AUTOMORPHISMS OF SPHERES, NUMBER FIELDS, MOTIVES AND SPECTRA

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June 3rd 2005

Spheres and discs.
Split fiber sequence

$$B\,\mathrm{Diff}(D^n) \to B\,\mathrm{Diff}(S^n) \to BO(n+1)$$

of diffeomorphisms $\mathrm{rel}\,\partial$, and fiber sequence

$$B\,\mathrm{Diff}(D^{n+1}) \to BC(D^n) \to B\,\mathrm{Diff}(D^n)$$

where $C(D^n) = \mathrm{Diff}(D^n \times I \,\mathrm{rel}\, D^n \times 0)$ has a geometrically significant involution. For n large, approximate fiber sequence

$$BC(D^n) \to Q(D^n_+) \to A(D^n)$$

and rational equivalence $A(D^n) \to K(\mathbb{Z})$. From Borel's theorem

$$K_i(\mathbb{Z}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } i = 0 \\ \mathbb{Q} & \text{for } i = 4j+1,\ j \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

so for n large

$$\pi_i BC(D^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } i = 4j,\ j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The sign of the involution depends on the parity of n:

$$\pi_i B\,\mathrm{Diff}(D^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } n \text{ odd and } i = 4j,\ j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Borel's theorem involves the real analysis of $SL_n(\mathbb{Z})$-invariant differential forms on the symmetric space $SL_n(\mathbb{R})/O(n)$. This explains $K(\mathbb{Z})$ from the "inside" of the matrix group $GL_n(\mathbb{Z})$. Can we understand it from the "outside", by descent from the extensions of $\mathbb{Z} \subset \mathbb{Q}$? The following is a brief extract from papers by Jack Morava, Alain Connes and Matilde Marcolli.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Number fields.

$\mathbb{Q} \subset \mathbb{Q}^{ab} \subset \overline{\mathbb{Q}}$

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab}) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

Kronecker–Weber: $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}^*$.

Shafarevich conjecture: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab})$ is a free profinite group.

Covering spaces.

Let X be a topological space, and let $\mathrm{Cov}(X)$ be the category of covering spaces $\pi\colon Y \to X$. Let

$$\omega\colon \mathrm{Cov}(X) \to \mathrm{Sets}$$

be the fiber functor at a point $x_0 \in X$: $\omega(Y) = \pi^{-1}(x_0)$. The set of natural automorphisms $g\colon \omega \to \omega$ of the functor ω forms a group under composition, naturally isomorphic to the fundamental group $\pi_1(X, x_0)$.

Motives.

$\mathbb{Q}$-linear category $MT(\mathbb{Z})$ of Tate motives of $\mathrm{Spec}(\mathbb{Z})$ is Tannakian (an abelian tensor category with rigid duality), thus equivalent to the category of representations of a pro-algebraic group, the motivic Tate Galois group $\mathrm{Gal}_{\mathrm{mot}}(\mathbb{Z})$ of $\mathbb{Z}$. The singular cohomology of the complex points (Betti cohomology) is a fiber functor

$$\omega\colon MT(\mathbb{Z}) \to \mathrm{Vect}$$

that respects direct sums and tensor products, and $\mathrm{Gal}_{\mathrm{mot}}(\mathbb{Z})$ is the pro-algebraic group of natural automorphisms of this functor ω.

$$\mathfrak{Fr}_{\mathrm{odd}} \to \mathrm{Gal}_{\mathrm{mot}}(\mathbb{Z}) \to \mathbb{G}_m$$

$\mathfrak{Fr}_{\mathrm{odd}}$ is pro-unipotent, with Lie algebra $\mathfrak{fr}_{\mathrm{odd}}$ the free Lie algebra with one generator Z_k in each odd weight (or half-degree) $k \geq 3$. The action by the multiplicative group $\mathbb{G}_m$ in the extension specifies the grading.

Motivic realization.

There is a motivic realization functor from commutative rings to motives. Symmetries of rings become symmetries of motives, defining a map of extensions from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\mathrm{Gal}_{\mathrm{mot}}(\mathbb{Z})$.

The profinite group $\hat{\mathbb{Z}}^*$ includes as a Zariski dense subset of the pro-algebraic group $\mathbb{G}_m$.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab})$ also injects into a pro-algebraic group GT, the Grothendieck–Teichmueller group of Drinfel'd, which conjecturally is the same as $\mathfrak{Fr}_{\mathrm{odd}}$. This concerns Deligne's action on the motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and his conjecture on Hochschild (co-)homology.

Algebraic K-theory.

The algebraic K-theory $K(\mathbb{Z})$ should only depend on the motive of $\mathbb{Z}$. Expect descent spectral sequence

$$E^2_{s,t} = H^{-s}_{\mathrm{mot}}(\mathbb{Z}; K^{\mathrm{mot}}_t) = H^{-s}(\mathrm{Gal}_{\mathrm{mot}}(\mathbb{Z}); K^{\mathrm{mot}}_t) \Longrightarrow K_{s+t}(\mathbb{Z})$$

that collapses rationally. Here $K_*^{\mathrm{mot}} = \mathbb{Z}[\beta]$ with $|\beta| = 2$. We compute the E^2-term with the Serre spectral sequence

$$E_2^{p,q} = H^p(\mathbb{G}_m; H^q(\mathfrak{Fr}_{\mathrm{odd}}; K_t^{\mathrm{mot}})) \Longrightarrow H^{p+q}(\mathrm{Gal}_{\mathrm{mot}}(\mathbb{Z}); K_t^{\mathrm{mot}})\,.$$

