GALOIS EXTENSIONS OF "BRAVE NEW RINGS"

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Contents

Typeset by $\mathcal{AMS}$-TEX

Integral group rings

First consider:

- X a based space;
- $\widetilde{X} \to X$ its universal covering space;
- $\pi = \pi_1(X)$ the fundamental group;
- $\mathbb{Z}[\pi]$ the integral group ring.

Many classical invariants of topological spaces are algebraic invariants of the integral group ring $\mathbb{Z}[\pi]$.

The cell structure on a finite CW complex X lifts to one on $\widetilde{X}$, with π permuting the cells freely. The cellular chain complex

$$0 \to C_n(\widetilde{X}) \xrightarrow{d_n} \ldots \xrightarrow{d_1} C_0(\widetilde{X}) \to 0$$

is a chain complex of free $\mathbb{Z}[\pi]$-modules.

The Whitehead group

When $C_*(\widetilde{X})$ is exact, its **Whitehead torsion** is an invariant in the **Whitehead group**

$$Wh_1(\pi) = K_1(\mathbb{Z}[\pi])/(\pm\pi)\,.$$

For any ring R the first **algebraic K-group** is the maximal abelian quotient

$$K_1(R) = GL(R)/[GL(R), GL(R)]$$

of the infinite general linear group of R. For $R = \mathbb{Z}[\pi]$, $\pm\pi$ sits inside $GL_1(R) \subset GL(R)$, and maps to $K_1(R)$.

The h-cobordism theorem

A **cobordism** on a closed n-manifold M is a compact $(n+1)$-manifold W containing M as a component of the boundary ∂W.

It is an h-**cobordism** if the inclusion $M \subset W$ is a homotopy equivalence.

Theorem (Smale; Barden; Mazur; Stallings). *Let M be a closed n-manifold, $n \geq 5$, with $\pi = \pi_1(M)$. There is a one-to-one correspondence between*

(1) *isomorphism classes of h-cobordisms on M, and*

(2) *elements of the Whitehead group $Wh_1(\pi)$,*

which takes $[W]$ to the Whitehead torsion of the exact complex $C_*(\widetilde{W}, \widetilde{M})$.

Implies Poincaré conjecture in dimensions ≥ 6.

Fundamental to classification of manifolds by surgery theory.

"Brave new rings"

Many modern invariant of topological spaces are invariants of "brave new rings", alias $\mathbb{S}$-algebras, which are a homotopy theoretic generalization of the classical notion of a ring.

- $\mathbb{S}$ the **sphere spectrum**;
- $\mathcal{M}_{\mathbb{S}}$ the category of **$\mathbb{S}$-modules** (= spectra).

$\mathcal{M}_{\mathbb{S}}$ has a symmetric monoidal pairing $\wedge$, called the **smash product**. An **$\mathbb{S}$-algebra** A is an $\mathbb{S}$-module with an associative and unital product

$$\mu\colon A \wedge A \to A$$

and unit $\eta\colon \mathbb{S} \to A$. These represent generalized cohomology theories with cup product.

Analogous to: the integers $\mathbb{Z}$, the category of $\mathbb{Z}$-modules (= abelian groups), the tensor product $\otimes$, and $\mathbb{Z}$-algebras (= rings).

$\mathbb{S}$-algebras generalize rings; $\mathbb{S}$ rather than $\mathbb{Z}$ is the initial object.

The Whitehead space

Now consider:

- X a based space;
- $PX \to X$ its path space fibration;
- ΩX the loop group;
- $\mathbb{S}[\Omega X]$ the group $\mathbb{S}$-algebra.

The **algebraic K-theory space** $K(A)$ can be defined for any $\mathbb{S}$-algebra A, extending algebraic K-theory of rings.

The **(smooth) Whitehead space** $Wh(X)$ sits in a (split) fiber sequence

$$Q(X_+) \to K(\mathbb{S}[\Omega X]) \to Wh(X)\,.$$

So $\pi_1(Wh(X)) \cong Wh_1(\pi_1(X))$.

