MAT1300
MANDATORY ASSIGNMENT II
DUE BY APRIL 24TH 2009

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The 8 parts I(a-e) and II(a-c) of this assignment all carry equal weight. To pass, at least 40% of the answers must be correct. In each case you may use the conclusions of previous questions even if you have not answered them.

Most parts ask for a proof that a certain claim is correct. The emphasis is on providing clear and valid arguments for these claims. No credit is given for only stating the given hypothesis and then asserting the desired conclusion, without indication of proof. Cite the main results from the textbook that you use.

Students may work together to solve these problems, but your written answers should reflect your own understanding. If it is unclear whether you understand the answers you hand in, you may be asked to give an oral presentation of your work.

Problem I

Let $\mathbf{0} = (0, \dots, 0)$ be the origin in $\mathbb{R}^m$, and let $E = \mathbb{R}^m \setminus \{\mathbf{0}\}$. Define a function $\mathbf{f} \colon E \to \mathbb{R}^m$ by

$$\mathbf{f}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^2}$$

where $\|\mathbf{x}\|$ denotes the Euclidean norm of $\mathbf{x}$.

[Remark: Geometrically, $\mathbf{f}$ is an inversion of E about the unit sphere in $\mathbb{R}^m$. The vectors $\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$ lie on the same ray from the origin, but $\|\mathbf{f}(\mathbf{x})\| = 1/\|\mathbf{x}\|$.]

(a) Show that the derivative of $\mathbf{f}$ at $\mathbf{x} \in E$ is the linear map $D\mathbf{f}(\mathbf{x}) \colon \mathbb{R}^m \to \mathbb{R}^m$ given by

$$D\mathbf{f}(\mathbf{x})(\mathbf{h}) = \frac{1}{\|\mathbf{x}\|^2}\Big(\mathbf{h} - 2\frac{\mathbf{x} \cdot \mathbf{h}}{\|\mathbf{x}\|^2}\mathbf{x}\Big)$$

for all $\mathbf{h} \in \mathbb{R}^m$, where $\mathbf{x} \cdot \mathbf{h}$ is the usual scalar product of $\mathbf{x}$ and $\mathbf{h}$.

[Hint: See Exercise 6.26.]

(b) Compute the (i,j)-th entry $\mathbf{f}_{i,j}(\mathbf{x})$ of the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x} = (x_1, \dots, x_m)$, for all $1 \le i, j \le m$.

[Note: Your answer may look different for $i = j$ and for $i \ne j$.]

(c) For a fixed $\mathbf{x} \in E$, let $\alpha = D\mathbf{f}(\mathbf{x})$ and consider two vectors $\mathbf{h}, \mathbf{k} \in \mathbb{R}^m$. Show that

$$\alpha(\mathbf{h}) \cdot \alpha(\mathbf{k}) = \frac{1}{\|\mathbf{x}\|^4}\mathbf{h} \cdot \mathbf{k}\,.$$

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[Remark: This implies that the angle between $\alpha(\mathbf{h})$ and $\alpha(\mathbf{k})$ equals the angle between $\mathbf{h}$ and $\mathbf{k}$, so that $\alpha = D\mathbf{f}(\mathbf{x})$ is an angle-preserving linear map. We say that the inversion $\mathbf{f}$ is a conformal map.]

(d) Compute the operator norm $\|D\mathbf{f}(\mathbf{x})\|$ as a function of $\mathbf{x} \in E$.

(e) Let $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$ be nonzero vectors in $\mathbb{R}^m$, let $R > 0$ be a positive real number, and assume that $\|\mathbf{x}\| \geq R$ for all points $\mathbf{x}$ on the line segment joining $\mathbf{a}$ and $\mathbf{b}$. Show that

$$\|\frac{\mathbf{a}}{\|\mathbf{a}\|^2} - \frac{\mathbf{b}}{\|\mathbf{b}\|^2}\| \leq \frac{1}{R^2}\|\mathbf{a} - \mathbf{b}\| .$$

Problem II

Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Explain why this function is bounded, but not continuous.

(b) Show that f is Riemann integrable.
[Hint: Use Lemma 8.13(i), and Theorem 8.32 for the restriction of f to $[c, 1]$, for suitable $0 < c < 1$.]

(c) Is it true that any bounded function $g\colon [a, b] \to \mathbb{R}$ that is continuous on $(a, b]$ is Riemann integrable? Give a short justification for your answer.