UiO : University of Oslo

ECON 4160: Econometrics-Modelling and Systems Estimation

Lecture 4: Difference equations (2 of 2)

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4 September 2018

The reference to this lecture is:

- Chapter 3.5 to end of chapter 3 in the textbook: *Dynamic Econometrics for Empirical Macroeconomic Modelling*.
- Chapter 3.5.3 can be skipped in this course (I will avoid making reference to factorization)
- In chapter 3.5.4, the math after (3.87) can be skipped. (For those who have printed Ch 3 last week: Note that eq (3.93) is to be deleted).

The lag operator

- The *lag operator*, denoted L, shifts the dating of a variable one period back in time.

$$LY_t \equiv Y_{t-1}. \tag{1}$$

- Applying the lag operator twice, written as L^2, shifts Y_t two periods back:

$$LLY_t = L^2 Y_t = LY_{t-1} = Y_{t-2}. \tag{2}$$

And in general:

$$L^p Y_t = Y_{t-p} \;\; \text{for} \;\; \text{p=0,1,2,....} \tag{3}$$

We also have conventions for the case of $p = 0$:

$$L^0 = 1, \tag{4}$$

$$L^0 Y_t = Y_t. \tag{5}$$

Multiplication:

$$L^p L^s = L^p L^k = L^{(p+s)}, \tag{6}$$

and:

$$(aL^p + bL^s)\, Y_t = aL^p Y_t + bL^s Y_t = aY_{t-p} + bY_{t-s}. \tag{7}$$

See that the lag operator follows the same rules as multiplication does in ordinary algebra.
Therefore we will sometimes write "multiply by L", instead of the more technically correct "operate L on the time series Y_t".

“Leads”

- Sometimes it is useful to allow the power p in L^p to be a negative integer:
$$L^{-1}Y_t = Y_{t+1}$$
- Just at Y_{t-1} is called a lagged variable, Y_{t+1} may be called a ”lead-variable”. In general:

$$L^{-s}Y_t = Y_{t+s}$$

- A final convention (from Ch. 3.5.2) is that for a constant a:

$$La = a$$

  since a constant can be thought of time series which is unaffected by being lagged one or more periods.
- This detail is going to be important in the expression of solutions of difference equations with the use of lag-operator notation.

The difference operator

In Lecture 1 we showed that the two final form equations could be expressed as

$$\Delta P_t = (\phi_1 - 1)\left[P_{t-1} - P^*\right],$$
$$\Delta Q_t = (\phi_1 - 1)\left[Q_{t-1} - Q^*\right],$$

and in Lecture 3, that for stable 1st order dynamics:

$$\Delta Y_t = \phi_0 + (\phi_1 - 1)[Y_{t-1} - Y^*] + \varepsilon_t.$$

- In these expressions of equilibrium correction dynamics, Δ indicates change in a variable, for example:

$$\Delta Y_t = Y_t - Y_{t-1}.$$

- We now formalize Δ as the **difference operator**:

$$\Delta \overset{\text{def}}{=} 1 - L. \tag{8}$$

Exercises:

$$(1 - 0.7L + 0.5L^2)x_t = 0 \Leftrightarrow ?$$

$$(1 - L)x_t + 2L^0 = 0 \Leftrightarrow ?$$

$$[(1 - L^4) - 0.5(L - L^5)]x_t = 0 \Leftrightarrow ?$$

$$(1 - L)^2 x_t = ?$$

Lag polynomial form of difference equation

By using the lag polynomial of order p:

$$\pi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p, \tag{9}$$

we can write the difference equation ((7) in Lecture 3) compactly:

$$\pi(L) Y_t = \phi_0 + \varepsilon_t. \tag{10}$$

Interpretation: The lag polynomial $\pi(L)$ "operates on Y_t" and transforms this variable to $\phi_0 + \varepsilon_t$.

It will be useful to be able to express $\pi(L)$ in terms of another polynomial $\phi(L)$:

$$\pi(L) \underset{def}{=} 1 - \phi(L) \tag{11}$$

where $\phi(L)$ is defined as:

$$\phi(L) \underset{def}{=} \sum_{i=1}^{p} \phi_i L^i \equiv \sum_{i=0}^{p} \phi_{i+1} L^i. \tag{12}$$

(The motivation for including the two equivalent, expressions for $\phi(L)$ is that sometimes we want to write $\sum_{i=1}^{p} \phi_i Y_{t-1}$ as $\phi(L)Y_t$ (the first expression), while in other situations $\phi(L)Y_{t-1}$ (the second) is easier to use.)

Sums of lag coefficients

By setting $L = 1$ in $\phi(L)$, we can define:

$$\phi(1) = \sum_{i=1}^{p} \phi_i,$$

the sum of the lag-coefficients, and:

$$\pi(1) = 1 - \phi(1),$$

one minus the sum of the lag-coefficients.
These two conventions will be referred to quite often as we go along.

Solution with lag-operator notation

Assume that $\pi(L)$ has an inverse $\pi(L)^{-1}$, meaning that

$$\pi(L)\pi(L)^{-1} = 1.$$

We can then express a solution of the difference equation:

$$\pi(L)Y_t = \phi_0 + \varepsilon_t$$

as:

$$Y_t = \pi(L)^{-1}(\phi_0 + \varepsilon_t). \tag{13}$$

Since lag operators do not operate on a constant we set $L = 1$ in:

$$\pi(L)^{-1}\phi_0 = \frac{\phi_0}{\pi(1)} \tag{14}$$

and the solution becomes:

$$Y_t = \frac{\phi_0}{\pi(1)} + \frac{\varepsilon_t}{\pi(L)}, \tag{15}$$

noting that this requires:

$$\pi(1) \neq 0$$

to make sense.