THE STABLE PARAMETRIZED h-COBORDISM THEOREM

Theorem (Waldhausen). *Let M be a compact manifold. There is a weak equivalence between*

(1) *the space of h-cobordisms on $M \times I^k$, for $k \gg 0$, and*
(2) *the looped Whitehead space $\Omega Wh(M)$.*

Enriches h-cobordism theorem from a statement about isomorphism classes to one about parameter spaces.

Leads to a description of the spaces of homeomorphisms, resp. diffeomorphisms, of manifolds.

Commutative $\mathbb{S}$-algebras

Desire to understand the algebraic K-theory of group $\mathbb{S}$-algebras. Like group rings, these are generally non-commutative.

Theorem (Farrell, Jones). *Let M be a closed Riemannian manifold, with sectional curvature $K \leq 0$ everywhere. (Other geometric hypotheses also suffice.) Then $Wh(M)$ can be assembled up to weak equivalence from*

(1) *$Wh(\{p\})$ for every point $p \in M$, and*
(2) *$Wh(\gamma)$ for every closed geodesic $\gamma \subset M$.*

Reduces to the cases $\mathbb{S}[\Omega\{p\}] \cong \mathbb{S}$ and $\mathbb{S}[\Omega\gamma] \cong \mathbb{S}[\Omega S^1] \simeq \mathbb{S}[z, z^{-1}]$, which are **commutative $\mathbb{S}$-algebras**.

Remark. Commutative $\mathbb{S}$-algebras are only commutative up to homotopy, but in a coherent way, hence are somewhat like non-commutative deformations of commutative rings.

Zariski localization

Algebraic K-theory $K(R)$ of a ring R is formed from

- finitely generated projective R-modules,

or when R is commutative,

- finite rank locally free sheaves over $\operatorname{Spec} R$.

The excision property of topological K-theory has an analogue in the Zariski localization property of algebraic K-theory, leading to a spectral sequence

$$E^2_{s,t} = H^{-s}_{Zar}(\operatorname{Spec} R; \mathcal{K}_t) \Longrightarrow K_{s+t}(R)$$

expressing $K(R)$ in terms of the K-theory sheaf $\mathcal{K}_*$. The sheaf is determined by its stalks at Zariski points, i.e., at local rings. (Can restrict further to Hensel local rings, and often to fields.)

Spectra as sheaves

Algebraic K-theory $K(A)$ of an $\mathbb{S}$-algebra A is formed from

- finite cell A-modules and their retracts.

View such modules as sheaves over some (yet undefined) geometric object $\operatorname{Spec} A$ when A is commutative, e.g. for $A = \mathbb{S}$.

The Hurewicz homomorphism $h\colon \mathbb{S} \to \mathbb{Z}$ induces a map

$$^a h\colon \operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{S}$$

which inflates each closed point (p) to a highly complex, infinite dimensional variety in $\operatorname{Spec} \mathbb{S}$!

Like $\operatorname{Spec} \mathbb{Z}$, $\operatorname{Spec} \mathbb{S}$ is a cofinite union of local pieces $\operatorname{Spec} \mathbb{S}_{(p)}$ for all rational primes p, glued along the generic point $\operatorname{Spec} \mathbb{Q}$.

What does $\operatorname{Spec} \mathbb{S}_{(p)}$ look like ?

The chromatic filtration

A **localization functor** is thought of as restricting a sheaf to an open subset.

Theorem (Devinatz, Hopkins, Smith). *The localization functors on the category of finite cell $\mathbb{S}_{(p)}$-modules form an infinite chain of functors L_n, for $0 \leq n < \infty$, with natural transformations between them:*

$$id \to \cdots \to L_n \to L_{n-1} \to \cdots \to L_1 \to L_0 \,.$$

Interpret L_n as restricting a sheaf over $\operatorname{Spec} \mathbb{S}_{(p)}$ to an open subset $U_n = \{v_0, v_1, \dots, v_n\}$ of chromatic points:

$$\operatorname{Spec} \mathbb{S}_{(p)} \supset \cdots \supset U_n \supset U_{n-1} \supset \cdots \supset U_1 \supset U_0 \,.$$

The infinite nested chain shows that $\operatorname{Spec} \mathbb{S}_{(p)}$ is not Noetherian.

