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Foreword

These are notes intended for the author’s Algebraic Topology II lectures at the University of Oslo in the fall term of 2011. The main references for the course will be:

- Allen Hatcher’s book “Algebraic Topology” [2], drawing on chapter 3 on cohomology and chapter 4 on homotopy theory.

Comments and corrections are welcome—please write to rognes@math.uio.no.
Introduction

Cohomology

(Co-)homology theories

There are various ways of associating to each topological object, like a topological space or a differentiable manifold, an algebraic object, like a group or a graded commutative ring. This can be interesting because of what the resulting algebraic object tells us about the topological object, or because known topological examples can produce novel algebraic examples. This is often the general framework of algebraic topology.

Usually the algebraic objects are constructed by comparing the given topological object, say a topological space $X$, with familiar topological objects, like the standard simplices $\Delta^n$ or the complex plane/line $\mathbb{C}$, or specially designed topological spaces, like the Eilenberg–Mac Lane spaces $K(G, n)$.

For example, to study singular homology, one considers the continuous maps $\sigma: \Delta^n \to X$ for all $n \geq 0$, assembles these into the singular chain complex $(C^\ast(X), \partial)$, and passes to homology, to obtain the singular homology groups $H_n(X)$ for $n \geq 0$. This is a standard approach in algebraic topology. The construction involves maps into $X$, and is covariant in $X$, in the sense that for a map $f: X \to Y$ there is an induced homomorphism $f^*: H_n(X) \to H_n(Y)$ in the same direction.

As another example, one may consider the commutative ring $C(X)$ of continuous maps $\varphi: X \to \mathbb{C}$ under pointwise addition and multiplication. These are the global sections in a sheaf of rings that to each open subset $U \subseteq X$ associates the ring $C(U)$ of continuous functions on $U$. Under suitable assumptions on $X$ one may consider refined versions of this: if $X$ is a complex variety one can consider the ring $\mathcal{O}(X)$ of holomorphic maps $\varphi: X \to \mathbb{C}$. Using sheaf cohomology one can associate cohomology groups $H^n(X)$ to these ringed spaces. This is a standard approach in algebraic geometry. The construction involves maps out of $X$, and is contravariant in $X$, in the sense that for a (regular) map $f: X \to Y$ there is an induced homomorphism $f^*: H^n(Y) \to H^n(X)$ in the opposite direction.

If $X$ is a smooth (infinitely differentiable) manifold one can consider the ring $C^\infty(X)$ of smooth maps $\varphi: X \to \mathbb{R}$. Each point $p \in X$ determines a maximal ideal $m_p$, and the rule $\varphi \mapsto d\varphi(p)$ induces an isomorphism $m_p/m_p^2 \cong T^*_p X$ to the cotangent space at $p$, dual to the tangent space. Gluing these vector spaces together one can view each differential $n$-form $\omega$ on $X$ as a section in a vector bundle over $X$. These can be assembled into the deRham complex $(\Omega^\ast X, d)$,
whose cohomology defines the de Rham cohomology $H^*_{dR}(X)$. This is a standard approach in differential topology. The differential forms on $X$ are again maps out of $X$, and the construction is contravariant in $X$.

There is a variant of singular homology, called singular cohomology, which is also contravariant. Its construction is of somewhat mixed variance, since it is given in terms of functions out of things given by maps into $X$. More precisely, one considers functions $\varphi: \{n\text{-simplices in } X\} \to G$ from the set of singular $n$-simplices $\sigma: \Delta^n \to X$ to a fixed abelian group $G$. This is equivalent to considering homomorphisms $\varphi: C_n(X) \to G$ from the free abelian group of singular $n$-chains on $X$. From these functions or homomorphisms one forms a cochain complex $C^*(X; G)$, whose cohomology groups are the singular cohomology groups $H^n(X; G)$.

And then there is a more directly contravariant construction, valid for all topological spaces $X$. For each abelian group $G$ and each $n \geq 0$ there exists a topological space $K(G, n)$, well-defined up to homotopy equivalence, such that the group $\pi_i K(G, n) = [S^i, K(G, n)]$ of homotopy classes of maps $S^i \to K(G, n)$ is trivial for $i \neq n$, and is identified with $G$ for $i = n$. Such a space is called an Eilenberg–Mac Lane complex of type $(G, n)$. The group $[X, K(G, n)]$ of homotopy classes of maps $X \to K(G, n)$ defines a cohomology theory in $X$, which is isomorphic to the singular cohomology group $H^n(X; G)$ for a large class of spaces $X$.

Given this wealth of possible constructions, the good news is that there are interesting uniqueness theorems: For large classes of reasonable topological spaces the various constructions agree. The formulation and proof of these theorems is best done in the language of category theory, in terms of functors and natural transformations, which was originally developed by Eilenberg and Mac Lane, largely for this purpose. The result is in some sense surprising, since it is not so clear that an abelian group built out of the continuous maps $\Delta^n \to X$ should have much to do with another abelian group built out of the continuous maps $X \to \mathbb{C}$ or $X \to K(G, n)$.

Consider for example the space $X = \mathbb{Q}$ of rational numbers, with the subspace topology from $\mathbb{R}$. Any continuous map $\Delta^n \to \mathbb{Q}$ is constant, so to the eyes of singular homology and cohomology, $\mathbb{Q}$ could equally well have had the discrete topology. On the other hand, not every map $\mathbb{Q} \to \mathbb{C}$ is continuous, so to the eyes of sheaf cohomology, the choice of topology on $\mathbb{Q}$ makes an essential difference.

The standard techniques of singular (co-)homology, like homotopy invariance, the long exact sequence of a pair, excision, behavior on sums, and the dimension axiom, suffice to prove uniqueness results for the homology and cohomology of CW spaces, i.e., spaces that can be given the structure of a CW complex, and more generally for all spaces that are of the homotopy type of a CW complex. Any manifold or complex variety is a CW space, so for geometric purposes, this class of spaces is usually fully adequate for topological work. On the other hand, the space of rational numbers mentioned above is not of the homotopy type of a CW complex. When going outside of this class of spaces, there are many variant (co-)homology theories, often with special properties that may be useful in particular settings.
Cup product

The (co-)homology groups of a topological space are useful in classification of general classes of spaces, and in answering questions about special classes of spaces. The classification problem concerns questions like: “what are the possible spaces of this type?” and “given a space, which one is it?” Since the (co-)homology groups of a space are usually quite easy to compute, and two abelian groups can usually quite easily be compared to each other, it is useful to try to answer these questions in terms of the (co-)homology groups of the space.

As a first step towards determining what possibilities there are for a class of topological objects, one should then determine what possibilities there are for the corresponding class of algebraic objects. Here it turns out to be fruitful to consider the (co-)homology groups as examples of a richer algebraic structure than just a sequence of abelian groups.

One extra structure comes from the same source as the commutative ring structure on the set $C(X)$ of continuous functions on a space $X$. This was given by the pointwise sum and product of functions, so given two maps $\varphi, \psi: X \to \mathbb{C}$, we can form the sum given by $(\varphi + \psi)(p) = \varphi(p) + \psi(p)$ and the product given by $(\varphi \cdot \psi)(p) = \varphi(p)\psi(p)$. To make it clearer what structures are involved, we might express these formulas in terms of diagrams. Since the right hand sides in these expressions involve evaluation at $p$ two times, we need to make two copies of that point. This is done using the diagonal map

$$\Delta: X \to X \times X$$

that takes $p \in X$ to $(p, p) \in X$. The sum of $\varphi$ and $\psi$ is then given by the composite map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\varphi \times \psi} \mathbb{C} \times \mathbb{C} \xrightarrow{+} \mathbb{C},$$

where the last map is the addition in $\mathbb{C}$, and similarly for the product. The commutativity of the product is derived from the fact that the composite

$$\tau\Delta: X \xrightarrow{\Delta} X \times X \xrightarrow{\tau} X \times X$$

is equal to $\Delta$, where $\tau: X \times X \to X \times X$ is the twist homeomorphism that takes $(p, q)$ to $(q, p)$.

What is the associated structure in (co-)homology? The diagonal map induces a homomorphism

$$\Delta_*: H_n(X) \to H_n(X \times X),$$

but this lands in the homology of $X \times X$, not the homology of $X$. With a little care it is possible to define a homology cross product map

$$\times: H_i(X) \otimes H_j(X) \to H_{i+j}(X \times X)$$

for all $i, j \geq 0$, and these assemble to a map

$$\bigoplus_{i+j=n} H_i(X) \otimes H_j(X) \to H_n(X \times X).$$
In general this map is not an isomorphism. If it were, we could have composed \( \Delta^* \) with the inverse isomorphism, and obtained a homomorphism

\[
H_n(X) \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(X)
\]

for all \( n \). As a convention, the tensor product of two graded abelian groups is defined so that the collection of all of these maps could be written as

\[
H_\ast(X) \longrightarrow H_\ast(X) \otimes H_\ast(X),
\]

which would make the homology groups \( H_\ast(X) \) into a “graded coring”.

The combined cross product map is an isomorphism if we work with homology with coefficients in a field, and this is one reason to consider homology groups with coefficients. However, the algebraic structures of corings or coalgebras are unfamiliar ones. It is therefore most often more convenient to dualize, and to consider cohomology instead of homology.

Let \( R \) be a commutative ring, for instance the ring of integers \( \mathbb{Z} \). The diagonal map induces a homomorphism

\[
\Delta^*: H^\ast(X \times X; R) \longrightarrow H^\ast(X; R)
\]

and there is a cohomology cross product map

\[
\times: H^i(X; R) \otimes H^j(X; R) \longrightarrow H^{i+j}(X \times X; R)
\]

for all \( i, j \geq 0 \). The composite is a homomorphism

\[
\cup = \times \circ \Delta^*: H^i(X; R) \times H^j(X; R) \rightarrow H^{i+j}(X; R)
\]

for all \( i, j \geq 0 \), called the cup product. We may assemble these cup product maps to a pairing

\[
\cup: H^\ast(X; R) \otimes H^\ast(X; R) \longrightarrow H^\ast(X; R),
\]

which makes \( H^\ast(X; R) \) a graded ring, or more precisely, a graded \( R \)-algebra. In fact, the cohomology cross product map, taking \( a \otimes b \) to \( a \times b \), is compatible with the twist homeomorphism \( \tau \), in the graded sense that \( \tau^*(a \times b) = (-1)^{ij} b \times a \), where \( i \) and \( j \) are the degrees of \( a \) and \( b \), respectively, so that the cohomology ring \( H^\ast(X; R) \) becomes graded commutative.

In much work in algebraic topology, it is therefore standard to consider the cohomology \( H^\ast(X; R) \) of a space \( X \), not as a graded abelian group, but as a graded commutative ring or algebra. This enriched algebraic structure is still manageable, but often carries much more useful information than the plain group structure.

**Poincaré duality**

Much geometric work is concerned with manifolds, or smooth varieties, rather than general topological spaces. In an \( n \)-dimensional manifold there is a certain duality between \( k \)-dimensional subobjects and suitable \( (n-k) \)-dimensional subobjects. For example, each compact, convex polyhedron in \( \mathbb{R}^3 \) determines a cell structure on its boundary, a topological 2-sphere, dividing it into vertices
(0-cells), edges (1-cells) and faces (2-cells). There is also a dual cell structure, with a 0-cell for each of the old faces, a 1-cell for each of the old edges, and a 2-cell for each of the old vertices. We can superimpose these cell structures, so that each of the old \( k \)-cells meets one of the new \( (2-k) \)-cells, in a single point.

Algebraically, this is reflected in a certain duality in the homology, or cohomology, of a manifold. It says that for suitable \( n \)-manifolds \( X \) (closed, connected and oriented) there is a preferred isomorphism \( H^n(X; R) \cong R \), and the cup product pairing

\[
H^k(X; R) \otimes H^{n-k}(X; R) \xrightarrow{\cup} H^n(X; R) \cong R
\]

defines a perfect pairing modulo torsion. This is the Poincaré duality theorem. If \( R = F \) is a field this means that the corresponding homomorphisms

\[
H^k(X; F) \rightarrow \text{Hom}(H^{n-k}(X; F), F) = H^{n-k}(X; F)^*
\]

are isomorphisms, for all \( k \). This homomorphism takes \( a \in H^k(X; F) \) to the homomorphism \( H^{n-k}(X; F) \rightarrow F \) that takes \( b \) to the image of \( a \cup b \in H^n(X; F) \) in \( F \), under the preferred isomorphism. In particular

\[
\dim_F H^k(X; F) = \dim_F H^{n-k}(X; F)
\]

for all \( k \).

This kind of symmetry, between dimension \( k \) and codimension \( k \) phenomena in the (co-)homology of an \( n \)-manifold, is the key feature taken as the starting point for the classification of manifolds, as a special class of topological objects among all topological spaces.

**Characteristic classes**

((More later))

**Bordism**

((More later))
Chapter 1

Singular homology and cohomology

Following Hatcher’s book “Algebraic Topology” [2], we first review the definition of singular homology, and then introduce singular homology with coefficients and singular cohomology.

1.1 Chain complexes

A chain complex is a diagram

\[ \cdots \rightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \rightarrow \cdots \]

of abelian groups (or \( R \)-modules, or objects in a more general abelian category), such that the composite

\[ \partial^2 = \partial \partial : C_{n+1} \rightarrow C_{n-1} \]

is the zero homomorphism, for each integer \( n \). We also use the abbreviated notation \( (C_*, \partial) \), or just \( C_* \), for the diagram above. The elements of \( C_n \) are called \( n \)-chains. We think of \( C_* \) as a graded abelian group, with \( C_n \) in degree \( n \) and \( \partial \) of degree \(-1\). This is the standard convention in algebraic topology.

Let

\[ B_n = B_n(C_*, \partial) = \text{im}(\partial : C_{n+1} \rightarrow C_n) \]

be the group of \( n \)-boundaries, and let

\[ Z_n = Z_n(C_*, \partial) = \ker(\partial : C_n \rightarrow C_{n-1}) \]

be the group of \( n \)-cycles. Then

\[ B_n \subseteq Z_n \subseteq C_n \]

since any element of \( B_n \) has the form \( x = \partial y \), and then \( \partial x = \partial^2 y = 0 \). In general, the inclusion \( B_n \subseteq Z_n \) may be a proper inclusion. To detect the possible difference, we form the quotient group

\[ H_n(C_*, \partial) = Z_n/B_n, \]
called the \( n \)-th homology group of \((C_*, \partial)\). A necessary and sufficient condition for an \( n \)-cycle \( x \in Z_n \) to be an \( n \)-boundary is then that its equivalence class (= coset) \([x] \in Z_n/B_n = H_n(C_*, \partial)\) is zero. We call the equivalence class \([x]\) the homology class of the cycle \( x \).

If there is no difference between cycles and boundaries, so that \( B_n = \text{im}(\partial) \) is equal to \( Z_n = \ker(\partial) \), as subgroups of \( C_n \), then we say that the chain complex is exact at \( C_n \). This is equivalent to the vanishing \( H_n(C_*, \partial) = 0 \) of the \( n \)-th homology group. A chain complex is exact if it is exact at each object in the diagram.

An exact chain complex is also called a long exact sequence. An exact chain complex of the form

\[
0 \to A \overset{i}{\to} B \overset{j}{\to} C \to 0
\]

(extended by 0’s in both directions) is called a short exact sequence. Exactness at \( A \) means that \( i \) is injective, exactness at \( B \) means that \( \text{im}(i) = \ker(j) \), and exactness at \( C \) means that \( j \) is surjective. If we identify \( A \) with its image \( i(A) \subseteq B \), this means that \( j \) induces an isomorphism \( B/A \cong C \).

Let \((C_*, \partial)\) and \((D_*, \partial)\) be two chain complexes. A chain map \( f_* : C_* \to D_* \) is a commutative diagram

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
C_{n+1} & \overset{\partial}{\longrightarrow} & C_n & \overset{\partial}{\longrightarrow} & C_{n-1} & \longrightarrow & \cdots \\
| & \downarrow{f_{n+1}} & | & \downarrow{f_n} & | & \downarrow{f_{n-1}} & \\
D_{n+1} & \overset{\partial}{\longrightarrow} & D_n & \overset{\partial}{\longrightarrow} & D_{n-1} & \longrightarrow & \cdots \\
\end{array}
\]

of abelian groups. In other words, it is a sequence of group homomorphisms \( f_n : C_n \to D_n \) such that \( \partial f_n = f_{n-1} \partial : C_n \to D_{n-1} \), for all \( n \). A chain map \( f_* : C_* \to D_* \) restricts to homomorphisms \( B_n(C_*, \partial) \to B_n(D_*, \partial) \) and \( Z_n(C_*, \partial) \to Z_n(D_*, \partial) \), hence induces a homomorphism of quotient groups:

\[
f_* : H_n(C_*, \partial) \to H_n(D_*, \partial)
\]

for each \( n \). If \( g_* : D_* \to E_* \) is another chain map, then we have the relation

\[(gf)_* = g_* f_* : H_n(C_*, \partial) \to H_n(E_*, \partial)\]

for each \( n \), saying that the homology groups \( H_n(C_*, \partial) \) are (covariant) functors of the chain complex \((C_*, \partial)\). (We omit to mention the identity condition.)

If the groups are reindexed by superscripts:

\[C^m = C_{-m}\]

we obtain a diagram

\[
\cdots \to C^{m-1} \overset{\delta}{\longrightarrow} C^m \overset{\delta}{\longrightarrow} C^{m+1} \to \cdots
\]

such that the composite

\[
\delta^2 = \delta \delta : C^{m-1} \longrightarrow C^{m+1}
\]

is the zero homomorphism, for each \( m \). This is called a cochain complex. We abbreviate this to \((C^*, \delta)\), or just \( C^* \). The elements of \( C^m \) are called \( m \)-cochains.
Again $C^*$ is a graded abelian group, with $C^m$ in degree $m$ and $\delta$ of degree +1. This is the standard convention in algebraic geometry.

Let $B^m = \text{im}(\delta: C^{m-1} \to C^m)$ and $Z^m = \ker(\delta: C^m \to C^{m+1})$ be the groups of $m$-coboundaries and $m$-cocycles, respectively. Then

$$B^m \subseteq Z^m \subseteq C^m$$

as before, and the quotient group

$$H^m(C^*, \delta) = \frac{Z^m}{B^m}$$

is called the $m$-th cohomology group of $(C^*, \delta)$. If $C^*$ is obtained from $C_*$ by the reindexing $C^m = C_{-m}$, then $B^m = B_{-m}$, $Z^m = Z_{-m}$ and $H^m(C^*, \delta) = H_{-m}(C_*, \partial)$.

### 1.2 Some homological algebra

((Short exact sequence of chain complexes, connecting homomorphism, long exact sequence in homology, naturality, chain homotopy, five-lemma.))

### 1.3 Singular homology

For each $n \geq 0$, let the standard $n$-simplex $\Delta^n$ be the subspace

$$\Delta^n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \text{each } t_i \geq 0, \sum_{i=0}^n t_i = 1\}$$

of $\mathbb{R}^{n+1}$ consisting of all convex linear combinations

$$(t_0, t_1, \ldots, t_n) = \sum_{i=0}^n t_i v_i$$

of the $(n+1)$ unit vectors $v_0, v_1, \ldots, v_n$, where

$$v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

has a single 1 in the $i$-th position, counting from 0. We call $t_i$ the $i$-th barycentric coordinate of the point $(t_0, t_1, \ldots, t_n)$. We call $v_i$ the $i$-th vertex of $\Delta^n$. Note that $\Delta^n$ has $(n+1)$ vertices.

For each $0 \leq i \leq n$, with $n \geq 1$, there is an affine linear embedding

$$\delta^i_n: \Delta^{n-1} \to \Delta^n,$$

called the $i$-th face map, that takes $(t_0, \ldots, t_{n-1}) \in \Delta^{n-1}$ to

$$\delta^i_n(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

In other words, it takes the $j$-th vertex of $\Delta^{n-1}$ to the $j$-th vertex of $\Delta^n$ for $0 \leq j < i$, and to the $(j+1)$-th vertex of $\Delta^n$ for $i \leq j \leq n-1$. In this way it omits the $i$-th vertex of $\Delta^n$, and induces the unique order-preserving
correspondence between the $n$ vertices of $\Delta^{n-1}$ with the remaining $n$ vertices of $\Delta^{n+1}$.

The image of $\delta^n_i$ is the subspace of $\Delta^n$ where the $i$-th barycentric coordinate $t_i$ is zero:

$$\delta^n_i(\Delta^{n-1}) = \{(t_0, \ldots, t_n) \in \Delta : t_i = 0\}$$

We call this part of the boundary of $\Delta^n$ the $i$-th face. The topological boundary of $\Delta^n$, as a subspace of the hyperplane in $\mathbb{R}^{n+1}$ where $\sum_{i=0}^n t_i = 1$, is the union of these faces:

$$\partial \Delta^n = \bigcup_{i=0}^n \delta^n_i(\Delta^{n-1}).$$

Let $X$ be any topological space. A map (= a continuous function) $\sigma : \Delta^n \rightarrow X$ is called a singular $n$-simplex in $X$. Let the singular $n$-chains $C_n(X) = \mathbb{Z}\{\sigma : \Delta^n \rightarrow X\}$ be the free abelian group generated by the set of singular $n$-simplices in $X$. Its elements are finite formal sums

$$\sum_{\sigma} n_\sigma \sigma,$$

where $\sigma$ ranges over the maps $\Delta^n \rightarrow X$, each $n_\sigma$ is an integer, and only finitely many of the $n_\sigma$ are different from zero. This abelian group can also be written as the direct sum

$$C_n(X) = \bigoplus_{\sigma : \Delta^n \rightarrow X} \mathbb{Z}$$

of one copy of the integers for each singular $n$-simplex.

For each singular $n$-simplex $\sigma : \Delta^n \rightarrow X$, and each face map $\delta^n_i : \Delta^{n-1} \rightarrow \Delta^n$, the composite map

$$\sigma \delta^n_i = \sigma \circ \delta^n_i : \Delta^{n-1} \rightarrow X$$

is a singular $(n-1)$-simplex in $X$. Under the identification of $\Delta^{n-1}$ with the $i$-th face in the boundary of $\Delta^n$, we can think of $\sigma \delta^n_i$ as the restriction of $\sigma$ to that subspace. We call this $(n-1)$-simplex the $i$-th face of $\sigma$, and use one of the notations

$$\sigma[[v_0, \ldots, \hat{v}_i, \ldots, v_n]] = \sigma[[v_0, \ldots, \hat{v}_i, \ldots, v_n]],$$

where the “hat” indicates a term to be omitted.

The restriction of $\sigma$ to the boundary of $\Delta^n$ is not itself a simplex, but $\partial \Delta^n$ is covered by the $(n+1)$ faces $\delta^n_i(\Delta^{n-1})$, and we define the boundary of $\sigma$ as a sum of the corresponding faces $\sigma \delta^n_i$. For reasons having to do with the ordering of the vertices of a simplex, or more precisely, with the orientation of a simplex, it turns out to be best to make this an alternating sum, with the $i$-th face taken with the sign $(-1)^i$.

For each singular $n$-simplex $\sigma : \Delta^n \rightarrow X$ in $X$, with $n \geq 1$, let the boundary $\partial \sigma$ be the singular $(n-1)$-chain

$$\partial \sigma = \sum_{i=0}^n (-1)^i \sigma \delta^n_i = \sum_{i=0}^n (-1)^i \sigma[[v_0, \ldots, \hat{v}_i, \ldots, v_n]].$$
More generally, define the boundary homomorphism 
\[ \partial: C_n(X) \to C_{n-1}(X) \]
to be the additive extension of this rule, so that
\[ \partial(\sum n_\sigma \sigma) = \sum n_\sigma \partial \sigma. \]

It is then a consequence of the relation
\[ \delta^i_{n+1} \circ \delta^i_n = \delta^i_{n+1} \circ \delta^{i-1}_n: \Delta^{n-1} \to \Delta^{n+1} \]
for \( 0 \leq i < j \leq n + 1 \) (both maps omit the \( i \)-th and \( j \)-th vertices), that
\[ \partial^2 = 0: C_{n+1}(X) \to C_{n-1}(X). \]

Hence the diagram

\[ \cdots \to C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_0(X) \to 0 \to \cdots \]
is a chain complex, called the \textit{singular chain complex of} \( X \). By convention, \( C_n(X) = 0 \) for \( n < 0 \).

1.4 Tensor product and Hom-groups

Let \( A \) and \( G \) be abelian groups. The \textit{tensor product} \( A \otimes G \) is the abelian group generated by symbols \( a \otimes g \), with \( a \in A \) and \( g \in G \), subject to the bilinearity relations

\[ (a + a') \otimes g = a \otimes g + a' \otimes g \]
and

\[ a \otimes (g + g') = a \otimes g + a \otimes g' \]
for \( a, a' \in A \) and \( g, g' \in G \). The \textit{Hom-group} \( \text{Hom}(A, G) \) is the abelian group of group homomorphisms \( f: A \to G \), with the group operation given by pointwise addition:

\[ (\varphi + \varphi')(a) = \varphi(a) + \varphi'(a) \]
for \( \varphi, \varphi': A \to G, a \in A \). The sum \( \varphi + \varphi' \) is a group homomorphism since \( G \) is abelian.

If \( f: A \to B \) is a homomorphism of abelian groups, then there are induced homomorphisms

\[ f_* = f \otimes 1: A \times G \to B \otimes G \]
given by $f_*(a \otimes g) = f(a) \otimes g$ for $a \in A, g \in G$, and

$$f^* = \text{Hom}(f, 1): \text{Hom}(B, G) \to \text{Hom}(A, G)$$

given by $f^*(\psi)(a) = \psi(f(a))$ for $\psi: B \to G, a \in A$. Note how the direction of the map $f^*$ is reversed, compared to that of $f$ and $f_*$. If $g: B \to C$ is a second homomorphism, then we have the relations

$$(gf)_* = g_* f_*: A \otimes G \to C \otimes G$$

and

$$(gf)^* = f^* g^*: \text{Hom}(C, G) \to \text{Hom}(A, G),$$

saying that $(-) \otimes G$ is a covariant functor and $\text{Hom}(-, G)$ is a contravariant functor (in the indicated variable).

If $A = \mathbb{Z}$, then there is a natural isomorphism $\mathbb{Z} \otimes G \cong G$, taking $n \otimes g$ to the multiple $ng$ formed in the group $G$. More generally, if $A = \mathbb{Z}\{S\} = \bigoplus_S \mathbb{Z}$ is the free abelian group generated by a set $S$, then

$$\mathbb{Z}\{S\} \otimes G \cong \bigoplus_S G$$

is the direct sum of one copy of $G$ for each element of $S$.

If $A = \mathbb{Z}$, then there is a natural isomorphism $\text{Hom}(\mathbb{Z}, G) \cong G$, taking $\varphi: \mathbb{Z} \to G$ to the value $\varphi(1)$ at $1 \in \mathbb{Z}$. If $A = \mathbb{Z}\{S\} = \bigoplus_S \mathbb{Z}$ then

$$\text{Hom}(\mathbb{Z}\{S\}, G) \cong \prod_S G$$

is the product of one copy of $G$ for each element of $S$. A homomorphism $\varphi: \mathbb{Z}\{S\} \to G$ corresponds to the sequence $(\varphi(s))_{s \in S}$ in $G$, of values of $\varphi$ at the generators $s \in S$ viewed as elements of $\mathbb{Z}\{S\}$.

