t - 1) \\
+ d_{bh+1} + x_{bh+1} - 2(k - b_{h+1} + 1) + \beta_{h+1} - 1 \\
- \sum_{t=1}^{\beta_h+1} (d_{bh} + x_{bh} - 2(k - b_h + 1) + t - 1) \\
= d_{bh+1} + x_{bh+1} - 2(k - b_{h+1} + 1) \\
- (d_{bh} + x_{bh} - 2(k - b_h + 1)) + \beta_{h+1} + 1
\]

Note that the resulting sequence may contradict condition (9) in Definition 3.1. Here we give the general rule for remedying this in a general context that applies whenever a sub-quadric is replaced with another sub-quadric.

**Rule 3.** Given the defining sequence of a restriction variety, consider the modified sequence where a sub-quadric is replaced with another sub-quadric. If \( n_j - r_i = 1 \) for an isotropic linear space \( L_{n_j} \) and a sub-quadric \( Q_{d_i}^{s_{d_i}} \), then let \( r_{bs_{d_{bs}}} := \max\{r_{bs} \leq r_i\} \), and replace \( L_{n_j} \) with \( L_{r_{bs_{d_{bs}}}} \).

The only nontrivial parameterizations in the inverse image of a general point \( \Lambda \) in \( \Sigma_{\partial_b} \), are in the row of \( T^{k-b_h+1} \) and once \( T^{k-b_h+1} \) is fixed, the rest of the row can be determined uniquely. The linear space

\( T^1 \subseteq L_2 \)
\( T^2 \subseteq L_4 \)
\( T^3 \subseteq Q_9^0 \)

\( Q_{20}^{3,\text{sing}} \)
\( T^1 \subseteq O_{1,\alpha_1} \subseteq L_5 \)
\( T^2 \subseteq O_{1,\alpha_2} \subseteq L_7 \)
\( T^3 \subseteq Z^1 \subseteq Q_{20}^{3} \)

Figure 20. \( \bar{V} \) for \( [L_2 \subseteq L_4 \subseteq Q_9^0] \)

Figure 21. \( \bar{V} \) for \( [L_5 \subseteq L_7 \subseteq Q_{20}^3] \)
\[ \begin{align*}
Q_7^{2,\text{sing}} & \\
T^1 & \subseteq O^{1,r_1} \subseteq Q_7^2 \\
T^2 & \subseteq Z^2 \subseteq Q_9^3
\end{align*} \]

Thus we have
\[
\text{codim}(\pi^{-1}(\Sigma_{d_h})) = d_{b_h+1} + x_{b_h+1} - 2(k - b_{h+1} + 1) \\
- (d_b + x_b - 2(k - b + 1)) + \beta_{h+1} + 1
\]

which is larger than one by the definition of \( \beta_{h+1} \).

**Example 5.22.** Let \( V = \left[ Q^2_4 \subseteq Q^3_9 \right] \), then \( \tilde{V} \) is given by the diagram Figure 22.

The subvariety \( \Sigma_{d_h} = \left[ Q^3_9 \subseteq Q^2_9 \right] \) has codimension 3. In the inverse image \( \pi^{-1}(\Lambda) \) of a general point \( \Lambda \) in \( \Sigma_{d_h} \), we have \( T^2 = \Lambda \) and \( Z^2 = \Lambda, Q_7^{2,\text{sing}} \). The linear space \( T^1 \) satisfies \( T^1 \subseteq \Lambda \cap Q_7^2 \) and hence can be parameterized by \( G(1,2) \). Then \( O^{1,r_1} \) is determined uniquely as \( O^{1,r_1} = \overline{Q_7^{2,\text{sing}}, T^1} \). Thus \( \dim(\pi^{-1}(\Lambda)) = 1 \) and \( \text{codim}(\pi^{-1}(\Sigma_{d_h})) = 3 - 1 = 2 \).

This concludes the computation behind Observation 5.7. The following lemma, which is based on Lemma 2.4, allows us to give a partial description of the singular locus of \( V \).

**Lemma 5.23.** A subvariety \( \Sigma \) of a restriction variety \( V \) satisfying \( \text{codim}(\pi^{-1}(\Sigma)) > 1 \) is in the singular locus of \( V \).