Chromatic localization

Let $L_{K(n)}$ be Bousfield localization with respect to the n-th Morava K-theory $K(n)$, for $0 \le n < \infty$.

Theorem (Hovey, Strickland). *The category of L_n-local $\mathbb{S}_{(p)}$-modules with trivial L_{n-1}-localization is equivalent to the category of $K(n)$-local $\mathbb{S}_{(p)}$-modules.*

Restricted to finite cell modules, this gives a fibration

$$K(L_{K(n)}\mathbb{S}_{(p)}) \to K(L_n\mathbb{S}_{(p)}) \to K(L_{n-1}\mathbb{S}_{(p)})\,.$$

Together with the Hopkins–Ravenel chromatic convergence theorem, this leads to a spectral sequence

$$E^2_{s,t} = H^{-s}_{chr}(\operatorname{Spec}\mathbb{S}; \mathcal{K}_t) \Longrightarrow K_{s+t}(\mathbb{S})$$

expressing $K(\mathbb{S})$ in terms of the K-theory sheaf $\mathcal{K}_*$. The sheaf is determined by its stalks at chromatic points, i.e., at the commutative $\mathbb{S}$-algebras $L_{K(n)}\mathbb{S}_{(p)}$.

Here $L_{K(0)}\mathbb{S}_{(p)} = \mathbb{Q}$, while $L_{K(1)}\mathbb{S}_{(p)} = J_p^\wedge$ is the well-known p-complete non-connective image-of-J spectrum.

Galois extensions of commutative rings

Consider:

- $R \to T$ a (pro-)finite Galois extension of commutative rings;
- $G = \mathrm{Gal}(T/R)$ its (pro-)finite Galois group;
- M a finitely generated projective R-module.

The tensor product $T \otimes_R M$ is a finitely generated projective T-module. Get a map

$$\eta \colon K(R) \to K(T)^{hG} = F(EG_+, K(T))^G$$

to the G-**homotopy fixed points** of $K(T)$. Here EG is a free, contractible G-space.

Galois descent for commutative rings

The Lichtenbaum–Quillen conjectures suppose that Galois descent holds for algebraic K-theory, i.e., that η induces an isomorphism on homotopy groups with finite coefficients in sufficiently high degrees. These conjectures have been verified in several cases.

Leads to spectral sequence

$$E^2_{s,t} = H^{-s}_{Gal}(\operatorname{Spec} R; \hat{\mathcal{K}}_t) \Longrightarrow \hat{K}_{s+t}(R)$$

converging in high degrees. Here Galois-cohomology is the continuous cohomology of the absolute Galois group $G_R = \mathrm{Gal}(\overline{R}/R)$. The "hat" denotes ℓ-adic completion, for some prime ℓ with $1/\ell \in R$. The ℓ-adically completed K-theory sheaf $\hat{\mathcal{K}}_*$ is determined by its stalks at Galois closed points, i.e., at separably closed fields, and is known:

Theorem (Suslin). *For a separably closed field $\overline{F}$ of characteristic $\neq \ell$, $\hat{K}_*(\overline{F}) \cong \pi_*(ku)^\wedge_\ell = \hat{\mathbb{Z}}_\ell[v]$ with* $\deg(v) = 2$. *Thus $\hat{\mathcal{K}}_t = \hat{\mathbb{Z}}_\ell(t/2)$ for $t \geq 0$ even, and $= 0$ otherwise.*

Galois extensions of commutative $\mathbb{S}$-algebras

Definition. Let $A \to B$ be a map of commutative $\mathbb{S}$-algebras, and let G be a finite group acting on B by A-algebra homomorphisms. Suppose that the natural maps $A \to B^{hG}$ and $B \wedge_A B \to F(G_+, B)$ are weak equivalences. Then $A \to B$ is a G**-Galois extension** of commutative $\mathbb{S}$-algebras.