### 1.5 Homology with coefficients

Let $X$ be any topological space and $G$ any abelian group. The singular chain complex of $X$ with coefficients in $G$ is the diagram

$$\cdots \to C_{n+1}(X) \otimes G \xrightarrow{\partial \otimes 1} C_n(X) \otimes G \xrightarrow{\partial \otimes 1} C_{n-1}(X) \otimes G \to \cdots.$$ 

Here $(\partial \otimes 1)(\partial \otimes 1) = \partial^2 \otimes 1 = 0$, by functoriality, so this is indeed a chain complex. Note that

$$C_n(X) \otimes G \cong \bigoplus_{\sigma: \Delta^n \to X} G$$

is the direct sum of one copy of the group $G$ for each singular $n$-simplex. Its elements are finite formal sums

$$\sum_{\sigma} g_\sigma \sigma,$$

where $\sigma$ ranges over the singular $n$-simplices in $X$, each $g_\sigma$ is an element of $G$, and only finitely many of them are nonzero.
We also use the notations $C_n(X; G) = C_n(X) \otimes G,$

$$B_n(X; G) = \text{im}(\partial \otimes 1 : C_{n+1}(X; G) \to C_n(X; G))$$

and

$$Z_n(X; G) = \ker(\partial \otimes 1 : C_n(X; G) \to C_{n-1}(X; G))$$

for the singular $n$-chains, $n$-boundaries and $n$-cycles in $X$ with coefficients in $G,$ respectively. We often abbreviate $\partial \otimes 1$ to $\partial.$ By definition, the $n$-th singular homology group of $X$ with coefficients in $G$ is the quotient group

$$H_n(X; G) = \frac{Z_n(X; G)}{B_n(X; G)} = H_n(C_*(X; G), \partial).$$

For example, let $X = \ast$ be a single point. Then there is a unique singular $n$-simplex $\sigma_n : \Delta^n \to \ast$ for each $n \geq 0,$ so $C_n(\ast) = \mathbb{Z}\{\sigma_n\}$ and $C_n(\ast; G) = G\{\sigma_n\}$ for each $n \geq 0.$ We have $\sigma_n \delta_n = \sigma_{n-1}$ for each $0 \leq i \leq n, n \geq 1,$ so $\partial \sigma_n = \sum_{i=0}^{n}(-1)^i \sigma_{n-1}$ equals $\sigma_{n-1}$ for $n \geq 2$ even, and equals 0 for $n \geq 1$ odd. Hence $C_*(\ast; G)$ appears as follows:

$$\ldots \xymatrix{ 1 \ar[r] & G\{\sigma_3\} \ar[r]^0 & G\{\sigma_2\} \ar[r]^1 & G\{\sigma_1\} \ar[r]^0 & G\{\sigma_0\} \ar[r] & 0 }$$

The boundary homomorphisms labeled 1 are isomorphisms and the ones labeled 0 are trivial. Hence $B_n(\ast; G)$ equals $G\{\sigma_n\}$ for $n \geq 1$ odd, and is zero otherwise, while $Z_n(\ast; G)$ equals $G\{\sigma_n\}$ for $n \geq 1$ odd, or for $n = 0,$ and is zero otherwise. Thus for $n \neq 0$ we have $B_n(\ast; G) = Z_n(\ast; G)$ and $H_n(\ast; G) = 0.$ In the case $n = 0$ we have

$$H_0(\ast; G) = Z_0(\ast; G)/B_0(\ast; G) = G\{\sigma_0\}/0 \cong G.$$

Let $f : X \to Y$ be any map of topological spaces. There is an induced chain map

$$f\# = C_*(f; G) : C_*(X; G) \to C_*(Y; G)$$

given by the formula

$$f\#(\sum_\sigma g_\sigma \sigma) = \sum_\sigma g_\sigma f\sigma.$$

Here $\sigma : \Delta^n \to X$ ranges over the singular $n$-simplices of $X,$ and the composite

$$f\sigma : \Delta^n \xymatrix{ \ar[r]^\sigma & X \ar[r]^f & Y }$$

is an $n$-simplex of $Y.$ This is a chain map because the associativity of composition, $(f\sigma)\delta_n = f(\sigma\delta_n),$ implies that $\partial(f\sigma) = f(\partial\sigma).$ Hence there is an induced homomorphism of homology groups,

$$f_* = H_n(f; G) : H_n(X; G) \to H_n(Y; G)$$

for all $n.$ If $g : Y \to Z$ is a second map, then the relation

$$(gf)_* = g_*f_* : H_n(X; G) \to H_n(Z; G)$$

holds.
1.6 Relative homology

Let $A \subseteq X$ be any subspace. Write $i: A \to X$ for the inclusion map. The chain map $i\# : C_\bullet(A; G) \to C_\bullet(X; G)$ is injective in each degree, identifying each simplex $\sigma: \Delta^n \to A$ with the composite $i\sigma: \Delta^n \to X$. Let the group of \textit{relative $n$-chains in $(X, A)$ with coefficients in $G$} be the quotient group

$$C_n(X, A; G) = \frac{C_n(X; G)}{C_n(A; G)}$$

of $n$-chains in $X$ modulo the $n$-chains in $A$. Since $i\#$ is a chain map, there is an induced boundary homomorphism

$$\partial: C_n(X, A; G) \to C_{n-1}(X, A; G)$$

given by taking the equivalence class of an $n$-chain $x$ in $X$ modulo $n$-chains in $A$ to the equivalence class of the $(n-1)$-chain $\partial x$ in $X$ modulo $(n-1)$-chains in $A$. Since $\partial^2 = 0$ in $C_\bullet(X; G)$, we must have $\partial^2 = 0$ in $C_\bullet(X, A; G)$, so $(C_\bullet(X, A; G), \partial)$ is a chain complex. We write

$$B_n(X, A; G) = \text{im}(\partial: C_{n+1}(X, A; G) \to C_n(X, A; G))$$

and

$$Z_n(X, A; G) = \ker(\partial: C_n(X, A; G) \to C_{n-1}(X, A; G))$$

like before, and define the $n$-th \textit{singular homology group of the pair $(X, A)$ with coefficients in $G$} to be the quotient group

$$H_n(X, A; G) = \frac{Z_n(X, A; G)}{B_n(X, A; G)} = H_n(C_\bullet(X, A; G), \partial).$$

Let $j\# : C_\bullet(X; G) \to C_\bullet(X, A; G)$ be the canonical quotient homomorphism. Then $j\#$ is a chain map. Drawing the chain complexes vertically and the chain maps horizontally, we have a commutative diagram

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
\downarrow & \downarrow & \downarrow & \ddots & \ddots & \ddots & \\
0 & i\# & j\# & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & i\# & j\# & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
\end{array}
\]

with exact rows. We usually draw this more compactly as the following short exact sequence of chain complexes:

$$0 \to C_\bullet(A; G) \xrightarrow{i\#} C_\bullet(X; G) \xrightarrow{j\#} C_\bullet(X, A; G) \to 0$$
Note that if $A = \emptyset$ is empty, then $j_\#$ is an isomorphism of chain complexes $C_\ast(X; G) \cong C_\ast(X, \emptyset; G)$, and $j_\ast$ is an isomorphism $H_n(X; G) \cong H_n(X, \emptyset; G)$ for all $n$, so (absolute) homology is a special case of relative homology.

There is a connecting homomorphism in homology

$$\partial: H_n(X, A; G) \to H_{n-1}(A; G)$$

defined by taking the homology class $[x]$ of a relative $n$-cycle $x \in Z_n(X, A; G)$ to the homology class $[\partial \tilde{x}]$ of the unique lift to $C_{n-1}(A; G)$ of the boundary in $C_n(X; G)$ of a representative $\tilde{x}$ in $C_n(X; G)$ of $x$. Here $j_\#(\tilde{x}) = x$, so the lift exists because $j_\#(\partial \tilde{x}) = \partial x = 0$ in $C_{n-1}(X, A; G)$. It is an $(n-1)$-cycle, since its boundary in $C_{n-2}(A; G)$ maps to $\partial^2 \tilde{x} = 0$ in $C_{n-2}(X; G)$ under the injective homomorphism $i_\#$. (Well defined, additive.)

We refer to the pair $(X, A)$, with $A$ a subspace of $X$, as a pair of spaces. Let $f: (X, A) \to (Y, B)$ be any map of pairs of spaces. This is a map $f: X \to Y$, subject to the condition that $f(A) \subseteq B$, so that we have a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow f' & & \downarrow f \\
B & \xrightarrow{i} & Y \\
\end{array}
\]

where $f'$ denotes the restriction of $f$. Then we have a commutative diagram of chain complexes and chain maps

\[
\begin{array}{c}
0 \longrightarrow C_\ast(A; G) \xrightarrow{i_\#} C_\ast(X; G) \xrightarrow{j_\#} C_\ast(X, A; G) \longrightarrow 0 \\
\downarrow f'_\# & & \downarrow f_\# & & \downarrow f'_\# \\
0 \longrightarrow C_\ast(B; G) \xrightarrow{i_\#} C_\ast(Y; G) \xrightarrow{j_\#} C_\ast(Y, B; G) \longrightarrow 0
\end{array}
\]

where the left hand square is induced by the square above, and the rows are short exact sequences of chain complexes. The chain map $f'_\#$ on the right hand side is then determined by $f_\#$ by the passage to a quotient.

In particular, we have an induced homomorphism

$$f_\ast = H_n(f; G): H_n(X, A; G) \to H_n(Y, B; G)$$

for each $n$. If $g: (Y, B) \to (Z, C)$ is a second map of pairs of spaces, then $(gf)_\ast = g_\ast f_\ast$. Under the isomorphism $C_\ast(X; G) \cong C_\ast(X, \emptyset; G)$ we can identify $j_\#: C_\ast(X; G) \to C_\ast(X, A; G)$ with the chain map $j'_\#$ for $j$ equal to the map of pairs $(X, \emptyset) \to (X, A)$ given by the identity on $X$.

The connecting homomorphism $\partial: H_n(X, A; G) \to H_{n-1}(A; G)$ is natural, in the sense that for any map of pairs $f: (X, A) \to (Y, B)$ the diagram

\[
\begin{array}{ccc}
H_n(X, A; G) & \xrightarrow{\partial} & H_{n-1}(A; G) \\
\downarrow f_\ast & & \downarrow f'_\ast \\
H_n(Y, B; G) & \xrightarrow{\partial} & H_{n-1}(B; G)
\end{array}
\]

commutes.
1.7 The Eilenberg–Steenrod axioms for homology

Theorem 1.7.1 (Eilenberg–Steenrod axioms). Let $G$ be a fixed abelian group and let $n$ range over all integers. We abbreviate $H_n(X, \varnothing; G)$ to $H_n(X; G)$.

(Functoriality) The rule that takes a pair of spaces $(X, A)$ to $H_n(X, A; G)$, and a map $f: (X, A) \to (Y, B)$ to the homomorphism $f_*: H_n(X, A; G) \to H_n(Y, B; G)$, defines a covariant functor from pairs of spaces to graded abelian groups.

(Naturality) The rule that takes a pair of spaces $(X, A)$ to the connecting homomorphism $\partial: H_n(X, A; G) \to H_{n-1}(A; G)$ is a natural transformation.

(Long exact sequence) The natural diagram

$$\ldots \to H_n(A; G) \xrightarrow{i_*} H_n(X; G) \xrightarrow{j_*} H_n(X, A; G) \xrightarrow{\partial} H_{n-1}(A; G) \xrightarrow{i_*} \ldots$$

is a long exact sequence, where $i_*$ is induced by the inclusion $i: A \to X$ and $j_*$ is induced by the inclusion $j: (X, \varnothing) \to (X, A)$.

(Homotopy invariance) If $f \simeq g: (X, A) \to (Y, B)$ are homotopic as maps of pairs, then $f_* = g_*: H_n(X, A; G) \to H_n(Y, B; G)$.

(Excision) If $Z \subseteq A \subseteq X$ are subspaces, so that the closure of $Z$ is contained in the interior of $A$, then the inclusion $(X - Z, A - Z) \to (X, A)$ induces isomorphisms

$$H_n(X - Z, A - Z; G) \cong H_n(X, A).$$

(Sum) If $(X, A) = \coprod \alpha (X_\alpha, A_\alpha)$ is a disjoint union of pairs of subspaces, then the inclusion maps $(X_\alpha, A_\alpha) \to (X, A)$ induce isomorphisms

$$\bigoplus \alpha H_n(X_\alpha, A_\alpha; G) \cong H_n(X, A).$$

(Dimension) Let $\ast$ be a one-point space. Then $H_0(\ast; G) = G$ and $H_n(\ast; G) = 0$ for $n \geq 0$.

The sum axiom is only interesting for infinite indexing sets, since the case of finite disjoint unions follows from the long exact sequence and excision. The dimension axiom implies that the homology of an $n$-dimensional disc, relative to its boundary, is concentrated in degree $n$. Hence for $(X, A) = (D^n, \partial D^n)$ the dimension $n$ can be recovered from the homology groups $H_*(X, A; G)$ (for $G \neq 0$).

Definition 1.7.2. A functor $(X, A) \mapsto h_*(X, A)$ and natural transformation $\partial: h_*(X, A) \to h_*^{-1}(A)$ satisfying all of the Eilenberg–Steenrod axioms for homology, except the dimension axiom, is called a generalized homology theory.

1.8 Singular cohomology

Let $X$ be any topological space and $G$ any abelian group. The singular cochain complex of $X$ with coefficients in $G$ is the diagram

$$\cdots \to \text{Hom}(C_{n-1}(X), G) \xrightarrow{\delta} \text{Hom}(C_n(X), G) \xrightarrow{\delta} \text{Hom}(C_{n+1}(X), G) \to \cdots$$
where \( \delta = \text{Hom}(\partial, 1) \) is called the \textit{coboundary homomorphism}. Here \( \delta^2 = \text{Hom}(\partial^2, 1) = 0 \), by contravariant functoriality, so this is indeed a cochain complex. Note that

\[
\text{Hom}(C_n(X), G) \cong \prod_{\sigma: \Delta^n \to X} G
\]

is the product of one copy of the group \( G \) for each singular \( n \)-simplex. Its elements are functions

\[
\varphi: \{ \sigma: \Delta^n \to X \} \to G
\]

where \( \sigma \) ranges over the singular \( n \)-simplices in \( X \), and each value \( \varphi(\sigma) \) lies in \( G \). Note that for \( \varphi \in C^{n-1}(X; G) \), \( \delta \varphi \in C^n(X; G) \) corresponds to the function given by the alternating sum

\[
(\delta \varphi)(\sigma) = \varphi(\delta \sigma) = \sum_{i=0}^{n} (-1)^i \varphi(\sigma \delta_i^n) = \sum_{i=0}^{n} (-1)^i \varphi(\sigma[v_0, \ldots, \hat{v}_i, \ldots, v_n]).
\]

We also use the notations \( C^n(X; G) = \text{Hom}(C_n(X), G) \),

\[
B^n(X; G) = \text{im}(\delta: C^{n-1}(X; G) \to C^n(X; G))
\]

and

\[
Z^n(X; G) = \ker(\delta: C^n(X; G) \to C^{n+1}(X; G))
\]

for the singular \( n \)-cochains, \( n \)-coboundaries and \( n \)-cocycles in \( X \) with coefficients in \( G \), respectively. By definition, the \( n \)-th singular cohomology group of \( X \) with coefficients in \( G \) is the quotient group

\[
H^n(X; G) = \frac{Z^n(X; G)}{B^n(X; G)} = H^n(C^*(X; G), \delta).
\]

For example, let \( X = \star \) be a single point. Then there is a unique singular \( n \)-simplex \( \sigma_n: \Delta^n \to \star \) for each \( n \geq 0 \), so \( C_n(\star) = \mathbb{Z}[\sigma_n] \) and \( C^n(\star; G) = G\{ \varphi^n \} \) for each \( n \geq 0 \), where \((g \varphi^n)(\sigma_n) = g \). We have \( \delta \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n(i) \) equals \( \sigma_{n-1} \) for \( n \geq 2 \) even, and equals 0 for \( n \geq 1 \) odd. Hence \( \delta \varphi^n \) equals \( \varphi^{n+1} \) for \( n \geq 1 \) odd, and equals 0 for \( n \geq 0 \) even. Hence \( C^*(\star; G) \) appears as follows:

\[
0 \to G\{ \varphi^0 \} \xrightarrow{0} G\{ \varphi^1 \} \xrightarrow{1} G\{ \varphi^2 \} \xrightarrow{0} G\{ \varphi^3 \} \to \ldots
\]

The boundary homomorphisms labeled 1 are isomorphisms and the ones labeled 0 are trivial. Hence \( B^n(\star; G) \) equals \( G\{ \varphi^n \} \) for \( n \geq 2 \) even, and is zero otherwise, while \( Z^n(\star; G) \) equals \( G\{ \varphi^n \} \) for \( n \geq 0 \) even, and is zero otherwise. Thus for \( n \neq 0 \) we have \( B^n(\star; G) = Z^n(\star; G) \) and \( H^n(\star; G) = 0 \). In the case \( n = 0 \) we have

\[
H^0(\star; G) = Z^0(\star; G)/B^0(\star; G) = G\{ \varphi^0 \}/0 \cong G.
\]

Let \( f: X \to Y \) be any map of topological spaces. There is an induced cochain map

\[
f^# = C^*(f; G): C^*(Y; G) \to C^*(X; G)
\]

given by the formula

\[
(f^# \varphi)(\sigma) = \varphi(f \sigma).
\]
Here $\sigma: \Delta^n \to X$ ranges over the singular $n$-simplices of $X$, and the composite

$$f \sigma: \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

is an $n$-simplex of $Y$, so that $\varphi(f \sigma)$ takes values in $G$. This is a cochain map because the associativity of composition, $(f \sigma) \delta_i = f(\sigma \delta_i)$, implies that $\delta(f^# \sigma) = f^#(\delta \sigma)$. Hence there is an induced homomorphism of cohomology groups,

$$f^*: H^n(Y; G) \to H^n(X; G)$$

for all $n$. If $g: Y \to Z$ is a second map, then the relation

$$(gf)^* = f^* g^*: H^n(Z; G) \to H^n(X; G)$$

holds, showing that $H^*(X; G)$ is a contravariant functor of $X$.

### 1.9 Relative cohomology

Let $A \subseteq X$ be any subspace. Write $i: A \to X$ for the inclusion map. The cochain map $i^*: C^*(X; G) \to C^*(A; G)$ is surjective in each degree, restricting each homomorphism $\varphi: C_n(X) \to G$ on $n$-chains in $X$ to the $n$-chains that happen to lie in $A$. Let the group of relative $n$-cochains in $(X, A)$ with coefficients in $G$ be the subgroup

$$C^n(X, A; G) = \ker(i^*: C^n(X; G) \to C^n(A; G)) \subseteq C^n(X; G)$$

of $n$-cochains in $X$ that are zero on all $n$-chains that lie in $A$. Since $i^*$ is a cochain map, there is an induced coboundary homomorphism

$$\delta: C^{n-1}(X, A; G) \to C^n(X, A; G)$$

given by taking an $(n-1)$-cochain $\varphi$ on $X$ that vanishes on $A$ to the $n$-cochain $\delta \varphi$, which vanishes on $A$ since the boundary of a chain in $A$ still lies in $A$. Since $\delta^2 = 0$ in $C^*(X; G)$, we must have $\delta^2 = 0$ in $C^*(X, A; G)$, so $(C^*(X, A; G), \delta)$ is a cochain complex. We write

$$B^n(X, A; G) = \operatorname{im}(\delta: C^{n-1}(X, A; G) \to C^n(X, A; G))$$

and

$$Z^n(X, A; G) = \ker(\delta: C^n(X, A; G) \to C^{n+1}(X, A; G))$$

like before, and define the $n$-th singular cohomology group of the pair $(X, A)$ with coefficients in $G$ to be the quotient group

$$H^n(X, A; G) = \frac{Z^n(X, A; G)}{B^n(X, A; G)} = H^n(C^*(X, A; G), \delta).$$

Let $j^*: C^n(X, A; G) \to C^n(X; G)$ be the canonical inclusion homomorphism. Then $j^*$ is a cochain map. Drawing the cochain complexes vertically
and the cochain maps horizontally, we have a commutative diagram

\[ \begin{array}{cccccc}
0 & \rightarrow & C^{n-1}(X, A; G) & \xrightarrow{j^\#} & C^{n-1}(X; G) & \xrightarrow{i^\#} & C^{n-1}(A; G) & \rightarrow & 0 \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \delta \\
0 & \rightarrow & C^n(X, A; G) & \xrightarrow{j^\#} & C^n(X; G) & \xrightarrow{i^\#} & C^n(A; G) & \rightarrow & 0 \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \delta \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array} \]

with exact rows. We usually draw this more compactly as the following short exact sequence of cochain complexes:

\[ 0 \rightarrow C^*(X, A; G) \xrightarrow{j^\#} C^*(X; G) \xrightarrow{i^\#} C^*(A; G) \rightarrow 0 \]

Note that if \( A = \emptyset \) is empty, then \( j^\# \) is an isomorphism of chain complexes \( C^*(X, \emptyset; G) \cong C^*(X; G) \), and \( j^\# \) is an isomorphism \( H^*(X, \emptyset; G) \cong H^*(X; G) \) for all \( n \), so (absolute) cohomology is a special case of relative cohomology.

There is a connecting homomorphism in cohomology

\[ \delta: H^{n-1}(A; G) \rightarrow H^n(X, A; G) \]

associated, as usual, to the short exact sequence of cochain complexes above. It is defined by taking the cohomology class \([\varphi]\) of an \((n-1)\)-cocycle \( \varphi \in Z^{n-1}(A; G) \) to the cohomology class \([\delta \tilde{\varphi}]\) of the unique lift to \( C^n(X, A; G) \) of the coboundary in \( C^n(X; G) \) of an extension \( \tilde{\varphi} \) in \( C^{n-1}(X; G) \) of \( \varphi \). Here \( \delta^\#(\tilde{\varphi}) = \varphi \), so the lift exists because \( \delta^\#(\delta \tilde{\varphi}) = \delta \varphi = 0 \) in \( C^n(A; G) \). It is an \( n \)-cocycle, since its coboundary in \( C^{n+1}(X, A; G) \) maps to \( \delta^2 \tilde{\varphi} = 0 \) in \( C^{n+1}(X; G) \) under the inclusion \( j^\# \). (Well defined, additive.)

We refer to the pair \((X, A)\), with \( A \) a subspace of \( X \), as a pair of spaces. Let \( f: (X, A) \rightarrow (Y, B) \) be any map of pairs of spaces. This is a map \( f: X \rightarrow Y \), subject to the condition that \( f(A) \subseteq B \), so that we have a commutative diagram

\[ \begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow f' & & \downarrow f \\
B & \xrightarrow{i} & Y \\
\end{array} \]

where \( f' \) denotes the restriction of \( f \). Then we have a commutative diagram of cochain complexes and chain maps

\[ \begin{array}{cccccc}
0 & \rightarrow & C^*(Y, B; G) & \xrightarrow{j^\#} & C^*(Y; G) & \xrightarrow{i^\#} & C^*(B; G) & \rightarrow & 0 \\
f'^\# & \downarrow & f^\# & \downarrow & f^\# & \downarrow & f^\# & \downarrow & f^\# \\
0 & \rightarrow & C^*(X, A; G) & \xrightarrow{j^\#} & C^*(X; G) & \xrightarrow{i^\#} & C^*(A; G) & \rightarrow & 0 \\
\end{array} \]
where the right hand square is induced by the square above, and the rows are short exact sequences of chain complexes. The cochain map \( f''\# \) on the left hand side is then determined by \( f\# \) by passage to subcomplexes.

In particular, we have an induced homomorphism

\[
f^* = H^n(f; G) : H^n(Y, B; G) \to H^n(X, A; G)
\]

for each \( n \). If \( g : (Y, B) \to (Z, C) \) is a second map of pairs of spaces, then \((gf)^* = f^*g^*\). Under the isomorphism \( C^*(X, \emptyset; G) \cong C^*(X; G) \) we can identify \( j^\#: C^*(X, A; G) \to C^*(X; G) \) with the chain map \( j''\# \) for \( j \) equal to the map of pairs \( (X, \emptyset) \to (X, A) \) given by the identity on \( X \).

The connecting homomorphism \( \delta : H^{n-1}(A; G) \to H^n(X, A; G) \) is natural, in the sense that for any map of pairs \( f : (X, A) \to (Y, B) \) the diagram

\[
\begin{array}{ccc}
H^{n-1}(B; G) & \xrightarrow{\delta} & H^n(Y, B; G) \\
\downarrow{f^*} & & \downarrow{f''} \\
H^{n-1}(A; G) & \xrightarrow{\delta} & H^n(X, A; G)
\end{array}
\]

commutes.

1.10 The Eilenberg–Steenrod axioms for cohomology

**Theorem 1.10.1 (Eilenberg–Steenrod axioms).** Let \( G \) be a fixed abelian group and let \( n \) range over all integers. We abbreviate \( H^n(X, \emptyset; G) \) to \( H^n(X; G) \).

(Functoriality) The rule that takes a pair of spaces \((X, A)\) to \( H^n(X, A; G) \), and a map \( f : (X, A) \to (Y, B) \) to the homomorphism \( f^* : H^n(Y, B; G) \to H^n(X, A; G) \), defines a contravariant functor from pairs of spaces to graded abelian groups.

(Naturality) The rule that takes a pair of spaces \((X, A)\) to the connecting homomorphism \( \delta : H^{n-1}(A; G) \to H^n(X, A; G) \) is a natural transformation.

(Long exact sequence) The natural diagram

\[
\cdots \to H^n-1(A; G) \xrightarrow{i^*} H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} \to \cdots
\]

is a long exact sequence, where \( i^* \) is induced by the inclusion \( i \) : \( A \to X \) and \( j^* \) is induced by the inclusion \( j \) : \( (X, \emptyset) \to (X, A) \).

(Homotopy invariance) If \( f \simeq g : (X, A) \to (Y, B) \) are homotopic as maps of pairs, then \( f^* = g^* : H^n(Y, B; G) \to H^n(X, A; G) \).

(Excision) If \( Z \subseteq A \subseteq X \) are subspaces, so that the closure of \( Z \) is contained in the interior of \( A \), then the inclusion \( (X - Z, A - Z) \to (X, A) \) induces isomorphisms

\[
H^n(X, A; G) \xrightarrow{\cong} H_n(X - Z, A - Z; G).
\]

(Product) If \( (X, A) = \bigsqcup_{\alpha} (X_\alpha, A_\alpha) \) is a disjoint union of pairs of subspaces, then the inclusion maps \( (X_\alpha, A_\alpha) \to (X, A) \) induce isomorphisms

\[
H^n(X, A) \cong \prod_{\alpha} H^n(X_\alpha, A_\alpha; G)
\]
(Dimension) Let \( \ast \) be a one-point space. Then \( H^0(\ast; G) = G \) and \( H^n(\ast; G) = 0 \) for \( n \geq 0 \).

**Definition 1.10.2.** A functor \((X, A) \mapsto h^*(X, A)\) and natural transformation \( \delta : h^{*-1}(A) \to h^*(X, A) \) satisfying all of the Eilenberg–Steenrod axioms for cohomology, except the dimension axiom, is called a generalized cohomology theory.

**Proof.** Contravariant functoriality of the cohomology groups, naturality of the connecting homomorphism and exactness of the long exact sequence are clear from the contravariant functoriality of the short exact sequence

\[
0 \to \mathcal{C}^n(X, A; G) \xrightarrow{\partial} \mathcal{C}^n(X; G) \xrightarrow{i_\#} \mathcal{C}^n(A; G) \to 0
\]
of cochain complexes, together with the standard construction of the connecting homomorphism and exactness of the long exact sequence for short exact sequences of chain complexes.

Homotopy invariance for singular cohomology follows from the proof of homotopy invariance for singular homology. Let \( F : (X, A) \times I = (X \times I, A \times I) \to (Y, B) \) be a homotopy of pairs from \( f : (X, A) \to (Y, B) \) to \( g : (X, A) \to (Y, B) \). For each \( n \)-simplex \( \Delta^n \) there is a triangulation of the cylinder \( \Delta^n \times I \), where \( I = [0, 1] \), which gives rise to a prism operator

\[
P : C_n(X, A) \to C_{n+1}(Y, B)
\]
such that

\[
\partial P + P \partial = g_\# - f_\# : C_n(X, A) \to C_n(Y, B)
\]
for each \( n \geq 0 \). Applying \( \text{Hom}(-, G) \), we get the dual prism operator

\[
P^* = \text{Hom}(P, 1) : C_{n+1}(Y, B; G) \to C^n(X, A; G)
\]
such that

\[
P^* \delta + \delta P^* = g^\# - f^\# : C^n(Y, B; G) \to C^n(X, A; G).
\]

For each \( n \)-cocycle \( \varphi \in Z^n(Y, B; G) \) we have \( \delta \varphi = 0 \), so \( \delta P^*(\varphi) = g^\#(\varphi) - f^\#(\varphi) \), which implies that \( f^\#(\varphi) \) and \( g^\#(\varphi) \) are cohomologous, i.e., they represent the same cohomology class:

\[
f^*[\varphi] = [f^\#(\varphi)] = [g^\#(\varphi)] = g^*[\varphi].
\]

Thus \( f^* = g^* : H^n(Y, B; G) \to H^n(X, A; G) \).

Excision for singular cohomology also follows from the proof of excision for singular homology. Let \( B = X - Z \), so that \( \text{Int}(A) \) and \( \text{Int}(B) \) cover \( X \). Let

\[
\iota : C_*(A + B) \subseteq C_*(X)
\]
be the inclusion of the subcomplex of simplices in \( A \) or \( B \). Using barycentric subdivision, there is a chain map

\[
\rho : C_*(X) \to C_*(A + B)
\]
such that $\rho = 1$ and $1 - \rho = \partial D + D\partial$ for a chain homotopy $D$. Applying \( \text{Hom}(-, G) \), we get dual cochain maps

$$\iota^*: C^*(X; G) \rightarrow C^*(A + B; G) = \text{Hom}(C_*(A + B), G)$$

and

$$\rho^*: C^*(A + B; G) \rightarrow C^*(X; G)$$

such that $\iota^* \rho^* = 1$ and $1 - \rho^* \iota^* = D^* \delta + \delta D^*$. Hence $\iota^*$ induces an isomorphism in cohomology. By the five-lemma for the map of long exact sequences induced by the map of short exact sequences of cochain complexes

$$0 \rightarrow C^*(A + B; A; G) \rightarrow C^*(A + B; G) \rightarrow C^*(A; G) \rightarrow 0$$

it follows that also the left hand map $\iota^*$ induces an isomorphism in cohomology. There is a natural identification $C^*(A + B; A; G) \cong C^*(B, A \cap B; G)$, and the composite of the induced isomorphism $H^*(A + B, A; G) \cong H^*(B, A \cap B; G)$ with $\iota^*$ is the excision isomorphism.

The product axiom is only interesting for infinite indexing sets, since the case of finite disjoint unions follows from the long exact sequence and excision. Since any simplex $\sigma: \Delta^n \rightarrow \coprod_{\alpha} X_\alpha$ lands in precisely one of the $X_\alpha$’s, there is a direct sum decomposition $C_*(X, A) \cong \bigoplus_{\alpha} C_*(X_\alpha, A_\alpha)$. Applying $\text{Hom}(-, G)$ we get a product factorization

$$C^*(X, A; G) \cong \prod_{\alpha} C^*(X, A; G).$$

Since each coface map $\delta$ factors as the product $\prod_{\alpha} \delta_{\alpha}$, we also get product factorizations of $B^*(X, A; G)$ and $Z^*(X, A; G)$, which induce the claimed product factorization of $H^*(X, A; G)$.

We discussed the dimension axiom for cohomology above. \qed

### 1.11 Cellular homology and cohomology

If $X$ is a CW complex, and $A \subseteq X$ a subcomplex, then the \textit{cellular complexes} $C_*^{\text{CW}}(X, A; G)$ and $C_*^{\text{CW}}(X, A; G)$ are smaller complexes than the singular ones, which can be used to compute the homology and cohomology groups.

Let

$$\emptyset = X(-1) \subseteq X^{(0)} \subseteq \cdots \subseteq X^{(n-1)} \subseteq X^{(n)} \subseteq \cdots \subseteq X$$

be the skeleton filtration of $X$, so that there is a pushout square

$$\begin{array}{ccc}
\coprod_{\alpha} \partial D^n & \longrightarrow & \coprod_{\alpha} D^n \\
\varphi \downarrow & & \downarrow \Phi \\
X^{(n-1)} & \longrightarrow & X^{(n)}
\end{array}$$

for each $n \geq 0$, and $X = \bigcup_{n \geq 0} X^{(n)}$ has the weak (colimit) topology. The index $\alpha$ runs over the set of $n$-cells in $X$, and we decompose $\varphi = \coprod_{\alpha} \varphi_{\alpha}$ and
Φ = \coprod \Phi_\alpha$, where $\varphi_\alpha: \partial D^n \to X^{(n-1)}$ is the attaching map and $\Phi_\alpha: D^n \to X^{(n)} \subseteq X$ is the characteristic map of the $\alpha$-th $n$-cell.

For each $n \geq 0$, let

$$C_n^{CW}(X) = H_n(X^{(n)}, X^{(n-1)}).$$

By excision, homotopy invariance and the sum axiom, there is an isomorphism

$$C_n^{CW}(X) \cong \bigoplus_\alpha H_n(D^n, \partial D^n) \cong \bigoplus_\alpha \mathbb{Z},$$

where $\alpha$ runs over the set of $n$-cells in $X$.

