UiO : University of Oslo

ECON 4160: Econometrics-Modelling and Systems Estimation

Lecture 7: Single equation models

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The reference to this lecture is:

- Chapter 6 in the textbook: *Dynamic Econometrics for Empirical Macroeconomic Modelling*.
- Chapter 6.5.2 is cursory reading. There will be one point later in the course (tests of cointegration) where it becomes important to know what a common factor restriction is. So will bring up the definition from this sub-chapter then.
- 6.6 and 6.8 are also cursory reading, and the paragraphs in 6.3 where complex numbers are used.

The simplest ADL model

The dynamic regression model with one explanatory variable and one lag:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + \epsilon_t, t = 1, 2, ..., T, \qquad (1)$$

where

- ϵ_t is *white noise*,
- and unpredictable by conditioning on X_t, X_{t-1}, Y_{t-1}, as well as any longer lags of the two variables.
- Is sum: $E(\varepsilon_t \mid Y_{t-1}, X_{t,} X_{t-1}) = 0$.
- Known as *Autoregressive Distributed Lag model* also in Stock and Watson and other introductory books

ADL as a conditional model of the VAR

Can write the VAR at the start of the Lecture 6 as:

$$Y_t = E(Y_t \mid Y_{t-1}, X_{t-1}) + \varepsilon_{yt} \tag{2}$$
$$X_t = E(X_t \mid Y_{t-1}, X_{t-1}) + \varepsilon_{xt} \tag{3}$$

where:

$$E(Y_t \mid Y_{t-1}, X_{t-1}) = \mu_{y,t-1} \quad = \phi_{10} + \phi_{11} Y_{t-1} + \phi_{12} X_{t-1}, \tag{4}$$
$$E(X_t \mid Y_{t-1}, X_{t-1}) = \mu_{x,t-1} \quad = \phi_{20} + \phi_{21} Y_{t-1} + \phi_{22} X_{t-1}, \tag{5}$$

and:

$$\begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix} \sim IIN \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}}_{\mathbf{\Sigma}} \right), t = 1, ..., T. \tag{6}$$

- Hence, the 2-variable VAR with Gaussian white noise disturbances is saying that Y and X has a joint distribution function which is normal:

$$\begin{pmatrix} Y_t \\ X_t \end{pmatrix} \sim IIN\left(\begin{pmatrix} \mu_{y,t-1} \\ \mu_{x,t-1} \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}\right), t = 1, ..., T. \quad (7)$$

- Therefore, the properties of the Gaussian regression model, see Chapter 2.4.2 (and Exercise 2.6) and Lecture 2, can be applied without any further remarks to show that (1) can be expressed as:

$$Y_t = E(Y_t \mid X_t, Y_{t-1}, X_{t-1}) + \epsilon_t, \quad (8)$$

- and the formulaes that link the ADL coefficients ϕ_0,ϕ_1,β_0,β_1 to the VAR parameters in Ch 6.2.2, also follows directly from the properties of the normal conditional distribution of Y_t given X_t.
- The common distribution of the ADL error-terms:

$$\varepsilon_t \sim N(0, \sigma^2 \mid X_t, X_{t-1}, Y_{t-1}), t = 1, ..., T. \quad (9)$$

$$\sigma^2 = \sigma_y^2(1 - \rho_{xy}^2), \quad (10)$$

Estimation of the conditional ADL

- It follows that:

$$E(\epsilon_t \mid X_t, X_{t-1}, Y_{t-1}) = 0, \tag{11}$$

  see equation (6.13) in the book.
- Therefore, the construction of the expression of the log-likelihood follows the same logical steps as for AR(1), to give (6.16) in the book:

$$\mathcal{L}(\phi_0, \phi_1, \beta_0, \beta_1, \sigma^2 \mid Y_0, X_0) = -\frac{T}{2}(\ln(2\pi/\sigma^2)) - \sum_{t=1}^{T} \frac{(Y_t - \phi_0 - \phi_1 Y_{t-1} - \beta_0 X_t - \beta_1 X_{t-1})^2}{2\sigma^2}.$$

- Therefore the MLE of ϕ_0,ϕ_1,β_0,β_1 are the OLS estimators. Again, for σ^2, the custom is to use the "degrees of freedom corrected" estimator ((6.20)) rather than the MLE estimators.

Example of ADL (cobweb model data)

- VAR estimation of cobweb model data in Ch. 5.3
- Because the economic theory of that system is that Q_t is inelastic in the short-run, a relevant conditional ADL model is for P_t regressed on Q_t and the lags Q_{t-1}, P_{t-1}: see equation (6.21) in the book:

$$P_t = \phi_0 + \phi_1 P_{t-1} + \beta_0 Q_t + \beta_1 Q_{t-1} + \gamma_1 S11_t + \epsilon_t, t = 5, 6, ..., 100 \quad (12)$$