What does it mean to require $\pi(1) \neq 0$?

- Write out the condition:

$$(1)^p - \phi_1(1)^{p-1} - \ldots - \phi_p \neq 0$$

- In other words: that the number 1 does not represent a root in the characteristic equation
- Hence (15) is an expression for the stable solution.

Impulse response function

- The same mathematics apply to the dynamics responses of Y to shocks in **observable** and **unobservable** exogenous variables in a dynamic model.
- We denote the former as **dynamic multipliers**, and come back to them in Ch 6. The responses to changes in ϵ and other **unobservables** are called **impulse responses**
- To analyse impulse responses we need expressions, or computer simulations, of solutions of difference equations.
- Consider first the solutions we worked with in Lecture 3 (after that, the use of lag polynomial notation).

Impulse responses using Lect 3 solutions

- Consider the general solution (using the companion form) in equation (17) in Lecture 3 slide set, (3.40) in the book, which becomes (3.41) for Y_t when we "unpack" the vector and matrices.
- An equivalent expression holds for Y_{t+j}, when we consider $Y_{t-1}, ..., Y_{t-p}$ as known initial conditions ,(3.94) in book:

$$\begin{aligned} Y_{t+j} &= b_{11}^{(j+1)} Y_{t-1} + b_{12}^{(j+1)} Y_{t-2} + \cdots + b_{1p}^{(j+1)} Y_{t-p} + \\ &\quad \phi_0 (b_{11}^{(j)} + b_{11}^{(j-1)} + \ldots + b_{11}^{(1)} + 1) + \\ &\quad b_{11}^{(j)} \varepsilon_t + b_{11}^{(j-1)} \varepsilon_{t+1} + \ldots + b_{11}^{(1)} \varepsilon_{t+j-1} + \varepsilon_{t+j}. \end{aligned} \tag{16}$$

- Impulse responses are defined as the partial derivatives of Y_{t+j} with respect to ε_t:

$$\delta_j = \frac{\partial Y_{t+j}}{\partial \varepsilon_t} = b_{11}^{(j)}, \;\; j = 0, 1, 2, \ldots, \;\; b_{11}^{(0)} = 1. \tag{17}$$

General, and first order case:

- General

$$b_{11}^{(j)} = c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j \, , j = 0, 1, 2, \ldots, \tag{18}$$

where c_i are constants that sum to 1.

- First order

$$b_{11}^{(j)} = \phi_1^j, j = 0, 1, 2, ... \tag{19}$$

Theorem (Asymptotic behaviour of impulse response functions)

Let δ_j denote the value of the impulse response function with argument j, of a difference equation with constant coefficients. The necessary and sufficient condition for:

$$\delta_j \longrightarrow 0 \;\; as \;\; j \longrightarrow \infty, \tag{20}$$

is that all the characteristic roots associated with the difference equation are less than one in magnitude (moduli less that one, all roots are located inside the unit-circle).

Impulse responses using lag polynomial

The solution:

$$Y_t = \pi(L)^{-1}(\phi_0 + \varepsilon_t).$$

We can write $\pi(L)^{-1}$ as the infinite lag polynomial with the dynamic multipliers as coefficients:

$$\pi(L)^{-1} = \delta_0 + \delta_1 L + \delta_2 L^2 + \dots .$$

The responses can be found from:

$$\pi(L)(\delta_0 + \delta_1 L + \delta_2 L^2 + \dots) = 1 + 0L + 0L^2 + ...$$

by multiplication on the left hand side, and collecting terms that have the same power of the lag operator, L^j:

$$
\begin{array}{llcl}
L^0: & 1 \cdot \delta_0 = 1 & \Rightarrow & \delta_0 = 1, \\
L^1: & (-\phi_1\delta_0 + \delta_1)\, L = 0 & \Rightarrow & \delta_1 = \phi_1\delta_0, \\
L^2: & (-\phi_2\delta_0 - \delta_1\phi_1 + \delta_2)\, L^2 = 0 & \Rightarrow & \delta_2 = \phi_2\delta_0 + \delta_1\phi_1, \\
 & \vdots & & \vdots
\end{array} \tag{21}
$$

- This approach shows how to derive the impulse responses without the use of the characteristic roots.
- If we imagine "permanent impulse", we find the long-run response as

$$
\frac{\partial Y^*}{\partial \varepsilon^*} = \frac{1}{1 - \phi_1 - \phi_2 - \ldots - \phi_p} \tag{22}
$$

- Interpretation: The change in the stationary situation Y^*, and only defined for the case of dynamic stability.

- In Lecture 1 we had examples of 2- and 3-equations systems.
- We then discovered that dynamics in one equation was "invasive": It made the solution become dynamic for the other endogenous variables of the model as well.
- Chapter 3.7 formalizes this important property for the bivariate system with two variables (3.108): X_t and Y_t, can be written as:

$$\begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix}, \tag{23}$$

  where a_{ij} denotes a constant coefficient and $\varepsilon_{y,t}$ and $\varepsilon_{x,t}$ are two random variables
- The system is a 2-variable VAR (Vector AutoRegressive)
- Follow the derivations in the chapter to convince yourself that the two final form equations have identical homogenous parts.

Companion form for multiple equations models

- The notation that we showed at the end of the Lecture 3 (from Chapter 3.4) can be re-used (and re-interpreted) to n-equations with p-order dynamics.
- The companion form matrix **B** has dimension $np \times np$ so can be huge!
- Nevertheless, as we shall see, if we have estimated the coefficients of the VAR, PcGive easily calculates all the np characteristic roots.