Let

$$d_n: C_n^{CW}(X) \to C_{n-1}^{CW}(X)$$

be the composite homomorphism

$$H_n(X^{(n)}, X^{(n-1)}) \xrightarrow{\partial} H_{n-1}(X^{(n-1)}) \xrightarrow{\iota_*} H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

This equals the connecting homomorphism in the long exact sequence of the triple $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$ (ETC). Then $d_n d_{n+1} = 0$. We call $(C_*^{CW}(X), d)$ the cellular chain complex of $X$, and define the cellular homology groups of $X$ to be its homology groups

$$H_n^{CW}(X) = \frac{\ker(d_n)}{\text{im}(d_{n+1})} = H_n(C_*^{CW}(X), d).$$

If $f: X \to Y$ is a cellular map of CW complexes, so that $f(X^{(n)}) \subseteq Y^{(n)}$ for all $n$, we get a homomorphism

$$f\# = C_n^{CW}(f) = H_n(f): H_n(X^{(n)}, X^{(n-1)}) \to H_n(Y^{(n)}, Y^{(n-1)}),$$

which defines a chain map $f\#: C_*^{CW}(X) \to C_*^{CW}(Y)$ and an induced homomorphism $f_*: H_*^{CW}(X) \to H_*^{CW}(Y)$. Hence the cellular complex and cellular homology groups are covariant functors from the category of CW complexes and cellular maps.

Let $A \subseteq X$ be a subcomplex, with skeleton filtration $\{A^{(n)}\}_n$, such that $A^{(n)}$ is built from $A^{(n-1)}$ by attaching a subset of the $n$-cells of $X$, with attaching maps landing in $A^{(n-1)} \subseteq X^{(n-1)}$.

The inclusion $i: A \to X$ is cellular, and identifies the cellular chain complex $C_*^{CW}(A)$ with a subcomplex of $C_*^{CW}(X)$. Let the relative cellular $n$-chains

$$C_n^{CW}(X, A) = \frac{C_n^{CW}(X)}{C_n^{CW}(A)} \cong H_n(X^{(n)}, X^{(n-1)} \cup A^{(n)})$$

be the quotient group. It is the free abelian group generated by the $n$-cells of $X$ that are not cells in $A$. There is a relative boundary homomorphism

$$d_n: C_n^{CW}(X, A) \to C_{n-1}^{CW}(X, A)$$

and a short exact sequence of cellular chain complexes

$$0 \to C_*^{CW}(A) \xrightarrow{i_*} C_*^{CW}(X) \xrightarrow{j_*} C_*^{CW}(X, A) \to 0.$$
CHAPTER 1. SINGULAR HOMOLOGY AND COHOMOLOGY

((Exercise: Give an explicit description of the relative boundary homomorphism in terms of the maps \( i_*, j_* \) and/or \( \partial \) of various pairs.))

Introducing coefficients, let

\[
C^*_n(X; A; G) = C_n^*(X; A) \otimes G \cong \bigoplus_{\alpha} G
\]

and

\[
C^n_{CW}(X; A; G) = \text{Hom}(C^n_{CW}(X; A; G)) \cong \prod_{\alpha} G,
\]

where \( \alpha \) runs over the set of \( n \)-cells in \( X \) that are not cells in \( A \). Using the boundary homomorphisms \( \partial = \partial \otimes 1 \) and \( \delta = \text{Hom}(\partial, 1) \) we get the cellular homology and cohomology groups

\[
H^C_n(X; A; G) = H_n(C_n^*(X; A; G), \partial)
\]

and

\[
H^n_{CW}(X; A; G) = H^n(C^n_{CW}(X; A; G), \delta).
\]

Lemma 1.11.1. There are isomorphisms

\[
C^*_n(X; A; G) \cong H_n(X^{(n)}, X^{(n-1)} \cup A^{(n)}; G)
\]

and

\[
C^n_{CW}(X; A; G) \cong H^n(X^{(n)}, X^{(n-1)} \cup A^{(n)}; G),
\]

compatible with the boundary homomorphisms.

Theorem 1.11.2 (Cellular (co-)homology). For all CW pairs \( (X, A) \) there are isomorphisms

\[
H_n(X; A; G) \cong H^C_n(X; A; G)
\]

and

\[
H^n(X; A; G) \cong H^n_{CW}(X; A; G),
\]

natural with respect to cellular maps of pairs.

If \( X \) is a CW complex of finite type, meaning that it has only a finite number of \( n \)-cells for each \( n \) (but may be infinite-dimensional), then the cellular complex \( C^*_n(X) \) is finitely generated in each degree. It follows that the cellular homology groups \( H^C_n(X) \) are finitely generated in each degree, hence this is also the case for the (isomorphic) singular homology groups \( H_*(X) \). Similarly for CW pairs of (relatively) finite type.

Exercise 1.11.3. Prove the lemma and the theorem.

Exercise 1.11.4. Consider the CW structure on the unit sphere \( S^k \subset \mathbb{R}^{k+1} \), with \( n \)-skeleton \( S^n \) for \( n \leq k \), and two \( n \)-cells \( e^n_+ \) and \( e^n_- \) for each \( 0 \leq n \leq k \). Determine the cellular complex \( C_{CW}^*(S^k) \), and compute the cellular homology groups \( H_{CW}^*(S^k; G) \) and the cellular cohomology groups \( H_{CW}^*(S^k; G) \). The cases \( k = -1 \) and \( k = 0 \) may be treated separately. Be careful with orientations and signs.
Exercise 1.11.5. Consider the CW structure on the real projective space $\mathbb{R}P^k = S^k/\sim$, where $p \sim -p$ for $p \in S^k$, with $n$-skeleton $\mathbb{R}P^n$ for $n \leq k$, and one $n$-cell $e^n$ for each $0 \leq n \leq k$. Use naturality for the cellular map $f: S^k \to \mathbb{R}P^k$ to determine the cellular complex $C_*^{CW}(\mathbb{R}P^k)$, and compute the cellular homology groups $H_*^{CW}(\mathbb{R}P^k; G)$ and the cellular cohomology groups $H_*^{CW}(\mathbb{R}P^k; G)$. You may concentrate on the cases $k \geq 1$, $G = \mathbb{Z}$, $G = \mathbb{Z}/2$ and $G = \mathbb{Z}/p$ with $p$ an odd prime.

Topologists often write $\mathbb{Z}/m$ where algebraists might write $\mathbb{Z}/m\mathbb{Z}$ or $\mathbb{Z}/(m)$. Hatcher [2] writes $\mathbb{Z}_m$, as topologists used to do, but this is easily confused with the ring of $p$-adic integers, especially when $m = p$. 


Chapter 2

The universal coefficient theorems

There is a natural homomorphism
\[ H_n(X) \otimes G \rightarrow H_n(X; G) \]
taking a tensor product \([x] \otimes g\), where \(x\) is a singular \(n\)-cycle in \(X\), to the homology class \([x \otimes g]\) of the \(n\)-cycle \(x \otimes g\) in the complex \(C_\ast(X; G)\). This homomorphism is injective, but is not an isomorphism in general.

Similarly, there is a natural homomorphism
\[ H^n(X; G) \rightarrow \text{Hom}(H_n(X), G) \]
taking the cohomology class \([\varphi]\) of an \(n\)-cocycle \(\varphi : C_\ast(X) \rightarrow G\) to the homomorphism \(\varphi_* : H_n(X) \rightarrow G\), taking \([x]\) to \(\varphi(x)\), where \(x\) is an \(n\)-cycle in \(X\). Note that if \(x\) is changed by a boundary \(\partial y\), then \(\varphi(x)\) changes by \(\varphi(\partial y) = (\delta \varphi)(y) = 0\), since \(\varphi\) was assumed to be a cocycle. This homomorphism is surjective, but is not an isomorphism in general.

For each pair of spaces \((X, A)\), the tensor product of the long exact sequence in homology with the abelian group \(G\) is a chain complex
\[ \cdots \rightarrow H_n(A) \otimes G \rightarrow H_n(X) \otimes G \rightarrow H_n(X, A) \otimes G \rightarrow \cdots . \]
but in general this is not an exact complex. On the other hand, the chain complex
\[ \cdots \rightarrow H_n(A; G) \rightarrow H_n(X; G) \rightarrow H_n(X, A; G) \rightarrow \cdots . \]
is exact. In this sense, the functor \(H_n(X; G)\) is better behaved than the functor \(H_n(X) \otimes G\).

Similarly, applying \(\text{Hom}(\_, G)\) to the long exact sequence in homology we get a cochain complex
\[ \cdots \rightarrow \text{Hom}(H_n(X, A), G) \rightarrow \text{Hom}(H_n(X), G) \rightarrow \text{Hom}(H_n(A), G) \rightarrow \cdots \]
but in general this is not an exact complex. On the other hand, the cochain complex
\[ \cdots \rightarrow H^n(X, A; G) \rightarrow H^n(X; G) \rightarrow H^n(A, G) \rightarrow \cdots . \]
is exact. In this sense, the functor $H^n(X; G)$ is better behaved than the functor $\text{Hom}(H_n(X), G)$.

2.1 Half-exactness

Let

$$0 \to A \overset{i}{\to} B \overset{j}{\to} C \to 0$$

be a short exact sequence of abelian groups.

**Lemma 2.1.1.** Let $G$ be an abelian group. Then

$$A \otimes G \overset{i \otimes 1}{\to} B \otimes G \overset{j \otimes 1}{\to} C \otimes G \to 0$$

is exact, but $i \otimes 1$ might not be injective, and

$$0 \to \text{Hom}(C, G) \overset{\text{Hom}(i, 1)}{\to} \text{Hom}(B, G) \overset{\text{Hom}(j, 1)}{\to} \text{Hom}(A, G)$$

is exact, but $\text{Hom}(i, 1)$ might not be surjective.

We say that $(-) \otimes G$ is right exact and that $\text{Hom}(-, G)$ is left exact.

**Proof.** ((ETC))

**Lemma 2.1.2.** The following are equivalent:

(a) There is a homomorphism $r: B \to A$ with $ri = 1: A \to A$.

(b) There is a homomorphism $s: C \to B$ with $js = 1: C \to C$.

**Proof.** Given $r$ we may choose $s$ so that $ir + sj = 1: B \to B$, and conversely. \(\square\)

In this case, we say that $0 \to A \to B \to C \to 0$ is a split (short) exact sequence. We call $r$ a retraction and $s$ a section. There are then preferred isomorphisms

$$i + s: A \oplus C \xrightarrow{\cong} B$$

and

$$(j, r): B \xrightarrow{\cong} A \times C.$$ 

**Lemma 2.1.3.** If $0 \to A \to B \to C \to 0$ is split, then $i \otimes 1: A \otimes G \to B \otimes G$ is split injective and $\text{Hom}(i, 1): \text{Hom}(B, G) \to \text{Hom}(A, G)$ is split surjective.

**Proof.** If $r: B \to A$ is a retraction, then $(r \otimes 1)(i \otimes 1) = 1$ shows that $i \otimes 1$ is (split) injective and $\text{Hom}(i, 1)\text{Hom}(r, 1) = 1$ shows that $\text{Hom}(i, 1)$ is (split) surjective. \(\square\)

**Lemma 2.1.4.** If $C = \mathbb{Z}\{T\}$ is a free abelian group, then any surjection $j: B \to C$ admits a section.

**Proof.** For each basis element $t \in T$, use surjectivity to choose an element $s(t) \in B$ with $js(t) = t$. Since $C$ is free, we can extend additively to obtain the desired homomorphism $s: C \to B$ with $js = 1$. \(\square\)

((Flat, projective.))
2.2 Free resolutions

Taken together, these lemmas show that when applied to short exact sequences

\[ 0 \to A \to B \to C \to 0 \]

with \( C \) a free abelian group, the functors \((-) \otimes G\) and \( \text{Hom}(-, G)\) are exact. To control the failure of exactness for general abelian groups \( C \), we resolve \( C \) by a free abelian groups. This means that we replace \( C \) by a chain complex

\[
\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0
\]

of free abelian groups \( F_n = \mathbb{Z}\{T_n\} \), such that \( H_0(F_\ast, f) \cong C \) and \( H_n(F_\ast, f) = 0 \) for \( n \neq 0 \). An isomorphism \( H_0(F_\ast, f) \cong C \) corresponds to a choice of a homomorphism \( \epsilon: F_0 \to C \) that makes the diagram

\[
\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} C \to 0
\]

exact at all points. In other words, \( \epsilon \) is surjective and \( \text{im}(f_1) = \ker(\epsilon) \). Such a diagram is called an augmented chain complex. We call the complex \((F_\ast, f)\) a free resolution of \( C \).

If we think of \( C \) is a chain complex concentrated in degree 0, then \( \epsilon \) can also be viewed as a chain map

\[
\epsilon: F_\ast \to C
\]

that induces an isomorphism in homology. A free resolution of \( C \) is thus a chain complex of free abelian groups, with homology isomorphic to \( C \) (concentrated in degree 0).

Any abelian group \( C \) admits a short free resolution. For by choosing any set of generators \( T_0 \subseteq C \) we can let \( F_0 = \mathbb{Z}\{T_0\} \), and obtain a surjective augmentation \( \epsilon: F_0 \to C \) by sending each element of the basis \( T_0 \) for \( F_0 \) to the corresponding element of \( C \). Then \( \ker(\epsilon) \subseteq F_0 \) is a subgroup of a free abelian group. It is an algebraic fact that any subgroup of a free abelian group is again a free group, so there is a set \( T_1 \) and an isomorphism \( \mathbb{Z}\{T_1\} \cong \ker(\epsilon) \). We let \( F_1 = \mathbb{Z}\{T_1\} \) and define \( f_1: F_1 \to F_0 \) by sending each generator of \( T_1 \) to the corresponding element in \( \ker(\epsilon) \subseteq F_0 \). Then

\[
0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} C \to 0
\]

(extended by 0’s to the left) is a free resolution of \( C \). Since \( F_n = 0 \) for \( n > 1 \) we say that this is a resolution of length 1.

For example, the free group \( C = \mathbb{Z} \) has a very short free resolution

\[
0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \to 0
\]

with \( F_0 = \mathbb{Z} \), and a finite cyclic group \( C = \mathbb{Z}/m \) has a short free resolution

\[
0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{\ epsilon} \mathbb{Z}/m \to 0
\]

with \( F_0 = F_1 = \mathbb{Z} \), where \( f_1 \) multiplies by \( m \).

((Discuss essential uniqueness and functoriality of free resolutions.))
2.3 Tor and Ext

The failure of exactness of the tensor product $C \otimes G$ is measured by the homology of the chain complex $F_* \otimes G$ obtained by replacing $C$ by a free resolution $F_*$. Similarly, the failure of exactness of $\text{Hom}(C, G)$ is measured by the cohomology of the cochain complex $\text{Hom}(F_*, G)$. These homology and cohomology groups are called Tor- and Ext-groups.

**Definition 2.3.1.** Let $C$ and $G$ be abelian groups. Choose any free resolution $\epsilon: F_* \rightarrow C$ of $C$. The Tor-groups of $C$ and $G$ are the homology groups

$$\text{Tor}_n^Z(C, G) = H_n(F_* \otimes G, f \otimes 1)$$

for $n \geq 0$. The Ext-groups of $C$ and $G$ are the cohomology groups

$$\text{Ext}_n^Z(C, G) = H^n(\text{Hom}(F^*, G), \text{Hom}(f, 1))$$

for $n \geq 0$.

This is the form of the definition that is interesting in the generality of $R$-modules over a general ring $R$. But in our case, of $\mathbb{Z}$-modules in the guise of abelian groups, only a few of these groups are relevant.

Since each abelian group admits a short free resolution $0 \rightarrow F_1 \rightarrow F_0 \rightarrow C \rightarrow 0$, the Tor-groups are the homology groups of the chain complex

$$0 \rightarrow F_1 \otimes G \xrightarrow{f_1 \otimes 1} F_0 \otimes G \rightarrow 0.$$

In view of the right exactness of $(-) \otimes G$, we get isomorphisms

$$\text{Tor}_0^Z(C, G) \cong C \otimes G$$

and

$$\text{Tor}_1^Z(C, G) \cong \ker(f_1 \otimes 1: F_1 \otimes G \rightarrow F_0 \otimes G),$$

while $\text{Tor}_n^Z(C, G) = 0$ for $n \geq 2$. Hence the 0-th Tor-group recovers the tensor product, and the only interesting, new, Tor-group is the 1-st one. We simplify its notation to

$$\text{Tor}(C, G) = \text{Tor}_1^Z(C, G).$$

For example, with $C = \mathbb{Z}$ we can use the very short resolution $F_* \rightarrow \mathbb{Z}$ with $F_1 = 0$, so $F_* \otimes G$ is 0 in degree 1 and $\text{Tor}(\mathbb{Z}, G) = 0$ for all $G$.

For a more interesting example, with $C = \mathbb{Z}/m$ we can use the short free resolution $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0$, with $f_1$ multiplying with the natural number $m$, so $F_* \otimes G$ is the complex

$$0 \rightarrow G \xrightarrow{m} G \rightarrow 0$$

considered in degrees 1 and 0. The homology in degree 0 is the tensor product $G \otimes \mathbb{Z}/m \cong G/mG$, while the homology in degree 1 is the Tor-group

$$\text{Tor}(\mathbb{Z}/m, G) = \ker(m: G \rightarrow G) = G[m].$$

Here we use the notation

$$G[m] = \{ x \in G \mid mx = 0 \}$$
for the exponent $m$-torsion in $G$. Another common notation for this subgroup is $mG$.

If $G$ is free abelian, so that $(-) \otimes G$ is exact, then $f_1 \otimes 1$ is injective, so that $\text{Tor}(C, G) = 0$ for all $C$. More generally, we say that $G$ is flat if $(-) \otimes G$ is exact, and then $\text{Tor}(C, G) = 0$ for all $C$. An abelian group $G$ is flat if and only if it is torsion free. For instance, $G = \mathbb{Q}$ is a torsion free group that is not free.

Returning to a general short free resolution $0 \to F_1 \to F_0 \to C \to 0$, the Ext-groups are the cohomology groups of the cochain complex

$$0 \to \text{Hom}(F_0, G) \xrightarrow{\text{Hom}(f_1, 1)} \text{Hom}(F_1, G) \to 0.$$ 

In view of the left exactness of $\text{Hom}(-, G)$, we get isomorphisms

$$\text{Ext}_0^Z(C, G) \cong \text{Hom}(C, G)$$

and

$$\text{Ext}_1^Z(C, G) \cong \text{cok}(\text{Hom}(f_1, 1): \text{Hom}(F_0, G) \to \text{Hom}(F_1, G)).$$

(Referring to the image of $\text{Tor}$- and Ext-groups and their functoriality.)

Theorem 2.4.1 (Universal coefficient theorem). Let $(X, A)$ be a pair of topological spaces, and let $G$ be an abelian group. There is a natural short exact sequence

$$0 \to H_n(X, A) \otimes G \xrightarrow{\alpha} H_n(X, A; G) \to \text{Tor}(H_{n-1}(X, A), G) \to 0$$

for each $n$. The sequence is split, but not naturally split.
Proposition 2.4.2. Let \((C_*, \partial)\) be a chain complex of free abelian groups, and let \(G\) be an abelian group. There is a natural short exact sequence
\[
0 \to H_n(C_*) \otimes G \xrightarrow{\alpha} H_n(C_*) \otimes G \to \text{Tor}(H_{n-1}(C_*), G) \to 0
\]
for each \(n\). The sequence is split, but not naturally split.

**Proof.** Let \(B_n = \text{im}(\partial) \subseteq Z_n = \ker(\partial) \subseteq C_n\), as usual. Since each \(C_n\) is free, so is each subgroup \(B_n\) and \(Z_n\). For each \(n\) there is a short exact sequence
\[
0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0.
\]
Since \(B_{n-1}\) is free, this sequence splits. Tensoring with \(G\), we get a (split) short exact sequence
\[
0 \to Z_n \otimes G \to C_n \otimes G \xrightarrow{\partial} B_{n-1} \otimes G \to 0
\]
for each \(n\), where we abbreviate \(\partial \otimes 1\) to \(\partial\). These fit together in a commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & Z_{n+1} \otimes G & \xrightarrow{\partial} & C_{n+1} \otimes G & \xrightarrow{\partial} & B_n \otimes G & \to 0 \\
0 & \to & Z_n \otimes G & \xrightarrow{\partial} & C_n \otimes G & \xrightarrow{\partial} & B_{n-1} \otimes G & \to 0 \\
0 & \to & Z_{n-1} \otimes G & \xrightarrow{\partial} & C_{n-1} \otimes G & \xrightarrow{\partial} & B_{n-2} \otimes G & \to 0 \\
& \vdots & & & & & \vdots \\
\end{array}
\]

which we view as a short exact sequence of chain complexes
\[
0 \to (Z_*, \otimes G, 0) \to (C_*, \otimes G, \partial) \xrightarrow{\partial} (B_{*-1} \otimes G, 0) \to 0.
\]
Here \((Z_*, \otimes G, 0)\) denotes the chain complex with \(Z_n \otimes G\) in degree \(n\) and zero maps as boundary homomorphisms, while \((B_{*-1} \otimes G, 0)\) denotes the chain complex with \(B_{n-1} \otimes G\) in degree \(n\) and zero boundaries.

The associated long exact sequence in homology contains the terms
\[
B_n \otimes G \xrightarrow{k_n} Z_n \otimes G \to H_n(C_*) \otimes G \to B_{n-1} \otimes G \xrightarrow{k_{n-1}} Z_{n-1} \otimes G.
\]
Hence there is a natural short exact sequence
\[
0 \to \text{cok}(k_n) \xrightarrow{\alpha} H_n(C_*) \otimes G \to \ker(k_{n-1}) \to 0.
\]
Chasing the definition of the connecting homomorphism, we see that \(k_n = \iota_n \otimes 1\) is equal to the tensor product of the inclusion \(\iota_n : B_n \subseteq Z_n\) with \(G\), in each degree \(n\).
By the definition of homology, there is a short exact sequence

\[ 0 \to B_n \xrightarrow{i_n} Z_n \xrightarrow{r_n} H_n(C_*) \to 0. \]

Since \( B_n \) and \( Z_n \) are free, this is a short free resolution \((F_*, f)\) of the homology group \( H_n(C_*) \). In the notation used above, \( F_1 = B_n, F_0 = Z_n \) and \( f_1 = i_n \). Hence the Tor-groups of \( H_n(C_*) \) and \( G \) are the homology groups of the complex \( F_* \otimes G \), so that there is an exact sequence

\[ 0 \to \text{Tor}(H_n(C_*), G) \to B_n \otimes G \xrightarrow{\alpha_{n+1}} Z_n \otimes G \to H_n(C_*) \otimes G \to 0. \]

In other words, there are natural isomorphisms \( \ker(\alpha_n) \cong \text{Tor}(H_n(C_*), G) \) and \( \cok(\alpha_n) \cong H_n(C_*) \otimes G \), for all \( n \). Hence we have a natural short exact sequence

\[ 0 \to H_n(C_*) \otimes G \xrightarrow{\alpha} H_n(C_*) \otimes G \to \text{Tor}(H_{n-1}(C_*), G) \to 0. \]

By inspection of the definitions, the left hand homomorphism \( \alpha \) takes \([x] \otimes g\) in \( H_n(C_*) \otimes G \) to \([x \otimes g]\) in \( H_n(C_* \otimes G) \), for each \( n \)-cycle \( x \in Z_n \).

To see that the universal coefficient short exact sequence admits a splitting, first choose a retraction \( r: C_n \to Z_n \) in the short exact sequence \( 0 \to Z_n \to C_n \to B_{n-1} \to 0 \). The composite \( \epsilon_n \circ r: C_n \to Z_n \to H_n(C_*) \) defines a chain map \((C_*, \partial) \to (H_*(C_*), 0)\), since \( r \) restricts to the identity on \( Z_n \), so that \( \epsilon_n \circ r \) is zero on \( B_n \). Tensoring with \( G \) we get a chain map

\[ (C_* \otimes G, \partial) \to (H_*(C_*) \otimes G, 0) \]

and an induced map in homology

\[ H_n(C_*; G) \to H_n(C_*) \otimes G \]

for each \( n \). This is a retraction for the universal coefficient sequence. \( \square \)

Note that we do not claim that the rejections \( r \) define a chain map \((C_*, \partial) \to (Z_*, 0)\).

### 2.5 The universal coefficient theorem in cohomology

**Theorem 2.5.1 (Universal coefficient theorem).** Let \((X, A)\) be a pair of topological spaces, and let \( G \) be an abelian group. There is a natural short exact sequence

\[ 0 \to \text{Ext}(H_{n-1}(X, A), G) \to H^n(X, A; G) \xrightarrow{\beta} \text{Hom}(H_n(X, A), G) \to 0 \]

for each \( n \). The sequence is split, but not naturally split.

Again, this follows from the following result in the case \( C_* = C_*(X, A) \).

**Proposition 2.5.2.** Let \((C_*, \partial)\) be a chain complex of free abelian groups, and let \( G \) be an abelian group. There is a natural short exact sequence

\[ 0 \to \text{Ext}(H_{n-1}(C_*), G) \to H^n(\text{Hom}(C_*), G) \xrightarrow{\beta} \text{Hom}(H_n(C_*), G) \to 0 \]

for each \( n \). The sequence is split, but not naturally split.
For an example of the failure of naturality of the splitting, see Section 3.1, Exercise 11 in [2].

Proof. Let $B_n = \text{im}(\partial) \subseteq \mathbb{Z}_n = \ker(\partial) \subseteq C_n$, as usual. Since each $C_n$ is free, so is each subgroup $B_n$ and $\mathbb{Z}_n$. For each $n$ there is a short exact sequence

$$0 \to \mathbb{Z}_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0.$$ 

Since $B_{n-1}$ is free, this sequence splits. Forming Hom-groups into $G$, we get a (split) short exact sequence

$$0 \to \text{Hom}(B_{n-1}, G) \xrightarrow{\delta} \text{Hom}(C_n, G) \to \text{Hom}(\mathbb{Z}_n, G) \to 0$$

for each $n$, where we abbreviate $\text{Hom}(\partial, 1)$ to $\delta$. These fit together in a commutative diagram

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
0 & \downarrow\delta & 0 \\
0 & \text{Hom}(B_{n-2}, G) & \text{Hom}(C_{n-1}, G) \\
& \downarrow\delta & \downarrow0 \\
0 & \text{Hom}(B_{n-1}, G) & \text{Hom}(C_n, G) \\
& \downarrow\delta & \downarrow0 \\
0 & \text{Hom}(B_n, G) & \text{Hom}(C_{n+1}, G) \\
& \downarrow\delta & \downarrow0 \\
& \vdots & \vdots \\
\end{array}
$$

which we view as a short exact sequence of cochain complexes

$$0 \to (\text{Hom}(B_{n-1}, G), 0) \xrightarrow{\delta} (\text{Hom}(C_n, G), \delta) \to (\text{Hom}(\mathbb{Z}_n, G), 0) \to 0.$$ 

Here $(\text{Hom}(B_{n-1}, G), 0)$ denotes the cochain complex with $\text{Hom}(B_{n-1}, G)$ in degree $n$ and zero maps as coboundary homomorphisms, while $(\text{Hom}(\mathbb{Z}_n, G), 0)$ denotes the cochain complex with $\text{Hom}(\mathbb{Z}_n, G)$ in degree $n$ and zero coboundaries.

The associated long exact sequence in cohomology contains the terms

$$\text{Hom}(\mathbb{Z}_{n-1}, G) \xrightarrow{k_{n-1}} \text{Hom}(B_{n-1}, G) \to H^n(\text{Hom}(C_n, G)) \to$$

$$\to \text{Hom}(\mathbb{Z}_n, G) \xrightarrow{k_n} \text{Hom}(B_n, G).$$

Hence there is a natural short exact sequence

$$0 \to \text{cok}(k_{n-1}) \to H^n(\text{Hom}(C_n, G)) \xrightarrow{\delta} \ker(k_n) \to 0.$$ 

Chasing the definition of the connecting homomorphism, we see that $k_n = \text{Hom}(i_n, 1)$ is Hom-dual to the inclusion $i_n: B_n \subseteq \mathbb{Z}_n$, in each degree $n$. 

By the definition of homology, there is a short exact sequence

$$0 \to B_n \xrightarrow{\epsilon_n} Z_n \xrightarrow{\epsilon_n} H_n(C_\ast) \to 0.$$  

Since $B_n$ and $Z_n$ are free, this is a short free resolution $(F_\ast, f)$ of the homology group $H_n(C_\ast)$. In the notation used above, $F_1 = B_n$, $F_0 = Z_n$ and $f_1 = \iota_n$. Hence the Ext-groups of $H_n(C_\ast)$ and $G$ are the cohomology groups of the cocomplex $\text{Hom}(F_\ast, G)$, so that there is an exact sequence

$$0 \to \text{Hom}(H_n(C_\ast), G) \xrightarrow{\iota_n} \text{Hom}(B_n, G) \xrightarrow{\text{Hom}(\epsilon_n, 1)} \text{Ext}(H_n(C_\ast), G) \to 0.$$  

In other words, there are natural isomorphisms $\ker(k_n) \cong \text{Hom}(H_n(C_\ast), G)$ and $\cok(k_n) \cong \text{Ext}(H_n(C_\ast), G)$, for all $n$. Hence we have a natural short exact sequence

$$0 \to \text{Ext}(H_{n-1}(C_\ast), G) \to H^n(\text{Hom}(C_\ast, G)) \xrightarrow{\beta} \text{Hom}(H_n(C_\ast), G) \to 0.$$  

By inspection of the definitions, the right hand homomorphism $\beta$ takes $[\varphi]$ in $H^n(\text{Hom}(C_\ast, G))$ to the homomorphism mapping $[x]$ in $H_n(C_\ast)$ to $\varphi(x)$ in $G$.

To see that the universal coefficient short exact sequence admits a splitting, first choose a retraction $r: C_n \to Z_n$ in the short exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$. The composite $\epsilon_n \circ r: C_n \to Z_n \to H_n(C_\ast)$ defines a chain map $(C_\ast, \partial) \to (H_\ast(C_\ast), 0)$, since $r$ restricts to the identity on $Z_n$, so that $\epsilon_n \circ r$ is zero on $B_n$. Hom'ing into $G$ we get a cochain map

$$(\text{Hom}(H_\ast(C_\ast), G), 0) \to (\text{Hom}(C_\ast, G), \delta)$$

and an induced map in cohomology

$$\text{Hom}(H_n(C_\ast), G) \to H^n(\text{Hom}(C_\ast, G))$$

for each $n$. This is a section for the universal coefficient sequence.  

