POSTNIKOV TOWERS AND k-INVARIANTS FOR S-MODULES, S-ALGEBRAS AND COMMUTATIVE S-ALGEBRAS

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Let A be a connective S-module, (associative) S-algebra or commutative S-algebra. We will discuss its **Postnikov tower**

$$A \to \cdots \to P^n A \xrightarrow{p_n} P^{n-1}A \to \cdots \to P^0 A = H\pi_0(A)$$

in each of these three categories. Here $A \to P^n A$ induces an isomorphism on π_i for $i \le n$, while $\pi_i(P^n A) = 0$ for $i > n$. Hence there is a homotopy cofiber sequence

$$\Sigma^n H\pi_n(A) \to P^n A \xrightarrow{p_n} P^{n-1}A$$

for each $n \ge 1$. We call $P^n A$ the n-th **Postnikov section** of A.

S-module k-invariants.

Suppose first that A is a connective S-module. In the category of S-modules, we can extend the homotopy cofiber sequence to the right by a map

$$k_{n-1} \colon P^{n-1}A \to \Sigma^{n+1} H\pi_n(A)$$

called the $(n-1)$-st (additive) k-invariant of A. Conversely, we can recover $P^n A$ from $P^{n-1}A$ and k_{n-1} as the homotopy fiber of k_{n-1}. By the theorem of Eilenberg and Mac Lane,

$$[P^{n-1}A, \Sigma^{n+1} H\pi_n(A)] \cong H^{n+1}(P^{n-1}A; \pi_n(A))$$

so we may view the S-module k-invariant k_{n-1} as an element in the spectrum cohomology group on the right.

For example, when $A = ku$, $\pi_*(ku) = \mathbb{Z}[u]$ with $|u| = 2$, we get $P^1 ku = P^0 ku = H\mathbb{Z}$, and $P^2 ku$ is determined by the homotopy cofiber sequence

$$\Sigma^2 H\mathbb{Z} \to P^2 ku \xrightarrow{p_2} H\mathbb{Z} \xrightarrow{k_1} \Sigma^3 H\mathbb{Z}$$

where the first S-module k-invariant lies in

$$k_1 \in H^3(H\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/2\{\tilde{Q}_1\}\,.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Here $\tilde{Q}_1$ is the composite operation

$$H\mathbb{Z} \xrightarrow{\pi} H\mathbb{Z}/2 \xrightarrow{Sq^2} \Sigma^2 H\mathbb{Z}/2 \xrightarrow{\beta} \Sigma^3 H\mathbb{Z}\,.$$

In the case of ku, $k_1 = \tilde{Q}_1$ is nontrivial.

((Proof using SU as third space in the Ω-spectrum of ku?))

The existence of the S-module Postnikov tower can be proved by induction on n. Assume for an $n \geq 1$ that we have the n-connected map $A \to P^{n-1}A$. Its homotopy cofiber $P^{n-1}A/A$ is then n-connected, with $\pi_{n+1}(P^{n-1}A/A) \cong \pi_n(A)$. By attaching cells of dimension $\geq (n+2)$, we may kill all the higher homotopy of $P^{n-1}A/A$, so as to get an $(n+2)$-connected map $P^{n-1}A/A \to \Sigma^{n+1}H\pi_n(A)$. The composite map

$$k_{n-1}\colon P^{n-1}A \to P^{n-1}A/A \to \Sigma^{n+1}H\pi_n(A)$$

is then the $(n-1)$-st k-invariant of A. We can define P^nA as the homotopy fiber of k_{n-1}. The composite map $A \to P^{n-1}A \to P^{n-1}A/A$ is null-homotopic, hence $A \to P^{n-1}A$ factors through P^nA. Since $P^{n-1}A/A \to \Sigma^{n+1}H\pi_n(A)$ is a π_{n+1}-isomorphism, $A \to P^nA$ must be a π_n-isomorphism, since $\pi_i(P^{n-1}A) = 0$ for $i \geq n$.

S-algebra k-invariants.