The classifying space of a free group is a wedge of circles, one for each generator. Its reduced cohomology is concentrated in degree 1, with one summand for each generator. Likewise for $\mathfrak{Fr}_{\mathrm{odd}}$:

$$H^q(\mathfrak{Fr}_{\mathrm{odd}}; \mathbb{Z}[\beta]) = \begin{cases} \mathbb{Z}[\beta] & \text{for } q = 0 \\ \mathbb{Z}\{e_{2j+1} \mid j \ge 1\}[\beta] & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Rationally the cohomology $H^p(\mathbb{G}_m; -)$ consists only of the invariants ($p = 0$, roughly). Here β has weight 1 and e_{2j+1} weight $-(2j+1)$, so these are spanned by 1 in $(s,t) = (0,0)$ and $e_{2j+1}\beta^{2j+1}$ in bidegree $(s,t) = (-1, 4j+2)$. Thus

$$K_i(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\{1\} & \text{for } i = 0 \\ \mathbb{Z}\{e_{2j+1}\beta^{2j+1}\} & \text{for } i = 4j+1,\ j \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

modulo torsion.

Renormalization of QFT.

Connes and Marcolli interpret an extension

$$\mathfrak{Fr} \to U^* \to \mathbb{G}_m$$

as a cosmic Galois group acting on the set of dimensionless coupling constants of physical theories, following Cartier. These "constants" are really sections in a bundle over a space of complexified dimensions, punctured at an integer dimension $D = 4$, and renormalization in quantum field theory amounts to extracting the regular part of meromorphic functions at D.

Here the pro-unipotent group $\mathfrak{Fr}$ has Lie algebra $\mathfrak{fr}$ the free Lie algebra with one generator Z_k in each weight ≥ 1. It is the affine variety

$$\mathfrak{Fr} = \mathrm{Spec}(\mathrm{QSymm})$$

of the commutative Hopf algebra of quasi-symmetric functions, which is dual to the cocommutative Hopf algebra

$$\mathrm{QSymm}^* = U(\mathfrak{fr}) = \mathbb{Z}\langle Z_k \mid k \ge 1\rangle$$

with coproduct

$$\Delta(Z_k) = \sum_{i+j=k} Z_i \otimes Z_j\,.$$

Here $U(-)$ denotes the enveloping algebra. The Z_k map to the Virasoro generators in the central extension of $\mathrm{Diff}(S^1)$.

The inclusion $\mathfrak{fr}_{\mathrm{odd}} \to \mathfrak{fr}$ corresponds to group homomorphisms $\mathfrak{Fr}_{\mathrm{odd}} \to \mathfrak{Fr}$ and

$$\mathrm{Gal}_{\mathrm{mot}}(\mathbb{Z}) \to U^*\,.$$

Complex cobordism.

The maximal abelian quotient of $U(\mathfrak{fr})$ is

$$\mathbb{Z}\langle Z_k \mid k \geq 1\rangle \to \mathbb{Z}[c_k \mid k \geq 1] = H^*(BU)$$

where $Z_k \mapsto c_k$ goes to the k-th Chern class. This is a cocommutative Hopf algebra homomorphism, by the Cartan formula

$$\Delta(c_k) = \sum_{i+j=k} c_i \otimes c_j \,.$$

Dually, there is an injection

$$\mathrm{QSymm} \leftarrow H_*(BU) = \mathbb{Z}[b_k \mid k \geq 1] = \mathrm{Symm}$$

of commutative Hopf algebras.

At the level of algebraic groups, we interpret this as a homomorphism

$$\mathfrak{Fr} \to \mathrm{Gal}(MU/S)\,.$$

If $S \to MU$ were a finite G-Galois extension, there would be an action map $\alpha\colon G_+ \wedge MU \to MU$. Or, by adjunction, we could write it as a coaction

$$\beta\colon MU \to MU \wedge DG_+\,,$$

where $DG_+ = F(G_+, S)$ is the functional dual. The diagonal and product on G make DG_+ a commutative S-Hopf algebra. There is no such group G, but its functional dual DG_+ exits as the commutative S-Hopf algebra $S[BU] = \Sigma^\infty BU_+$, with product coming from the infinite loop space structure on BU.

The coaction map is then the Thom diagonal

$$\beta\colon MU \to MU \wedge S[BU]$$

derived from the space-level analogue

$$Th(\xi) \to Th(\xi) \wedge B_+$$

for a vector bundle $\xi\colon E \to B$. This makes $S \to MU$ a Hopf–Galois extension, since the composite

$$MU \wedge MU \xrightarrow{1\wedge\beta} MU \wedge MU \wedge S[BU] \xrightarrow{\mu\wedge 1} MU \wedge S[BU]$$

is a weak equivalence, namely the Thom isomorphism in MU-homology. So we should interpret DG_+ as $S[BU]$. In particular $H_*(BU) = H_*(DG_+) = H^*(G)$ is the ring of graded functions on G.

The last two homomorphisms

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab}) \to \mathfrak{Fr}_{\mathrm{odd}} \to \mathfrak{Fr} \to \mathrm{Gal}(MU/S)$$

suggest that there is an S-algebraic realization of motives, lifting the motive of $\mathbb{Z}$ to the sphere spectrum S, up to roots of unity.