2.6 Some calculations

Consider $X = \mathbb{R}P^3$ with the minimal CW structure, having one cell $e^n$ in each dimension for $0 \leq n \leq 3$. The cellular complex $C_\ast = C_{CW}^*(\mathbb{R}P^3)$ is

$$0 \to \mathbb{Z}\{e^3\} \xrightarrow{0} \mathbb{Z}\{e^2\} \xrightarrow{2} \mathbb{Z}\{e^1\} \xrightarrow{0} \mathbb{Z}\{e^0\} \to 0.$$  

Hence the integral homology groups are

$$H_n(\mathbb{R}P^3) = \begin{cases} 
\mathbb{Z} & \text{for } n = 0 \\
\mathbb{Z}/2 & \text{for } n = 1 \\
\mathbb{Z} & \text{for } n = 2 \\
0 & \text{for } n > 3.
\end{cases}$$
It follows that

\[ H_n(\mathbb{R}P^3) \otimes G = \begin{cases} 
G & \text{for } n = 0, n = 3 \\
G/2G & \text{for } n = 1 \\
0 & \text{for } n = 2, n > 3
\end{cases} \]

and

\[ \text{Tor}(H_n(\mathbb{R}P^3) \otimes G) = \begin{cases} 
G[2] & \text{for } n = 1 \\
0 & \text{otherwise.}
\end{cases} \]

Hence

\[ H_n(\mathbb{R}P^3; G) \cong \begin{cases} 
G & \text{for } n = 0 \\
G/2G & \text{for } n = 1 \\
G[2] & \text{for } n = 2 \\
G & \text{for } n = 3 \\
0 & \text{for } n > 3.
\end{cases} \]

Similarly,

\[ \text{Hom}(H_n(\mathbb{R}P^3), G) = \begin{cases} 
G & \text{for } n = 0, n = 3 \\
G[2] & \text{for } n = 1, n > 3 \\
0 & \text{for } n = 2
\end{cases} \]

and

\[ \text{Ext}(H_n(\mathbb{R}P^3), G) = \begin{cases} 
G/2G & \text{for } n = 1 \\
0 & \text{otherwise.}
\end{cases} \]

Hence

\[ H^n(\mathbb{R}P^3; G) \cong \begin{cases} 
G & \text{for } n = 0 \\
G[2] & \text{for } n = 1 \\
G/2G & \text{for } n = 2 \\
G & \text{for } n = 3 \\
0 & \text{for } n > 3.
\end{cases} \]

For instance,

\[ H^n(\mathbb{R}P^3; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{for } n = 0 \\
0 & \text{for } n = 1 \\
\mathbb{Z}/2 & \text{for } n = 2 \\
\mathbb{Z} & \text{for } n = 3 \\
0 & \text{for } n > 3.
\end{cases} \]

Notice how the torsion in \( H_n(X) \) is shifted up to \( H^{n+1}(X; \mathbb{Z}) \), while the free part of \( H_n(X) \) reappears in \( H^n(X; \mathbb{Z}) \).

Since \( H_0(X) \) is always free, \( \text{Tor}(H_0(X), G) = 0 \) and \( \text{Ext}(H_0(X), G) = 0 \), so for \( n \leq 1 \) there are isomorphisms

\[ \alpha : H_n(X) \otimes G \xrightarrow{\cong} H_n(X; G) \]

and

\[ \beta : H^n(X; G) \xrightarrow{\cong} \text{Hom}(H_n(X), G). \]

Similarly for relative (co-)homology.
CHAPTER 2. THE UNIVERSAL COEFFICIENT THEOREMS

The group $\mathbb{Q}$ is torsion free and divisible, so $\text{Tor}(H_{n-1}(X), \mathbb{Q}) = 0$ and $\text{Ext}(H_{n-1}(X), \mathbb{Q}) = 0$, and there are isomorphisms

$$\alpha: H_n(X) \otimes \mathbb{Q} \xrightarrow{\cong} H_n(X; \mathbb{Q})$$

and

$$\beta: H^n(X; \mathbb{Q}) \xrightarrow{\cong} \text{Hom}(H_n(X), \mathbb{Q})$$

for all $n$. It follows that there is an isomorphism

$$H^n(X; \mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}}(H_n(X; \mathbb{Q}), \mathbb{Q})$$

identifying $H^n(X; \mathbb{Q})$ with the dual $\mathbb{Q}$-vector space of $H_n(X; \mathbb{Q})$.

If $H_*(X)$ is of finite type, meaning that $H_n(X)$ is finitely generated for each $n$, then we can write

$$H_n(X) = T_n \oplus F_n$$

where $T_n$ is a finite abelian group, and $F_n$ is a finitely generated free abelian group. If $F_n \cong \mathbb{Z}^r$ we say that $H_n(X)$ has rank $r$. Note that $T_n \otimes \mathbb{Q} = 0$ and $\mathbb{Z}^r \otimes \mathbb{Q} = \mathbb{Q}^r$, so the rank of $H_n(X)$ equals the dimension of $H_n(X; \mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space. The rank of $H_n(X)$ is also known as the $n$-th Betti number of $X$.

2.7 Field coefficients

((Recall reduced homology.))

**Proposition 2.7.1.** Let $X$ be any space. Then $\tilde{H}_*(X) = 0$ if and only if $\tilde{H}_*(X; \mathbb{Q}) = 0$ and $\tilde{H}_*(X; \mathbb{Z}/p) = 0$ for all primes $p$.

**Corollary 2.7.2.** A map $f: X \to Y$ induces isomorphisms

$$f_*: H_*(X) \xrightarrow{\cong} H_*(Y)$$

in integral homology if and only if it induces isomorphisms

$$f_*: H_*(X; F) \xrightarrow{\cong} H_*(Y; F)$$

with coefficients in the fields $F = \mathbb{Q}$ and $F = \mathbb{Z}/p$, for all primes $p$.

This follows by passage to the mapping cone $C_f$, using the long exact sequence

$$\cdots \to H_n(X; G) \xrightarrow{f_*} H_n(Y; G) \to \tilde{H}_n(C_f; G) \to \cdots$$

for $G = \mathbb{Z}$ and $G = F$.

**Proof.** The forward implication is clear from the universal coefficient theorem in homology. For the converse, assume that $\tilde{H}_*(X; \mathbb{Q}) = 0$ and $\tilde{H}_*(X; \mathbb{Z}/p)$ for all primes $p$. From the short exact sequence

$$0 \to \tilde{H}_n(X)/p \to \tilde{H}_n(X; \mathbb{Z}/p) \to \tilde{H}_{n-1}(X)[p] \to 0$$

we deduce that $\tilde{H}_n(X)/p = 0$ and $\tilde{H}_{n-1}(X)/[p] = 0$, so multiplication by $p$ on $\tilde{H}_n(X)$ is an isomorphism. Hence $\tilde{H}_n(X) \to \tilde{H}_n(X) \otimes \mathbb{Q}$, which inverts every prime, is already an isomorphism. But

$$\tilde{H}_n(X) \otimes \mathbb{Q} \cong \tilde{H}_n(X; \mathbb{Q})$$

is zero by assumption, so $\tilde{H}_n(X) = 0$. \qed
Proposition 2.7.3. Let $(X, A)$ be a pair of topological spaces, and let $F$ be a field. There is a natural isomorphism

$$\beta: H^n(X, A; F) \cong \text{Hom}_F(H_n(X, A; F), F) = H_n(X, A; F)^*$$

There is a more general version of the universal coefficient theorems, for a principal ideal domain $R$ and an $R$-module $M$. Replacing $C_n(X)$ by $C_n(X; R)$ one is led to work with a chain complex $C_\ast$ of free $R$-modules. The assumption that $R$ is a PID ensures that the submodules $B_\ast$ and $Z_\ast$ are still free. This leads to the split short exact sequences

$$0 \to H_n(X; R) \otimes_R M \to H_n(X; M) \to \text{Tor}^R_1(H_{n-1}(X; R), M) \to 0$$

and

$$0 \to \text{Ext}^1_R(H_{n-1}(X; R), M) \to H^n(X; M) \to \text{Hom}_R(H_n(X; R), M) \to 0,$$

and similarly for relative (co-)homology. In the case where $R$ is a field $F$ the derived functors $\text{Tor}^F_1$ and $\text{Ext}^1_F$ vanish. This leads to the stated isomorphism.

Corollary 2.7.4. $\hat{H}_\ast(X) = 0$ if and only if $\hat{H}^\ast(X; \mathbb{Q}) = 0$ and $\hat{H}^\ast(X; \mathbb{Z}/p) = 0$ for all primes $p$. 
Chapter 3

Cup product

We turn to a method of introducing a product structure on $C^*(X; R)$ and $H^*(X; R)$, for $R$ a ring, induced from the diagonal map $\Delta: X \to X \times X$.

3.1 The Alexander–Whitney diagonal approximation

Let $k, \ell \geq 0$. Inside the standard $(k + \ell)$-simplex

$$\Delta^{k+\ell} = [v_0, \ldots, v_k, \ldots, v_{k+\ell}]$$

there is a front $k$-simplex

$$\Delta^k \simeq [v_0, \ldots, v_k] \subset \Delta^{k+\ell}$$

where $t_{k+1} = \cdots = t_{k+\ell} = 0$, and a back $\ell$-simplex

$$\Delta^\ell \simeq [v_k, \ldots, v_{k+\ell}] \subset \Delta^{k+\ell}$$

where $t_0 = \cdots = t_{k-1} = 0$. These meet in the single vertex $v_k$, where $t_k = 1$. Let $\lambda_{k+\ell}^k: \Delta^k \to \Delta^{k+\ell}$ and $\rho_{k+\ell}^\ell: \Delta^\ell \to \Delta^{k+\ell}$ be the two affine linear embeddings.

To each singular $(k + \ell)$-simplex

$$\sigma: \Delta^{k+\ell} \to X$$

in a topological space $X$, we can associate the front $k$-face

$$\sigma \lambda_{k+\ell}^k = \sigma|[v_0, \ldots, v_k]: \Delta^k \to X$$

and the back $\ell$-face

$$\sigma \rho_{k+\ell}^\ell = \sigma|[v_k, \ldots, v_{k+\ell}]: \Delta^\ell \to X$$

Their tensor product defines a homomorphism

$$\Psi_{k,\ell}: C_{k+\ell}(X) \to C_k(X) \otimes C_\ell(X)$$

that takes $\sigma$ to

$$\sigma \lambda_{k+\ell}^k \otimes \sigma \rho_{k+\ell}^\ell = \sigma|[v_0, \ldots, v_k] \otimes \sigma|[v_k, \ldots, v_{k+\ell}]$$.
For \( k = \ell = 0 \), the homomorphism \( \Psi_{0,0} : C_0(X) \to C_0(X) \otimes C_0(X) \) corresponds to the diagonal map \( \Delta : X \to X \times X \) taking a point \( p \in X \) to \( (p, p) \in X \times X \), under the correspondences \( C_0(X) \cong \mathbb{Z}\{X\} \) and \( C_0(X) \otimes C_0(X) \cong \mathbb{Z}\{X\} \otimes \mathbb{Z}\{X\} \cong \mathbb{Z}\{X \times X\} \).

For two chain complexes \( (C_\ast, \partial) \) and \( (D_\ast, \partial) \), we define the tensor product chain complex \( (C_\ast \otimes D_\ast, \partial) \) to be given in degree \( n \) by

\[
(C \times D)_n = \bigoplus_{k + \ell = n} C_k \otimes D_\ell
\]

with boundary homomorphism given by

\[
\partial(x \otimes y) = \partial x \otimes y + (-1)^k x \otimes \partial y
\]

for \( x \in C_k \) and \( y \in D_\ell \). Note that

\[
\partial^2(x \otimes y) = \partial(\partial x \otimes y + (-1)^k x \otimes \partial y) = \partial^2 x \otimes y + (-1)^{k-1} \partial x \otimes \partial y + (-1)^k \partial x \otimes \partial y + (-1)^{2k-1} x \otimes \partial^2 y = 0
\]

so that \( (C_\ast \otimes D_\ast, \partial) \) is a chain complex. The sign \((-1)^k\) can be justified geometrically, since we are commuting the passage to a boundary past the \( k \)-dimensional object \( k \), or algebraically, to make sure that the two middle terms in the above sum cancel.

For each \( n \geq 0 \), we can form the sum over all \((k, \ell)\) with \( k + \ell = n \) of the homomorphisms \( \Psi_{k,\ell} \) to get the homomorphism

\[
\Psi_n : C_n(X) \to \bigoplus_{k + \ell = n} C_k(X) \otimes C_\ell(X)
\]

taking \( \sigma : \Delta^n \to X \) to

\[
\bigoplus_{k + \ell = n} = \sigma[[v_0, \ldots, v_k]] \otimes \sigma[[v_k, \ldots, v_{k+\ell}]].
\]

**Lemma 3.1.1.** The identity

\[
\Psi_{k,\ell} \circ \partial = (\partial \otimes 1) \Psi_{k+1,\ell} + (-1)^k (1 \otimes \partial) \Psi_{k,\ell+1}
\]

holds, so the homomorphisms \( (\Psi_n)_n \) define a chain map

\[
\Psi_\# : C_\ast(X) \to C_\ast(X) \otimes C_\ast(X).
\]

Since \( \Psi_0 \) is compatible with the diagonal map, we call \( \Psi_\# \) a diagonal approximation.

**Proof.** We must prove that the diagram

\[
\begin{align*}
(C_\ast(X) \otimes C_\ast(X))_{n+1} & \xrightarrow{\partial} (C_\ast(X) \otimes C_\ast(X))_n \\
(C_\ast(X) \otimes C_\ast(X))_{n+1} & \xrightarrow{\Psi_{n+1}} C_{n+1}(X) \\
C_n(X) & \xrightarrow{\Psi_n} C_n(X)
\end{align*}
\]
commutes, for each \( n \). We check that for each \( \sigma : \Delta^{n+1} \to X \) and each pair \((k, \ell)\) with \( k + \ell = n \), the images of \( \sigma \) under \( \Psi_n \partial \) and \( \partial \Psi_{n+1} \) have the same components in \( C_k(X) \otimes C_\ell(X) \).

The \((k, \ell)\)-th component of \( \partial \Psi_n(\sigma) \) is the sum of two contributions. One comes from the composite

\[
(\partial \otimes 1)\Psi_{k+1,\ell} : C_{n+1}(X) \to C_{k+1}(X) \otimes C_\ell(X) \to C_k(X) \otimes C_\ell(X)
\]

and the other comes from the composite

\[
(-1)^k(1 \otimes \partial)\Psi_{k,\ell+1} : C_{n+1}(X) \to C_k(X) \otimes C_{\ell+1}(X) \to C_k(X) \otimes C_\ell(X).
\]

The first takes \( \sigma : \Delta^{n+1} \to X \) to

\[
(\partial \otimes 1)\sigma[[v_0, \ldots, v_{k+1}] \otimes \sigma[[v_{k+1}, \ldots, v_{n+1}]]
= \sum_{i=0}^{k+1} (-1)^i \sigma[[v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}] \otimes \sigma[[v_{k+1}, \ldots, v_{n+1}]]
\]

and the second takes \( \sigma \) to

\[
(-1)^k(1 \otimes \partial)\sigma[[v_0, \ldots, v_k] \otimes \sigma[[v_{k+1}, \ldots, v_{n+1}]]
= (-1)^k \sum_{j=0}^{\ell+1} (-1)^j \sigma[[v_0, \ldots, v_k] \otimes \sigma[[v_{k+j}, \ldots, v_{n+1}]]
= \sum_{i=k}^{n+1} (-1)^i \sigma[[v_0, \ldots, v_k] \otimes \sigma[[v_k, \ldots, \hat{v}_i, \ldots, v_{n+1}]]
\]

Notice that the term \( i = k + 1 \) in the first sum is equal to the term \( i = k \) in the second sum, up to a sign. Hence these two terms cancel when we add the expressions together, so that the \((k, \ell)\)-th component of \( \partial \Psi_n(\sigma) \) is

\[
\sum_{i=0}^{k} (-1)^i \sigma[[v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}] \otimes \sigma[[v_{k+1}, \ldots, v_{n+1}]]
+ \sum_{i=k+1}^{n+1} (-1)^i \sigma[[v_0, \ldots, v_k] \otimes \sigma[[v_k, \ldots, \hat{v}_i, \ldots, v_{n+1}]].
\]

On the other hand, the \((k, \ell)\)-th component of \( \Psi_n \partial(\sigma) \) is

\[
\Psi_{k,\ell}(\sum_{i=0}^{n+1} (-1)^i \sigma[[v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}]]
= \sum_{i=0}^{k} (-1)^i \sigma[[v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}] \otimes \sigma[[v_{k+1}, \ldots, v_{n+1}]]
+ \sum_{i=k+1}^{n+1} (-1)^i \sigma[[v_0, \ldots, v_k] \otimes \sigma[[v_k, \ldots, \hat{v}_i, \ldots, v_{n+1}]].
\]

These expressions are the same, proving the claim. \qed
3.2 The cochain cup product

Let $R$ be a ring, and consider cochains and cohomology with coefficients in $R$.

The cochain cup product is a pairing

$$C^k(X; R) \otimes_R C^\ell(X; R) \xrightarrow{\cup} C^{k+\ell}(X; R).$$

For cochains $\varphi: C_k(X) \to R$ and $\psi: C_\ell(X) \to R$ the cup product is defined to be the $(k + \ell)$-cochain $\varphi \cup \psi: C_{k+\ell}(X) \to R$ given as the composite

$$C_{k+\ell}(X) \xrightarrow{\Psi_{k,\ell}} C_k(X) \otimes C_\ell(X) \xrightarrow{\varphi \otimes \psi} R \otimes R \xrightarrow{\cdot} R$$

where $\Psi_{k,\ell}$ is as in the previous subsection and $\cdot: R \otimes R \to R$ is the ring multiplication.

More explicitly, the cup product $\varphi \cup \psi$ takes the value

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma[v_0, \ldots, v_k]) \cdot \varphi(\sigma[v_k, \ldots, v_{k+\ell}])$$
on a $(k + \ell)$-simplex $\sigma: \Delta^{k+\ell} \to X$.

**Lemma 3.2.1.** The cochain cup product is unital and associative, with unit element $1 \in C^0(X; R)$ the cochain $\epsilon: C_0(X) \to R$ that sends each 0-simplex to the ring unit 1.

A graded ring is a graded abelian group $A_\ast = (A_n)_n$ with a unital and associative pairing

$$A_k \otimes A_\ell \to A_{k+\ell}$$

for all $k, \ell$, which we can also write as a homomorphism

$$A_\ast \otimes A_\ast \to A_\ast.$$

By the lemma above the cochains $C^\ast(X; R)$ constitute a graded ring.

3.3 The cohomology cup product

The cochain cup product satisfies a Leibniz formula.

**Lemma 3.3.1.** The identity

$$\delta(\varphi \cup \psi) = \delta \varphi \cup \psi + (-1)^k \varphi \cup \delta \psi$$

holds in $C^{k+\ell+1}(X; R)$, for $\varphi \in C^k(X; R)$ and $C^\ell(X; R)$, so the cup product defines a cochain map

$$\cup: C^\ast(X; R) \otimes_R C^\ast(X; R) \to C^\ast(X; R).$$

**Proof.** Let $n = k + \ell$. For each $(n+1)$-simplex $\sigma$, we have

$$\delta(\varphi \cup \psi)(\sigma) = (\varphi \cup \psi)(\partial \sigma) = (\varphi \otimes \psi)(\Psi_{k,\ell} \circ \partial)(\sigma)$$
which by the lemma of the previous subsection is the sum of
\[(\varphi \otimes \psi)(\partial \otimes 1)\Psi_{k+1,\ell}(\sigma) = (\delta \varphi \otimes \psi)\Psi_{k+1,\ell}(\sigma) = (\delta \varphi \cup \psi)(\sigma)\]
and
\[(-1)^k(\varphi \otimes \psi)(1 \otimes \partial)\Psi_{k,\ell+1}(\sigma) = (-1)^k(\varphi \otimes \delta \psi)\Psi_{k,\ell+1}(\sigma) = (-1)^k(\varphi \cup \delta \psi)(\sigma)\].

\[\blacksquare\]

**Corollary 3.3.2.** If \(\varphi \in C^k(X; R)\) and \(\psi \in C^\ell(X; R)\) are cocycles, then \(\varphi \cup \psi \in C^{k+\ell}(X; R)\) is a cocycle. If furthermore \(\varphi\) is a coboundary, or \(\psi\) is a coboundary, then \(\varphi \cup \psi\) is a coboundary.

**Proof.** If \(\delta \varphi = 0\) and \(\delta \psi = 0\) then \(\delta(\varphi \cup \psi) = 0 \cup \psi + (-1)^k \varphi \cup 0 = 0\). If also \(\varphi = \delta \xi\) then \(\delta(\xi \cup \psi) = \varphi \cup \psi + \xi \cup 0 = \varphi \cup \psi\). If instead \(\psi = \delta \eta\) then \(\delta(\varphi \cup \eta) = 0 \cup \eta + (-1)^k \varphi \cup \psi = (-1)^k \varphi \cup \psi\).

The cohomology cup product is the induced pairing
\[H^k(X; R) \otimes_R H^\ell(X; R) \xrightarrow{\cup} H^{k+\ell}(X; R)\]
given by the formula
\[[\varphi] \cup [\psi] = [\varphi \cup \psi]\]
for each \(k\)-cocycle \(\varphi\) and each \(\ell\)-cocycle \(\psi\). It is well-defined by the corollary above. The cup product makes \(H^*(X; R)\) a graded ring.

**Lemma 3.3.3.** The cohomology cup product is unital and associative, with unit element \(1 \in H^0(X; R)\) the cohomology class of the cocycle \(\epsilon: C_0(X) \to R\) that sends each 0-simplex to the ring unit 1.

A cup product for simplicial cohomology can be defined by the same formula as for singular cohomology. Hence the isomorphism between singular cohomology and simplicial cohomology is compatible with the cup products, so that for simplicial complexes, or more generally, for \(\Delta\)-complexes, the cup products in singular cohomology can be computed using simplicial cochains.

**Example 3.3.4.** The closed orientable surface \(M_g\) of genus \(g \geq 1\) has a triangulation as a \(\Delta\)-complex obtained by triangulating a regular \(4g\)-gon by starring with an interior point, and identifying the boundary edges pairwise according to the pattern
\[a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}\]
The integral homology groups are \(H_0(M_g) = \mathbb{Z}\), \(H_1(M_g) = \mathbb{Z}\{a_1, b_1, \ldots, a_g, b_g\}\) and \(H_2(M_g) \cong \mathbb{Z}\). A generator of \(H_2(M_g)\) is represented by the 2-cycle given by the signed sum of all of the 2-simplices in the triangulation, with sign +1 for the 2-simplices spanned by the center and one of the positively oriented edges \(a_i\) or \(b_i\), and sign −1 for the 2-simplices spanned by the center and one of the negatively oriented edges \(a_i^{-1}\) or \(b_i^{-1}\).
Dually, the integral cohomology groups are
\[ H^0(M_g) = H^0(M_g; \mathbb{Z}) = \mathbb{Z}\{1\}, \]
\[ H^1(M_g) = H^1(M_g; \mathbb{Z}) = \mathbb{Z}\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\} \]
and \[ H^2(M_g) = H^2(M_g; \mathbb{Z}) \cong \mathbb{Z}\{\gamma\}, \]
with \(\alpha_i\) dual to \(a_i\) and \(\beta_i\) dual to \(b_i\). ((Recall what duality means for a basis.))

A generator \(\gamma \in H^2(M_g)\) is represented by a 2-cochain/cocycle that takes the value +1 on the 2-cycle representing a generator of \(H_2(M_g)\). The interesting cup product is the pairing
\[ \cup : H^1(M_g) \otimes H^1(M_g) \to H^2(M_g). \]

To compute cup products, we must represent the cohomology classes \(\alpha_i\) and \(\beta_j\) by 1-cocycles, say \(\varphi_i\) and \(\psi_j\).

The condition \(\delta \varphi_i = 0\) asserts that the alternating sum of values of \(\varphi_i\) on the three edges of each 2-simplex in \(M_g\) must be 0. To represent \(\alpha_i\), \(\varphi_i\) must evaluate to 1 on the edge \(a_i\). By inspection, we can let \(\varphi_i\) evaluate to 1 on the two edges leading from the center to the end-points of \(a_i\) and \(a_i^{-1}\), and to 0 on all other edges.

Similarly, \(\psi_j\) evaluates to 1 on the edge \(b_j\), as well as on the two edges leading from the center to the end-points of \(b_j\) and \(b_j^{-1}\), and to 0 on all other edges.

The cup product of two 1-cocycles \(\varphi\) and \(\psi\) is the 2-cocycle whose value on a 2-simplex \(\sigma\) is the product
\[ (\varphi \cup \psi)(\sigma) = \varphi(\sigma[v_0, v_1]) \cdot \psi(\sigma[v_1, v_2]). \]

The 2-simplices of \(M_g\) fall into \(g\) groups of four triangles each. The cocycles \(\varphi_i\) and \(\psi_i\) are zero outside of the \(i\)-th group, so if \(i \neq j\) the cup product of \(\varphi_i\) or \(\psi_i\) with \(\varphi_j\) or \(\psi_j\) is zero on all 2-simplices. Hence these cup products are zero on the simplicial cochain level, and
\[ a_i \cup a_j = a_i \cup b_j = b_i \cup a_j = b_i \cup b_j = 0 \]
for \(i \neq j\).

Fortunately, the case \(i = j\) is more interesting. The cup product \(\varphi_i \cup \psi_i\) takes the value
\[ \varphi_i(a_i) \cdot \psi_i(b_i) = 1 \cdot 1 = 1 \]
on the 2-simplex spanned by the center and the edge \(b_i\), and is zero on the other 2-simplices. Hence this cup product evaluates to +1 on the 2-cycle representing the generator of \(H_2(M_g)\), so the cohomology cup product
\[ \alpha_i \cup \beta_i = \gamma \]
equals the dual generator of \(H^2(M_g)\).

The cup product \(\psi_i \cup \varphi_i\) takes the value
\[ \psi_i(b_i) \cdot \varphi_i(a_i) = 1 \cdot 1 = 1 \]
on the 2-simplex spanned by the center and the edge \(a_i^{-1}\), and is zero on the other 2-simplices. Hence this cup product evaluates to −1 on the 2-cycle representing the generator of \(H_2(M_g)\), so the cohomology cup product
\[ \beta_i \cup \alpha_i = -\gamma \]
equals the negative of the dual generator of $H^2(M_g)$.

The cup products $\varphi_i \cup \varphi_i$ and $\psi_i \cup \psi_i$ are zero on all 2-simplices, so the cohomology cup products $\alpha_i \cup \alpha_i$ and $\beta_i \cup \beta_i$ are both zero.

The bilinear pairing $H^1(M_g) \times H^1(M_g) \to H^2(M_g)$ is thus identified with the bilinear pairing $\mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}$ represented by the skew-symmetric $2g \times 2g$ matrix

\[
\begin{bmatrix}
0 & 1 & \ldots & 0 & 0 \\
-1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & -1 & 0
\end{bmatrix}
\]

with $g$ copies of the hyperbolic form $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ along the diagonal, and zeroes elsewhere.

With a different choice of basis, the cup product pairing corresponds to a different matrix. The natural choice made corresponds to a matrix that has as many vanishing entries as is possible. We assumed that $g \geq 1$. The conclusion holds as stated in the case $g = 0$ with $M_0 = S^2$, in a somewhat trivial way.

### 3.4 Relative cup products, naturality

Let $A, B \subseteq X$ be subspaces. If $\varphi \in C^k(X, A; R)$ vanishes on chains in $A$ and $\psi \in C^\ell(X, B; R)$ vanishes on chains in $B$, then

$\varphi \cup \psi \in C^{k+\ell}(X, A + B; R)$

vanishes on chains in $A$ or in $B$. Hence there is a relative cup product

$H^k(X, A; R) \otimes_R H^\ell(X, B; R) \xrightarrow{\cup} H^{k+\ell}(X, A + B; R)$.

If $\{A, B\}$ is excisive, so that $H_*(A + B) \to H_*(A \cup B)$ is an isomorphism, then $H^*(X, A + B; R) \to H^*(X, A; R) \otimes_R H^*(X, B; R)$ is an isomorphism by the long exact sequence and universal coefficient theorem. (This applies, for instance, when $A$ and $B$ are open subsets, or $X$ is a CW complex and $A$ and $B$ are subcomplexes.) Then the cup product lifts through the isomorphism to

$H^k(X, A; R) \otimes_R H^\ell(X, B; R) \xrightarrow{\cup} H^{k+\ell}(X, A \cup B; R)$.

Some important special cases are the relative cup products

$H^k(X, A; R) \otimes_R H^\ell(X, R) \xrightarrow{\cup} H^{k+\ell}(X, A; R)$

$H^k(X; R) \otimes_R H^\ell(X, A; R) \xrightarrow{\cup} H^{k+\ell}(X, A; R)$

$H^k(X, A; R) \otimes_R H^\ell(X, A; R) \xrightarrow{\cup} H^{k+\ell}(X, A; R)$

for all pairs $(X, A)$.

For each map $f : X \to Y$ the cup product satisfies

$f^\#(\varphi \cup \psi) = f^\#(\varphi) \cup f^\#(\psi)$.
in $C^{k+\ell}(X; R)$, for $\varphi \in C^k(Y; R)$ and $\psi \in C^\ell(Y; R)$, so the cochain cup product is natural in the sense that the diagram

$$
\begin{array}{ccc}
C^k(Y; R) \otimes_R C^\ell(Y; R) & \xrightarrow{\cup} & C^{k+\ell}(Y; R) \\
\downarrow f^# \otimes f^# & & \downarrow f^# \\
C^k(X; R) \otimes_R C^\ell(X; R) & \xrightarrow{\cup} & C^{k+\ell}(X; R)
\end{array}
$$

commutes. It follows that the cohomology cup product satisfies

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

in $H^{k+\ell}(X; R)$, for $\alpha \in H^k(X; R)$ and $\beta \in H^\ell(X; R)$, so the cohomology cup product is natural in the same way. Hence the graded ring $H^*(X; R)$ is functorial for all spaces $X$. Similarly, the relative cohomology ring $H^*(X, A; R)$ is functorial for all pairs $(X, A)$.