**Proof.** Suppose \( \text{codim}(\pi^{-1}(\Sigma)) > 1 \) and \( \Lambda \in \Sigma \) is a point such that \( \pi^{-1}(\Lambda) \) is positive dimensional. If \( \Lambda \) is smooth, then in order to check that \( \pi \) is a local isomorphism, it suffices to check that the Jacobian does not vanish. Since \( \text{codim}(\pi^{-1}(\Sigma)) > 1 \) and the vanishing locus of the Jacobian is a divisor, we conclude that the Jacobian does not vanish. On the other hand, since \( \pi \) is not a local isomorphism around \( \pi^{-1}(\Lambda) \), we conclude that \( \Lambda \) is a singular point. \( \square \)

**Corollary 5.24.** Let \( V(L, Q_s) \) be a restriction variety and \( \pi : \tilde{V}(L, Q_s) \to V(L, Q_s) \) the resolution of singularities in Theorem 4.7. The components of the exceptional locus whose images are of the form

- \( \Sigma_{r_{bh}} \) with \( r_{bh} < n_s \)
- \( \Sigma_{n_{sg}} \) for all \( 1 \leq g \leq t - 1 \)
- \( \Sigma_{n_{kg}} \) with \( d_{k-s} + x_{k-s} - s - n_s > 2 \)
- \( \Sigma_{d_{bh}} \) for all \( 1 \leq h \leq u - 1 \)

are in the singular locus of \( V(L, Q_s) \).

Our results so far give a partial description of the singular locus of a restriction variety, but are inconclusive about the remaining types of loci:

- \( \Sigma_{r_{bh}} \) with \( r_{bh} \geq n_s \), and
- \( \Sigma_{n_{gs}} \) with \( d_{k-s} + x_{k-s} - s - n_s = 2 \).
Studying the tangent space of a restriction variety at a point will allow us to observe these loci further in the following.

Now we study the tangent space of a restriction variety \( V \) at a point \( \Lambda \) starting with the one-step case. We refer the reader to [7] for a different approach to tangent spaces to Schubert varieties, and to [6] for general information on tangent spaces to Grassmannians.

If \( V = L_e \) is an isotropic linear space, the tangent space at a point \( v \) can be identified with the quotient 
\[
L_e/\Lambda = L_e / <v>.
\]

If \( V = Q_d^r \) is a quadric, the tangent space at a point \( v \) can be obtained by evaluating the kernel of the Jacobian of the polynomial \( F_{Q_d^r} \) at \( v \), and taking the quotient with \( v \). More concretely, \( F_{Q_d^r} \) can be taken to be
\[
\sum_{i=r+1}^{r+m} x_i y_i \text{ if } d - r = 2m \quad \text{and} \quad x_i y_i \text{ if } d - r = 2m + 1,
\]
and the kernel of the Jacobian is of the form
\[
\text{Ker } \begin{bmatrix} \cdots & 0 & y_{r+1} & x_{r+1} & \cdots \end{bmatrix},
\]
where the last nonzero term is \( x_{r+m} \) or \( 2x_{r+m+1} \) depending on the rank of \( Q_d^r \). The quotient of the kernel with \( \Lambda \) has dimension \( d - 2 \) if evaluated at a smooth point \( v \), but has dimension \( d - 1 \) if evaluated at a singular point \( v \).

Now consider a general restriction variety \( V \) defined by
\[
L_{n_1} \subseteq \ldots \subseteq L_{n_s} \subseteq Q_{d-k-s}^{r-s} \subseteq \ldots \subseteq Q_{d}^{r}.
\]
and let \( \Lambda = <v_1, \ldots, v_k> \) be a general point in \( V \) with \( v_j \in L_{n_j} \) for \( 1 \leq j \leq s \), and \( v_{k-i+1} \in Q_{d_i}^{r_i} \) for \( 1 \leq i \leq k-s \). An arc \( \Gamma(t) \) through \( \Lambda \) contained in \( V \) is obtained by moving \( \Lambda \)'s intersection with each step of the sequence inside that step. Explicitly, \( \Gamma(t) = <\gamma_1(t), \ldots, \gamma_k(t)> \) where \( \gamma_j \) is an arc through \( v_j \) contained in \( L_{n_j} \) for \( 1 \leq j \leq s \), and \( \gamma_{k-i+1} \) is an arc through \( v_{k-i+1} \) contained in \( Q_{d_i}^{r_i} \) for \( 1 \leq i \leq k-s \). Therefore the tangent space of \( \Gamma \) can be studied by considering the tangent spaces of \( \gamma_i \).