Suppose next that A is a connective S-algebra. In the category of S-algebras, we shall see that we can express P^nA as a homotopy pullback

$$\begin{array}{ccc} P^nA & \longrightarrow & P^{n-1}A \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle in_1} \\ P^{n-1}A & \xrightarrow{a_{n-1}} & P^{n-1}A \vee \Sigma^{n+1}H\pi_n(A) \end{array}$$

where the composite of a_{n-1} with $pr_1\colon P^{n-1}A \vee \Sigma^{n+1}H\pi_n(A) \to P^{n-1}A$ is the identity.

For each S-algebra B and B-bimodule M, let the **square-zero** extension $B \vee M$ be the S-algebra with product

$$(B\vee M)\wedge(B\vee M) \cong (B\wedge B)\vee(B\wedge M)\vee(M\wedge B)\vee(M\wedge M) \to B\vee M\vee M\vee * \to B\vee M$$

where the middle map uses the multiplication in B, the left and right B-module actions on M, and the zero map $M \wedge M \to *$. The right hand map folds the two copies of M together. The inclusion and projection maps

$$B \xrightarrow{in_1} B \vee M \xrightarrow{pr_1} B$$

are maps of S-algebras. We write $pr_2\colon B \vee M \to M$ for the other projection. A map $a\colon B \to B \vee M$ of S-algebras that makes the diagram

$$\begin{array}{ccc} & & B \vee M \\ & {\scriptstyle a}\nearrow & \downarrow{\scriptstyle pr_1} \\ B & \xrightarrow[=]{} & B \end{array}$$

commute is called an **associative derivation** of B with values in M. The space of such maps a is denoted $\mathrm{ADer}(B, M)$.

In the pullback diagram above, $P^{n-1}A \vee \Sigma^{n+1} H\pi_n(A)$ is the S-algebra given as a square-zero extension of $P^{n-1}A$ by the $P^{n-1}A$-bimodule $\Sigma^{n+1} H\pi_n(A)$. The map a_{n-1} is an associative derivation of $P^{n-1}A$ with values in $\Sigma^{n+1} H\pi_n(A)$. The bimodule structure is given by the projection $P^{n-1}A \to P^0 A = H\pi_0(A)$, followed by the left and right module actions of $\pi_0(A)$ on $\pi_n(A)$.

When $B = \mathbb{T}X \simeq \bigvee_{j \geq 0} X^{\wedge j}$ is the free S-algebra generated by an S-module X, the space $\mathrm{ADer}(\mathbb{T}X, M)$ is equivalent to the space of S-module maps

$$a|X \colon X \to \mathbb{T}X \vee M$$

with $pr_1 \circ (a|X) \colon X \to \mathbb{T}X$ equal to the inclusion for $j = 1$, which in turn is equivalent to the space of S-module maps $pr_2 \circ (a|X) \colon X \to M$. These are equivalent to the space $\mathrm{Mod}_{\mathbb{T}X - \mathbb{T}X}(\mathbb{T}X \wedge X \wedge \mathbb{T}X, M)$ of $\mathbb{T}X$-bimodule maps

$$\mathbb{T}X \wedge X \wedge \mathbb{T}X \to M\,.$$

Here the $\mathbb{T}X$-bimodule $\mathbb{T}X \wedge X \wedge \mathbb{T}X$ also arises as the homotopy fiber of the multiplication map on $\mathbb{T}X$, giving the homotopy cofiber sequence

$$\mathbb{T}X \wedge X \wedge \mathbb{T}X \to \mathbb{T}X \wedge \mathbb{T}X \xrightarrow{\mu} \mathbb{T}X\,.$$

For a general S-algebra B, let I_B be the B-bimodule defined by the homotopy cofiber sequence

$$I_B \to B \wedge B \xrightarrow{\mu} B\,.$$

Then there is a homotopy equivalence

$$(*) \qquad \mathrm{ADer}(B, M) \simeq \mathrm{Mod}_{B-B}(I_B, M)$$

between the (derived) space of B-bimodule maps $I_B \to M$ and the space of associative derivations of B with values in M. Andrei Lazarev [L] proves this by resolving B by a simplicial S-algebra that is free in each degree, and using the equivalence above in each simplicial degree.

