UiO : University of Oslo

ECON 4160: Econometrics-Modelling and Systems Estimation

Lecture 3 (4 incl. E5106): Difference equations (1 of 2)

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The reference to this lecture is:

- Chapter 3.1-3.4 in the textbook: *Dynamic Econometrics for Empirical Macroeconomic Modelling*.
  - The mathematical details of Ch 3.3.4 (unit-circle)
  - the Appendix about complex numbers can be skipped.
- Suggested chapters about (deterministic) difference equations in math textbooks:
  - Sydsæter, Hammond, Seierstad and Strøm: *Further mathematics for Economic Analysis*. Ch 11.
  - Sydsæter, Seierstad and Strøm: *Matematisk analyse, Bind 2*. Kap 9.

Back to Lecture 1: Cobweb: Final form equations

- For the *deterministic version* of cobweb-model, the two final form equations became:

$$P_t = \phi_1 P_{t-1} + \frac{d - b_t}{a} \tag{1}$$

$$Q_t = \phi_1 Q_{t-1} + \frac{da - cb_{t-1}}{a}, \tag{2}$$

  ϕ_1 is the **autoregressive parameter**.

$$\phi_1 = \frac{c}{a}. \tag{3}$$

- Note: $\phi_1 < 0$ is implied by the assumptions of the model.

- The final form equations for Price and Quantity have the same mathematical form as the **difference equation** (3.7) in the book.
- Focus on the final form equation for price:

$$x_t = P_t$$

and set $p = 1$ in equation (3.7) for **first order dynamics**:

$$\underbrace{a_0}_{1}\underbrace{x_t}_{P_t} + \underbrace{a_1}_{-\phi_1}\underbrace{x_{t-1}}_{P_{t-1}} = \underbrace{b^{(3.7)}}_{\frac{d-b_t}{a}}, \text{ for } p = 1 \qquad (4)$$

- We can now use the theorems include in Chapter 3.2 to find a solution to (4).

Homogenous and particular solution

The solution of (4) can be expressed as:

$$x_t = x_t^h + x_t^s$$

where

- x_t^h is a solution of the **homogenous difference equation**:

$$a_0 x_t + a_1 x_{t-1} = 0,$$

- and x_t^s is a **special solution** of (4).

A particular solution

- Since we consider the case with $-1 < \phi_1 < 0$, it is relevant to look at the stable solution
- Obtain it by repeated substitution backwards in time from period t :

$$
\begin{aligned}
x_t^s &= \phi_1 x_{t-1}^s + b_t = \phi_1^2 x_{t-2}^s + \frac{d - b_t}{a} + \phi_1 \frac{d - b_{t-1}}{a} = \\
&\quad \text{continue repeated substitution...} \\
&= \sum_{i=0}^{\infty} \phi_1^i \frac{d - b_{t-i}}{a} \text{ since } -1 < \phi_1 < 0
\end{aligned}
$$

- DIY exercise 1: Show that this solution fits in $x_t - \phi_1 x_{t-1} = \frac{d-b_t}{a}$

The homogenous solution

- From equation (3.6) in the book, x_t^h, is given by:

$$x_t^h = C\lambda^t$$

where λ is the **root** of the **characteristic equation**:

$$p(\lambda) = 0 \Leftrightarrow \lambda + a_1 = 0$$

- In the example in Lecture 1 (Ch 1), we used:

$$a_1 = -\frac{1}{-1.3} = 0.77$$

and λ is therefore determined as:

$$\lambda = -0.77.$$

The full solution:

- The solution is given by $x_t^h + x_t^s$:

$$x_t = x_t^h + x_t^s = C(-0.77)^t + \sum_{i=0}^{\infty}(-0.77)^i \frac{d - b_{t-i}}{a}$$

- $\frac{d-b_{t-i}}{a}$ are fixed numbers , so the only remaining task is to determine C.
- Assume that we know the value of x_t in period $t = 0$. Since the solution holds for x_0, we determine C from:

$$x_0 = C + \sum_{i=0}^{\infty}(-0.77)^i \frac{d - b_{-i}}{a},$$

$$C = x_0 - \sum_{i=0}^{\infty}(-0.77)^i \frac{d - b_{-i}}{a}$$

- Re-introducing P_t we get the full solution as:

$$P_t = \underbrace{\left[P_0 - \sum_{i=0}^{\infty}(-0.77)^i \frac{d - b_{-i}}{a}\right](-0.77)^t}_{x_t^h} + \underbrace{\sum_{i=0}^{\infty}(-0.77)^i \left[\frac{d - b_{t-i}}{a}\right]}_{x_t^s}$$