The tangent space of \( \gamma_j \) for \( 1 \leq j \leq s \) is given by the quotient
\[
L_{n_j}/\Lambda = L_{n_j}/<v_1, \ldots, v_j>,
\]
which has dimension \( n_j - j \).

The arc \( \gamma_{k-i+1} \) for \( 1 \leq i \leq k-s \) lies in the orthogonal complement of \( \Lambda \cap Q_{d_i}^{r_i} \), and is contained in \( Q_{d_i}^{r_i} \). Let \( Q_{d_i}^{r_i} \) be the sub-quadric obtained by specializing the hyperplane section of \( Q_{d_i}^{r_i} \) until it is tangent to \( Q \) along \( Q_{d_i}^{r_i,sing} \), \( \Lambda \cap Q_{d_i}^{r_i} \). Note that
\[
\dim(Q_{d_i}^{r_i,sing}, \Lambda \cap Q_{d_i}^{r_i}) = r_i + k - i + 1 - x_i, \quad \text{therefore } \quad Q_{d_i}^{r_i} = Q_{d_i}^{r_i, -(k-i+1-x_i)}.
\]
The tangent space can be identified with the quotient
\[
\text{Ker } \begin{bmatrix} \cdots & 0 & y_{r+1} \end{bmatrix} / \Lambda = \text{Ker } \begin{bmatrix} \cdots & 0 & x_{r+1} \end{bmatrix} < v_1, \ldots, v_{k-i+1} >,
\]
which has dimension \( d_i + x_i - 2(k-i+1) \).

Note that for a general point \( \Lambda \) in \( V \), these dimensions are the expressions appearing in the formula for \( \dim V \), hence unsurprisingly \( \dim V = \dim T_\Lambda V \) at a general point. In other orbits this equality does not necessarily hold, and this is what we inspect for the two types of loci for which our previous results are inconclusive.

**Proposition 5.25.** The loci of type \( \Sigma_{r_{bh}} \) with \( r_{bh} \geq n_s \) are in the singular locus of \( V \).

**Proof.** Let \( \Lambda \) be a general point in the locus of the form \( \Sigma_{r_{bh}} \) for some \( r_{bh} \geq n_s \). The only arcs affected by the increase of \( \dim(\Lambda \cap Q_{d_i}^{r_i,sing}) \) are the group of \( \beta_h \) arcs \( \gamma_i \) with \( b_h - \beta_h + 1 \leq i \leq b_h \). For each \( i \), we have
\[
\dim(\Lambda \cap Q_{d_i}^{r_i,sing}) = x_i + 1, \quad \text{and} \quad \dim(Q_{d_i}^{r_i,sing}, \Lambda \cap Q_{d_i}^{r_i}) = r_i + k - i + 1 - (x_i + 1), \quad \text{therefore } \quad Q_i^{r_i} = Q_{d_i, -(k-i+1-x_i+1)}.\]
Consequently, the tangent space $\text{Ker} \left[ JF_{Q_i} \right]_{v_{k-i}} \Lambda$ has dimension $d_i + x_i - 2(k - i + 1) + 1$ for each $i$ in $b_h - \beta_h + 1 \leq i \leq b_h$. Hence

$$\dim T_\Lambda V = \dim V + \beta_h,$$

which shows that $\Lambda$ is in the singular locus of $V$.

\[ \square \]

**Proposition 5.26.** The loci of type $\Sigma_{n_s}$ with $d_{k-s} + x_{k-s} - s - n_s = 2$ are in the smooth locus of $V$.