Topological Hochschild (co-)homology.

We can view $\mathrm{Mod}_{B-B}(I_B, M)$ as the underlying space of the bimodule function spectrum $F_{B-B}(I_B, M)$. The **topological Hochschild cohomology** of B with values in M is defined as the (derived) bimodule function spectrum

$$\mathrm{THH}^\bullet(B, M) = F_{B-B}(B, M) = \mathrm{Ext}_{B-B}(B, M)\,,$$

with homotopy groups

$$\mathrm{THH}^q(B, M) = \pi_{-q}\,\mathrm{THH}^\bullet(B, M)\,.$$

The homotopy cofiber sequence above the gives rise to the homotopy cofiber sequence

$$\mathrm{THH}^\bullet(B, M) \to M \to F_{B-B}(I_B, M)\,.$$

In particular, when $B = P^{n-1}A$ and $M = \Sigma^{n+1} H\pi_n(A)$ with $n \geq 1$ the connecting map

$$\begin{aligned} \pi_0 \operatorname{ADer}(P^{n-1}A, \Sigma^{n+1} H\pi_n(A)) &\cong \pi_0 F_{B-B}(I_B, M) \\ &\xrightarrow{\cong} \pi_{-1} \operatorname{THH}^\bullet(B, M) = \operatorname{THH}^{n+2}(P^{n-1}A, \pi_n(A)) \end{aligned}$$

is a bijection. Hence we may view the S-algebra k-invariant a_{n-1} as an element in the topological Hochschild cohomology group on the right.

The **topological Hochschild homology** of B with values in M is defined as the (derived) bimodule smash product

$$\operatorname{THH}_\bullet(B, M) = M \wedge_{B-B} B = \operatorname{Tor}^{B-B}(M, B)\,.$$

Replacing the right hand B with the two-sided bar construction $\beta(B, B, B)$, with $B \wedge B^{\wedge q} \wedge B$ in simplicial degree q, we get the usual Hochschild complex model for $\operatorname{THH}_\bullet(B, M)$, with $M \wedge B^{\wedge q}$ in simplicial degree q. When $M = B$, we abbreviate $\operatorname{THH}_\bullet(B, B)$ to $\operatorname{THH}(B)$.

When B is a commutative S-algebra and M a B-module, we may view B as an S-algebra and M as a symmetric B-bimodule. In that case there is a base change equivalence

$$F_{B-B}(B, M) \simeq F_B(B \wedge_{B-B} B, M) \simeq F_B(\operatorname{THH}(B), M)\,.$$

For example, when $A = ku$, $n = 2$, $B = P^1 A = H\mathbb{Z}$ and $M = \Sigma^3 H\mathbb{Z}$, the first S-algebra k-invariant lies in

$$a_1 \in \operatorname{THH}^4(H\mathbb{Z}; \mathbb{Z}) \cong \operatorname{Ext}^1_{\mathbb{Z}}(\operatorname{THH}_3(H\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/2\,,$$

as we shall compute later.

Existence of the S-algebra Postnikov tower.

To express $P^n A$ as the homotopy pullback along an associative derivation a_{n-1}, we need an extension of Lazarev's result to S-algebras B under a fixed S-algebra A. For each map $\phi\colon A \to B$ of S-algebras define the B-bimodule $\Omega_{B/A}$ by the homotopy fiber sequence

$$I_{B/A} \to B \wedge_A B \xrightarrow{\mu} B$$

where μ is the multiplication map. By an associative derivation of B over A with values in M we mean a map $a\colon B \to B \vee M$ of S-algebras, under A and over B, meaning that the S-algebra diagram

$$\begin{array}{ccc} A & \xrightarrow{in_1 \phi} & B \vee M \\ {\scriptstyle \phi}\downarrow & \overset{a}{\nearrow} & \downarrow{\scriptstyle pr_1} \\ B & \xrightarrow[=]{} & B \end{array}$$

commutes. Let $\operatorname{ADer}(B/A, M)$ be the space of such maps a. Then there is a homotopy equivalence

$$\operatorname{ADer}(B/A, M) \simeq \operatorname{Mod}_{B-B}(I_{B/A}, M)\,.$$

The identity map of $I_{B/A}$ corresponds to a universal associative derivation $d\colon B \to B \vee I_{B/A}$.