- The same steps for quantity, Q_t, gives the solution:

$$Q_t = \left[Q_0 - \sum_{i=0}^{\infty}(-0.77)^i \frac{da - cb_{-i-1}}{a}\right](-0.77)^t + \sum_{i=0}^{\infty}(-0.77)^i \left[\frac{da - cb_{t-i-1}}{a}\right]$$

- The solutions are relevant, for example they imply the cobweb-oscillations, and that P_t "react first" when there is a Demand-shift.
- Also quite general, since $\lambda = -0.77$ can be replaced by any number between -1 and +1 without changing the mathematical form of the solution.
- Solutions for $\lambda = - \pm 1$ exist. However, the detailed mathematical form must necessarily be different for that case.

- Since a solution exists for difference equations under quite mild technical assumptions, the most important decision we have to take is about the **relevance** of the solution
- In the cobweb model, we were led by the economic interpretation of the model to chose a relevant solution based on:
  - Known values of P_0,Q_0. This assumption is called **known initial condition**
  - Asymptotic global stability of the homogenous difference equation, meaning $-1 < \lambda < 1$
- Known initial conditions is the most important of these. Based on that we can find a relevant solution also for $\lambda = - \pm 1$. Such solutions are called non-stationary solutions, or "random walk" solutions. As we will see, they will play central roles in this course

"...I turn to my computer"

- That was a lot of ado to find the solution of a very simple model.
- Usually we can use the computer to find the solution. This method is called computer simulation.
- For example, to solve the 1st order deterministic difference equation we have just worked with, with the aid of OxMetrics, the code is:

```
algebra {

b = dummy(1980,1,2020,4) ;   // creates a step-dummy variable: zero until 1979(4)
                             //  and 1 from 1980(1)

xh= year() >= 1961 ? -0.7*lag(xh,1)    : 2 ; //Solution of the homogeneous
                                             //Known initial condition: xh=2
x= year() >=  1961 ? -0.7*lag(x,1) + b : 2 ; // The full solution
}
```

Look at, and run, computer code for simulation of solution for

- Deterministic 1st order dynamics ($p = 1$)

$$x_t + a_1 x_{t-1} = b_t.$$

  Note that we set $a_0 = 1$ without loss of generality.

- Deterministic 2nd order dynamics ($p = 2$))

$$x_t + a_1 x_{t-1} = b_t.$$

before reviewing theory of 2nd order dynamics.

Real and complex roots

The 2nd order equation:

$$x_t + a_1 x_{t-1} + a_2 x_{t-2} = b_t,$$

has associated characteristic polynomial:

$$p(\lambda) = \lambda^2 + a_1 \lambda + a_2,$$

and characteristic equation, $p(\lambda) = 0$:

$$\lambda^2 + a_1 \lambda + a_2 = 0. \tag{5}$$

(5) has two roots: λ_1 and λ_2. We assume they are distinct. x_t^h is then given by:

$$x_t^h = C_1 \lambda_1^t + C_2 \lambda_2^t. \tag{6}$$

Real and complex roots, cont'd

- As above we assume known initial conditions: x_0, and x_{-1} if we choose $t = 1$ as the first solution period.
- This will determine C_1 and C_2, as numbers.
- It remains only the to determine the two characteristic roots, which will depend on the number a_1 and a_2
- The distinct roots can be
  - Two real numbers.
  - A pair of complex numbers.

Magnitude and Modulus

- A concept which covers both types of numbers is **magnitude** of a root:

$$|\lambda|, \text{magnitude of a characteristic root}$$

  - Real root: $|\lambda|$ is the absolute value
  - **Complex root**: $|\lambda|$ is the **modulus** (norm).
- The requirement that a 1st order equation is global asymptotically dynamically stable if and only if:

$$-1 < \lambda < 1$$

generalizes to:

$$|\lambda_i| < 1 \text{ for } i = 1, 2$$

and further to:

$$|\lambda_i| < 1 \text{ for } i = 1, 2, ..., p$$

Examples of 2nd order dynamics

$$x_t + 0.7x_{t-1} - 0.5x_{t-2} = 0$$
$$\lambda^2 + 0.7\lambda - 0.5 = 0$$

Roots:

$$\lambda_1 = -1.139 \quad \lambda_2 = 0.438\,99$$

$$x_t + 0.4x_{t-1} + 0.3x_{t-2} = 0$$
$$\lambda^2 + 0.4\lambda + 0.3 = 0$$

Roots:

$$\lambda_1 = -0.2 - 0.5099i \quad \lambda_2 = -0.2 + 0.509\,9i$$

$i = \sqrt{-1}$ is the *imaginary* number

$$\text{modulus} = \sqrt{(-0.2)^2 + (0.509\,9)^2} = 0.54772$$

A stochastic difference equation

of order p with constant coefficients is given by:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p}, + \varepsilon_t \tag{7}$$

where $Y_{t-i}, i = 0, 1, \ldots, p$ and ε_t are random variables and ϕ_i are the coefficients.
$\varepsilon_1, \varepsilon_2, \ldots \varepsilon_T$ are independent and identically distributed variables with expectation 0 and variance σ_ε^2. In line with the notation in used in Chapter 2 we write is as:

$$\varepsilon_t \sim \text{IID}\left(0, \sigma_\varepsilon^2\right), \ \forall\, t. \tag{8}$$

Mathematical equivalence with deterministic equations

- Mathematically speaking, (7) has the same properties as the deterministic equation:

$$a_0 x_t + a_1 x_{t-1} + \ldots + a_p x_{t-p} = b_t,$$

- This must be true, since we can set:
  - $x_t = Y_t$,
  - $a_0 = 1$,
  - $b_t = \phi_0 + \epsilon_t$, (non-homogenous part)
  - $a_i = \phi_i$, $i = 1, 2, \ldots, p$. (homogenous part)

  and "get back" (7).
- As a consequence, we can "re-use" the same methods to find formal mathematical solution of (7).

Characteristic equation and solution

- Characteristic polynomial

$$p(y) = y^p - \phi_1 y^{p-1} - \ldots - \phi_p, \tag{9}$$

  for an arbitrary number y.

- λ is a **root** in the associated **characteristic equation** $p(\lambda) = 0$:

$$\lambda^p - \phi_1 \lambda^{p-1} - \ldots - \phi_p = 0. \tag{10}$$

- Therefore:

$$Y_t^h = C\lambda^t,$$

  is a solution of the homogenous part of (7):

$$Y_t^h - \phi_1 Y_{t-1}^h - \ldots - \phi_p Y_{t-p}^h = 0.$$

General solution

- The general solution Y_t is also analogous to the deterministic case above:

$$Y_t = Y_t^h + Y_t^s$$

  where Y_t^s represents a special solution of (7).
- In Chapter 3.3.2 of the book, the solution of the 1st order model:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \varepsilon_t. \tag{11}$$

  is derived by using the same method that we have just used for the deterministic version.

- Chapter 3.3.2 shows in detail that the **stable solution**:

$$
\begin{aligned}
Y_t &= Y_t^h + Y_t^s \\
&= C\phi_1^t + \frac{\phi_0}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}, \text{ (3.17) in book}
\end{aligned} \tag{12}
$$

is equivalent to the solution obtained by **repeated substitution** backwards to period Y_0:

$$
Y_t = \phi_0 \sum_{i=0}^{t-1} \phi_1^i + \phi_1^t Y_0 + \sum_{i=0}^{t-1} \phi_1^i \varepsilon_{t-i} \text{ (3.20) in book} \tag{13}
$$

if and only if:

$$
-1 < \phi_1 < 1 \Leftrightarrow -1 < \lambda < 1.
$$

(13) is a more general expression, and is valid for unstable and explosive models.

Stability of a solution

The definition of, and condition for stability, are in terms of **homogenous solution**:

$$Y_t^h = C_1\lambda_1^t + C_2\lambda_2^t + ... + C_p\lambda_p^t. \quad (14)$$

Theorem (Condition for global asymptotic stability)

A necessary and sufficient condition for global asymptotic stability of a difference equation with constant coefficients is that all the roots of the associated characteristic equation have moduli less than 1.

It follows that:

$$Y_t^h \longrightarrow 0 \text{ for } t \longrightarrow \infty,$$

implying that the full solution approaches the chosen particular solution asymptotically:

$$Y_t \longrightarrow Y_t^s \text{ as } t \longrightarrow \infty.$$

Instable and explosive solution

Theorem (Instability of solution)

The solution of a difference equation with constant coefficients is dynamically unstable if at least one root of the associated characteristic equation has modulus (norm) equal to 1.