## 3.5 Cross product

For a pair of spaces $X$ and $Y$, the natural projection maps

$$p_1: X \times Y \longrightarrow X$$

$$p_2: X \times Y \longrightarrow Y$$

induce cochain homomorphisms

$$p_1^#: C^*(X; R) \longrightarrow C^*(X \times Y; R)$$

$$p_2^#: C^*(Y; R) \longrightarrow C^*(X \times Y; R).$$

When combined with the cochain cup product for $X \times Y$ we get a pairing

$$\times: C^k(X; R) \otimes_R C^\ell(Y; R) \xrightarrow{p_1^# \otimes p_2^#} C^k(X \times Y; R) \otimes_R C^\ell(X \times Y; R) \xrightarrow{\cup} C^{k+\ell}(X \times Y; R)$$

called the \textit{cochain cross product}, denoted $\times$. It takes a $k$-cocycle $\varphi: C_k(X) \to R$ on $X$ and an $\ell$-cocycle $\psi: C_\ell(Y) \to R$ on $Y$ to the cup product of their respective pullbacks to $X \times Y$:

$$\varphi \times \psi = p_1^#(\varphi) \cup p_2^#(\psi).$$

Its value on a $(k + \ell)$-simplex $(\sigma, \tau): \Delta^{k+\ell} \to X \times Y$ is, by definition,

$$(\varphi \times \psi)(\sigma, \tau) = \varphi(\sigma|v_0, \ldots, v_k) \cdot \psi(\tau|v_k, \ldots, v_{k+\ell}).$$

\textbf{Lemma 3.5.1.} \textit{The identity}

$$\delta(\varphi \times \psi) = \delta \varphi \times \psi + (-1)^k \varphi \times \delta \psi$$

\textit{holds in $C^{k+\ell+1}(X \times Y; R)$, for $\varphi \in C^k(X; R)$ and $C^\ell(Y; R)$, so the cross product defines a cochain map}

$$\times: C^*(X; R) \otimes_R C^*(Y; R) \longrightarrow C^*(X \times Y; R).$$
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Proof. This follows from the cup product Leibniz formula by naturality:

\[ \delta(\varphi \times \psi) = \delta(p^\#_1(\varphi) \cup p^\#_2(\psi)) \]
\[ = \delta p^\#_1(\varphi) \cup p^\#_2(\psi) + (-1)^k p^\#_1(\varphi) \cup \delta p^\#_2(\psi) \]
\[ = p^\#_1(\delta \varphi) \cup p^\#_2(\psi) + (-1)^k p^\#_1(\varphi) \cup p^\#_2(\delta \psi) \]
\[ = \delta \varphi \times \psi + (-1)^k \varphi \cup \delta \psi \]

\[ \square \]

The *cohomology cross product* is the induced pairing

\[ H^k(X; R) \otimes_R H^\ell(Y; R) \xrightarrow{\times} H^{k+\ell}(X \times Y; R) \]

given by the formula

\[ [\varphi] \times [\psi] = [\varphi \times \psi] \]

for each \( k \)-cocycle \( \varphi \) in \( X \) and each \( \ell \)-cocycle \( \psi \) in \( Y \). It is well-defined by the cross product Leibniz formula. The cross product can be computed in terms of the cup product by the formula

\[ \alpha \times \beta = p^\#_1(\alpha) \cup p^\#_2(\beta) . \]

To compute the cross product in some interesting examples, we must first discuss some of its formal properties.

### 3.6 Relative cross products, naturality

Let \( (X, A) \) and \( (Y, B) \) be pairs. If \( \varphi \in C^k(X, A; R) \) vanishes on chains in \( A \) and \( \psi \in C^\ell(Y, B; R) \) vanishes on chains in \( B \), then \( \varphi \times \psi \in C^{k+\ell}(X \times Y, A \times Y + X \times B; R) \) vanishes on chains in \( A \times Y \) or in \( X \times B \). Hence there is a relative cross product

\[ H^k(X, A; R) \otimes_R H^\ell(Y, B; R) \xrightarrow{\times} H^{k+\ell}(X \times Y, A \times Y + X \times B; R) . \]

If \( A \) and \( B \) are open, or if \( X \) and \( Y \) are CW complexes and \( A \) and \( B \) are subcomplexes, then \( H_*(A \times Y + X \times B) \to H_*(A \times Y \cup X \times B) \) is an isomorphism, so \( H^*(X \times Y, A \times Y \cup X \times B; R) \cong H^*(X \times Y, A \times Y + X \times B; R) \). Then the cross product lifts to

\[ H^k(X, A; R) \otimes_R H^\ell(Y, B; R) \xrightarrow{\times} H^{k+\ell}(X \times Y, A \times Y \cup X \times B; R) . \]

The target group is often written as \( H^{k+\ell}((X, A) \times (Y, B); R) \), using the notation

\[ (X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B) . \]

As regards naturality, for each pair of maps \( f: X \to X' \) and \( g: Y \to Y' \) there is a map \( f \times g: X \times Y \to X' \times Y' \), and the cochain cross product satisfies

\[ (f \times g)^\#(\varphi \times \psi) = f^\#(\varphi) \times g^\#(\psi) \]
in $C^{k+\ell}(X \times Y; R)$, for $\varphi \in C^k(X'; R)$ and $\psi \in C^\ell(Y'; R)$. Hence the cohomology cross product satisfies

$$(f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times g^*(\beta)$$

in $H^{k+\ell}(X \times Y; R)$, for $\alpha \in H^k(X'; R)$ and $\beta \in H^\ell(Y'; R)$, and the diagram

$$
\begin{array}{ccc}
H^k(X'; R) \otimes_R H^\ell(Y'; R) & \xrightarrow{\times} & H^{k+\ell}(X' \times Y'; R) \\
\downarrow f^* \otimes g^* & & \downarrow (f \times g)^* \\
H^k(X; R) \otimes_R H^\ell(Y; R) & \xrightarrow{\times} & H^{k+\ell}(X \times Y; R)
\end{array}
$$

commutes. Similarly, the relative cross product is natural for all pairs $(X, A)$ and $(Y, B)$.

Note that the cup product can be recovered from the cross product, by pullback along the diagonal map $\Delta: X \to X \times X$. The composite

$$H^\ell(X; R) \otimes_R H^\ell(X; R) \xrightarrow{\times} H^{k+\ell}(X \times X; R) \xrightarrow{\Delta^*} H^{k+\ell}(X; R)$$

is equal to the cup product, since $p_1\Delta = 1 = p_2\Delta$, so $\Delta^*p_1^* = 1 = \Delta^*p_2^*$ and

$$\Delta^*(\alpha \times \beta) = \Delta^*(p_1^*(\alpha) \cup p_2^*(\beta)) = \Delta^*p_1^*(\alpha) \cup \Delta^*p_2^*(\beta) = \alpha \cup \beta.$$

The same result holds at the cochain level.

Naturality with respect to the connecting homomorphisms is a bit more subtle.

**Lemma 3.6.1.** For all pairs $(X, A)$ and spaces $Y$ the natural square

$$
\begin{array}{ccc}
H^k(A; R) \otimes_R H^\ell(Y; R) & \xrightarrow{\times} & H^{k+\ell}(A \times Y; R) \\
\downarrow \delta \otimes 1 & & \downarrow \delta \\
H^{k+1}(X, A; R) \otimes_R H^\ell(Y; R) & \xrightarrow{\times} & H^{k+\ell+1}(X \times Y, A \times Y; R)
\end{array}
$$

commutes, so

$$\delta(\alpha \times \eta) = \delta \alpha \times \eta$$

for $\alpha \in H^k(A; R)$ and $\eta \in H^\ell(Y; R)$.


**Lemma 3.6.2.** For all spaces $X$ and pairs $(Y, B)$ the natural square

$$
\begin{array}{ccc}
H^k(X; R) \otimes_R H^\ell(B; R) & \xrightarrow{\times} & H^{k+\ell}(X \times B; R) \\
\downarrow 1 \otimes \delta & & \downarrow \delta \\
H^k(X; R) \otimes_R H^{\ell+1}(Y, B; R) & \xrightarrow{\times} & H^{k+\ell+1}(X \times Y, X \times B; R)
\end{array}
$$

commutes up to the sign $(-1)^k$, so

$$\delta(\xi \times \beta) = (-1)^k \xi \times \delta \beta$$

for $\xi \in H^k(X; R)$ and $\beta \in H^\ell(B; R)$.  

Proof. Let \( \varphi \in C^k(X; R) \) and \( \psi \in C^\ell(B; R) \) be cocycles representing \( \xi \) and \( \eta \), respectively. Choose an extension \( \tilde{\varphi} \in C^k(Y; R) \) of \( \varphi \). Then \( \varphi \times \tilde{\varphi} \in C^{k+\ell}(X \times Y; R) \) is an extension of \( \varphi \times \psi \in C^{k+\ell}(X \times B) \), and \( \delta(\xi \times \beta) \) is represented by \( \delta(\varphi \times \tilde{\varphi}) \in \delta(\varphi \times \psi) \in C^{k+\ell+1}(X \times Y, X \times B; R) \). Since \( \varphi \) is a cocycle of degree \( k \), this equals \( (-1)^k \varphi \times \delta \tilde{\varphi} \) by the Leibniz formula, which represents \( (-1)^k \xi \circ \delta \beta \). \( \square \)

Recall that \( H^1(I, \partial I; R) = R\{\alpha\} \), where \( \alpha \) is dual to the generator of \( H_1(I, \partial I; R) \) represented by the 1-cycle \( \Delta^1 \cong I \). The other cohomology groups \( H^m(I, \partial I; R) \) vanish.

Lemma 3.6.3. Let \( Y \) be any space. The cross product

\[
H^1(I, \partial I; R) \otimes_R H^{n-1}(Y; R) \xrightarrow{\times} H^n(I \times Y, \partial I \times Y; R)
\]

is an isomorphism. Hence each element of \( H^n(I \times Y, \partial I \times Y; R) \) can be written uniquely as \( \alpha \times \beta \), where \( \beta \in H^{n-1}(Y; R) \). Similarly for pairs \((Y, B)\).

Proof. The long exact sequence in cohomology for the pair \((I \times Y, \partial I \times Y)\) breaks up into short exact sequences, since the inclusion \( \partial I \to I \) admits a section up to homotopy. Similarly for the pair \((I, \partial I)\). By naturality of the cross product, and flatness of \( H^1(I, \partial I; R) = R\{\alpha\} \), we have a map of vertical short exact sequences

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
H^0(I; R) \otimes_R H^{n-1}(Y; R) & \xrightarrow{\times} & H^{n-1}(I \times Y; R) \\
\downarrow & & \downarrow \\
H^0(\partial I; R) \otimes_R H^{n-1}(Y; R) & \xrightarrow{\times} & H^{n-1}(\partial I \times Y; R) \\
\downarrow_{\delta \otimes 1} & & \downarrow_{\delta} \\
H^1(I, \partial I; R) \otimes_R H^{n-1}(Y; R) & \xrightarrow{\times} & H^n(I \times Y, \partial I \times Y; R) \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

It is clear from unitality and a decomposition \( \partial I \times Y \cong Y \sqcup Y \) that the upper and middle cross product maps are isomorphisms, hence so is the lower cross product. \( \square \)

Let \( k, \ell \geq 0 \) and \( n = k + \ell \). Note that \( (I^k, \partial I^k) \times (I^\ell, \partial I^\ell) = (I^n, \partial I^n) \), since \( I^k \times I^\ell = I^n \) and \( \partial I^k \times I^\ell \cup I^k \times \partial I^\ell = \partial I^n \).

Corollary 3.6.4. For \( k, \ell \geq 0 \) and \( n = k + \ell \), the cross product

\[
H^k(I^k, \partial I^k; R) \otimes_R H^\ell(I^\ell, \partial I^\ell; R) \xrightarrow{x} H^n(I^n, \partial I^n; R)
\]

is an isomorphism. Hence \( H^n(I^n, \partial I^n; R) \) is the free \( R \)-module generated by the \( n \)-fold cross product

\[
\alpha \times \cdots \times \alpha
\]

where \( \alpha \in H^1(I, \partial I; R) \) is the standard generator. The remaining cohomology groups \( H^m(I^n, \partial I^n; R) \) are zero.
Recall that $H^0(S^1; R) = R\{1\}$ and $H^1(S^1; R) = R\{\alpha\}$, where $\alpha$ is dual to the generator of $H_1(S^1; R)$ represented by the 1-cycle $\Delta^1 \to \Delta^1/\partial\Delta^1 \cong S^1$.

The other cohomology groups $H^m(S^1; R)$ vanish.

**Proposition 3.6.5.** Let $Y$ be any space. The cross product

$$H^*(S^1; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(S^1 \times Y; R)$$

is an isomorphism, and similarly for pairs $(Y, B)$. Hence each element of $H^n(S^1 \times Y; R)$ can be written uniquely as a sum $\alpha \times \beta + 1 \times \gamma$, with $\beta \in H^{n-1}(Y; R)$ and $\gamma \in H^n(Y; R)$.

**Proof.** We use the pushout square

$$\begin{array}{ccc}
\partial I & \to & I \\
\downarrow & & \downarrow \\
* & \longrightarrow & S^1
\end{array}$$

where $* = \{s_0\}$ is the base-point of $S^1$. The map $(I, \partial I) \to (S^1, *)$ induces a cohomology isomorphism, and similarly when multiplied by $Y$. In view of the commutative square

$$\begin{array}{ccc}
H^1(S^1, *; R) \otimes_R H^{n-1}(Y; R) & \longrightarrow & H^n(S^1 \times Y, * \times Y; R) \\
\downarrow & & \downarrow \\
H^1(I, \partial I; R) \otimes_R H^{n-1}(Y; R) & \longrightarrow & H^n(I \times Y, \partial I \times Y; R)
\end{array}$$

and the previous lemma, it follows that the upper cross product is an isomorphism.

The long exact sequence for the pair $(S^1 \times Y, * \times Y)$ also breaks up, since the inclusion $* \to S^1$ admits a retraction, and we have another map of vertical short exact sequences

$$\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
H^1(S^1, *; R) \otimes_R H^{n-1}(Y; R) & \longrightarrow & H^n(S^1 \times Y, * \times Y; R) \\
\downarrow & & \downarrow \\
[H^*(S^1; R) \otimes_R H^*(Y; R)]^n & \longrightarrow & H^n(S^1 \times Y; R) \\
\downarrow & & \downarrow \\
H^0(*; R) \otimes_R H^0(Y; R) & \longrightarrow & H^n(* \times Y; R) \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}$$

We have seen that the upper cross product is an isomorphism. Since the lower one is obviously an isomorphism, it follows that the middle map is also an isomorphism.\qed
CHAPTER 3. CUP PRODUCT

Example 3.6.6. Let $T^n = S^1 \times \cdots \times S^1$ be the $n$-dimensional torus. The $n$-fold cross product

$$H^*(S^1; R) \otimes_R \cdots \otimes_R H^*(S^1; R) \xrightarrow{\times} H^*(T^n; R)$$

is an isomorphism. Hence $H^k(T^n; R)$ is a free $R$-module with basis the set of $k$-fold cup products

$$\alpha_{i_1} \cup \cdots \cup \alpha_{i_k}$$

for $1 \leq i_1 < \cdots < i_k \leq n$, where $\alpha_i = p_i^*(\alpha) \in H^1(T^n; R)$ is the pullback of the generator $\alpha \in H^1(S^1; R)$ along the $i$-th projection map $p_i: T^n \to S^1$.

This is clear by induction on $n$, using the proposition above, which tells us that a basis is given by the set of $n$-fold cross products

$$\beta_1 \times \cdots \times \beta_n \in H^k(T^n; R)$$

where $k$ of the classes $\beta_i$ are equal to $\alpha$, and the remaining $(n-k)$ of the classes $\beta_i$ are equal to 1. Numbering the $\beta_i$ that are equal to $\alpha$ as $\beta_{i_1}, \ldots, \beta_{i_k}$, we get the asserted formula.

3.7 Projective spaces

Let $\mathbb{R}P^n$ be the $n$-dimensional real projective space, and let $\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n$. Recall that the cellular complex $C^*_{CW}(\mathbb{R}P^n)$ has one generator $e^k$ in each degree $0 \leq k \leq n$, with boundary homomorphism $\partial(e^k) = (1 + (-1)^k)e^{k-1}$. Hence $C^*_{CW}(\mathbb{R}P^n; \mathbb{Z}/2)$ has trivial coboundary, so $H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for each $0 \leq k \leq n$, where the generator in degree $k$ evaluates to 1 in $\mathbb{Z}/2$ on $e^k$.

Proposition 3.7.1.

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$$

and

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x],$$

where $|x| = 1$.

Proof. We simplify notation by writing $P^n$ for $\mathbb{R}P^n$ and omitting the coefficient ring $\mathbb{Z}/2$. By induction on $n$ and naturality with respect to the inclusions $P^{n-1} \to P^n \to P^\infty$, it suffices to prove that the cup product of a generator of $H^{n-1}(P^n)$ and a generator of $H^1(P^n)$ is a generator of $H^n(P^n)$, for $n \geq 2$. It is no more difficult to prove that the cup product

$$H^i(P^n) \otimes H^j(P^n) \xrightarrow{\cup} H^n(P^n)$$

is an isomorphism, for $i + j = n$.

Consider the subspaces $\mathbb{R}^{i+1+0} \subset \mathbb{R}^{i+1+j} \supset \mathbb{R}^{0+1+j}$, which meet in $\mathbb{R}^{0+1+0}$. Passing to the spaces of lines through the origin we have the subspaces $P^n \supset P^i \supset P^j$ meeting in a single point $P^i \cap P^j = \{q\}$. Inside $P^n$ we have an affine $n$-space $\mathbb{R}^n \cong U \subset P^n$ where the $i$-th coordinate is nonzero (counting from 0 to $i + j = n$), whose complement is a copy of $P^{n-1}$. The intersection $U \cap P^n \cong \mathbb{R}^i$ is an affine $i$-space, with complement $P^{n-1}$ in $P^n$. Similarly, the intersection $U \cap P^j \cong \mathbb{R}^j$ is an affine $j$-space, with complement $P^{n-1}$ in $P^j$. 
We have a commutative diagram

\[
\begin{array}{ccc}
H^i(P^n) \otimes H^j(P^n) & \overset{\cup}{\longrightarrow} & H^{i+j}(P^n) \\
\downarrow & & \downarrow \\
H^i(P^n, P^n - P^j) \otimes H^j(P^n, P^n - P^j) & \overset{\cup}{\longrightarrow} & H^{i+j}(P^n, P^n - \{q\}) \\
\simeq & & \simeq \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \otimes H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) & \overset{\cup}{\longrightarrow} & H^{i+j}(\mathbb{R}^n, \mathbb{R}^n - \{q\}) \\
& \overset{p_1 \otimes p_2}{\longrightarrow} & H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \otimes H^j(\mathbb{R}^j, \mathbb{R}^j - \{0\})
\end{array}
\]

The downward arrows are isomorphisms by excision. The cross product is an isomorphism by earlier calculations (replacing \((\mathbb{R}^n, \mathbb{R}^n - \{0\})\) by \((P^n, \partial P^n)\), etc.). The projection \(p_1: (\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \to (\mathbb{R}^i, \mathbb{R}^i - \{0\})\) away from a copy of \(\mathbb{R}^j\) is a homotopy equivalence, and similarly for \(p_2\), so \(p_1^*\) and \(p_2^*\) are isomorphisms. This proves that the middle horizontal cup product is an isomorphism.

The rest is maneuvering from the relative to the absolute case. The complement \(P^n - P^j\) deformation retracts to \(P^{n-1}\), since it consists of points \([x_0 : \cdots : x_n]\) where at least one of the homogeneous coordinates \(x_0, \ldots, x_{i-1}\) is nonzero, and a deformation retraction to the subspace \(P^{n-1}\), where all of the homogeneous coordinates \(x_i, \ldots, x_n\) are zero, is given by the formula

\[
(t, [x_0 : \cdots : x_n]) \mapsto [x_0 : \cdots : x_{i-1} : tx_i : \cdots : tx_n].
\]

Hence the homomorphism \(H^i(P^n, P^n - P^j) \to H^i(P^n)\) factors as the composite

\[
H^i(P^n, P^n - P^j) \to H^i(P^n, P^{n-1}) \to H^i(P^n)
\]

where the first arrow is an isomorphism because of the deformation retraction, and the second arrow is an isomorphism by consideration of the cellular complexes. The same conclusion holds for \(i\) replaced by \(j\) or \(n\). Hence the upper vertical arrows in the big diagram are isomorphisms, so that the upper horizontal cup product is an isomorphism.

Let \(x \in H^1(P^n)\) be the generator. Let

\[
x^n = x \cup \cdots \cup x \in H^n(P^n)
\]

denote the \(n\)-th cup power. By induction on \(n\), we know that \(x^{n-1} \in H^{n-1}(P^n)\) restricts to the generator of \(H^{n-1}(P^{n-1})\), hence is the generator of \(H^{n-1}(P^n)\). By what we have just shown,

\[
x^n = x \cup x^{n-1}
\]

is the generator of \(H^n(P^n)\).

We return to integer coefficients. Let \(\mathbb{C}P^n\) be the \(n\)-dimensional complex projective space, of real dimension \(2n\), and let \(\mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n\). The cellular complex \(C_*^{CW}(\mathbb{C}P^n)\) has one generator \(e^{2k}\) in each even degree \(0 \leq 2k \leq 2n\), with trivial boundary homomorphisms. Hence \(C_*^{CW}(\mathbb{C}P^n)\) has trivial coboundary, so \(H^{2k}(\mathbb{C}P^n) \cong \mathbb{Z}\) for each \(0 \leq k \leq n\), and the other cohomology groups are 0.
Proposition 3.7.2.  
\[ H^*(\mathbb{C}P^n) \cong \mathbb{Z}[y]/(y^{n+1}) \]
and  
\[ H^*(\mathbb{C}P^\infty) \cong \mathbb{Z} \]
where \(|y| = 2\).

Let \(\mathbb{H}P^n\) be the \(n\)-dimensional quaternionic projective space, of real dimension \(4n\), and let \(\mathbb{H}P^\infty = \bigcup_n \mathbb{H}P^n\). The cellular complex \(C^\text{CW}(\mathbb{H}P^n)\) has one generator \(e^{4k}\) in degree \(4k\), for \(0 \leq k \leq n\), and trivial boundary homomorphisms. Hence \(C^\text{CW}^*(\mathbb{H}P^n)\) has trivial coboundary, so \(H^{4k}(\mathbb{H}P^n) \cong \mathbb{Z}\) for each \(0 \leq k \leq n\), and the other cohomology groups are 0.

Proposition 3.7.3.  
\[ H^*(\mathbb{H}P^n) \cong \mathbb{Z}[z]/(z^{n+1}) \]
and  
\[ H^*(\mathbb{H}P^\infty) \cong \mathbb{Z}[z] \]
where \(|z| = 2\).

3.8 Hopf maps

One way to detect whether a map \(f: X \to Y\) is null-homotopic or not is to consider the cup product structure in the cohomology of the mapping cone \(C_f = Y \cup CX\).

\[ \xymatrix{ X \ar[r]^f & Y \ar@{..>}[rr]_r & & C_f } \]

If \(f\) is null-homotopic, then there is a retraction \(r: C_f \to Y\), so that the ring homomorphisms  
\[ H^*(Y) \xrightarrow{r^*} H^*(C_f) \xrightarrow{j^*} H^*(Y) \]
split off \(H^*(Y)\) as a graded subring of \(H^*(C_f)\). Therefore, if \(H^*(Y)\) does not split off from \(H^*(C_f)\), then \(f\) cannot be null-homotopic, i.e., it must be an essential map.

For example, in the CW structure on \(\mathbb{C}P^2\), the 4-cell is attached to the 2-skeleton \(\mathbb{C}P^1 = S^2\) by the complex Hopf map  
\[ \eta: S^3 \to S^2 \]
taking a point in \(S^3 \subset \mathbb{C}^2\) to the complex line that goes though it. The mapping cone is \(C_\eta = \mathbb{C}P^2\). Here  
\[ H^*(\mathbb{C}P^2) = \mathbb{Z}[y]/(y^3) = \mathbb{Z}\{1, y, y^2\} \]
restricts by \(j^*\) to  
\[ H^*(S^2) = \mathbb{Z}[y]/(y^2) = \mathbb{Z}\{1, y\}, \]
but since \(y^2 = 0\) in \(H^*(S^2)\) and \(y^2 \neq 0\) in \(H^*(\mathbb{C}P^2)\) there is no ring homomorphism \(r^*: H^*(S^2) \to H^*(\mathbb{C}P^2)\) that would be a section to \(j^*\). Hence \(\eta\) cannot be null-homotopic.
As a similar example, in the CW structure on $\mathbb{HP}^2$, the 8-cell is attached to the 4-skeleton $\mathbb{HP}^1 = S^H$ by the quaternionic Hopf map

$$\nu: S^7 \to S^4$$

taking a point in $S^7 \subset \mathbb{HP}^2$ to the quaternionic line that goes though it. The mapping cone is $C_\nu = \mathbb{HP}^2$. Here

$$H^*(\mathbb{HP}^2) = \mathbb{Z}[z]/(z^3) = \mathbb{Z}\{1, z, z^2\}$$

restricts by $j^*$ to

$$H^*(S^4) = \mathbb{Z}[z]/(z^2) = \mathbb{Z}\{1, z\},$$

but since $z^2 = 0$ in $H^*(S^4)$ and $z^2 \neq 0$ in $H^*(\mathbb{HP}^2)$ there is no ring homomorphism $r^*: H^*(S^4) \to H^*(\mathbb{HP}^2)$ that would be a section to $j^*$. Hence $\nu$ cannot be null-homotopic.

There is also an octonionic plane, denoted $\mathbb{OP}^2$, with cohomology ring

$$H^*(\mathbb{OP}^2) \cong \mathbb{Z}[w]/(w^3)$$

with $|w| = 8$, and the attaching map

$$\sigma: S^{15} \to S^8$$

is an essential map known as the octonionic Hopf map.

A more careful argument shows that $\eta$ has infinite order in $\pi_3(S^2)$, $\nu$ has infinite order in $\pi_7(S^4)$ and $\sigma$ has infinite order in $\pi_{15}(S^8)$.

### 3.9 Graded commutativity

Recall that

$$H^*(T^2) = \mathbb{Z}\{1, \alpha, \beta, \gamma\}$$

with $|\alpha| = |\beta| = 1$ and $|\gamma| = 2$, with $\alpha \cup \beta = \gamma = -\beta \cup \alpha$. This commutativity up to a sign is typical.

**Theorem 3.9.1.** Let $(X, A)$ be a pair of space and let $R$ be a commutative ring. Then

$$\beta \cup \alpha = (-1)^{k\ell} \alpha \cup \beta$$

in $H^{k+\ell}(X, A; R)$, for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$.

We say that $H^*(X, A; R)$ is graded commutative, or that it is a commutative graded ring. Note that if $H^*(X, A; R)$ is concentrated in even degrees, then the sign $(-1)^{k\ell}$ is always +1.

**Proof.** Let $\rho: C_n(X) \to C_n(X)$ be the (natural) chain map that takes $\sigma: \Delta^n \to X$ to

$$\rho(\sigma) = \epsilon_n \sigma[[v_n, \ldots, v_0]],$$

where $\sigma[[v_n, \ldots, v_0]]$ is the composite of the affine linear map $\rho_n: \Delta^n \to \Delta^n$ that reverses the ordering of the vertices, and $\epsilon_n = (-1)^{n(n+1)/2}$ is the sign of the associated permutation.

((Check that $\partial \rho = \rho \partial$, by calculation.))

There is a (natural) chain homotopy $P: C_n(X) \to C_{n+1}(X)$ from the identity $1$ to $\rho$.

((Define and check.))
CHAPTER 3. CUP PRODUCT

We get an induced chain map \( \rho^* : C^n(X; R) \to C^n(X; R) \) and chain homotopy \( P^* \) from 1 to \( \rho^* \).

Recall the definition of the cochain level cup product of \( \varphi : C_k(X) \to R \) and \( \psi : C_\ell(X) \to R \):

\[
(\varphi \cup \psi)(\sigma) = \varphi(\sigma[v_0, \ldots, v_k]) \cdot \psi(\sigma[v_k, \ldots, v_n])
\]

for \( \sigma : \Delta^n \to X \) as above, \( n = k + \ell \). Then

\[
\rho^*(\psi \cup \varphi)(\sigma) = \epsilon_n(\psi \cup \varphi)(\sigma[v_n, \ldots, v_0])
\]

while

\[
(\rho^* \varphi \cup \rho^* \psi)(\sigma) = \varphi(\epsilon_k(\psi \cup \varphi)(\sigma[v_0, \ldots, v_k])) \cdot \psi(\epsilon_\ell(\psi \cup \varphi)(\sigma[v_k, \ldots, v_n]))
\]

Using the relation \( \epsilon_n = (-1)^{k\ell} \epsilon_k \epsilon_\ell \) and commutativity of \( R \), we get that

\[
\rho^*(\psi \cup \varphi) = (-1)^{k\ell} \rho^* \varphi \cup \rho^* \psi.
\]

Hence, at the level of cohomology groups,

\[
\beta \cup \alpha = [\psi \cup \varphi] = [\rho^*(\psi \cup \varphi)] = (-1)^{k\ell}[\rho^* \varphi \cup \rho^* \psi]
\]

\[
= (-1)^{k\ell} [\rho^* \varphi] \cup [\rho^* \psi] = (-1)^{k\ell} [\varphi] \cup [\psi] = (-1)^{k\ell} \alpha \cup \beta
\]

when \( \varphi \) and \( \psi \) are cocycles representing \( \alpha \) and \( \beta \). \( \square \)

3.10 Tensor products of graded rings

If \( A_* \) and \( B_* \) are graded rings, we define their tensor product \( A_* \otimes B_* \) to be the tensor product of graded abelian groups, with

\[
[A_* \otimes B_*]_n = \bigoplus_{k+\ell=n} A_k \otimes B_\ell
\]

in degree \( n \), with the graded multiplication

\[
[A_* \otimes B_*]_n \otimes [A_* \otimes B_*]_{n'} \longrightarrow [A_* \otimes B_*]_{n+n'}
\]

given by

\[
(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') = (-1)^{k\ell} \alpha \alpha' \otimes \beta \beta'
\]

where \( |\beta| = k \) and \( |\alpha'| = \ell' \). In terms of diagrams, the multiplication on \( A_* \otimes B_* \) is the composite

\[
A_* \otimes B_* \otimes A_* \otimes B_* \xrightarrow{1 \otimes \tau \otimes 1} A_* \otimes A_* \otimes B_* \otimes B_* \xrightarrow{\mu \otimes \mu} A_* \otimes B_* ,
\]

where \( \tau : B_* \otimes A_* \to A_* \otimes B_* \) is the graded twist isomorphism that takes \( \beta \otimes \alpha' \) to \( (-1)^{k\ell} \alpha' \otimes \beta \), with notation as above, and \( \mu : A_* \otimes A_* \to A_* \) and \( \mu : B_* \otimes B_* \to B_* \) are the multiplications in \( A_* \) and \( B_* \).