When $A \to B = P^{n-1}A$ is the standard n-connected map, with homotopy cofiber B/A, we compute that $I_{B/A} \simeq B \wedge_A (B/A)$ is n-connected with

$$\pi_{n+1}(I_{B/A}) \cong \pi_0(B) \otimes_{\pi_0(A)} \pi_{n+1}(B/A) \cong \pi_n(A)\,.$$

Hence there is a B-bimodule map

$$I_{B/A} \to \Sigma^{n+1} H\pi_n(A)$$

that corresponds to the associative derivation

$$a_{n-1}\colon P^{n-1}A \to P^{n-1}A \vee \Sigma^{n+1} H\pi_n(A)$$

under A. The homotopy pullback of a_{n-1} and in_1 is then readily seen to be a model for P^nA.

Commutative S-algebra k-invariants.

Suppose now that A is a connective commutative S-algebra. Also in the category of commutative S-algebras, we can express P^nA as a homotopy pullback

$$\begin{array}{ccc} P^nA & \longrightarrow & P^{n-1}A \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle in_1} \\ P^{n-1}A & \xrightarrow{d_{n-1}} & P^{n-1}A \vee \Sigma^{n+1}H\pi_n(A) \end{array}$$

where the composite of d_{n-1} with $pr_1\colon P^{n-1}A \vee \Sigma^{n+1}H\pi_n(A) \to P^{n-1}A$ is the identity.

For each commutative S-algebra B and B-module M, let the square-zero extension $B \vee M$ be defined as before. The maps

$$B \xrightarrow{in_1} B \vee M \xrightarrow{pr_1} B$$

are maps of commutative S-algebras. Let $pr_2\colon B \vee M \to M$ be the other projection. A map $d\colon B \to B \vee M$ of commutative S-algebras that makes the diagram

$$\begin{array}{ccc} & & B \vee M \\ & {\scriptstyle d}\nearrow & \downarrow{\scriptstyle pr_1} \\ B & \xrightarrow[=]{} & B \end{array}$$

commute is called a (commutative) **derivation** of B with values in M. The space of such maps d is denoted $\mathrm{Der}(B, M)$.

In the pullback diagram above, d_{n-1} is a derivation of $P^{n-1}A$ with values in $\Sigma^{n+1}H\pi_n(A)$.

When $B = \mathbb{P}X \simeq \bigvee_{j\geq 0} X^{\wedge j}_{h\Sigma_j}$ is the free commutative S-algebra generated by an S-module X, $\mathrm{Der}(B, M)$ is equivalent to the space of S-module maps

$$d|X\colon X \to \mathbb{P}X \vee M$$

with $pr_1 \circ (d|X)$ equal to the inclusion, which in turn is equivalent to the space of S-module maps $pr_2 \circ (d|X)\colon X \to M$, and to the space of B-module maps $B \wedge X \to M$. The B-module $B \wedge X$ can be obtained from the commutative S-algebra $B = \mathbb{P}X$ by a process of stabilization, which we discuss next

Topological André–Quillen (co-)homology.

First we pass to a based situation, by base changing along $S \to B$, taking B to the augmented commutative B-algebra

$$B \wedge B \,.$$

The B-algebra unit map is the composite $B \cong B \wedge S \to B \wedge B$. The augmentation is the multiplication $\mu \colon B \wedge B \to B$.