- Remember that modulus is the same as absolute value for a real number.
- A **unit-root** denotes a characteristic root with modulus $= 1$

Theorem (Explosive solution)

The solution of a difference equation with constant coefficients, when we consider the case of given initial conditions, is explosive if at least one root of the characteristic equation has modulus larger than 1.

Note the qualification:

...given initial conditions

- This indicates that, maybe there may be solutions that are non-explosive even if a characteristic root has modulus larger than 1.
- In economics such solutions are called rational expectations forward looking models.
- We will have something to say about them later, but for the time being we focus on the **causal** solution, which is a relevant one in most cases.

Equilibrium correcting dynamics

- Consider the 1st order case with stable dynamics.
- We can choose the stationary solution Y^*:

$$Y^* = \frac{\phi_0}{1-\phi_1},$$

as the special solution Y_t^s:

$$Y_t^s = Y^* = \frac{\phi_0}{1-\phi_1} \text{ for } t = 1, 2, \ldots.$$

when we assume:

$$\varepsilon_t = 0 \text{ for } t = 1, 2, ..\text{(all shocks switched off).}$$

Hence:

$$Y_t \longrightarrow Y^* \text{ as } t \longrightarrow \infty, \text{ when } \varepsilon_t = 0 \text{ for } t = 1, 2, \ldots.$$

and teh case of stable dynamics is seen to be equilibrium correcting:

$$Y_t = Y^* + (Y_0 - Y^*)\phi_1^t, \text{ for } t = 1, 2, \ldots. \quad (15)$$

(use (3.21) and (3.36) in the book.)

ECM and stable dynamics

DIY exercise: show that:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \varepsilon_t$$

can be equivalently be written as:

$$\Delta Y_t = (\phi_1 - 1)[Y_{t-1} - Y^*] + \varepsilon_t,$$

where $\Delta Y_t = Y_t - Y_{t-1}$, in the stable case.

Generalization of solution

- By the use of matrix notation, the dynamic equation with pth order dynamics can be written as a first order equation.

$$\mathbf{y}_t = \mathbf{B}\mathbf{y}_{t-1} + \mathbf{e}_t \tag{16}$$

- The time series $(Y_t, Y_{t-1}, ..., Y_{t-1+p})$ is stacked in the vector $\mathbf{y}_t$.
- See equation (3.38) in the book for definitions of the symbols.
- $\mathbf{B}$ is called the **companion matrix** and (16) as the **companion form**.
- Solution for $\mathbf{y}_t$ by repeated insertion backwards:

$$\mathbf{y}_t = \mathbf{B}^t\mathbf{y}_0 + \mathbf{B}^{t-1}\mathbf{e}_1 + \mathbf{B}^{t-2}\mathbf{e}_2 + \cdots + \mathbf{B}\mathbf{e}_{t-1} + \mathbf{e}_t, \tag{17}$$

Eigenvalues of B and stability

- If **B** was a number the requirement for stability would be: $-1 < \mathbf{B} < 1$
- The generalization to matrix form is that all p **eigenvalues** of **B** are less that one in magnitude.
- The p eigenvalues are defined by the determinant equation:

$$\underbrace{|\mathbf{B} - \lambda \mathbf{I}|}_{\text{Determinant}} = 0, \tag{18}$$

I is the identity matrix. By "writing out" the determinant we see that the eigenvalues of **B** must be the same numbers as the root of the characteristic polynomial of the homogenous difference equation:

$$|\mathbf{B} - \lambda \mathbf{I}| = 0 \iff \lambda^p - \phi_1 \lambda^{p-1} - \ldots - \phi_p = 0. \tag{19}$$

Eigenvalues, roots and unit-circle

- We will use eigenvalues of the companion form matrix, and characteristic roots interchangeably during the rest of the courses.
- The conditions for asymptotic global stability of difference equations are often heard expressed at as **all p roots inside the unit-circle**.
- Likewise, instability is often expressed as **at least one root on the unit-cicle**.
- But this is equivalent with saying:
  - Stability: All roots have modulus less than one.
  - Instability: At least one root has modules equal to one
- Chapter 3.3.4 contains more about the unit-circle for those interested.