  Interpretation: A demand equation, with $\beta_0 < 0$
- The estimation results, using OLS, are:

$$\begin{aligned} P_t \;=\; & \underset{(0.0406)}{3.457} - \underset{(0.0251)}{0.001558}\, P_{t-1} - \underset{(0.0239)}{0.7641}\, Q_t \\ & - \underset{(0.00928)}{0.0006913}\, Q_{t-1} + \underset{(0.00849)}{0.7708}\, S11_t \qquad (13) \\ & \mathcal{L}_{P-ADL,U} = 332.861, T = 95 \end{aligned}$$

To complete the model of the VAR consisting of the conditonal ADL model for P_t and a marginal model of Q_t, we repeat the the VAR estimation results for the row normalized on Q_t: equation model:

$$\begin{aligned} Q_t &= -\underset{(0.15)}{0.83} + \underset{(0.05)}{0.94}\, P_{t-1} - \underset{(0.04)}{0.03}\, Q_{t-1} \\ &\quad + \underset{(0.04)}{0.06}\, S11_t \qquad (14) \\ &\mathcal{L}_{Q-ADL,U} = 190.944, T = 95 \end{aligned}$$

- Sum of the likelihoods of the two model equations: $\mathcal{L}_{P-ADL,U}$ and $\mathcal{L}_{Q-ADL,U}$:

  $$332.861 + 190.944 = 523.805 \equiv \text{the likelihood of the VAR.}$$

- meaning that the 2-equation model of the VAR does not "throw away" any statistical information.

Structural ADL equations

- In econometrics, the ADL "label" is used more widely for model equations where

$$E(\epsilon_t \mid X_t, X_{t-1}, Y_{t-1}) \neq 0,$$

  because of simultaneity between Y_t and X_t, or a measurement error in X_t.
- The ADL is then **not** a conditional model.
- The consumption function in the Keynesian macro model is an example of a structural equation in a simultanous equations model
- OLS is **not** MLE for the ADL parameters in this case. And OLS is not consistent either
- Instead IV and 2SLS can be used, as demonstrated in CC,
- and in the example on page 165 in the book. (The difference between OLS and IV is not large there, but that is note a general result).
- Lecture 7-10 will give (more of) the theory for estimation of simultaneous equations.

General notation for ADL model

- An ADL with p-lags in both Y and X:

$$Y_t - \phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p} = \phi_0 + \beta_0 X_t + \beta_1 X_{t-1} + \ldots + \beta_p X_{t-p} + \epsilon_t. \tag{15}$$

  where we now focus on the interpretation with conditioning, so that ϵ_t is uncorrelated with X_t.
- As showed in the book, any number of conditioning variables can be added, without changing the statistical interpretation.
- (15) expressed with lag-polynomials:

$$\pi(L) Y_t = \phi_0 + \beta(L) X_t + \epsilon_t, \tag{16}$$

$$\pi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p, \tag{17}$$

$$\beta(L) = \beta_0 + \beta_1 L + \cdots + \beta_p L^p, \tag{18}$$

Estimation of general ADL

- The estimation theory for ADL(1,1) extends to ADL(p,p) and ADS with several explanatory variables.
- OLS gives approximate ML estimation for ADL model equations with approximately Gaussian error-terms.
- We do not go through the algebra in this course, but the intuition is that is unproblematic to conditioning on longer time lags,
- and that conditioning on "period t variables" can be with respect to more than a single variable.

Dynamic multipliers

- The dynamic responses of Y_t with respect to a change in X_{t-j} are called *dynamic multipliers*.
- Mathematically, they are partial derivatives (equation (6.34) in book):

$$\frac{\partial Y_t}{\partial X_{t-j}} \equiv w_j = \begin{cases} \delta_0\beta_0, & j = 0 \\ \delta_0\beta_1 + \delta_1\beta_0, & j = 1 \\ \delta_{j-2}\beta_2 + \delta_{j-1}\beta_1 + \delta_j\beta_0, & j = 2, 3, \ldots \end{cases} \tag{19}$$

- for an ADL(2,2) model equation
- δ_j are the **impulse responses** (for an AR(p) model) from the previous lecture ($\delta_0 = 1$)

Dynamic multiplier calculation

- The book first shows a (rather complicated) manual calculation with complex number (it can can be skipped)
- The second manual method is to use standard rules for derivation on the ADL equation. The example is

$$Y_t = 1.1Y_{t-1} - 0.4Y_{t-2} + 0.4X_t + 0.8X_{t-1} + 1.1X_{t-2} + \epsilon_t,$$

and we can find the partial derivatives directly:

$$\begin{aligned}
w_0 &\equiv \frac{\partial Y_t}{\partial X_t} = ?, \\
w_1 &\equiv \frac{\partial Y_t}{\partial X_{t-1}} = 1.1\frac{\partial Y_{t-1}}{\partial X_{t-1}} + 0.8 = ?, \\
w_2 &\equiv \frac{\partial Y_t}{\partial X_{t-2}} = 1.1\frac{\partial Y_{t-1}}{\partial X_{t-2}} - 0.4\frac{\partial Y_{t-2}}{\partial X_{t-2}} + 1.1 = ?, \\
w_3 &\equiv \frac{\partial Y_t}{\partial X_{t-3}} = ?, \\
w_4 &\equiv \frac{\partial Y_t}{\partial X_{t-3}} = ?
\end{aligned}$$