(Example with products of spheres?)
Chapter 4

Künneth theorems

4.1 A Künneth formula in cohomology

Let \((X, A)\) and \((Y, B)\) be pairs of spaces, and let \(R\) be a commutative ring. Recall the notation \((X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)\).

**Theorem 4.1.1 (Künneth formula).** The cross product

\[
H^*(X, A; R) \otimes_R H^*(Y, B; R) \xrightarrow{x} H^*((X, A) \times (Y, B); R)
\]

is an isomorphisms of graded rings, if \((X, A)\) and \((Y, B)\) are pairs of CW complexes and \(H^\ell(Y, B; R)\) is a finitely generated projective \(R\)-module, for each \(\ell\).

4.2 The Künneth formula in homology

Let \(R\) be a PID, throughout this section.

**Theorem 4.2.1 (Künneth formula).** There is a natural short exact sequence

\[
0 \rightarrow \bigoplus_{k+\ell=n} H_k(X; R) \otimes_R H_\ell(Y; R) \xrightarrow{x} H_{k+\ell}(X \times Y; R) \rightarrow
\]

\[
\bigoplus_{k+\ell=n-1} \text{Tor}_1^R(H_k(X; R), H_\ell(Y; R)) \rightarrow 0
\]

for each \(n\), and these sequences split.

The hypothesis of the following consequence is automatic if \(R\) is a field.

**Corollary 4.2.2.** Suppose that \(H_\ell(Y; R)\) is flat over \(R\), for each \(\ell\). There is a natural isomorphism

\[
\times : H_*(X; R) \otimes_R H_*(Y; R) \xrightarrow{\cong} H_*(X \times Y; R).
\]

One proof of the theorem goes in two parts. One is the Eilenberg–Zilber theorem, relating the chains on \(X \times Y\) to the algebraic tensor product of the chains on \(X\) and the chains on \(Y\). The other is the algebraic Künneth theorem, computing the homology of a tensor product of chain complexes.
CHAPTER 4. KÜNNETH THEOREMS

((Reference to Spanier’s “Algebraic topology” or Mac Lane’s “Homology”.)
Here is the external version of the Alexander–Whitney diagonal approximation.

Definition 4.2.3. The Alexander–Whitney homomorphism

\[ AW^n : C_n(X \times Y; R) \longrightarrow \bigoplus_{k+\ell=n} C_k(X; R) \otimes_R C_\ell(Y; R) \]

takes \((\sigma, \tau) : \Delta^n \to X \times Y\) to

\[ \sigma[v_0, \ldots, v_k] \otimes \tau[v_k, \ldots, v_n]. \]

Theorem 4.2.4 (Eilenberg–Zilber theorem). The Alexander–Whitney homomorphism is a chain homotopy equivalence

\[ AW^\#: C^*(X \times Y; R) \congto C^*(X; R) \otimes_R C^*(Y; R). \]

To prove this, one can construct a chain homotopy inverse

\[ EZ^\#: C^*(X; R) \otimes R C^*(Y; R) \congto C^*(X \times Y; R). \]

This can either be done by the method of acyclic models, or by an explicit formula, known as the Eilenberg–Zilber shuffle homomorphism. ((ETC))

Theorem 4.2.5 (Algebraic K"unneth formula). Let \((C_* , \partial)\) and \((D_* , \partial)\) be chain complexes of free \(R\)-modules. Then there is a natural short exact sequence

\[ 0 \longrightarrow \bigoplus_{k+\ell=n} H_k(C_*) \otimes_R H_\ell(D_*) \longrightarrow H_n(C_* \otimes D_* ; R) \longrightarrow \bigoplus_{k+\ell=n-1} \text{Tor}_1^R(H_k(C_*), H_\ell(D_*)) \longrightarrow 0 \]

for each \(n\), and these sequences split.

The proof is similar to that of the universal coefficient theorem.

Under the assumption that \(H_k(X; R)\) and \(H_\ell(Y; R)\) are finitely generated projective \(R\)-modules, for each \(k\) and \(\ell\), we can dualize the homological K"unneth isomorphism

\[ H_* (X; R) \otimes_R H_* (Y; R) \congto H_*(X \times Y; R) \]

and use the universal coefficient theorem to get a cohomological K"unneth isomorphism

\[ H^* (X \times Y; R) \congto \text{Hom}_R(H_*(X \times Y; R), R) \congto \text{Hom}_R(H_*(X; R), R) \otimes_R \text{Hom}_R(H_*(Y; R), R) \congto H^*(X; R) \otimes_R H^*(Y; R). \]

Notice how finite generation is needed in the middle, using that the homomorphism

\[ \text{Hom}_R(M, R) \otimes_R \text{Hom}_R(N, R) \longrightarrow \text{Hom}_R(M \otimes_R N, R), \]

taking the tensor product of \(\varphi : M \to R\) and \(\psi : N \to R\) to the composite

\[ M \otimes_R N \xrightarrow{\varphi \otimes \psi} R \otimes_R R \congto R, \]

is an isomorphism when \(M\) and \(N\) are finitely generated projective \(R\)-modules.
4.3 Proof of the cohomology K"unneth formula

We will instead give a different proof of the cohomological K"unneth isomorphism, based on the study of generalized cohomology theories, which leads more directly to a result with weaker hypotheses.

The main part of the proof deals with the absolute case when \( B = \emptyset \), saying that the cross product
\[
H^*(X, A; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(X \times Y, A \times Y; R)
\]
is an isomorphism of graded rings, if \((X, A)\) is a pair of CW complexes and \(H^\ell(Y; R)\) is a finitely generated projective \(R\)-module, for each \(\ell\). The relative case, when \((Y, B)\) is a pair of CW complexes and \(H^\ell(Y, B; R)\) is a finitely generated projective \(R\)-module, for each \(\ell\), follows by naturality with respect to the map \((Y, B) \to (Y/B, B/B)\) inducing isomorphisms
\[
H^\ell(Y/B, B/B; R) \xrightarrow{\cong} H^\ell(Y, B; R)
\]
for all \(\ell\), and the splittings
\[
H^*(Y/B; R) \cong H^*(Y/B, B/B; R) \oplus H^*(B/B; R)
\]
and
\[
H^*((X, A) \times Y/B; R) \cong H^*((X, A) \times (Y/B, B/B); R) \oplus H^*((X, A) \times B/B; R).
\]

Consider the functors of CW pairs \((X, A)\) given by
\[
h^n(X, A) = \bigoplus_{k+\ell=n} H^k(X, A; R) \otimes_R H^\ell(Y; R)
\]
and
\[
k^n(X, A) = H^n(X \times Y, A \times Y; R).
\]
The cross product defines a natural transformation
\[
\mu: h^n(X, A) \to k^n(X, A)
\]
for all \(n\).

We prove that \(h^*\) and \(k^*\) are cohomology theories on the category of CW pairs, and that \(\mu\) is a map of cohomology theories, i.e., a natural transformation that commutes with the connecting homomorphisms.

**Proposition 4.3.1.** If a map \(\mu: h^* \to k^*\) of cohomology theories on the category of CW pairs is an isomorphism on the pair \((*, \emptyset)\), then it is an isomorphism for all CW pairs.

**Proof.** By the map of long exact sequences
\[
\begin{array}{ccccccccc}
h^{n-1}(X) & \to & h^{n-1}(A) & \to & h^n(X, A) & \to & h^n(X) & \to & h^n(A) \\
\mu & & & & & & & & \\
k^{n-1}(X) & \to & k^{n-1}(A) & \to & k^n(X, A) & \to & k^n(X) & \to & k^n(A)
\end{array}
\]
and the five-lemma, it suffices to prove the proposition in the case when \( A = \emptyset \).

In this absolute case first we proceed by induction on the dimension \( m \) of \( X \).

When \( X \) is 0-dimensional, it is the disjoint union \( X = \coprod \alpha \star \) of a set of points, so by the commutative diagram

\[
\begin{array}{ccc}
h^*(\coprod \alpha \star) & \xrightarrow{\cong} & \prod_\alpha h^*(\star) \\
\mu \downarrow & & \downarrow \prod_\alpha \mu \\
k^*(\coprod \alpha \star) & \xrightarrow{\cong} & \prod_\alpha k^*(\star)
\end{array}
\]

and the hypothesis for \( X = \star \), it follows that \( \mu \) is an isomorphism for \( X = \coprod \alpha \star \).

Let \( m \geq 1 \), assume that \( \mu \) is an isomorphism for all \( X \) of dimension less than \( m \), and suppose that \( X = X^{(m)} \) has dimension \( m \). By the map of long exact sequences above in the case \( (X, A) = (X^{(m)}, X^{(m-1)}) \), the inductive hypothesis and the five-lemma, it suffices to prove that \( \mu \) is an isomorphism for this CW pair. Let

\[
\Phi: \prod_\alpha (D^m, \partial D^m) \to (X^{(m)}, X^{(m-1)})
\]

be the characteristic maps of the \( m \)-cells of \( X \). In the commutative diagram

\[
\begin{array}{ccc}
h^*(X^{(m)}, X^{(m-1)}) & \xrightarrow{\Phi^*} & h^*(\coprod_\alpha (D^m, \partial D^m)) \\
\mu \downarrow & & \downarrow \prod_\alpha \mu \\
k^*(X^{(m)}, X^{(m-1)}) & \xrightarrow{\Phi^*} & k^*(\coprod_\alpha (D^m, \partial D^m))
\end{array}
\]

the homomorphisms labeled \( \Phi^* \) are isomorphisms by excision, and the right-hand horizontal arrows are isomorphisms by the product axiom. Hence it suffices to prove that \( \mu \) is an isomorphism for the CW pair \( (D^m, \partial D^m) \).

By the map of long exact sequences above, in the case \( (X, A) = (D^m, \partial D^m) \), it suffices to know that \( \mu \) is an isomorphism for \( X = D^m \) and for \( X = \partial D^m \). The first follows from the case \( X = \star \) naturality with respect to the map \( D^m \to \star \) and homotopy invariance. The second follows by induction, since the dimension of \( \partial D^m \) is less than \( m \).

The case of infinite-dimensional \( X \) remains. For this we use that \( X \) is the colimit of its skeleta, in the sense that there is a sequence of cellular inclusions of CW complexes

\[
X^{(m-1)} \subset X^{(m)} \subset \cdots \subset X = \bigcup_m X^{(m)}.
\]

There is a mapping telescope

\[
T = \bigcup_m [m, m + 1] \times X^{(m)} \subset \mathbb{R} \times X
\]

and the composite projection \( T \subset \mathbb{R} \times X \to X \) is a homotopy equivalence. See [2, Lemma 2.34].

We can write this mapping telescope as the homotopy coequalizer of two maps

\[
1, i: \prod_m X^{(m)} \longrightarrow \prod_m X^{(m)}
\]
where 1 is the coproduct of the identity maps $X^{(m)} \to X^{(m)}$, while $i$ is the coproduct of the inclusion maps $X^{(m-1)} \to X^{(m)}$. Hence there is a natural long exact sequence

$$h^{n-1}(\prod_m X^{(m)}) \xrightarrow{1-i^*} h^{n-1}(\prod_m X^{(m)}) \longrightarrow h^n(T) \longrightarrow h^n(\prod_m X^{(m)}) \xrightarrow{1-i^*} h^n(\prod_m X^{(m)})$$

which we can rewrite, using the product axiom and the homotopy equivalence $T \simeq X$, as

$$\prod_m h^{n-1}(X^{(m)}) \xrightarrow{1-i^*} \prod_m h^{n-1}(X^{(m)}) \longrightarrow h^n(X) \longrightarrow \prod_m h^n(X^{(m)}) \xrightarrow{1-i^*} \prod_m h^n(X^{(m)})$$

The kernel of the right hand $1 - i^*$ consists of the compatible sequences $(x_m)_m$ with $x_m \in h^n(X^{(m)})$ and $i^*(x_m) = x_{m-1}$ for all $m$, i.e., it equals the limit group

$$\ker(1 - i^*) = \lim_m h^n(X^{(m)}).$$

By definition, the cokernel of the left hand $1 - i^*$ is the derived limit group

$$\cok(1 - i^*) = \operatorname{Rlim}_m h^{n-1}(X^{(m)}).$$

It vanishes if the homomorphisms $i^*: h^n(X^{(m)}) \to h^n(X^{(m-1)})$ are surjective for sufficiently large $m$.

These considerations are natural in the cohomology theory $h$, so there is a map of short exact sequences

$$0 \longrightarrow \operatorname{Rlim}_m h^{n-1}(X^{(m)}) \longrightarrow h^n(X) \longrightarrow \operatorname{lim}_m h^n(X^{(m)}) \longrightarrow 0$$

We have already shown that $\mu$ is an isomorphism for each finite-dimensional $X^{(m)}$, hence is induces an isomorphism of limits and derived limits. Thus $\mu$ is also an isomorphism for the general CW complex $X$. \hfill \Box

**Proof of the cohomology Künneth formula.** We must exhibit $h^*$ and $k^*$ as cohomology theories, check that $\mu$ is a map of such, and that $\mu$ is an isomorphism for the one-point space $\star$.

The connecting homomorphism $\delta: h^{n-1}(A) \to h^n(X, A)$ is defined as the direct sum of the tensor products

$$\delta \otimes 1 : H^{k-1}(A; R) \otimes_R H^\ell(Y; R) \longrightarrow H^k(X, A; R) \otimes_R H^\ell(Y; R)$$

as $k$ ranges over the integers and $\ell = n - k$.

The connecting homomorphism $\delta: k^{n-1}(A) \to k^n(X, A)$ is the usual connecting homomorphism

$$\delta: H^{n-1}(A \times Y; R) \longrightarrow H^n(X \times Y, A \times Y; R)$$

of the pair $(X \times Y, A \times Y)$. 

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The tensor product of the long exact sequence
\[ H^{k-1}(X; R) \xrightarrow{\iota} H^{k-1}(A; R) \xrightarrow{\delta} H^k(X, A; R) \xrightarrow{\iota^*} H^k(X; R) \xrightarrow{\iota^*} H^k(A; R) \]
with \( H^\ell(Y; R) \) over \( R \) is still exact, because \( H^\ell(Y; R) \) is projective, hence flat. Sums over all \( k + \ell = n \) we get the long exact sequence for \( h^* \).

The long exact sequence for \( k^* \) at \((X, A)\) is just the usual long exact sequence for \( H^n(-; R) \) at \((X \times Y, A \times Y)\).

Homotopy invariance for \( h^* \) and \( k^* \) follows immediately from homotopy invariance for ordinary cohomology.

Excision, either in the form for general topological pairs, or in the form for subcomplexes of a CW complex, is also obvious for \( h^* \). The case of \( k^* \) is about as easy, since if \( Z \subseteq A \subseteq X \) with the closure of \( Z \) contained in the interior of \( A \), the \( Z \times Y \subset A \times Y \subset X \times Y \) with the closure of \( Z \times Y \) contained in the interior of \( A \times Y \), and similarly for products of subcomplexes of \( X \) with \( Y \).

The product axiom is clear for \( h^* \), since if \( X = \coprod \alpha X_\alpha \) then \( X \times Y = \coprod \alpha (X_\alpha \times Y) \), and similarly in the relative case. The product axiom for \( k^* \) is more subtle. It amounts to the assertion that
\[ \left( \prod_\alpha H^k(X_\alpha, A_\alpha; R) \otimes_R H^\ell(Y; R) \right) \longrightarrow \prod_\alpha (H^k(X_\alpha, A_\alpha; R) \otimes_R H^\ell(Y; R)) \]
is an isomorphism, for all \( k \) and \( \ell \). This is clear if \( H^\ell(Y; R) = R \), hence also if \( H^\ell(Y; R) \) is finitely generated and free, since finite sums of \( R \)-modules are also finite products. By naturality in \( H^\ell(Y; R) \), it also follows when \( H^\ell(Y; R) \) is finitely generated and projective.

The assertion that \( \mu \) is a map of generalized cohomology theories is clear from the naturality of the cross product, together with the previously proved formula \( \delta(\alpha \times \eta) = \delta\alpha \times \eta \), relating the connecting homomorphism to the cross product.

The assertion that \( \mu \) is an isomorphism for \((X, A) = (\ast, \varnothing)\) is the assertion that
\[ \times: H^0(\ast; R) \otimes_R H^\ell(Y; R) \longrightarrow H^\ell(\ast \times Y; R) \]
is an isomorphism for all \( \ell \), which is clear.

\[ \square \]

**Theorem 4.3.2 (Hopf).** If there is a real division algebra structure on \( \mathbb{R}^n \) then \( n \) is a power of 2.

**Proof.** A division algebra structure on \( \mathbb{R}^n \) is a bilinear pairing \( : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( x \cdot y = 0 \) only if \( x = 0 \) or \( y = 0 \). Given such a pairing, we have a map \( g: S^{n-1} \times S^{n-1} \rightarrow S^{n-1} \) given by \( g(x, y) = x \cdot y/|x \cdot y| \), such that
\[ g(-x, y) = -g(x, y) = g(x, -y) \, . \]

Passing to quotients we get a map \( h: \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1} \). We may assume that \( n \geq 2 \), in which case \( \pi_1(\mathbb{R}P^{n-1}) \cong \mathbb{Z}/2 \). The displayed formula, and a consideration of covering spaces, implies that \( h \) induces the sum homomorphism on \( \pi_1(\mathbb{R}P^{n-1}) \).

Passing to cohomology, we have a graded ring isomorphism
\[ H^*(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\gamma]/(\gamma^n = 0) \, . \]
with \( \deg(\gamma) = 1 \), and by the Künneth formula,

\[
H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha, \beta]/(\alpha^n = 0, \beta^n = 0)
\]

The formula for \( h \) on \( \pi_1 \) implies that \( h^*(\gamma) = \alpha + \beta \) in \( H^1 \), so

\[
0 = h^*(\gamma^n) = (\alpha + \beta)^n = \sum_{i=1}^{n-1} \binom{n}{i} \alpha^i \beta^{n-i}
\]

in \( H^n \). It is a number-theoretic fact that \( \binom{n}{i} \equiv 0 \mod 2 \) for all \( 0 < i < n \) (if and) only if \( n \) is a power of 2. \( \square \)
Chapter 5

Poincaré duality

Definition 5.0.3. An $n$-dimensional manifold is a Hausdorff space $M$ such that each point has an open neighborhood that is homeomorphic to $\mathbb{R}^n$. If $M$ is also compact as a topological space, then we call $M$ a closed $n$-manifold.

Poincaré duality asserts that for a closed, orientable $n$-manifold $M$ there is an isomorphism

$$H_k(M; \mathbb{Z}) \cong H^{n-k}(M; \mathbb{Z})$$

for each integer $k$. Without the orientability hypothesis, there is an isomorphism

$$H_k(M; \mathbb{Z}/2) \cong H^{n-k}(M; \mathbb{Z}/2)$$

for each $k$.

5.1 Orientations

Let $R$ be any commutative ring. Let $x \in M$ be any point, and let $U \subseteq M$ be an open neighborhood of $x$ with $(U, x) \cong (\mathbb{R}^n, 0)$. Then there are isomorphisms

$$H_k(M, M - \{x\}; R) \cong H_k(U, U - \{x\}; R)$$
$$\cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}; R)$$
$$\cong \tilde{H}_{k-1}(\mathbb{R}^n - \{0\}; R)$$

for each integer $k$. Hence

$$H_k(M, M - \{x\}; R) \cong \begin{cases} R & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

It follows that the dimension $n$ is an intrinsic property of the manifold $M$ (when nonempty).

Definition 5.1.1. By a local orientation of $M$ at $x$ we mean a choice $\mu_x \in H_n(M, M - \{x\}; \mathbb{Z})$ of one of the two possible generators of $H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$, as an abelian group. Under the isomorphisms above, this corresponds to a choice of a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})$, which in turn can be identified with a choice of vector space basis for $\mathbb{R}^n$, up to the equivalence relation that
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considers two bases to be equivalent if the change-of-basis matrix has positive determinant.

More generally, by a local $R$-orientation of $M$ at $x$ we mean a choice of a generator $\mu_x$ of $H_n(M, M - \{x\}; R) \cong R$, as an $R$-module. When $R = \mathbb{Z}/2$ there is one one possible such choice, namely the nonzero element in $H_n(M, M - \{x\}; \mathbb{Z})$. A local orientation is thus the same as a local $\mathbb{Z}$-orientation.

Definition 5.1.2. The following notational convention is helpful: For $A \subseteq X$ we let

$$H_k(X|A) = H_k(X, X - A)$$

for all integers $k$, and similarly with arbitrary coefficient groups. We may read $X|A$ as $X$ relative to the complement of $A$. In the special case when $A = \{y\}$ we simply write $H_k(X|y)$ for $H_k(X|\{y\})$. If $A \subseteq B \subseteq X$ the inclusion $X - B \subseteq X - A$ induces homomorphisms

$$H_k(X|B) \longrightarrow H_k(X|A)$$

which we say are given by restriction along $A \subseteq B$. If $U$ is a neighborhood of the closure of $A$ in $X$, then the inclusion $U \subseteq X$ induces isomorphisms

$$H_k(U|A) \cong H_k(X|A)$$

by excision, so $H_* (X|A)$ only depends on this neighborhood. We may therefore call $H_k(X|A)$ the local homology of $X$ at $A$.

A local $R$-orientation $\mu_x$ of an $n$-manifold $M$ at a point $x$ determines a local $R$-orientation $\mu_y$ of $M$ at all points $y$ in a neighborhood of $x$, by the following prescription: Let $U \subseteq M$ be an open neighborhood of $x$ with $(U, x) \cong (\mathbb{R}^n, 0)$, and let $B \subseteq U$ be a smaller open neighborhood that corresponds to an open ball $B^n(r) \subseteq \mathbb{R}^n$ of finite radius $r$, centered at the origin. Then the inclusion $U - B \subseteq U - \{y\}$ is a homotopy equivalence for each point $y \in B$, so the restriction map

$$H_n(M|B; R) \xrightarrow{=} H_n(M|y; R)$$

is an isomorphism, for each point $y \in B$. Let $\mu_B \in H_n(M|B; R)$ be the preimage of the generator $\mu_x \in H_n(M|x; R)$, in the special case $y = x$, and let $\mu_y \in H_n(M|y; R)$ be the image of $\mu_B$, for all $y \in B$. Then each $\mu_y$ is a local $R$-orientation of $M$ at $y$.

These local $R$-orientations at points $y \neq x$ may depend on the choice made of the neighborhood $U \cong \mathbb{R}^n$. If it is possible to make a consistent choice of local $R$-orientations, at all points of $M$, we say that $M$ is $R$-orientable.

Definition 5.1.3. An $R$-orientation of an $n$-manifold $M$ is a choice of a local $R$-orientation $\mu_x \in H_n(M|x; R)$ at each point $x \in M$, such that for each $x \in M$ there are open neighborhoods $B \subseteq U$ with $(U, B, x) \cong (\mathbb{R}^n, B^n(r), 0)$, and the class $\mu_B \in H_n(M|B; R)$ that restricts to $\mu_x \in H_n(M|x; R)$ restricts to $\mu_y \in H_n(M|y; R)$ for each $y \in B$.

An orientation of $M$ is the same as a $\mathbb{Z}$-orientation. Any $n$-manifold $M$ has a unique $\mathbb{Z}/2$-orientation, with $\mu_x \in H_n(M|x; \mathbb{Z}/2) \cong \mathbb{Z}/2$ equal to the nonzero element for each $x \in M$. This is clear, since the restriction isomorphisms
$H_n(M[B;\mathbb{Z}/2]) \cong H_n(M[y;\mathbb{Z}/2])$ always take the (unique) nonzero element to the nonzero element.

The consistency condition on local $R$-orientations at points can be re-expressed as a continuity condition for a section in a covering space projection $\pi: M_R \to M$ called the $R$-orientation covering of $M$.

**Definition 5.1.4.** Let $M$ be an $n$-manifold and let $R$ be any commutative ring. Let

$$M_R = \coprod_{x \in M} H_n(M[x; R])$$

be the disjoint union of the local homology groups $H_n(M[x; R]) = H_n(M, M - \{x\}; R)$, for all points $x \in M$. For each pair of open neighborhoods $B \subset U \subseteq M$, with $(U, B) \cong (\mathbb{R}^n, B^n(r))$ for some finite $r$, and each class $\alpha \in H_n(M[\bar{B}; R])$, let $V(B, \alpha) \subseteq M_R$ be the set of elements

$$\alpha_x \in H_n(M[x; R]) \subseteq M_R$$

with $x \in B$ and $\alpha_x$ the restriction of $\alpha$. The collection of subsets $V(B, \alpha)$ for all such $B$ and $\alpha$ is a basis for a topology on $M_R$, making $M_R$ a topological space.

The map $\pi: M_R \to M$ taking all of $H_n(M[x; R]) \subseteq M_R$ to $x \in M$ is a covering space projection. For each pair $B \subset U$ as above, the preimage $\pi^{-1}(B)$ decomposes as

$$\pi^{-1}(B) \cong B \times H_n(M[\bar{B}; R])$$

over $B$, where $H_n(M[\bar{B}; R]) \cong R$ has the discrete topology. The isomorphism takes $(x, \alpha)$ for $x \in B$, $\alpha \in H_n(M[\bar{B}; R])$ to the restriction $\mu_x \in H_n(M[x; R]) \subseteq \pi^{-1}(B)$ of $\alpha$ along $\{x\} \subset B$.

Each fiber $\pi^{-1}(x) = H_n(M[x; R])$ is isomorphic to $R$, as an $R$-module. An $R$-orientation of $M$ is then the same as a continuous section $\mu: M \to M_R$ (with $\pi \circ \mu = 1_M$), taking $x \in M$ to an $R$-module generator $\mu_x \in \pi^{-1}(x)$ for each $x$.

**Definition 5.1.5.** Let $\Gamma_R(M)$ be the set of (continuous) sections $\alpha: M \to M_R$, mapping $x \in M$ to $\alpha_x \in M_R$. It is an $R$-module, with $(r \cdot \alpha)_x = r \cdot \alpha_x$ in $H_n(M[x; R])$, for $r \in R$ and $x \in M$.

**Theorem 5.1.6.** Let $M$ be a closed, connected $n$-manifold.

(a) $H_k(M; R) = 0$ for $k > n$.

(b) If $M$ is $R$-orientable, then the restriction map

$$H_n(M; R) \cong H_n(M[x; R]) \cong R$$

is an isomorphism, for each $x \in X$.

**Definition 5.1.7.** An element $[M] \in H_n(M; R)$ whose image in $H_n(M[x; R])$ is a generator for all $x \in M$ is called a fundamental class or an orientation class for $M$, with coefficients in $R$. Specifying a fundamental class for $M$ is equivalent to specifying an $R$-orientation for $M$.

The theorem follows from the following more precise statement.

**Lemma 5.1.8.** Let $M$ be an $n$-manifold and let $A \subseteq M$ be a compact subset.
(a) $H_k(M;A;R) = 0$ for $k > n$.

(b) If $\alpha \in \Gamma_R(M)$ is a section of the covering space $M_R \to M$, then there exists a unique class $\alpha_A \in H_n(M;A;R)$ whose restriction to $H_n(M;x;R)$ is $\alpha_x$ for each $x \in A$.

In other words, part (b) of the lemma says that the natural homomorphism

$$H_n(M;A;R) \longrightarrow \{\text{lifts } A \to M_R \text{ of } A \to M\}$$

is injective, and that it image contains the image of the homomorphism

$$\Gamma_R(M) \longrightarrow \{\text{lifts } A \to M_R \text{ of } A \to M\}$$

that takes a section $\alpha: M \to M_R$ to its restriction $\alpha|A$.

**Proof of theorem.** When $M$ is compact, the case $A = M$ of the lemma says that $H_k(M;R) = 0$ for $k > n$, and that there is an isomorphism

$$H_n(M;R) \cong \Gamma_R(M).$$

If $M$ is R-orientable, we can choose an $R$-orientation $x \mapsto \mu_x$, which is a section $\mu \in \Gamma_R(M)$ such that $\mu_x$ is a generator of $H_n(M;x;R)$ for each $x \in M$. Hence any section $\alpha \in \Gamma_M(R)$ can be uniquely written as $\{(\text{ETC})\}$

**Proof of lemma.** We omit $R$ from the notation. The proof goes in four steps.

Step (1): If the lemma is true for compact subsets $A$, $B$ and $A \cap B$ of $M$, then it is true for $A \cup B$.

There is a long exact Mayer–Vietoris sequence

$$\ldots \longrightarrow H_k(M;A \cup B) \xrightarrow{\Phi} H_k(M;A) \oplus H_k(M;B) \xrightarrow{\Psi} H_k(M;A \cap B) \xrightarrow{\partial} \ldots$$

associated to the covering of $M$ by the open subsets $M - A$ and $M - B$. Here $\Phi$ takes $\alpha \in H_k(M;A \cup B)$ to $(\alpha, \alpha)$ in $H_k(M;A) \oplus H_k(M;B)$, and $\Psi$ takes $(\alpha, \beta) \in H_k(M;A) \oplus H_k(M;B)$ to $\alpha - \beta \in H_k(M;A \cap B)$ (omitting notation for the restriction maps).

Part (a) of the lemma follows by exactness, since each group $H_k(M;A \cup B)$ sits between two trivial groups, for $k > n$.

For $k = n$, we have the half-exact sequence

$$0 \to H_n(M;A \cup B) \xrightarrow{\Phi} H_n(M;A) \oplus H_n(M;B) \xrightarrow{\Psi} H_n(M;A \cap B).$$

Suppose that $x \mapsto \alpha_x$ is a section in $\Gamma_R(M)$. By assumption there are unique classes $\alpha_A \in H_n(M;A)$, $\alpha_B \in H_n(M;B)$ and $\alpha_{A \cap B} \in H_n(M;A \cap B)$ such that $\alpha_A$ restricts to $\alpha_x$ for each $x \in A$, $\alpha_B$ restricts to $\alpha_y$ for each $y \in B$, and $\alpha_{A \cap B}$ restricts to $\alpha_z$ for each $z \in A \cap B$. By the uniqueness of $\alpha_{A \cap B}$, both $\alpha_A$ and $\alpha_B$ restrict to $\alpha_{A \cap B}$ in $H_n(M;A \cap B)$. Hence $\Psi(\alpha_A, \alpha_B) = \alpha_{A \cap B} - \alpha_{A \cap B} = 0$.