**Proof.** Let $\Lambda$ be a general point in the locus of type $\Sigma_{n_s}$. As a result of $\dim(\Lambda \cap L_{n_s}) = s + 1$, both arcs $\gamma_s$ and $\gamma_{s+1}$ are contained in $L_{n_s}$, and hence the tangent space of $\gamma_s$ can be identified with

$$\text{Ker} \left[ JF_{Q_{k-s}} \right]_{v_s} \Lambda = \text{Ker} \left[ JF_{Q_{k-s}} \right]_{v_s} / \langle v_1, \ldots, v_{k+1} \rangle,$$

the construction as the one for $\gamma_{s+1}$, but evaluated at $v_s$. We observe the difference in dimensions as

$$\dim T_\Lambda V - \dim V = (d_{k-s} + x_{k-s} - 2(s + 1)) - (n_s - s)$$

$$= d_{k-s} + x_{k-s} - s - n_s - 2.$$

Hence follows the result. \[ \square \]

In particular, the image of the exceptional locus is not equal to the singular locus in general. The following corollary summarizes the results of this chapter.

**Corollary 5.27.** Let $V(L_s, Q_s)$ be a restriction variety, $\pi$ the resolution of singularities, $E_\pi$ the exceptional locus of $\pi$, and $\Sigma_\pi$ the components of $\pi(\Sigma_{n_s})$ as above. The singular locus of $V$ is the union of

- $\Sigma_{r_{nj}}$
- $\Sigma_{n_{as}}$ for all $1 \leq g \leq t - 1$
- $\Sigma_{n_s}$ with $d_{k-s} + x_{k-s} - s - n_s > 2$
- $\Sigma_{d_{bh}}$ for all $1 \leq h \leq u - 1$.

Equivalently,

$$V^{\text{sing}} = \begin{cases} 
\pi(E_\pi) & \text{if } d_{k-s} + x_{k-s} - s - n_s > 2 \\
\pi(E_\pi) \setminus \Sigma_{n_s} & \text{if } d_{k-s} + x_{k-s} - s - n_s = 2.
\end{cases}$$

### 6. The Algorithm and Examples

We present an algorithm for finding the singular locus of a restriction variety that is based on our study of the exceptional locus of $\pi$. The three rules introduced before will be used in the algorithm, we repeat them here for convenience.

**Rule 1.** Given the defining sequence of a restriction variety, consider the modified sequence where an isotropic linear space $L_{n_j}$ is replaced with a smaller dimensional isotropic linear space. If there are sub-quadratics $Q'_{d_i}$ in the sequence satisfying $r_i < n_j$, then let $r_{i_0} := \max\{r_i < n_j\}$, and replace $Q'_{d_{i_0}}$ with $Q^{n_{i_0}}_{d_{i_0} - (n_j - r_{i_0})}$.

**Rule 2.** Given the defining sequence of a restriction variety, consider the modified sequence where a sub-quadratic is replaced with an isotropic linear space. If $n_j - r_i = 1$ for an isotropic linear space $L_{n_j}$, and a sub-quadratic $Q'_{d_i}$ in the modified sequence, then let $n_{a_0} := \min\{n_{as} \geq n_j\}$, and replace $Q'_{d_i}$ with $Q^{n_{a_0}}_{d_i - (n_{a_0} - r_i)}$.

**Rule 3.** Given the defining sequence of a restriction variety, consider the modified sequence where a sub-quadratic is replaced with another sub-quadratic. If $n_j - r_i = 1$ for an isotropic linear space $L_{n_j}$ and a sub-quadratic $Q'_{d_i}$, then let $r_{b_{i_0}} := \max\{r_b \leq r_i\}$, and replace $L_{n_j}$ with $L_{r_{b_{i_0}}}$.
Algorithm 6.1. Let $V$ be defined by the sequence

$$L_{n_1} \subseteq \ldots \subseteq L_{n_s} \subseteq Q^r_{d_k-s-1} \subseteq \ldots \subseteq Q^r_{d_1},$$

or equivalently, by the partitions

$$(n_1^{a_1}, \ldots, n_s^{a_s}), (d_1^{b_1}, \ldots, d_s^{b_s}), (r_1, \ldots, r_k-s).$$

(1) Steps for $r_b \geq n_s$. If $r_b > x_b$ then proceed, otherwise $\Sigma_{rb} = \emptyset$.

