THE FRACTION FIELD OF TOPOLOGICAL K-THEORY

John Rognes

Contents

This poster provides motivation for my talk on "Topological logarithmic structures", scheduled for Thursday at 16:30.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. The p-adic integers and the p-adic numbers

Fix a prime p, work in a p-complete setting for simplicity, and consider Quillen's localization sequence

$$K(\mathbb{Z}/p) \xrightarrow{i_*} K(\mathbb{Z}_p) \xrightarrow{j^*} K(\mathbb{Q}_p)\,.$$

It is a cofiber sequence of spectra, where i_* is the transfer associated to $i\colon \mathbb{Z}_p \to \mathbb{Z}/p$ and j^* is the natural map associated to $j\colon \mathbb{Z}_p \to \mathbb{Q}_p$. We use algebro-geometric conventions for the variance of these maps.

The proven Lichtenbaum–Quillen conjectures concern the étale descent properties of p-completed algebraic K-theory, here denoted $K(A)_p$. The main case is that of Galois descent: If $A \to B$ is a G-Galois extension, then the induced map

$$K(A)_p \to K(B)_p^{hG}$$

is an equivalence in sufficiently high degrees.

This is more interesting in the case of the fraction field $\mathbb{Q}_p$ than in the case of its valuation ring $\mathbb{Z}_p$, simply because there are many more Galois extensions of the former than of the latter. For example, the p-th cyclotomic extension

$$\mathbb{Q}_p \to \mathbb{Q}_p(\zeta_p)$$

is Δ-Galois, where $\Delta = (\mathbb{Z}/p)^\times \cong \mathbb{Z}/(p{-}1)$, whereas the map of valuation rings

$$\mathbb{Z}_p \to \mathbb{Z}_p[\zeta_p]$$

is not Galois, since it is ramified at (p). The canonical map $h\colon \mathbb{Z}_p[\zeta_p] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p] \to \prod_\Delta \mathbb{Z}_p[\zeta_p]$ has cokernel a nontrivial finite $\mathbb{Z}_p$-module.

In the limit, we compare $K(\mathbb{Q}_p)_p$ with continuous homotopy fixed points of $K(\overline{\mathbb{Q}}_p)_p$, and in the algebraically closed case we have Suslin's theorem saying that

$$K(\overline{\mathbb{Q}}_p)_p \simeq ku_p\,.$$

Note that $K(\overline{\mathbb{Q}}_p)_p$ is significantly simpler than $K(\mathbb{Q}_p)_p$ or $K(\mathbb{Q}_p^{nr})_p$, where $\mathbb{Q}_p^{nr}$ is the maximal unramified extension of $\mathbb{Q}_p$.

2. Tate–Poitou arithmetic duality

The local field $\mathbb{Q}_p$ also has the exceptional feature that its Galois cohomology satisfies a form of Poincaré duality, known as Tate–Poitou arithmetic duality. The mod p Galois cohomology is related to the mod p algebraic K-theory by an Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H^{-s}_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(t/2)) \Longrightarrow K_{s+t}(\mathbb{Q}_p; \mathbb{Z}/p)$$

(for $2s + t \geq 0$), and the cup product

$$H^r_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(i)) \otimes H^{2-r}_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(1-i)) \xrightarrow{\cup} H^2_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(1)) \cong [p]\,\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Z}/p$$

is a perfect pairing. Here $\mathrm{Br}(F)$ is the Brauer group of central simple F-algebras up to Morita equivalence.

Notation

ku_p	connective p-complete topological K-theory
	$\pi_* ku_p = \mathbb{Z}_p[u], \quad \lvert u\rvert = 2$;
KU_p	periodic p-complete topological K-theory
	$\pi_* ku_p = \mathbb{Z}_p[u, u^{-1}]$;
ℓ_p	the connective p-complete Adams summand
	$\pi_* \ell_p = \mathbb{Z}_p[v_1], \quad \lvert v_1\rvert = 2p-2$;
L_p	the periodic p-complete Adams summand
	$\pi_* L_p = \mathbb{Z}_p[v_1, v_1^{-1}]$.

3. The connective and the periodic p-adic Adams summands

There is a similar localization sequence

$$K(\mathbb{Z}_p) \xrightarrow{\pi_*} K(\ell_p) \xrightarrow{\rho^*} K(L_p)$$

established by Andrew Blumberg and Mike Mandell. Here π_* is the transfer associated to the zero-th Postnikov section $\pi\colon \ell_p \to H\mathbb{Z}_p$, and ρ^* is the natural map associated to the localization map $\rho\colon \ell_p \to L_p$.