By exactness and injectivity of $\Phi$, it follows that there is a unique class $\alpha_{A \cup B} \in H_n(M;A \cup B)$ that restricts to $\alpha_A$ in $H_n(M;A)$ and to $\alpha_B \in H_n(M;B)$. This is equivalent to saying that it is the unique class that restricts to $\alpha_x$ for all $x \in A$ and to $\alpha_y$ for all $y \in B$. This, in turn, is equivalent to part (b) of the lemma for $A \cup B$.

Step (2): It suffices to prove the lemma for $M = \mathbb{R}^n$.

Step (3):

Step (4): $\{(\text{ETC})\}$

\[\square\]
5.2 Cap product

Definition 5.2.1. Let $X$ be a space and $R$ a commutative ring. The cap product

$$C_{k+\ell}(X; R) \otimes_R C^k(X; R) \xrightarrow{\cap} C_{\ell}(X; R)$$

sends a $(k+\ell)$-simplex $\sigma: \Delta^{k+\ell} \to X$ and a $k$-cochain $\varphi \in C^k(X; R)$ to the $\ell$-chain

$$\sigma \cap \varphi = \varphi(\sigma[v_0, \ldots, v_k])\sigma[v_k, \ldots, v_{k+\ell}].$$

Lemma 5.2.2. $\partial \sigma \cap \varphi = \sigma \cap \delta \varphi + (-1)^{\ell}\partial(\sigma \cap \varphi)$.

Hence there is an induced cap product

$$H_{k+\ell}(X; R) \otimes_R H^k(X; R) \xrightarrow{\cap} H_{\ell}(X; R).$$

For pairs $(X, A)$ there are relative forms

$$H_{k+\ell}(X, A; R) \otimes_R H^k(X, A; R) \xrightarrow{\cap} H_{\ell}(X, A; R)$$

$$H_{k+\ell}(X, A; R) \otimes_R H^k(X, A; R) \xrightarrow{\cap} H_{\ell}(X; R)$$

and, more generally,

$$H_{k+\ell}(X, A + B; R) \otimes_R H^k(X, A; R) \xrightarrow{\cap} H_{\ell}(X, B; R)$$

for subsets $A, B \subset X$. If $A$ and $B$ are excisive in $X$, the source can be rewritten as

$$H_{k+\ell}(X, A + B; R) \cong H_{k+\ell}(X, A \cup B; R).$$

Lemma 5.2.3 (Projection formula). Let $f: X \to Y$ be any map. Then

$$f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*(\varphi))$$

in $C_{\ell}(Y; R)$ for $\alpha \in C_{k+\ell}(X; R)$ and $\varphi \in C^k(Y; R)$, hence also in $H_\ell(Y; R)$ for $\alpha \in H_{k+\ell}(X; R)$ and $\varphi \in H^k(Y; R)$.

In other words, $f_*: H_*(X; R) \to H_*(Y; R)$ is an $H^*(Y; R)$-module homomorphism, where $H^*(Y; R)$ acts on $H_*(Y; R)$ by cap product, and on $H_*(X; R)$ via the ring homomorphism $f^*: H^*(Y; R) \to H^*(X; R)$ followed by cap product.

Theorem 5.2.4 (Poincaré duality). Let $M$ be a closed, $R$-oriented $n$-manifold with fundamental class $[M] \in H_n(M; R)$. Then the homomorphism

$$D_M: H^k(M; R) \to H_{n-k}(M; R),$$

given by $D_M(\varphi) = [M] \cap \varphi$, is an isomorphism for all $k$.

5.3 Cohomology with compact supports

Fix a space $X$. The partially ordered set of compact subsets $A \subset X$ is directed, in the sense that it is nonempty, and any two compact subsets $A, B \subset X$ are contained in a larger compact subset, such as their union $A \cup B$. 


For each compact $A$, there is a subcomplex
\[ C^*(X|A) = C^*(X, X - A) \subset C^*(X) \]
of cochains that vanish on simplices in the complement of $A$. If $A \subset B$ then restriction along $i: (X, X - B) \subset (X, X - A)$ induces an inclusion $i^*: C^*(X|A) \subset C^*(X|B)$ of subcomplexes. The union of all of these subcomplexes, the colimit
\[ C^*_c(X) = \bigcup_{A \text{ compact}} C^*(X|A) = \colim_{A \text{ compact}} C^*(X|A), \]
is the subcomplex of compactly supported cochains in $C^*(X)$.

(Co-)homology commutes with directed colimits, so
\[ H^*_c(X) = H_*(C^*_c(X), \delta) \cong \colim_{A \text{ compact}} H^*(X|A) \]
defines the cohomology with compact supports of $X$. There is a natural homomorphism
\[ H^*_c(X) \rightarrow H^*(X), \]
which is an isomorphism for $X$ compact.

Similarly with coefficients in any abelian group $G$.

**Lemma 5.3.1.**
\[ H^k_c(\mathbb{R}^n; G) \cong \begin{cases} G & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** Each compact subset $A \subset \mathbb{R}^n$ is contained in a closed ball $\bar{B}^n(r)$ of positive radius, centered at the origin. We say that these balls form a cofinal family within the directed set of compact subsets. Hence
\[ H^*_c(\mathbb{R}^n; G) \cong \colim_r H^*(\mathbb{R}^n|\bar{B}^n(r); G). \]
Here $H^k(\mathbb{R}^n|\bar{B}^n(r); G)$ is $G$ for $k = n$ and $0$ for $k \neq n$, since
\[ (\mathbb{R}^n, \mathbb{R}^n - \bar{B}^n(r)) \sim (\mathbb{R}^n, \mathbb{R}^n - \{0\}) \]
is a homotopy equivalence for all positive $r$. Furthermore, the restriction homomorphisms for varying $r$ are all isomorphisms, so the colimit system is constant, with colimit $G$ for $k = n$ and $0$ otherwise. \[ \square \]

### 5.4 Duality for noncompact manifolds

**Definition 5.4.1.** Let $M$ be an $R$-oriented $n$-manifold, possibly noncompact.

For each compact $A \subset M$ there is a unique element $\mu_A \in H_n(M|A; R)$ that restricts to the given orientation of $M$ at all points $x \in A$. Cap product defines a homomorphism
\[ \mu_A \cap (-): H^k(M|A; R) \rightarrow H_{n-k}(M; R) \]
taking $\varphi \in H^k(M|A; R)$ to $\mu_A \cap \varphi \in H_{n-k}(M; R)$. For compact $A \subset B \subset M$, the identity $\mu_A \cap \varphi = \mu_B \cap i^*(\varphi)$ holds, since $i^*(\mu_B) = \mu_A$, so these homomorphism combine to define a homomorphism
\[ D_M: H^k_c(M; R) \rightarrow H_{n-k}(M; R) \]
for each integer $k$. 
Theorem 5.4.2. Let $M$ be an $R$-oriented $n$-manifold. Then the homomorphism
\[ D_M : H^k_c(M; R) \to H_{n-k}(M; R) \]
that extends $\mu_A \cap (-)$ on $H^k(M|A; R)$ for each compact $A \subset M$, is an isomorphism for each integer $k$.

This result contains the Poincaré duality theorem for closed manifolds, in the case when $M$ is compact.

**Lemma 5.4.3.** Suppose that $M = U \cup V$ is a union of two open subsets. Then there is a diagram of Mayer–Vietoris sequences

\[
\begin{array}{cccccc}
H^k_c(U \cap V) & \to & H^k_c(U \cap H^k_c(V) \to & H^k_c(M) & \to & H^k_c(U \cap V) \\
D_{U \cap V} & & D_{U \cap V} & & D_M & & D_{U \cap V} \\
H_{n-k}(U \cap V) & \to & H_{n-k}(U \cap H^k_c(V) \to & H_{n-k}(M) & \to & H_{n-k}(U \cap V) \\
& & & & &
\end{array}
\]

that commutes up to sign.

**Proof.** Let $A \subset U$ and $B \subset V$ be compact. Then $A \cap B \subset U \cap V$ is also compact. There are excision isomorphisms $H^k(M|A) \cong H^k(U|A)$, etc. We get a diagram, where the upper row is the Mayer–Vietoris sequence in cohomology for $(M, M - A)$ and $(M, M - B)$, and the lower row is the Mayer–Vietoris sequence in homology for $U$ and $V$.

\[
\begin{array}{cccccc}
H^k(M|A \cap B) & \to & H^k(M|A) \oplus H^k(M|B) & \to & H^k(M|A \cup B) & \to & H^{k+1}(M|A \cap B) \\
\cong & & \cong & & \cong & & \\
H^k(U \cap V|A \cap B) & \to & H^k(U|A) \oplus H^k(V|B) & \to & H^k(U \cap V|A \cup B) & \to & H^{k+1}(U \cap V|A \cap B) \\
\mu_{A \cap B} & & \mu_{A \cap (-)} \oplus \mu_{B \cap (-)} & & \mu_{A \cap B} & & \\
H_{n-k}(U \cap V) & \to & H_{n-k}(U) \oplus H^k(V) & \to & H_{n-k}(M) & \to & H_{n-k}(U \cap V) \\
& & & & & &
\end{array}
\]

This is straightforward the check for the left hand and middle square, less so for the right hand square. We refer to [2, pp. 246–247] for the details.

As $A$ and $B$ range over all compact subsets of $U$ and $V$, respectively, the intersection $A \cap B$ ranges over all compact subsets of $U \cap V$, and the union $A \cup B$ ranges over a cofinal family of compact subsets of $M$. Hence the colimit over the directed set of all such $A$ and $B$, of the diagrams above, is the diagram of the lemma, with exact rows and squares that commute up to sign.

**Proof of the duality theorem.** Step (1): If $M = U \cup V$ with $U$ and $V$ open, and the theorem holds for $U$, $V$ and $U \cap V$, then it holds for $M$. This is clear by the previous lemma and the 5-lemma.

Step (2): If $M = \bigcup U_i$ is the union of an increasing sequence of open subsets, and the theorem holds for each $U_i$, then it holds for $M$. Each compact subset of $M$ is contained in some $U_i$, so $\text{colim}_i H^*_c(U_i) \cong H^*_c(M)$. Similarly, each simplex in $M$ factors through some $U_i$, so $\text{colim}_i H_*c(U_i) \cong H_*(M)$. The homomorphism
\[ D_M : H^k_c(M) \cong H_{n-k}(M) \]
is then the colimit of the assumed isomorphisms
\[ D_{U_i} : H^k_c(U_i) \xrightarrow{\cong} H_{n-k}(U_i), \]
hence is an isomorphism.

Step (3): The theorem holds for \( M = \mathbb{R}^n \). This is clear from the lemma computing \( H^*_c(\mathbb{R}^n; G) \) for \( G = R \), since for each closed ball \( B = B^n(r) \) the cap product \( \mu_B \) and a generator of \( H^n(\mathbb{R}^n; R) \) takes the orientation class \( \mu_B \) and a generator of \( H^n(\mathbb{R}^n; R) \) to a unit times a point in \( \mathbb{R}^n \), which represents a generator of \( H_0(\mathbb{R}^n; R) \).

Step (4): The theorem holds for \( M \) an open subset of \( \mathbb{R}^n \). ((Write \( M \) as a countable union of convex open subsets. Induction over the number of convex subsets, and passage to a colimit.))

Step (5): The theorem holds for \( M \) a countable union of open subsets homeomorphic to \( \mathbb{R}^n \). ((ETC))

5.5 Connection with cup product

Lemma 5.5.1. Let \( X \) be a space and \( R \) a commutative ring. The relation
\[ \psi(\alpha \cap \varphi) = (\varphi \cup \psi)(\alpha) \]
holds for \( \varphi \in C^k(X; R) \), \( \psi \in C^\ell(X; R) \) and \( \alpha \in C^{k+\ell}(X; R) \). Hence the diagram
\[ \begin{array}{ccc}
H^\ell(X; R) & \xrightarrow{\beta} & \text{Hom}_R(H_\ell(X; R), R) \\
\alpha \cup (-) & \downarrow & \downarrow (\alpha \cap \varphi)^* \\
H^{k+\ell}(X; R) & \xrightarrow{\beta} & \text{Hom}_R(H_{k+\ell}(X; R), R)
\end{array} \]
commutes.

An \( R \)-bilinear pairing \( A \times B \to R \) is said to be nonsingular if both of the induced homomorphisms \( A \to \text{Hom}_R(B, R) \) and \( B \to \text{Hom}_R(A, R) \) are isomorphisms.

Lemma 5.5.2. Let \( M \) be an \( R \)-orientable \( n \)-manifold, with fundamental class \( [M] \in H_n(M; R) \). The cup pairing
\[ H^k(M; R) \times H^{n-k}(M; R) \to R \]
taking \( (\varphi, \psi) \) to \( (\varphi \cup \psi)[M] \) is nonsingular when \( R \) is a field. When \( R = \mathbb{Z} \) the induced pairing
\[ H^k(M)/(\text{torsion}) \times H^{n-k}(M)/(\text{torsion}) \to \mathbb{Z} \]
(of torsion-free quotient groups) is nonsingular.
Lemma 5.5.3. Given an ordered basis $\alpha_1, \ldots, \alpha_r$ for $H^k(M; R)$ when $R$ is a field (resp. for $H^k(M)/\text{torsion}$ for $R = \mathbb{Z}$), there is a unique dual basis $\beta_1, \ldots, \beta_r$ for $H^{n-k}(M; R)$ (resp. for $H^{n-k}(M)/\text{torsion}$), such that

$$(\alpha_i \cup \beta_j)[M] = \delta_{ij}$$

is equal to 1 for $i = j$ and 0 for $i \neq j$.

In the middle-dimensional case $n = 2k$, the $r \times r$ matrix with $(i,j)$-th entry $$(\alpha_i \cup \beta_j)[M]$$

is an invertible matrix, hence has nonzero determinant when $R$ is a field, and determinant $\pm 1$ when $R = \mathbb{Z}$. It is symmetric if $k$ is even, so that $n \equiv 0 \mod 4$, and it is skew-symmetric if $k$ is odd, so that $n \equiv 2 \mod 4$.

Remark 5.5.4. The classification of (skew-)symmetric integer matrices with determinant $\pm 1$ is interesting. These are known as (skew-)symmetric unimodular forms on $\mathbb{Z}^r$. A change of basis for $\mathbb{Z}^r$ changes the integer matrix by conjugation, which gives an isomorphic unimodular form. In the skew-symmetric case, each unimodular matrix is conjugate to a block sum of copies of the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(realized by $H^1(T^2)$). In particular, the rank $r$ must be even. In the symmetric case, the simplest examples are

$$\begin{bmatrix} 1 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix}$$

(realized by $H^2(\mathbb{C}P^2)$) and the form of rank 8 known as $E_8$

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(which cannot be realized as $H^2(M)$ for a closed, simply-connected smooth 4-manifold $M$, by a theorem of Rochlin).

Corollary 5.5.5. $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[y]/(y^{n+1} = 0)$ for $n \geq 0$, and $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$, where $\deg(y) = 2$.

Proof. This is clear for $n \leq 1$. Consider $n \geq 2$ and assume that the result holds for $\mathbb{C}P^{n-1}$. The inclusion $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ induces an isomorphism in cohomology except in degree $2n$, so if $y \in H^2(\mathbb{C}P^n)$ is a generator, then $y^k \in H^{2k}(\mathbb{C}P^n)$ is a generator for all $0 \leq k < n$. It remains to prove that it is also a generator for $k = n$. But this follows from Poincaré duality for the orientable $2n$-manifold $\mathbb{C}P^n$, since the pairing $H^2(\mathbb{C}P^n) \times H^{2n-2}(\mathbb{C}P^n) \to \mathbb{Z}$.
is nonsingular, hence must take \((y, y^{n-1})\) to a generator of \(\mathbb{Z}\). This means that 
\(y^n\) evaluates to \(\pm 1\) on the fundamental class of \(CP^n\), hence is a generator of 
\(H^{2n}(CP^n)\).

The inclusion \(CP^n \to CP^\infty\) induces an isomorphism in cohomology in degrees 
\(\ast \leq 2n+1\), which proves the formula for \(H^\ast(CP^\infty)\) in that range of 
degrees. Let \(n\) grow to infinity. \(\square\)

**Corollary 5.5.6.**

\[H^\ast(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1} = 0)\]

for \(n \geq 0\), and \(H^\ast(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]\), where \(\deg(x) = 1\).

**Proof.** Same proof as for \(CP^n\), using Poincaré duality for the \(\mathbb{Z}/2\)-orientable 
\(n\)-manifold \(\mathbb{R}P^n\). \(\square\)

Let \(m \geq 2\) be a natural number. Let \(C_m \subset S^1\) be the cyclic group of 
order \(m\). Consider \(S^{2n+1}\) to be the unit sphere in \(\mathbb{C}^{n+1}\). The diagonal action by 
the complex numbers on \(\mathbb{C}^{n+1}\) restricts to a free action of \(S^1\) on \(S^{2n+1}\), which 
otherwise restricts to a free action of \(C_m\) on \(S^{2n+1}\). Let the lens space 
\[L^{2n+1}_m = S^{2n+1}/C_m\]

be the orbit space. There is a CW structure on \(S^{2n+1}\) with \((2k+1)\)-skeleton 
equal to \(S^{2k+1}\), and with \(2k\)-skeleton equal to the join of \(S^{2k-1}\) and \(C_m\) inside of 
\(S^{2k-1} \ast S^1 \cong S^{2k+1}\)

for each \(0 \leq k \leq n\). This structure has exactly \(m\) cells of each dimension from 
0 to \(2n+1\), and the \(C_m\)-action permutes the cells freely. Hence the orbit space 
\(L^{2n+1}_m\) has a CW structure with exactly one cell of each dimension from 0 to 
\(2n+1\), with \((2k+1)\)-skeleton equal to \(L^{2k+1}_m\), for each \(0 \leq k \leq n\).

**Corollary 5.5.7.** Let \(p\) be an odd prime.

\[H^\ast(L^{2n+1}_p; \mathbb{Z}/p) \cong \mathbb{Z}/p[x, y]/(x^2 = 0, y^{n+1} = 0)\]

for \(n \geq 0\), and \(H^\ast(L^\infty_p; \mathbb{Z}/p) \cong \mathbb{Z}/p[x, y]/(x^2)\), where \(\deg(x) = 1\) and \(\deg(y) = 2\).

**Proof.** We omit the subscript \(p\). The result is clear for \(n = 0\), when \(L^1 = S^1/C_p\) 
is a circle, with cohomology \(H^\ast(L^1; \mathbb{Z}/p) \cong \mathbb{Z}/p[1, x]\). Let \(n \geq 1\) and suppose that 
the formula holds for \(L^{2n-1}\). The inclusion \(L^{2n-1} \to L^{2n+1}\) induces an 
isomorphism in cohomology with \(\mathbb{Z}/p\)-coefficients, except in degrees \(2n\) and 
\(2n+1\), so if \(x \in H^1(L^{2n+1}; \mathbb{Z}/p)\) and \(y \in H^2(L^{2n+1}; \mathbb{Z}/p)\) are generators then 
\(y^k \in H^{2k}(L^{2n+1}; \mathbb{Z}/p)\) and \(xy^k \in H^{2k+1}(L^{2n+1}; \mathbb{Z}/p)\) are generators for all 
\(0 \leq k < n\). We have \(x^2 = 0\) by graded commutativity, since \(p\) is odd. It 
remains to prove that \(y^n\) and \(xy^n\) are generators. The second claim follows from 
Poincaré duality for the \(\mathbb{Z}/p\)-orientable \((2n+1)\)-manifold \(L^{2n+1}\), since the pairing 
\[H^2(L^{2n+1}; \mathbb{Z}/p) \times H^{2n-1}(L^{2n+1}; \mathbb{Z}/p) \to \mathbb{Z}/p\]
is nonsingular, hence must take \((y, xy^{n-1})\) to a generator. This means that 
\(y \cdot xy^{n-1} = xy^n\) is a generator. The first claim follows by writing this generator 
as the product \(x \cdot y^n\). \(\square\)
The cup product structure in $H^*(L^\infty_m; \mathbb{Z}/m)$ is a little more complicated for non-prime $m$.

Since $S^\infty$ is contractible, the spaces $CP^\infty$, $RP^\infty$, and $L^\infty_p$ are classifying spaces for the groups $S^1$, $C_2$, and $C_p$, respectively. They are representatives for the homotopy types denoted $BS^1 = BU(1)$, $BC_2 = BO(1)$ and $BC_p$, and the corollaries above compute the cohomology algebras of the spaces of these homotopy types.

5.6 Other forms of duality

Let $\mathbb{R}_+^n \subseteq \mathbb{R}^n$ be the half-space of points $(x_1, \ldots, x_n)$ where $x_n \geq 0$. An $n$-manifold with boundary is a Hausdorff space $M$ such that each point has an open neighborhood that is homeomorphic to $\mathbb{R}^n$ or to $\mathbb{R}_+^n$.

For points $x \in M$ that correspond to a point in $\mathbb{R}_+^n$ with $x_n = 0$ (which we may assume is the origin), the local homology group

$$H_n(M|\{x\}) = H_n(M, M - \{x\}) \cong H_n(\mathbb{R}_+^n, \mathbb{R}_+^n - \{0\}) \cong 0$$

is trivial, unlike the local homology groups for $x \in M$ that correspond to points in $\mathbb{R}_+^n$ with $x_n > 0$ (or to points in $\mathbb{R}^n$). Hence the former points constitute a well-defined subspace $\partial M \subset M$, called the boundary of $M$. The subspace $\partial M$ is an $(n - 1)$-dimensional manifold (without boundary).

We say that a compact manifold $M$ with boundary is $R$-orientable if $M - \partial M$ is orientable as a manifold without boundary.

((Explain fundamental class in $H_n(M, \partial M)$.)

**Theorem 5.6.1 (Lefschetz duality).** Let $M$ be a compact $R$-oriented $n$-manifold with boundary, and suppose that $\partial M = A \cup B$ is the union of two compact $(n-1)$-manifolds $A$ and $B$, with common boundary $\partial A = \partial B$. Then cap product with the fundamental class $[M] \in H_n(M, \partial M; R)$ gives an isomorphism

$$D_M: H^k(M, A; R) \xrightarrow{\cong} H_{n-k}(M, B; R)$$

for all $k$.

((Deduce for $A = \partial M$ from Poincaré duality, etc.))
Chapter 6

Vector bundles and classifying spaces

We now follow Milnor and Stasheff’s book “Characteristic Classes” [3].

6.1 Real vector bundles

Definition 6.1.1. A family $\xi$ of (real) vector spaces is a projection map

$$\pi: E \to B$$

from the total space $E = E(\xi)$ to the base space $B = B(\xi)$, together with the structure of a (real) vector space on the fiber $F_b = F_b(\xi) = \pi^{-1}(b)$, for each $b \in B$. If each vector space $F_b$ has dimension $n$ we say that $\xi$ is a family of $n$-dimensional vector spaces.

Example 6.1.2. Let $B$ be any space. The trivial family $\epsilon^n_B$ of $n$-dimensional vector spaces is the projection

$$\pi: B \times \mathbb{R}^n \to B$$

taking $(b, x)$ to $b$ for $x \in \mathbb{R}^n$.

Definition 6.1.3. A map $\xi \to \eta$ of families of vector spaces is a pair of maps $g: E(\xi) \to E(\eta)$ and $\bar{g}: B(\xi) \to B(\eta)$, such that

$$\begin{array}{ccc}
E(\xi) & \xrightarrow{g} & E(\eta) \\
\pi & & \pi \\
B(\xi) & \xrightarrow{\bar{g}} & B(\eta)
\end{array}$$

commutes and $g$ restricts to a linear homomorphism $F_b(\xi) \to F_{\bar{g}(b)}(\eta)$ for each $b \in B(\xi)$.

If $\xi$ and $\eta$ have the same base space $B = B(\xi) = B(\eta)$, then a map $\xi \to \eta$ over $B$ is a map of families $(g, \bar{g})$ such that $\bar{g} = id_B$. 

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Lemma 6.1.4. A map \((g, \bar{g})\) of families of vector spaces is an isomorphism if and only if \(g\) and \(\bar{g}\) are homeomorphisms.

In particular, an isomorphism \(\xi \cong \eta\) over \(B\) is a homeomorphism \(g: E(\xi) \to E(\eta)\) such that
\[
\begin{array}{ccc}
E(\xi) & \xrightarrow{g} & E(\eta) \\
\pi & \cong & \pi \\
B & \downarrow & B
\end{array}
\]
commutes and \(g\) restricts to a linear isomorphism \(F_b(\xi) \cong F_b(\eta)\) for each \(b \in B\).

Definition 6.1.5. A trivialization of a family \(\xi\) of vector spaces is an isomorphism \(\epsilon^n_B \cong \xi\) over \(B = B(\xi)\).

More explicitly, this is a homeomorphism \(h: B \times \mathbb{R}^n \to E(\xi)\) such that \(x \mapsto h(b, x)\) is a linear isomorphism from \(\mathbb{R}^n\) to \(F_b(\xi)\), for each \(b \in B\).

Definition 6.1.6. If \(\pi: E \to B\) is a family of vector spaces \(\xi\), and \(A \subset B\) is a subspace, then the restriction \(\xi|A\) is the family of vector spaces with total space
\[E(\xi|A) = \pi^{-1}(A),\]
base space \(B(\xi|A) = A\) and projection map \(\pi^{-1}(A) \to A\) given by restricting \(\pi\). There is a canonical map \(\xi|A \to \xi\), given by the inclusions \(i\) and \(\bar{i}\) that make the diagram
\[
\begin{array}{ccc}
E(\xi|A) & \xrightarrow{i} & E(\xi) \\
\pi & \downarrow & \pi \\
A & \xrightarrow{\bar{i}} & B
\end{array}
\]
commute.

Definition 6.1.7. A (real) vector bundle \(\xi\) is a locally trivial family of (real) vector spaces \(\pi: E \to B\), meaning that \(B\) has a cover by open subsets \(U\) such that each restriction \(\xi|U\) admits a trivialization.

More explicitly, this trivialization is a homeomorphism \(h: U \times \mathbb{R}^n \to \pi^{-1}(U)\) such that \(x \mapsto h(b, x)\) is a linear isomorphism from \(\mathbb{R}^n\) to \(F_b(\xi)\), for each \(b \in U\).

An \(n\)-dimensional real vector bundle is also called an \(\mathbb{R}^n\text{-bundle}\). A 1-dimensional bundle is often called a line bundle.

If \(\xi\) and \(\eta\) are vector bundles, then a bundle map \(\xi \to \eta\) is the same as a map of the underlying families of vector spaces.

Example 6.1.8. The trivial family \(\epsilon^n_B\) of vector spaces is a vector bundle, called the trivial vector bundle over \(B\). A vector bundle that is isomorphic to the trivial bundle is said to be trivialized, or trivializable, or just a trivial bundle.

Example 6.1.9. The tangent bundle \(\tau_M\) of a smooth \(n\)-manifold \(M\) is an \(n\)-dimensional real vector bundle \(\pi: TM \to M\), with fiber \(\pi^{-1}(p) = T_pM\) for \(p \in M\) given by the tangent space to \(M\) at that point.

If \(\tau_M\) admits a trivialization, we say that \(M\) is parallelizable. An open submanifold of \(\mathbb{R}^n\) is parallelizable. The 2-sphere \(S^2\) is not parallelizable.
Example 6.1.10. Suppose that $M \subset \mathbb{R}^{n+k}$ is embedded as a smooth submanifold of $\mathbb{R}^{n+k}$. The embedding identifies each tangent space $T_pM$ with a subspace of $T_p\mathbb{R}^{n+k} \cong [0, \mathbb{R}^{n+k}]$. The normal bundle $\nu_M$ is the $k$-dimensional real vector bundle $\pi: E(\nu_M) \to M$, with fiber

$$\pi^{-1}(p) = (T_pM)^{\perp}$$

consisting of the orthogonal complement of $T_pM$ in $\mathbb{R}^{n+k}$.

The normal bundle for the usual embedding $S^n \subset \mathbb{R}^{n+1}$ admits a trivialization $e^{1}_{\nu_{n}} \cong \nu_{S^{n}}$, taking $(p, t) \in S^{n} \times \mathbb{R}$ to $(p, tp)$ in $E(\nu_{S^{n}})$, i.e., $t$ times the outward pointing unit normal to $S^n$ at $p$.

Example 6.1.11. Let $\mathbb{R}P^n$ be the $n$-dimensional real projective space, given as the orbit space for the antipodal action on $S^n$, or equivalently as the space of lines through the origin in $\mathbb{R}^{n+1}$. The canonical line bundle $\gamma^{1}_{n} = \gamma^{1}(\mathbb{R}^{n+1})$ over $\mathbb{R}P^n$ has total space

$$E(\gamma^{1}_{n}) = \{(L, v) \mid v \in L \subset \mathbb{R}^{n+1}\}$$

$$\subset \mathbb{R}P^n \times \mathbb{R}^{n+1}$$

consisting of the pairs $(L, v)$ with $L \subset \mathbb{R}^{n+1}$ a 1-dimensional subspace, and $v \in L$ a point on that line.

The space $\mathbb{R}P^n$ is covered by the open subspaces $U_i \cong \mathbb{R}^n$, consisting of the lines through the points $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ with $x_i = 1$, for each of the cases $0 \leq i \leq n$. The restriction of $\gamma^{1}_{n}$ to $U_i$ is trivialized by the homeomorphism

$$U_i \times \mathbb{R} \overset{\cong}{\to} \pi^{-1}(U_i)$$

over $U_i$, taking a pair $(L, t)$, for $L \subset U_i$ the line through $x$ with $x_i = 1$ and $t \in \mathbb{R}$, to the pair $(L, tx)$ in $E(\gamma^{1}_{n})$. Hence $\gamma^{1}_{n}$ is a line bundle. Similarly, there is a canonical line bundle $\gamma^{1}$ over $\mathbb{R}P^\infty$, and the restriction of $\gamma^{1}$ to $\mathbb{R}P^n \subset \mathbb{R}P^\infty$ is $\gamma^{1}_{n}$.

Definition 6.1.12. A section in a vector bundle $\pi: E \to B$ is a map $s: B \to E$ such that $\pi \circ s = id_B$. It takes each $b \in B$ to the fiber $F_b = \pi^{-1}(b)$ over that point.