In the pointed category $\mathcal{C}_B/B$ of commutative B-algebras over B, there is a suspension operator E taking C to the (derived) smash product

$$E(C) = B \wedge_C B = \mathrm{Tor}^C(B, B) \,.$$

This is the homotopy pushout in $\mathcal{C}_B/B$ of the diagram $B \leftarrow C \to B$, and can be computed by replacing the right hand B by the two-sided bar construction $\beta(B, B, B)$. When $C = B \wedge B$, we see that $E(B \wedge B) = \mathrm{THH}(B)$.

For comparison, the suspension ΣY of a based space Y is the homotopy pushout in based spaces of the diagram $* \leftarrow Y \to *$, which is the pushout of $* \leftarrow Y \to CY$.

To stabilize, we iterate the suspension process, forming a suspension E-spectrum

$$\{m \mapsto E^m(B \wedge B)\} \,.$$

Such E-spectra form a stable category $\mathcal{S}pt(\mathcal{C}_B/B)$. By a theorem of Maria Basterra and Mike Mandell [BM], there are equivalences

$$\mathcal{S}pt(\mathcal{C}_B/B) \xrightarrow{\simeq} \mathcal{S}pt(\mathrm{Mod}_B) \xleftarrow{\simeq} \mathrm{Mod}_B \,.$$

The first equivalence is given by taking an E-spectrum

$$\{m \mapsto D_m\}$$

in $\mathcal{C}_B/B$ to the Σ-spectrum

$$\{m \mapsto D_m/B\}$$

in Mod_B. Here D_m/B denotes the unit cofiber, or equivalently, the augmentation fiber, of D_m.

((Discuss how $ED_m \to D_{m+1}$ induces $\Sigma(D_m/B) \to D_{m+1}/B$.))

The second equivalence takes a B-module M to its suspension Σ-spectrum $\{m \mapsto \Sigma^m M\}$.

The composite construction takes a commutative S-algebra B first to $B \wedge B$ in $\mathcal{C}_B/B$, then to its E-suspension spectrum $\{m \mapsto E^m(B \wedge B)\}$, then to its unit cofibers $\{m \mapsto E^m(B \wedge B)/B\}$, and then to the B-module

$$\mathrm{TAQ}(B) = \mathrm{hocolim}_m \, \Sigma^{-m} E^m(B \wedge B)/B \,.$$

This provides one definition of the **topological André–Quillen homology** of B, also denoted Ω^1_B.

Another definition is given by the (left derived) commutative B-algebra indecomposables Q_B of the (right derived) augmentation ideal I_B of the (left derived) smash product $B \wedge B$ in $\mathcal{C}_B/B$:

$$\mathrm{TAQ}(B) = Q_B I_B(B \wedge B) \,.$$

When $B = \mathbb{P}X$ is free, we get that $B \wedge B = B \wedge \mathbb{P}X$. The free functor $\mathbb{P}$ takes homotopy pushouts of S-modules to homotopy pushouts of augmented commutative S-algebras, so there are equivalences $S \wedge_{\mathbb{P}X} S \simeq \mathbb{P}\Sigma X$ and $E(B \wedge \mathbb{P}X) \simeq B \wedge \mathbb{P}\Sigma X$. Hence

$$E^m(B \wedge B) \simeq B \wedge \mathbb{P}\Sigma^m X$$

for all $m \geq 0$. When X is connective, the inclusion

$$B \wedge (S \vee \Sigma^m X) \to B \wedge \mathbb{P}\Sigma^m X$$

of the $0 \leq j \leq 1$ summands is about $2m$-connected, so

$$B \wedge X = \Sigma^{-m} B \wedge \Sigma^m X \to \Sigma^{-m} E^m(B \wedge B)/B$$

is about m-connected. In the colimit, we get an equivalence

$$B \wedge X \simeq \mathrm{TAQ}(\mathbb{P}X)$$

of B-modules, for $B = \mathbb{P}X$.

For general B, there is a homotopy equivalence

$$\text{(**)} \qquad \mathrm{Der}(B, M) \simeq \mathrm{Mod}_B(\mathrm{TAQ}(B), M)$$

between the (derived) space of B-module maps $\mathrm{TAQ}(B) \to M$ and the space of derivations of B with values in M. Basterra [B] proved this by means of the second definition of $\mathrm{TAQ}(B)$, which is equivalent to the first one by the joint paper of Basterra and Mandell.