Also in this case there are more Galois extensions for the periodic commutative S-algebras than for their connective covers. There is a Δ-Galois extension

$$L_p \to KU_p \,,$$

with Δ as above, but

$$\ell_p \to ku_p$$

is not Galois. The canonical map $h\colon ku_p \wedge_{\ell_p} ku_p \to \prod_\Delta ku_p$ has cofiber a nontrivial ku_p-module with finite Postnikov tower. In other words, $\ell_p \to ku_p$ is ramified at (u) over (v_1).

Still, it is hard to find many G-Galois extensions of L_p. Andy Baker and Birgit Richter have recently shown that the largest connected one is

$$\overline{KU}_p = KU_p^{nr} \,,$$

with

$$\pi_* KU_p^{nr} = \mathbb{W}\overline{\mathbb{F}}_p[u, u^{-1}] \,,$$

where $\mathbb{W}$ denotes the Witt ring functor.

A TC-calculation, like the one mentioned in the next section, shows that the algebraic K-theory $K(\overline{KU}_p)_p$ of this algebraic closure in commutative S-algebras is not any simpler than that of L_p and KU_p. This is in contrast to Suslin's theorem, and suggests that the algebraic closure should be formed in a category where also ramified Galois extensions exist.

4. A weak duality

If we assume that there is a motivic cohomology theory for these commutative S-algebras, then we also expect a spectral sequence relating it to algebraic K-theory. This time we prefer to work with $V(1)$-coefficients, where

$$V(1) = S/(p, v_1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$$

is the Smith–Toda complex, and $V(1)_* X = \pi_*(V(1) \wedge X)$. The Atiyah–Hirzebruch spectral sequence may then take the form

$$E^2_{s,t} = H^{-s}_{mot}(\ell_p; \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t} K_{s+t}(\ell_p)$$

(for $2s + t \geq 0$), and similarly for L_p in place of ℓ_p.

By computing with topological cyclic homology, Christian Ausoni and the author determined the abutment of this expected spectral sequence for $p \geq 5$, in which case its E^2-term has a more-or-less obvious form. There is a v_2-self map of degree $(2p^2 - 2)$ acting on $V(1)$, so v_2 acts on the spectral sequence by a shift of $(p^2 - 1)$ in $t/2$, as well as on the abutment. The remarkable outcome of the computations is that the algebraic K-theory abutment is v_2-periodic, i.e., it is v_2-torsion free.

In this case, once the motivic cohomology is made periodic in the t-grading (and christened Galois cohomology), there is no difference between the ℓ_p- and the L_p-case. Furthermore, there is an apparent perfect cup product pairing

$$H^r_{Gal}(L_p; \mathbb{F}_{p^2}(i)) \otimes H^{3-r}_{Gal}(L_p; \mathbb{F}_{p^2}(d-i)) \xrightarrow{\cup} H^3_{Gal}(L_p; \mathbb{F}_{p^2}(d)) \cong \mathbb{Z}/p\,,$$

where $d = p^2 + p$. However, since in this case the target group of the pairing depends on the prime p (in the twist d), there is little hope of an intrinsic understanding of this group, like the identification of $H^2_{Gal}(F; \mathbb{G}_m)$ with the Brauer group $\mathrm{Br}(F)$ mentioned above.

5. The fraction field of the p-adic Adams summand

It appears that we should not view L_p as a (fraction) field in commutative S-algebras, due to the possibility of ramification over (p). So we seek some form of localization of L_p away from p, which we will formally denote as

$$\mathit{ff}L_p = p^{-1}L_p\,.$$

By this we do not mean the usual algebraic Bousfield localization $L_p[p^{-1}] = L\mathbb{Q}_p$, with $\pi_* L\mathbb{Q}_p = \mathbb{Q}_p[v_1, v_1^{-1}]$, since this is an $H\mathbb{Q}_p$-algebra, so $K(L\mathbb{Q}_p)$ is a $K(\mathbb{Q}_p)$-module, and $V(1)_*K(\mathbb{Q}_p)$ is finite, so $V(1)_*K(L\mathbb{Q}_p)$ will only consist of v_2-torsion. This completely loses the connection with the v_2-periodicity that we think is the essential feature in this story. We are looking for a milder, less drastic, localization.