(a) If $r_b > n_s$ then replace $Q^r_{d_k-s}$ with $L_nb$. The resulting sequence gives $\Sigma_{rb}$.

(b) If $r_b = n_s$ then replace $Q^r_{d_k-s-\alpha}$ with $L_n$, and replace $L_\tau$, where $n_s - \alpha_t + 1 \leq \tau \leq n_s$, with $L_{\tau-1}$. Apply Rule 2. The resulting sequence gives $\Sigma_{rb}$.

(c) Otherwise $\Sigma_{rb} = \emptyset$.

(2) Steps for each $r_bh < n_s$, where $1 \leq h \leq u$. For each $h$, if $r_bh > x_bh$ then proceed, otherwise $\Sigma_{rbh} = \emptyset$.

(a) If $r_bh < n_s$ and $r_bh \neq n_j$ for any $j$, then let $j_b = \min \{ n_j \mid r_bh < n_j \}$, let $r_bh = \max \{ r_bh \mid r_bh < n_j \}$, and replace $L_n$ with $L_{rbh}$. Apply Rule 1. The resulting sequence is $\Sigma_{rbh}$.

(b) If $r_bh = n_s < n_s$ then replace $L_{n_s+1}$ with $L_{n_s}$, and replace $L_\tau$, where $n_s - \alpha_t + 1 \leq \tau \leq n_s$, with $L_{\tau-1}$. Apply Rule 1. The resulting sequence is $\Sigma_{rbh}$.

(c) Otherwise $\Sigma_{rbh} = \emptyset$.

(3) Steps for $n_s$. If $n_s > s > d_1 + x_s - s - n_s > 2$ then proceed, otherwise $\Sigma_{ns} = \emptyset$.

(a) If $n_s > r_b$ and the proposition $b_1$ is a special index $\left[ b_1 \in \text{a special index} \right] \wedge \left[ 2n_s = d_1 + r_b \right]$ is false, then replace $Q^r_{d_k-s}$ with $L_{n_s}$, and replace $L_\tau$, where $n_s - \alpha_t + 1 \leq \tau \leq n_s$, with $L_{\tau-1}$. Apply Rule 2. The resulting sequence gives $\Sigma_{ns}$.

(b) If $b_1$ is a special index and $2n_s = d_1 + r_b$ and $k \geq s + 2$, then replace $Q^r_{d_k-s-1}$ with $L_{n_s}$, replace $Q^r_{d_k-s}$ with $L_{n_s-1}$, and replace $L_\tau$, where $n_s - \alpha_t + 2 \leq \tau \leq n_s$, with $L_{\tau-2}$. Apply Rule 2. The resulting sequence gives $\Sigma_{ns}$.

(c) Otherwise $\Sigma_{ns} = \emptyset$.

(4) Steps for each $n_s$, where $1 \leq g \leq t - 1$.

(a) If $n_s > d_g$ and $r_b < n_s < n_s$, then replace $L_{n_s+1}$ with $L_{n_s+1}$, and replace $L_\tau$, where $n_s - \alpha_t + 1 \leq \tau \leq n_s$, with $L_{\tau-1}$. Apply Rule 2. The resulting sequence gives $\Sigma_{ns}$.

(b) Otherwise $\Sigma_{ns} = \emptyset$.

(5) Steps for each $d_bh$, where $1 \leq h \leq u - 1$.

(a) If $d_bh - r_bh - 2\beta_h \geq 3$ then replace $Q^r_{d_k-s-1} + \beta_h - 1$ with $Q^r_{d_k-s}$, replace $Q^r_{d_k-s}$ with $Q^r_{d_k-s-1}$, where $r_bh + \beta_h - 1 \geq \rho \geq r_bh$ and $d_bh - \beta_h + 1 \leq \delta \leq d_bh$, with $Q^r_{d_k-s-1}$. Apply Rule 3. The resulting sequence gives $\Sigma_{dbh}$.