Let

$$\begin{aligned} \mathrm{TAQ}_q(B, M) &= \pi_q(M \wedge_B \mathrm{TAQ}(B)) \\ \mathrm{TAQ}^q(B, M) &= \pi_{-q}(F_B(\mathrm{TAQ}(B), M) \end{aligned}$$

be the topological André–Quillen homology and cohomology groups of B with coefficients in M. We abbreviate $\mathrm{TAQ}_q(B, B)$ to $\mathrm{TAQ}_q(B) = \pi_q \, \mathrm{TAQ}(B)$.

In particular, when $B = P^{n-1}A$ and $M = \Sigma^{n+1} H\pi_n(A)$ with $n \geq 1$ the commutative S-algebra k-invariant d_{n-1} is an element in the topological André–Quillen cohomology group on the right.

$$\begin{aligned} &\pi_0 \, \mathrm{Der}(P^{n-1}A, \Sigma^{n+1} H\pi_n(A)) \\ &\qquad\qquad \cong \pi_0 F_B(\mathrm{TAQ}(B), M) = \mathrm{TAQ}^{n+1}(P^{n-1}A, \pi_n(A)) \, . \end{aligned}$$

For example, when $A = ku$, $n = 2$, $B = P^1 A = H\mathbb{Z}$ and $M = \Sigma^3 H\mathbb{Z}$, the first commutative S-algebra k-invariant lies in

$$d_1 \in \mathrm{TAQ}^3(H\mathbb{Z}; \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{TAQ}_2(H\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/2 \, ,$$

as we shall soon compute.

Comparison maps.

We may view a commutative S-algebra A as an (associative) S-algebra or just an S-module. The extension $P^nA \to P^{n-1}A = B$ is the classified by an element in one of the sets

$$\pi_0 \operatorname{Der}(B, \Sigma^{n+1} H\pi_n(A)) \to \pi_0 \operatorname{ADer}(B, \Sigma^{n+1} H\pi_n(A)) \to [B, \Sigma^{n+1} H\pi_n(A)]$$

where the maps correspond to forgetting some of the structure. As discussed above, this diagram is isomorphic to

$$\mathrm{TAQ}^{n+1}(B, \pi_n(A)) \to \mathrm{THH}^{n+2}(B, \pi_n(A)) \to H^{n+1}(B, \pi_n(A))\,.$$

These groups consist of the homotopy classes of B-module maps to $\Sigma^{n+1} H\pi_n(A)$ from the diagram

$$B \wedge B \to \Sigma^{-1} \mathrm{THH}(B) \to \mathrm{TAQ}(B)\,,$$

in reverse order. Since $n \geq 1$, we may replace $B \wedge B$ by $(B \wedge B)/B$, and $\mathrm{THH}(B)$ by $\mathrm{THH}(B)/B$, so that this diagram consists of the terms $m = 0$, $m = 1$ and the homotopy colimit as $m \to \infty$ of the diagram

$$\cdots \to \Sigma^{-m} E^m (B \wedge B)/B \to \dots\,.$$

By an inspection of the various identifications made, the forgetful maps above correspond to the stabilization map

$$\Sigma E^m (B \wedge B)/B \to E^{m+1}(B \wedge B)/B$$

for $m = 0$, and their composites as $m \to \infty$.

Calculations for $H\mathbb{Z}$.

We consider the case $B = H\mathbb{Z}$ and $n = 2$, and show that the homomorphisms

$$\mathrm{TAQ}^3(H\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cong} \mathrm{THH}^4(H\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cong} H^3(H\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/2\{\tilde{Q}_1\}$$

are all isomorphisms, by showing that the maps

$$(H\mathbb{Z} \wedge H\mathbb{Z})/H\mathbb{Z} \to \Sigma^{-1} \mathrm{THH}(H\mathbb{Z})/H\mathbb{Z} \to \mathrm{TAQ}(H\mathbb{Z})$$

induce isomorphisms on π_2 (where all groups are $\mathbb{Z}/2$) and on π_3 (where all groups are 0).