Galois extensions are faithful: if M is an A-module with $B \wedge_A M \simeq *$, then $M \simeq *$ is contractible.

Galois extensions are formally étale: the topological André–Quillen homology $TAQ(B/A) \simeq *$ vanishes.

The fundamental theorem holds: if $H \subset G$ is a subgroup, then $B^{hH} \to B$ is an H-Galois extension. If H is normal, then $A \to B^{hH}$ is G/H-Galois.

Minkowski's theorem on discriminants extends: there are no non-trivial G-Galois extensions $\mathbb{S} \to B$ with B connected (not split as a product of non-trivial $\mathbb{S}$-algebras).

Examples

(1) A G-Galois extension $R \to T$ of commutative rings is a G-Galois extension of commutative $\mathbb{S}$-algebras.

(2) The complexification map $KO \to KU$ from real to complex topological K-theory is a C_2-Galois extension.

(3) The map $J_p^\wedge \to KU_p^\wedge$ is a $\mathbb{Z}_p^*$-pro-Galois extensions. This is the case $n = 1$ of the following more general example:

(4) E_n is the Hopkins–Miller commutative $\mathbb{S}$-algebra with coefficient ring

$$\pi_*(E_n) = \mathbb{W}_{\mathbb{F}_{p^n}}[[u_1, \dots, u_{n-1}]][u, u^{-1}]$$

representing the Lubin–Tate universal deformation of the Honda formal group law F_n of height n, over $\mathbb{F}_{p^n}$. The pro-finite group $C_n \ltimes \mathbb{S}_n$ acts on E_n, where $C_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ and $\mathbb{S}_n$ is the Morava stabilizer group of automorphisms of F_n. There is a map

$$L_{K(n)}\mathbb{S}_{(p)} \to E_n$$

which is a $(C_n \ltimes \mathbb{S}_n)$-pro-Galois extension of commutative $\mathbb{S}$-algebras.

Galois descent for commutative $\mathbb{S}$-algebras

Can extend Lichtenbaum–Quillen conjectures to suppose that Galois descent holds for algebraic K-theory of commutative $\mathbb{S}$-algebras.

Conjecture. *Let $A \to B$ be a G-Galois extension of $K(n)$-local commutative $\mathbb{S}$-algebras. Let V be a finite cell $\mathbb{S}$-module of chromatic filtration $(n+1)$, so $L_n V \simeq *$. Then the natural map*

$$\eta\colon K(A) \to K(B)^{hG}$$

induces an isomorphism on homotopy groups in sufficiently high degrees, after smashing with V.

This leads to a spectral sequence

$$E^2_{s,t} = H^{-s}_{Gal}(\operatorname{Spec} A; \mathcal{K}_t(-;V)) \Longrightarrow K_{s+t}(A;V)$$

converging in high degrees to the algebraic K-theory of A with finite coefficients in V, i.e., to $K_*(A;V) = \pi_*(K(A) \wedge V)$.

Conclusion

This would explain $K(A)$, with suitable finite coefficients, in terms of

(1) the group cohomology of the absolute Galois group $G_A = \mathrm{Gal}(\overline{A}/A)$ of A, where the separable closure $\overline{A}$ is a maximal connected pro-Galois extension of A, and
(2) the values of the K-theory sheaf at Galois closed points, i.e., at separably closed $K(n)$-local commutative $\mathbb{S}$-algebras.

In the case $A = L_{K(n)}\mathbb{S}_{(p)}$ it is plausible that the Hopkins–Miller spectrum E_n is close to $\overline{A}$ (after extending $\mathbb{W}_{\mathbb{F}_{p^n}}$ to $\mathbb{W}_{\overline{\mathbb{F}}_p}$).

It is also plausible that the fairly formal techniques proving Suslin's theorem for separably closed fields can be applied to separably closed $K(n)$-local commutative $\mathbb{S}$-algebras.

Such results for $K(L_{K(n)}\mathbb{S}_{(p)})$ would then assemble by the chromatic filtration to explain the global object $K(\mathbb{S})$, and thus the Whitehead space $Wh(*)$.