A section $s$ is nowhere zero if the vector $s(b)$ is not the zero vector in $F_b$, for each $b \in B$.

A $k$-tuple of sections $s_1, \ldots, s_k$ in $\pi: E \to B$ is nowhere dependent if the vectors $s_1(b), \ldots, s_k(b)$ are linearly independent in $F_b$, for each $b \in B$.

Theorem 6.1.13. The bundle $\gamma^{1}_{n}$ over $\mathbb{R}P^n$ is not trivial, for each $n \geq 1$.

Proof. A trivial line bundle admits a section that is nowhere zero. We show that every section of $\gamma^{1}_{n}$ is zero somewhere in $\mathbb{R}P^n$.

Let $s: \mathbb{R}P^n \to E(\gamma^{1}_{n})$ be a section, and consider the composite

$$S^n \overset{q}{\to} \mathbb{R}P^n \overset{s}{\to} E(\gamma^{1}_{n})$$

that first maps $x \in S^n$ to the line $L$ through $x$, then to the value $s(L) = (L, t(x)x)$ of the section at that line, for some real number $t(x)$. The function $t: S^n \to \mathbb{R}$ is continuous, and satisfies $t(-x) = -t(x)$, since $x$ and $-x$ both map to $s(L)$, so that $t(-x)(-x) = t(x)x$. For $n \geq 1$ the space $S^n$ is connected, so by the intermediate value theorem there has to be a point $x \in S^n$ with $t(x) = 0$, which means that $s(L)$ is the zero vector for $L$ equal to the line though $x$. \qed
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Theorem 6.1.14. An $\mathbb{R}^n$-bundle $\xi$ is trivial if and only if it admits $n$ sections $s_1, \ldots, s_n$ that are nowhere dependent.

Proof. A trivialization

$$h: B \times \mathbb{R}^n \longrightarrow E(\xi)$$

over $B$ determines a nowhere dependent $n$-tuple of sections by the formulas

$$s_i(b) = h(b, e_i)$$

for $b \in B$, where $e_i \in \mathbb{R}^n$ is the $i$-th standard basis vector, for $1 \leq i \leq n$.

Conversely, the sections determine the trivialization by the formula

$$h(b, x) = \sum_{i=1}^n x_i s_i(b)$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The resulting map $h$ is continuous, and restricts to a linear isomorphism on each fiber. It follows that $h$ is a homeomorphism, by using that the inverse $A^{-1}$ of an invertible $n \times n$ matrix $A$ depends continuously on $A$. \hfill \Box

Example 6.1.15. A section in the tangent bundle $TM \to M$ of a smooth manifold is the same as a vector field on $M$. Hence an $n$-manifold $M$ is parallelizable if and only if it admits $n$ vector fields that are nowhere dependent.

Example 6.1.16. The circle $S^1$ is parallelizable. One nowhere zero vector field is given by $s(x) = (x, \bar{s}(x))$, where

$$\bar{s}(x_1, x_2) = (-x_2, x_1).$$

Note that $i(x_1 + ix_2) = -x_2 + ix_1$ in $\mathbb{C}$.

The 3-sphere $S^3$ is also parallelizable. Three nowhere dependent vector fields are given by $s_i(x) = (x, \bar{s}_i(x))$, where

$$\bar{s}_1(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$$

$$\bar{s}_2(x_1, x_2, x_3, x_4) = (-x_3, x_4, x_1, -x_2)$$

$$\bar{s}_3(x_1, x_2, x_3, x_4) = (-x_4, -x_3, x_2, x_1).$$

Note that $i(x_1 + ix_2 + jx_3 + kx_4) = -x_2 + ix_1 - jx_4 + kx_3$ etc. in the quaternions.

More generally, any Lie group $G$ is parallelizable, since left multiplication $L_g: G \to G$ by an element $g$ induces an isomorphism $(L_g)_*: T_eG \to T_gG$ of tangent spaces. These combine to a trivialization

$$G \times T_eG \xrightarrow{\cong} TG$$

over $G$. Here $T_eG = g$ is the Lie algebra of $G$.

6.2 Other kinds of vector bundles

Definition 6.2.1. We define families of complex vector spaces and complex vector bundles in the same way as in the real case, by replacing real vector spaces, real linear homomorphisms and the standard $n$-dimensional real vector space $\mathbb{R}^n$ by complex vector spaces, complex linear homomorphisms and the standard $n$-dimensional complex vector space $\mathbb{C}^n$, respectively.
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Example 6.2.2. Let $\mathbb{C}P^n$ be the $n$-dimensional complex projective space, given as the orbit space for the free action of the complex numbers of unit length $S^1 \subset \mathbb{C}$ on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, or equivalently as the space of complex lines through the origin in $\mathbb{C}^{n+1}$. The canonical complex line bundle $\gamma^1_n = \gamma^1(\mathbb{C}^{n+1})$ over $\mathbb{C}P^n$ has total space $E(\gamma^1_n) = \{ (L, v) \mid v \in L \subset \mathbb{C}^{n+1} \}$ consisting of the pairs $(L, v)$ with $L \subset \mathbb{C}^{n+1}$ a 1-dimensional complex subspace, and $v \in L$ a point on that complex line. Similarly, there is a canonical complex line bundle $\gamma^1_\infty = \gamma^1(\mathbb{C}^\infty)$ over $\mathbb{C}P_\infty$, and the restriction of $\gamma^1_\infty$ to $\mathbb{C}P^n \subset \mathbb{C}P_\infty$ is $\gamma^1_n$.

Definition 6.2.3. A real inner product space is a real vector space $V$ with a symmetric, bilinear and positive definite pairing $V \times V \to \mathbb{R}$, taking $(v, w)$ to $\langle v, w \rangle$. A complex inner product space is a complex vector space $V$ with a pairing $V \times V \to \mathbb{C}$, taking $(v, w)$ to $\langle v, w \rangle$, that is conjugate symmetric, complex linear in the first variable, and positive definite.

A linear homomorphism $f : V \to W$ of (real or complex) inner product spaces is an isometry if $\langle f(v), f(v') \rangle = \langle v, v' \rangle$ for all $v, v' \in V$. Such a homomorphism is necessarily injective.

Example 6.2.4. The Euclidean real inner product on $\mathbb{R}^n$ takes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ to $\sum_{i=1}^n x_i y_i$.

The Hermitian complex inner product on $\mathbb{C}^n$ takes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ to $\sum_{i=1}^n x_i \bar{y}_i$.

Definition 6.2.5. Euclidean vector bundles are defined like real vector bundles, replacing real vector spaces by real inner products spaces, real linear homomorphisms by isometries, and equipping the standard vector space $\mathbb{R}^n$ with the Euclidean inner product.

Hermitian vector bundles are defined like complex vector bundles, replacing complex vector spaces by complex inner products spaces, complex linear homomorphisms by isometries, and equipping the standard vector space $\mathbb{C}^n$ with the Hermitian inner product.

Example 6.2.6. Suppose again that $M \subset \mathbb{R}^{n+k}$ is embedded as a smooth $n$-dimensional submanifold. The Euclidean inner product on $T_p \mathbb{R}^{n+k} \cong \mathbb{R}^{n+k}$ restricts to an Euclidean inner product on the subspace $T_p M \subset T_p \mathbb{R}^{n+k}$, for each $p \in M$. Hence $\tau_M$ becomes a Euclidean vector bundle, and the bundle map $\tau_M \to \tau_{\mathbb{R}^{n+k}}$ is an isometry.
6.3 Constructing new bundles out of old

**Definition 6.3.1.** Let $\pi: E \rightarrow B$ be a vector bundle. If $f: A \rightarrow B$ is any map, then the pullback $f^*\xi$ is the vector bundle $\pi: E(f^*\xi) \rightarrow A$, where

$$E(f^*\xi) = \{ (a, e) \in A \times E \mid f(a) = \pi(e) \}$$

is the fiber product of $A$ and $E = E(\xi)$ over $B = B(\xi)$, and $\pi(a, e) = a$. We give $F_a(E(f^*\xi)) = \pi^{-1}(a)$ the vector space structure that makes the homeomorphism $F_a(E(f^*\xi)) \sim \pi^{-1}(a)$ given by $(a, e) \mapsto e$ a linear isomorphism. To prove that $f^*\xi$ is a vector bundle, one can check that if the restriction of $\xi$ to $U \subset B$ admits a trivialization, then the restriction of $f^*\xi$ to $f^{-1}(U) \subset A$ also admits a trivialization.

There is a canonical map $f^*\xi \rightarrow \xi$, given by the maps $\hat{f}: E(f^*\xi) \rightarrow E(\xi)$ taking $(a, e)$ to $e$ and $f: A \rightarrow B$, making the diagram

$$
\begin{array}{ccc}
E(f^*\xi) & \xrightarrow{\hat{f}} & E(\xi) \\
\downarrow{\pi} & & \downarrow{\pi} \\
A & \xrightarrow{f} & B
\end{array}
$$

commute.

If $\eta \rightarrow \xi$ is any bundle map $(g, \bar{g})$, with $\bar{g} = f: A \rightarrow B$, then it factors uniquely as the composite of of a bundle map $(k, \text{id}_A): \eta \rightarrow f^*\xi$ over $A$ and the canonical map $(\hat{f}, f): f^*\xi \rightarrow \xi$:

$$
\begin{array}{ccc}
E(\eta) & \xrightarrow{k} & E(f^*\xi) & \xrightarrow{f} & E(\xi) \\
\downarrow{\pi} & & \downarrow{\pi} & & \downarrow{\pi} \\
A & \xrightarrow{\text{id}_A} & A & \xrightarrow{f} & B
\end{array}
$$

Here $k(e) = (\pi(e), g(e)) \in E(f^*\xi) \subset A \times E(\xi)$ for $e \in E(\eta)$.

**Example 6.3.2.** If $i: A \rightarrow B$ is an inclusion, then the pullback $i^*\xi$ is naturally isomorphic to the restriction $\xi|_A$.

**Example 6.3.3.** If $f: A \rightarrow B$ is a constant map to a point $b \in B$, then $f^*\xi$ is isomorphic to the trivial bundle $\epsilon^n_A$, since $E(f^*\xi) = A \times F_b(\xi)$ and $\pi: E(f^*\xi) \rightarrow B(f^*\xi) = A$ takes $(a, e)$ to $a$.

**Example 6.3.4.** The pullback of the canonical line bundle $\gamma^1_n$ over $\mathbb{R}P^n$ along the covering map $q: S^n \rightarrow \mathbb{R}P^n$ is the normal bundle of $S^n \subset \mathbb{R}^{n+1}$, which is trivial:

$$q^*\gamma^1_n \cong \nu_{S^n} \cong \epsilon^1_{S^n}.$$

**Example 6.3.5.** The pullbacks of complex, Euclidean or Hermitian vector bundles are similarly defined.
Definition 6.3.6. The Cartesian product $\xi_1 \times \xi_2$ of two vector bundles $\xi_1$ and $\xi_2$, with projection maps $\pi_1 : E(\xi_1) \to B(\xi_1)$ and $\pi_2 : E(\xi_2) \to B(\xi_2)$, is the bundle with projection map

$$\pi_1 \times \pi_2 : E(\xi_1) \times E(\xi_2) \longrightarrow B(\xi_1) \times B(\xi_2).$$

The fiber $F_{(b_1, b_2)}(\xi_1 \times \xi_2) = F_{b_1}(\xi_1) \times F_{b_2}(\xi_2)$ has the vector space structure given by the product on the right.

Definition 6.3.7. Let $\xi$ and $\eta$ be vector bundles over the same base space $B$. The Whitney sum $\xi \oplus \eta$ is the vector bundle over $B$ defined by the pullback

$$\xi \oplus \eta = \Delta^*(\xi \times \eta)$$

of the Cartesian product $\xi \times \eta$ along the diagonal map $\Delta : B \to B \times B$. The fiber $F_b(\xi \oplus \eta) = F_b(\xi) \times F_b(\eta)$ is isomorphic to the direct sum $F_b(\xi) \oplus F_b(\eta)$, for each $b \in B$.

Example 6.3.8. There are natural isomorphisms $\xi \oplus \eta \cong \eta \oplus \xi$ and $\epsilon^m_B \oplus \epsilon^n_B \cong \epsilon^{m+n}_B$.

Example 6.3.9. We can recover the Cartesian product from the Whitney sum, as there is a canonical isomorphism

$$\xi_1 \times \xi_2 \cong p_1^*(\xi_1) \oplus p_2^*(\xi_2)$$

where $p_i : B(\xi_1) \times B(\xi_2) \to B(\xi_i)$ is the projection to the $i$-th factor, for $i = 1, 2$.

Example 6.3.10. The Cartesian product and Whitney sum of complex, Euclidean and Hermitian vector bundles are similarly defined. If $V$ and $W$ are inner product spaces, then the inner product on $V \oplus W \cong V \times W$ is given by

$$\langle (v, w), (v', w') \rangle = \langle v, v' \rangle + \langle w, w' \rangle.$$

This is also known as the orthogonal sum $V \perp W$.

Definition 6.3.11. Let $\mathcal{V}$ be the topological category of finite dimensional vector spaces and linear isomorphisms. This means that for each pair of vector spaces $V, W$ the set $\mathcal{V}(V, W)$ of linear isomorphisms $V \cong W$ is topologized as an open subspace of the vector space of linear homomorphisms $V \to W$. The composition rule $\mathcal{V}(V, W) \times \mathcal{V}(U, V) \to \mathcal{V}(U, W)$ is then continuous.

Let $T : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{V}$ be a continuous functor in $k$ variables. This means that the rule

$$\mathcal{V}(V_1, W_1) \times \cdots \times \mathcal{V}(V_k, W_k) \longrightarrow \mathcal{V}(T(V_1, \ldots, V_k), T(W_1, \ldots, W_k))$$

that assigns the induced isomorphism

$$T(\varphi_1, \ldots, \varphi_k) : T(V_1, \ldots, V_k) \cong T(W_1, \ldots, W_k)$$

to each $k$-tuple of isomorphisms $\varphi_1 : V_1 \cong W_1, \ldots, \varphi_k : V_k \cong W_k$, is a continuous map.
Let $\xi_1, \ldots, \xi_k$ be a $k$-tuple of vector bundles over the same base space $B$. We define the vector bundle $T(\xi_1, \ldots, \xi_k)$ over $B$ to have fibers

$$F_b = T(F_b(\xi_1), \ldots, F_b(\xi_k)),$$

total space $E = \coprod_{b \in B} F_b$, and projection $\pi : E \to B$ taking all of $F_b$ to $b$. The topology on $E$ is determined as follows. Each point $b \in B$ has an open neighborhood $U$ such that each bundle $\xi_i|U$ admits a trivialization $h_i : \xi_i|U \cong \xi_i|U$, for some non-negative integers $n_1, \ldots, n_k \geq 0$. The linear isomorphisms $\phi_i = h_i, b : \mathbb{R}^{n_i} \cong F_b(\xi_i)$ for $b \in U$ induce a linear isomorphism

$$T(\phi_1, \ldots, \phi_k) : T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \cong F_b$$

for each $b \in U$, and these combine to a bijective function

$$h : U \times T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \longrightarrow \pi^{-1}(U)$$

over $U$. We give $\pi^{-1}(U)$ the topology that makes $h$ a homeomorphism. Since $T$ is a continuous functor, this topology does not depend on the choices of trivializations $h_i$. These subsets $\pi^{-1}(U)$ cover $E$, and we give $E$ the finest topology that makes each inclusion $\pi^{-1}(U) \to E$ continuous.

**Definition 6.3.12.** Let $\eta$ be a Euclidean vector bundle, and let $\xi \subset \eta$ be a subbundle over the same base space $B$. Let $F_b(\xi^\perp) \subset F_b(\eta)$ be the orthogonal complement of $F_b(\xi)$, for each $b \in B$, and let $E(\xi^\perp) \subset E(\eta)$ be the disjoint union of the fibers $F_b(\xi^\perp)$. Then $\xi^\perp \subset \eta$ is a subbundle, and the canonical map

$$\xi \oplus \xi^\perp \xrightarrow{\cong} \eta$$

given by the sum $F_b(\xi) \oplus F_b(\xi^\perp) \to F_b(\eta)$ in each fiber, is an isomorphism. We call $\xi^\perp$ the **orthogonal complement** of $\xi$ in $\eta$.

**Example 6.3.13.** Let $\xi$ be a real vector bundle, and let

$$F_b(\xi^*) = \text{Hom}_\mathbb{R}(F_b(\xi), \mathbb{R})$$

be the linear dual to $F_b(\xi)$, for each $b \in B$. Let $E(\xi^*)$ be the disjoint union of the fibers $F_b(\xi^*)$. Then $\xi^*$ is a vector bundle.

If $\xi$ is a Euclidean bundle, then the inner product on $F_b(\xi)$ induces a linear isomorphism $F_b(\xi) \to \text{Hom}_\mathbb{R}(F_b(\xi), \mathbb{R})$ in each fiber. These combine to an isomorphism

$$\xi \xrightarrow{\cong} \xi^*$$

of vector bundles.

**Example 6.3.14.** Let $\xi$ be a complex vector bundle, and let

$$F_b(\xi^*) = \text{Hom}_\mathbb{C}(F_b(\xi), \mathbb{C})$$

be the linear dual to $F_b(\xi)$, for each $b \in B$. Let $E(\xi^*)$ be the disjoint union of the fibers $F_b(\xi^*)$. Then $\xi^*$ is a complex vector bundle.

Also let

$$F_b(\xi) = \overline{F_b(\xi)}$$
be the complex conjugate of $F_b(\xi)$, meaning that multiplication by $z = x + iy \in \mathbb{C}$ acts like multiplication by $\bar{z} = x - iy$ in $F_b(\xi)$. Then $\bar{\xi}$ is a complex vector bundle.

If $\xi$ is a Hermitian bundle, then the inner product on $F_b(\xi)$ induces a linear isomorphism $F_b(\xi) \to \text{Hom}(F_b(\xi), \mathbb{C})$ in each fiber, taking $w$ to the homomorphism $v \mapsto \langle v, w \rangle$. These combine to an isomorphism

$$\bar{\xi} \cong \xi^*$$

of complex vector bundles.

((It is not generally the case that $\xi \cong \xi^*$ for complex $\xi$.)

### 6.4 Grassmann manifolds and universal bundles

**Definition 6.4.1.** An $n$-tuple of linearly independent vectors $x_1, \ldots, x_n$ in $\mathbb{R}^{n+k}$ called an $n$-frame. The set of $n$-frames in $\mathbb{R}^{n+k}$ is an open subspace

$$V_n(\mathbb{R}^{n+k}) \subset \mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$$

of Euclidean $n(n+k)$-space, called a *Stiefel manifold*.

Let the *Grassmann manifold* $G_n(\mathbb{R}^{n+k})$ be the set of all $n$-dimensional vector subspaces $X$ of $\mathbb{R}^{n+k}$. There is a canonical map

$$q: V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$$

that takes an $n$-frame $x_1, \ldots, x_n$ to the $n$-dimensional subspace $X$ that it spans.

We give $G_n(\mathbb{R}^{n+k})$ the quotient topology determined by $q$, so a subset $U \subset G_n(\mathbb{R}^{n+k})$ is open if and only if the preimage $q^{-1}(U)$ is open in $V_n(\mathbb{R}^{n+k})$.

**Lemma 6.4.2.** $G_n(\mathbb{R}^{n+k})$ is a closed $nk$-dimensional manifold.

**Proof.** Let $X \in G_n(\mathbb{R}^{n+k})$ be any $n$-dimensional subspace of $\mathbb{R}^{n+k}$, with $k$-dimensional orthogonal complement $X^\perp$. We can identify $X \times X^\perp$ with $X \oplus X^\perp \cong \mathbb{R}^{n+k}$. The graph of any linear homomorphism $f: X \to X^\perp$ is an $n$-dimensional subspace of $X \times X^\perp \cong \mathbb{R}^{n+k}$. The $n$-dimensional subspaces of $\mathbb{R}^{n+k}$ that arise in this way form an open neighborhood of $X$ in $G_n(\mathbb{R}^{n+k})$, and this space is homeomorphic to the space of linear homomorphisms $\mathbb{R}^n \to \mathbb{R}^k$, which is homeomorphic to $\mathbb{R}^{nk}$.

The subspace $V^O_n(\mathbb{R}^{n+k}) \subset V_n(\mathbb{R}^{n+k})$ of orthonormal $n$-tuples is a closed subspace

$$V^O_n(\mathbb{R}^{n+k}) \subset S^{n+k-1} \times \cdots \times S^{n+k-1}$$

of the compact space on the right, hence is compact. The restriction

$$q^O: V^O_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$$

defines the same topology as $q$, hence $G_n(\mathbb{R}^{n+k})$ is compact. \qed

**Example 6.4.3.** $G_1(\mathbb{R}^{n+1})$ is the same as $\mathbb{R}P^n$. 


Definition 6.4.4. The canonical $\mathbb{R}^n$-bundle $\gamma_n(\mathbb{R}^{n+k})$ over $G_n(\mathbb{R}^{n+k})$ has total space

$$E(\gamma_n(\mathbb{R}^{n+k})) = \{(X, v) \mid v \in X \subset \mathbb{R}^{n+k}\}$$

$$\subset G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$$

consisting of the pairs $(X, v)$ with $X \subset \mathbb{R}^{n+k}$ an $n$-dimensional subspace, and $v \in X$ a vector in that subspace. The projection map

$$\pi : E(\gamma_n(\mathbb{R}^{n+k})) \to G_n(\mathbb{R}^{n+k})$$

is given by $\pi(X, v) = X$, and the fiber $F_X = \pi^{-1}(X)$ is identified with the vector space $X$ itself, via the linear isomorphism $v \mapsto (X, v)$.

Lemma 6.4.5. $\gamma_n(\mathbb{R}^{n+k})$ is a vector bundle.

((ETC))

6.5 Oriented bundles and the Euler class

6.6 The Thom isomorphism theorem

6.7 Chern classes

6.8 Pontryagin classes

((ETC))

6.9 Bordism

6.9.1 Thom complexes and the Thom isomorphism

\[
\begin{array}{ccc}
S(\xi) & \longrightarrow & D(\xi) \\
\downarrow \cong & & \downarrow \cong \\
E_0(\xi) & \longrightarrow & E(\xi) \\
\downarrow & & \downarrow \\
B & & (E(\xi), E_0(\xi))
\end{array}
\]
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6.9.2 Tubular neighborhoods and the Pontryagin–Thom construction

$M$ smooth, oriented $n$-manifold.

\[ E(\tilde{\gamma}^n) \to E(\tilde{\gamma}^{n+k}) \]

\[ TM \]

\[ \tilde{G}_n \to \tilde{G}_n^{(R^n+k)} \]

\[ \tau_M \]

\[ \nu_M \]

\[ S^{n+k} \]

\[ S^{n+k} \]

\[ \nu \]

\[ S(\nu) \]

\[ D(\nu) \]

\[ NM \]

\[ E(\tilde{\gamma}^k) \to E(\tilde{\gamma}^{k+n}) \]

\[ S^{n+k} \to S^{n+k} - Int D(\nu_M) \cong \frac{D(\nu_M)}{S(\nu_M)} = Th(\nu_M) \to Th(\tilde{\gamma}^k) = MSO_k \]

6.9.3 Transversality and Thom’s theorem

$s_0: BSO(k) = \tilde{G}_k \to D(\tilde{\gamma}_k) \to Th(\tilde{\gamma}_k) = MSO_k$ has normal bundle $\tilde{\gamma}_k$.

Any base-point preserving map $S^{n+k} \to Th(\tilde{\gamma}_k)$ is homotopic to one that is transverse to $s_0: \tilde{G}_k \to Th(\tilde{\gamma}_k)$. The preimage $M = g^{-1}(\tilde{G}_k)$ is then an $n$-dimensional oriented submanifold of $\mathbb{R}^{n+k} \subset S^{n+k}$.

\[ M \]

\[ \tilde{\gamma}^k \]

\[ \tilde{G}_k \]

\[ g \]

\[ Th(\tilde{\gamma}^k) \]

If $H: S^{n+k} \times I \to Th(\tilde{\gamma}_k)$ is a base-point preserving homotopy from $g$ to $g'$, all of which are transverse to $\tilde{G}_k$, then the preimage $W = H^{-1}(\tilde{G}_k)$ is a cobordism from $M$ to $M' = (g')^{-1}(\tilde{G}_k)$.

Oriented bordism:

\[ \Omega_n^{SO} \cong \text{colim}_k \pi_{n+k} Th(\tilde{\gamma}^k) = \pi_n(MSO) . \]
6.9.4 Homology of Thom spectra

\[ \pi_{n+k} \text{Th}(\nu_M) \to \pi_{n+k} \text{Th}(\tilde{\gamma}^k) \to \pi_n \text{MSO} \]

\[ \tilde{H}_{n+k}(S^{n+k}) \overset{\cong}{\to} \tilde{H}_{n+k}(\text{Th}(\nu_M)) \to \tilde{H}_{n+k}(\text{Th}(\tilde{\gamma}^k)) \to H_n(\text{MSO}) \]

\[ (-) \cap u_M \overset{\cong}{\to} (-) \cap u_k \]

\[ H_n(M) \overset{f_*}{\to} H_n(\tilde{G}_k) \to H_n(\text{BSO}) \]

The Hurewicz image in \( H_n(\text{MSO}) \) of homotopy class in \( \pi_n(\text{MSO}) \cong \Omega^n_{\text{SO}} \) is the image of fundamental class \([S^{n+k}] \in \tilde{H}_{n+k}(S^{n+k})\), which under the Thom isomorphism \( H_n(\text{MSO}) \cong H_n(\text{BSO}) \) corresponds to the image of the fundamental class \([M] \in H_n(M)\) under the classifying map \( f: M \to \tilde{G}_k \to \text{BSO} \) for the normal bundle \( \nu_M \).

Serre:

\[ \pi_n(\text{MSO}) \otimes \mathbb{Q} \overset{\cong}{\to} H_n(\text{MSO}) \otimes \mathbb{Q} \]

6.9.5 Pontryagin numbers

Consider (co-)homology with coefficients in a ring \( R \) containing \( \mathbb{Z}[1/2] \). Recall that

\[ H^*(\tilde{G}_{2m}; R) \cong R[p_1, \ldots, p_{m-1}, e] \]

with \( p_m = e^2 \), and

\[ H^*(\tilde{G}_{2m+1}; R) \cong R[p_1, \ldots, p_m] \]

with \( e = 0 \).

The limit systems

\[ \text{colim}_k H_n(\tilde{G}_k; R) \overset{\cong}{\to} H_n(\text{BSO}; R) \]

and

\[ H^n(\text{BSO}; R) \overset{\cong}{\to} \lim_k H^n(\tilde{G}_k; R) \]

stabilize for a finite \( k \), so the isomorphism \( H_n(\tilde{G}_k; R) \cong \text{Hom}(H^n(\tilde{G}_k; R), R) \), adjoint to the evaluation pairing, stabilizes to an isomorphism

\[ H_n(\text{BSO}; R) \cong \text{Hom}(H^n(\text{BSO}; R), R) \).

Here

\[ H^*(\text{BSO}; R) \cong R[p_i \mid i \geq 1] \).

Hence \( H^n(\text{BSO}; R) = 0 \) unless \( n \) is divisible by four. When \( n = 4k \), it is the free \( R \)-module generated by the monomials

\[ p^I = p_1^{i_1} \cdots p_k^{i_k} \]

where \( I = (i_1, \ldots, i_k) \) is a \( k \)-tuple of non-negative integers such that

\[ i_1 + 2i_2 + \cdots + ki_k = k \).
These correspond to partitions of \( k \), i.e., unordered ways of writing \( k \) as a sum of natural numbers, with \( i_j \) counting how often \( j \) occurs in the sum.

An element \( \alpha \in H_n(BSO) \) is thus in one-to-one correspondence with the collection of numbers

\[
\{ p^I, \alpha \} \in R
\]

where \( I \) ranges over these \( k \)-tuples indexing partitions of \( k \).

Let \( f : M \to BSO \) be the classifying map for the stable normal bundle of \( M \), so that \( p_i(\nu_M) = f^*(p_i) \) for each \( i \). When \( \alpha = f_*[M] \), we can rewrite these numbers as

\[
\{ p^I, f_*[M] \} = \{ f^*(p^I), [M] \} = \{ p^I(\nu_M), [M] \}
\]

where

\[
p^I(\nu_M) = p^i_1(\nu_M) \cdot \ldots \cdot p^i_k(\nu_M)
\]

is a monomial in the Pontryagin classes of the normal bundle of \( M \).

The numbers

\[
\{ p^I(\nu_M), [M] \} \in R
\]

are called the normal Pontryagin numbers of \( M \).

**Theorem 6.9.1.** \( \Omega_n^{SO} \otimes \mathbb{Q} = 0 \) for \( n \) not divisible by four. When \( n = 4k \geq 0 \), the rule taking the oriented cobordism class of \( M \) to the collection of normal Pontryagin numbers

\[
\{ p^I(\nu_M), [M] \} \in \mathbb{Q}
\]

for \( I = (i_1, \ldots, i_k) \) with \( \sum_j j i_j = k \), defines an isomorphism

\[
\Omega_n^{SO} \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_I \mathbb{Q}.
\]

In view of the formula \( \tau_M \oplus \nu_M \cong \epsilon^{n+k} \) and its consequence

\[
p(\tau_M) \cup p(\nu_M) = 1
\]

in \( H^*(M; R) \), it follows that each monomial \( p^I(\nu_M) \) can be written as a polynomial in the tangential Pontryagin classes \( p_i(\tau_M) \), and vice versa.

We can therefore use the monomials

\[
p^I(\tau_M) = p^i_1(\tau_M) \cdot \ldots \cdot p^i_k(\tau_M)
\]

in the tangential Pontryagin classes of \( M \) as alternative operators to detect \( H_n(M; R) \), in place of the normal classes.

**Theorem 6.9.2.** \( \Omega_n^{SO} \otimes \mathbb{Q} = 0 \) for \( n \) not divisible by four. When \( n = 4k \geq 0 \), the rule taking the oriented cobordism class of \( M \) to the collection of (tangential) Pontryagin numbers

\[
p^I(M) = \{ p^I(\tau_M), [M] \} \in \mathbb{Q}
\]

for \( I = (i_1, \ldots, i_k) \) with \( \sum_j j i_j = k \), defines an isomorphism

\[
\Omega_n^{SO} \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_I \mathbb{Q}.
\]

### 6.9.6 Computations in low dimensions

### 6.9.7 The index formula
Bibliography