(b) Otherwise $\Sigma_{dbh} = \emptyset$.

(6) Take the union of the restriction varieties obtained from the first five steps. The resulting restriction variety gives the singular locus of $V$.

Here are some examples illustrating Algorithm 6.1 in a few different cases. We refer the reader to [1] for the permutation notation, and for more examples on singularities of Schubert varieties.

Example 6.2. Let $V$ be defined by the sequence $\left[ Q^3_6 \subseteq Q^0_9 \right]$. This is a Schubert variety in $OG(2, 9)$. The locus $\Sigma_{r_3} = \left[ L_3 \subseteq Q^0_9 \right]$ is obtained by step (1)(a) in Algorithm 6.1. Note that the locus $\Sigma_{d_3}$ does not exist since $d_3 - r_3 - 2\beta_1 \geq 3$. Therefore

$$V^{sing} = \left[ L_3 \subseteq Q^0_9 \right].$$

Equivalently, in permutation notation we have

$$968753241^{sing} = (938654271).$$
Example 6.3. Let $V$ be the Schubert variety in $OG(2, 9)$ defined by $[Q_7^2 \subseteq Q_9^0]$. Using the steps (1)(a) and (5)(a), we have

$$V^{sing} = [L_2 \subseteq Q_9^0] \cup [Q_6^3 \subseteq Q_7^2] ,$$

equivalently, in permutation notation

$$(978654231)^{sing} = (9276543811) \cup (769852143) .$$

Example 6.4. Let $V$ be the Schubert variety in $OG(3, 9)$ defined by $[L_3 \subseteq Q_6^3 \subseteq Q_9^0]$. Using the step (1)(b), we have

$$V^{sing} = [L_2 \subsetneq L_3 \subseteq Q_9^0] ,$$

equivalently, in permutation notation

$$(963852741)^{sing} = (932654871) .$$

Example 6.5. Let $V$ be the Schubert variety in $OG(2, 8)$ defined by $[L_3 \subseteq Q_7^6 \subseteq Q_9^0]$. The locus $\Sigma_{r_{a_1}}$ is obtained by step (2)(a). Rule 1 replaces $Q_7^6$ with the union of $L_4$ and $L_4'$. Therefore the locus $\Sigma_{r_{a_1}}$ is the union of $[L_1 \subseteq L_4]$ and $[L_1 \subseteq L_4']$. Furthermore, step (3)(a) gives the locus $[L_2 \subseteq L_3]$. Hence

$$V^{sing} = [L_2 \subseteq L_3] \cup [L_1 \subseteq L_4] \cup [L_1 \subseteq L_4'] ,$$

equivalently, in permutation notation

$$(73845162)^{sing} = (32854176) \cup (41763285) \cup (51736284) .$$

Example 6.6. Let $V$ be the Schubert variety in $OG(3, 9)$ defined by $[L_2 \subseteq L_4 \subseteq Q_7^2]$. The locus $\Sigma_{r_{a_1}} = [L_1 \subseteq L_2 \subseteq L_4]$ is obtained by applying step (4)(a) and in particular Rule 1. Note that $\Sigma_{n_{a_2}}$ is in the smooth locus of $V$ since $d_{b_1} + x_{b_1} - s - n_s = 2$. We have

$$V^{sing} = [L_1 \subseteq L_2 \subseteq L_4] ,$$

equivalently, in permutation notation

$$(742951863)^{sing} = (421753986) .$$

Example 6.7. Let $V$ be the Schubert variety in $OG(3, 9)$ defined by $[L_3 \subseteq Q_7^4 \subseteq Q_9^0]$. The locus $\Sigma_{n_{a_1}} = [L_2 \subseteq L_3 \subseteq Q_9^0]$ is obtained by step (3)(a). We have

$$V^{sing} = [L_2 \subseteq L_3 \subseteq Q_9^0] ,$$

equivalently, in permutation notation

$$(983654721)^{sing} = (932654871) .$$

Example 6.8. Let $V$ be the Schubert variety in $OG(3, 9)$ defined by $[L_4 \subseteq Q_7^2 \subseteq Q_8^4]$. The locus $\Sigma_{r_{a_1}} = [L_1 \subseteq L_4 \subseteq Q_8^4]$ is obtained by applying step (2)(a). Furthermore, step (3)(a) is applied to obtain the locus $\Sigma_{n_{a_1}} = [L_2 \subseteq L_3 \subseteq L_4]$. Thus