Long-run multiplier

- We assume dynamic stability and stationarity of time series.
- It then follows that in the ADL(p,p):

$$\pi(1) \neq 0 \Leftrightarrow \lambda = 1 \text{ is not a root of } p(\lambda) = 0.$$

- A simple way of finding a stationary solution of ADL(p,p) is to chose the particular solution defined by setting $\epsilon_t = 0 \; \forall t$:

$$Y^* = \frac{\phi_0}{\pi(1)} + \frac{\beta(1)}{\pi(1)} X^*, \tag{20}$$

  and let X^* denote a constant value of X.
- The long-run multiplier is defined as:

$$\frac{\partial Y^*}{\partial X^*} = \frac{\beta(1)}{\pi(1)} \equiv \beta_*,$$

$$\sum_{j=0}^{\infty} w_j \equiv \beta_*.$$

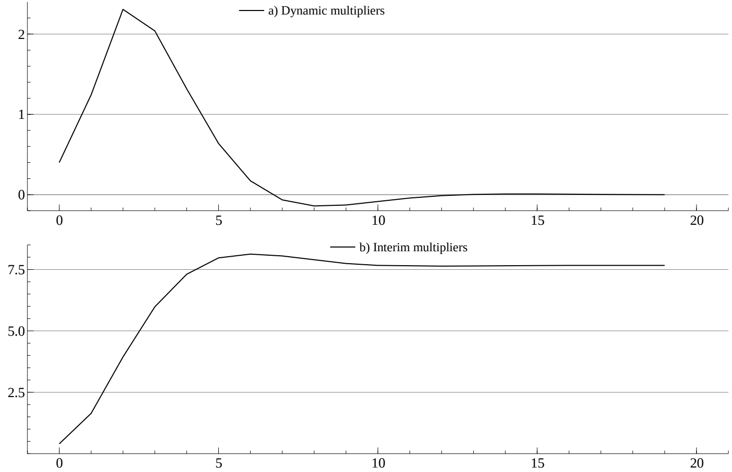

Figure 1: Dynamic multipliers and interim multipliers of the example ADL(2,2)

Estimation of long-run multiplier

- It is easy to calculate estimates of long-run multiplier for any ADL:
- Just sum the DL-coefficients and divide by one minus the sum of the AR-coefficietns.
- To obtain the estimated standard error of long-run multiplier:
  - Computer: *Dynamic analysis* $\rightarrow$ *Static long run solution* in PcGive
  - Manual: Use the Delta-method (Ch. 2), but it is then it is easiest to use the ECM form)

- Chapter 6.4 goes through the Equilibrium Correction Model form of ADL model is detail. Starting from:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + \epsilon_t, t = 1, 2, ..., T, \tag{21}$$

which can be *re-parameterized* as:

$$\Delta Y_t = \phi_0 + \beta_0 \Delta X_t + (\phi_1 - 1) Y_{t-1} + (\beta_0 + \beta_1) X_{t-1} + \epsilon_t \tag{22}$$

- which is often called the ECM-form, and a couple of further steps makes that interpretation even clearer, ending with:

$$\begin{aligned} \Delta Y_t &= \beta_0 \Delta X_t \\ &\quad + (\phi_1 - 1) \left\{ Y_{t-1} - \mu^*_{Y_{t-1}|X_{t-1}} \right\} + \epsilon_t, -1 < \phi_1 < 1. \end{aligned} \tag{23}$$

where $\mu^*_{Y_{t-1}|X_{t-1}} = Y^*$

- The ECM of AR(1), which we have mentioned several times already, is a special case: $\beta_0 = \beta_1 = 0$ and $Y^* = \frac{\phi_0}{1-\phi_1}$.

Attractive features of the ECM form

- If the primary parameters of interest are the impact effect β_0, and the long-run effect β^*, those two parameters are (almost) directly estimated from the ECM form
- And if we need the standard error of the estimated β^*, we can use the Delta-method and using the estimation results for only $\pi(1)$ and $\beta(1)$.
- Chapter 6.4.1 shows an example of estimation of the slope of the long-run Phillips curve:

$$INF = \frac{\phi_0}{\phi(1)} + \underbrace{\frac{\beta(1)}{\phi(1)}}_{\beta_*} U, \tag{24}$$

- Straight forward generalization to ADL equations with several X-variables and higher order dynamics: Chapter 6.42 contains the details.

Chapter 6.5 presents a selection of model equations that are used in econometrics that are special case of the ADL(1,1) model.

- *Common factor model*, cursory reading, but included for completeness
- *Distributed lag model*: Valid simplification of ADL if $\phi_1 = 0$ is true (ie not rejected by a valid statistical test).
- *Model in differences*: Valid simplification of ADL if $\phi_1 = 1$ and $\beta_1 = -\beta_0$ are both true.
- *AR(1) model*: Valid simplification of ADL if ....
- *Null model*: Valid simplification of ADL if ...