Write $H_*(X)$ for $H_*(X; \mathbb{F}_p)$ and consider $p = 2$. Then

$$H_*(H\mathbb{Z}) \cong P(\bar{\xi}_1^2, \bar{\xi}_k \mid k \geq 2)$$

with $|\bar{\xi}_1^2| = 2$ and $|\bar{\xi}_k| = 2^k - 1$. Hence

$$H_*(H\mathbb{Z} \wedge H\mathbb{Z}) \cong H_*(H\mathbb{Z}) \otimes H_*(H\mathbb{Z}) \cong H_*(H\mathbb{Z})\{1, \bar{\xi}_1^2, \bar{\xi}_2, \dots\}$$

with the remaining terms in dimensions ≥ 4. Here $\beta(\bar{\xi}_2) = \bar{\xi}_1^2$, so

$$\pi_*(H\mathbb{Z} \wedge H\mathbb{Z}) \cong (\mathbb{Z}\{1\}, 0, \mathbb{Z}/2\{\bar{\xi}_1^2\}, 0, \dots)\,.$$

We use the bar spectral sequence

$$E^2_{**} = \mathrm{Tor}^{H_*(C)}_{**}(H_*(B), H_*(B)) \Longrightarrow H_*(E(C))$$

to compute the homology of the E-suspension $E(C) = B \wedge_C B$. As before, C is an augmented commutative B-algebra. Each algebra indecomposable x in $H_i(C)$ contributes a term $[x] \in E^2_{1,i}$. For dimension reasons it is an infinite cycle (but it may be hit by a differential). It therefore represents an element $\sigma(x) \in H_{i+1}(E(C))$, where

$$\sigma\colon S^1 \wedge (C/B) \to E(C)/B$$

is the stabilizing B-module map.

For $B = H\mathbb{Z}$ and $C = H\mathbb{Z} \wedge H\mathbb{Z}$ we get $E(C) = \mathrm{THH}(H\mathbb{Z})$. The bar spectral sequence

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}^{H_*(H\mathbb{Z}\wedge H\mathbb{Z})}(H_*(H\mathbb{Z}), H_*(H\mathbb{Z})) \\ &= HH_*(H_*(H\mathbb{Z})) \Longrightarrow H_*(\mathrm{THH}(H\mathbb{Z})) \end{aligned}$$

was studied by Marcel Bökstedt. Here

$$E^2_{**} = H_*(H\mathbb{Z}) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_k \mid k \geq 2)\,.$$

There are no differentials, for bidegree reasons, but there are multiplicative extensions $(\sigma\bar{\xi}_k)^2 = \sigma\bar{\xi}_{k+1}$ for $k \geq 2$, while $(\sigma\bar{\xi}_1^2)^2 = 0$. Hence

$$H_*(\mathrm{THH}(H\mathbb{Z})) = H_*(H\mathbb{Z}) \otimes E(\sigma\bar{\xi}_1^2) \otimes P(\sigma\bar{\xi}_2)$$

with $|\sigma\bar{\xi}_1^2| = 3$ and $|\sigma\bar{\xi}_2| = 4$, so

$$\pi_*(\mathrm{THH}(H\mathbb{Z})) \cong (\mathbb{Z}\{1\}, 0, 0, \mathbb{Z}/2\{\sigma\bar{\xi}_1^2\}, 0, \dots)\,.$$

The stabilization map takes $\bar{\xi}_1^2$ to $\sigma\bar{\xi}_1^2$, hence

$$(H\mathbb{Z} \wedge H\mathbb{Z})/H\mathbb{Z} \to \Sigma^{-1}\,\mathrm{THH}(H\mathbb{Z})/H\mathbb{Z}$$

is a π_i-isomorphism for all $i \leq 3$.