$$V^{sing} = [L_1 \subseteq L_4 \subseteq Q_8^4] \cup [L_2 \subseteq L_3 \subseteq L_4] ,$$

equivalently, in permutation notation

$$(874951632)^{sing} = (841753986) \cup (432951876) .$$
Example 6.9. Let $V$ be the Schubert variety in $OG(3, 9)$ defined by $\left[ L_2 \subseteq L_4 \subseteq Q_6^5 \right]$. By steps (4)(a) and (3)(a), we have

$$V^{sing} = \left[ L_1 \subseteq L_2 \subseteq Q_5^2 \right] \cup \left[ L_2 \subseteq L_3 \subseteq L_4 \right],$$
equivalently, in permutation notation

$$(942753861)^{sing} = (721654983) \cup (432951876).$$

Example 6.10. Let $V$ be the Schubert variety in $OG(4, 9)$ defined by $\left[ L_2 \subseteq L_4 \subseteq Q_7^2 \subseteq Q_9^0 \right]$. Step (2)(b) is applied to obtain the locus $\Sigma_{d_{b_1}} = \left[ L_1 \subseteq L_2 \subseteq Q_3^3 \subseteq Q_7^2 \right]$. Note that $\Sigma_{n_{a_1}}$ is contained in $\Sigma_{d_{b_1}}$, and $\Sigma_{n_{a_2}}$ is contained in the smooth locus of $V$ since $d_{b_1} + x_{b_1} - s - n_s = 2$. Hence

$$V^{sing} = \left[ L_1 \subseteq L_2 \subseteq Q_6^0 \subseteq Q_7^2 \right],$$
equivalently, in permutation notation

$$(974258631)^{sing} = (762159843).$$

Example 6.11. Let $V$ be the restriction variety in $OG(6, 21)$ defined by the sequence

$$\left[ L_3 \subseteq L_8 \subseteq L_9 \subseteq Q_{12}^6 \subseteq Q_{13}^5 \subseteq Q_{20}^1 \right].$$
The loci $\Sigma_{r_{b_1}}$ and $\Sigma_{r_{b_2}}$ are obtained by applying step (2)(a). When applied to $Q_{20}^1$, we have

$$\Sigma_{r_{b_2}} = \left[ L_1 \subseteq L_8 \subseteq L_9 \subseteq Q_{12}^6 \subseteq Q_{13}^5 \subseteq Q_{18}^3 \right],$$
and when applied to $Q_{13}^5$, the sub-quadric $Q_{12}^6$ is replaced with the isotropic linear subspaces $L_9$ and $L_9'$. Thus

$$\Sigma_{r_{b_1}} = \left[ L_3 \subseteq L_5 \subseteq L_8 \subseteq L_9 \subseteq Q_{13}^5 \subseteq Q_{20}^1 \right] \cup \left[ L_3 \subseteq L_5 \subseteq L_8 \subseteq L_9' \subseteq Q_{13}^5 \subseteq Q_{20}^1 \right].$$
Applying step (4)(a) gives the locus

$$\Sigma_{n_{a_1}} = \left[ L_2 \subseteq L_3 \subseteq L_9 \subseteq Q_{12}^6 \subseteq Q_{13}^5 \subseteq Q_{20}^1 \right].$$
Since $b_1$ is a special index and $2n_s = d_{b_1} + r_{b_1}$, step (3)(b) is applied to obtain the locus

$$\Sigma_{n_{a_2}} = \left[ L_3 \subseteq L_6 \subseteq L_7 \subseteq L_8 \subseteq L_9 \subseteq Q_{20}^1 \right].$$
As a result, we have

$$V^{sing} = \Sigma_{n_{a_1}} \cup \Sigma_{n_{a_2}} \cup \Sigma_{r_{b_1}} \cup \Sigma_{r_{b_2}}.$$

References