In the next step, we consider $E(C) = \mathrm{THH}(H\mathbb{Z})$ as an augmented commutative $H\mathbb{Z}$-algebra. The bar spectral sequence

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}^{H_*(\mathrm{THH}(H\mathbb{Z}))}_{**}(H_*(H\mathbb{Z}), H_*(H\mathbb{Z})) \\ &\Longrightarrow H_*(E^2(H\mathbb{Z} \wedge H\mathbb{Z})) \end{aligned}$$

begins with

$$E^2_{**} = H_*(H\mathbb{Z}) \otimes \Gamma(\sigma^2\bar{\xi}_1^2) \otimes E(\sigma^2\bar{\xi}_2)$$

where Γ denotes the divided power algebra. The generator $\sigma^2\bar{\xi}_1^2$ is in bidegree $(1,3)$, its k-th divided power $\gamma_k\sigma^2\bar{\xi}_1^2$ is in bidegree $(k, 3k)$, and the generator $\sigma^2\bar{\xi}_2$ is in bidegree $(1,4)$. Since $H\mathbb{Z}$ splits off from $E^2(H\mathbb{Z} \wedge H\mathbb{Z})$, there cannot be any differentials on $\sigma^2\bar{\xi}_1^2$, $\gamma_2\sigma^2\bar{\xi}_1^2$ or $\sigma^2\bar{\xi}_2$, but perhaps on $\gamma_4\sigma^2\bar{\xi}_1^2$.

((More must be known!))

Hence

$$\pi_*(E^2(H\mathbb{Z} \wedge H\mathbb{Z})) \cong (\mathbb{Z}\{1\}, 0, 0, 0, \mathbb{Z}/2\{\sigma^2\bar{\xi}_1^2\}, 0, 0, 0, \dots)$$

(modulo odd torsion).

Assume inductively, for $m \geq 1$, that

$$H_*(E^m(C)) \cong H_*(H\mathbb{Z})\{1, \sigma^m\bar{\xi}_1^2, \sigma^m\bar{\xi}_2, \dots\}\,,$$

where $|\sigma^m\bar{\xi}_1^2| = m+2$, $|\sigma^m\bar{\xi}_2| = m+3$ and the remaining terms start in dimension $m+6$.

The m-th bar spectral sequence

$$E^2_{**} = \mathrm{Tor}^{H_*(E^m(C))}(H_*(H\mathbb{Z}), H_*(H\mathbb{Z})) \Longrightarrow H_*(E^{m+1}(C))$$

starts with

$$E^2_{**} = H_*(H\mathbb{Z})\{1, \sigma^{m+1}\bar{\xi}_1^2, \sigma^{m+1}\bar{\xi}_2, \dots\}$$

with $\sigma^{m+1}\bar{\xi}_1^2$ in bidegree $(1, m+2)$ and $\sigma^{m+1}\bar{\xi}_2$ in bidegree $(1, m+3)$, while and the remaining terms start in dimension $m+7$. Here $2(m+3) \geq m+7$ since $m \geq 1$. For bidegree reasons there is no room for differentials in this range of total degrees, hence the inductive hypothesis is verified for $m+1$.

It follows that

$$H_*(\mathrm{TAQ}(H\mathbb{Z})) \cong H_*(H\mathbb{Z})\{\bar{\xi}_1^2, \bar{\xi}_2, \dots\}$$

where the remaining terms start in dimension 6. Hence

$$\mathrm{TAQ}_*(H\mathbb{Z}) \cong (0, 0, \mathbb{Z}/2\{\bar{\xi}_1^2\}, 0, 0, 0, \dots)\,.$$

Since the stabilization maps take $\sigma^m\bar{\xi}_1^2$ to $\sigma^{m+1}\bar{\xi}_1^2$ for all $m \geq 1$, it is also clear that

$$\Sigma^{-1}\,\mathrm{THH}(H\mathbb{Z})/H\mathbb{Z} \to \mathrm{TAQ}(H\mathbb{Z})$$

is a π_i-isomorphism for all $i \leq 5$.

((Discuss where odd torsion enters?))

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