Instead we can show that $\ell/p = k(1)$ admits the structure of a non-commutative ℓ_p-algebra, so $L/p = K(1)$ is an L_p-algebra, and we can form the following commutative 3×3-diagram of horizontal and vertical cofiber sequences:

$$\begin{array}{ccccc}
K(\mathbb{Z}/p) & \xrightarrow{i_*} & K(\mathbb{Z}_p) & \xrightarrow{j^*} & K(\mathbb{Q}_p) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
K(\ell/p) & \xrightarrow{i_*} & K(\ell_p) & \xrightarrow{j^*} & K(p^{-1}\ell_p) \\
\downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*} \\
K(L/p) & \xrightarrow{i_*} & K(L_p) & \xrightarrow{j^*} & K(\mathit{ff}L_p)
\end{array}$$

For now, the terms $K(p^{-1}\ell_p)$ and $K(\mathit{ff}L_p)$ are just notation for the homotopy cofiber spectra that fit in the diagram. The Thursday talk will start to give some intrinsic sense to these expressions.

We thus view $\mathrm{Spec}(\mathit{ff}L_p)$ as a kind of complement of $\mathrm{Spnc}(L/p)$ in $\mathrm{Spec}(L_p)$, and similarly for $p^{-1}\ell_p$.

6. Topological higher arithmetic duality

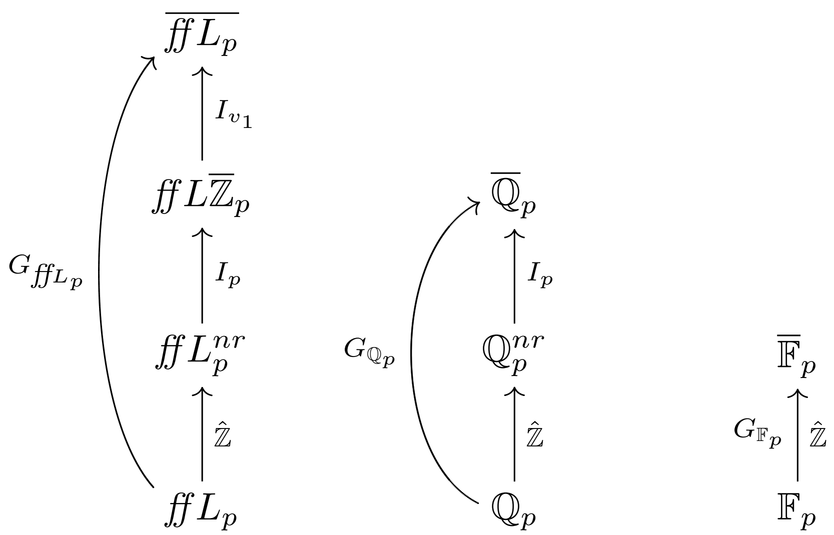
Further topological cyclic homology calculations by Ausoni and Rognes determine $V(1)_*K(\ell/p)$, $V(1)_*K(p^{-1}\ell_p)$ and $V(1)_*K(f\!f L_p)$. These remain v_2-periodic, and now the expected motivic and Galois cohomology groups of $f\!f L_p$ are better behaved than those of L_p, in that there is again an apparent perfect cup product pairing

$$H^r_{Gal}(f\!f L_p; \mathbb{F}_{p^2}(i)) \otimes H^{3-r}_{Gal}(f\!f L_p; \mathbb{F}_{p^2}(2-i)) \xrightarrow{\cup} H^3_{Gal}(f\!f L_p; \mathbb{F}_{p^2}(2)) \cong \mathbb{Z}/p\,,$$

where the twist in the target group is now independent of p.

This is exactly the kind of arithmetic duality as is seen in the Galois cohomology of "higher" local fields, like $\mathbb{F}_p((s,t))$ and $\mathbb{Q}_p((t))$, for which the local class field theory is expressed in terms of Milnor K-theory. The calculations therefore suggest that the fraction field $f\!f L_p$, and its analogously defined Δ-Galois extension $f\!f KU_p$, are brave new 2-dimensional "higher" local fields, invisible to classical algebra.

The valuation ring of $f\!f L_p$ is then $p^{-1}\ell_p$, with residue field $\mathbb{Q}_p = p^{-1}\mathbb{Z}_p$. We expect that there are Galois extensions



and group extensions

$$I_{v_1} \to G_{f\!f L_p} \to G_{\mathbb{Q}_p} \quad \text{and} \quad I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p}\,,$$

where $G_{\mathbb{F}_p}$ has p-cohomological dimension 1, $G_{\mathbb{Q}_p}$ has p-cohomological dimension 2, and the hypothetical $G_{f\!f L_p}$ has p-cohomological dimension 3.

Department of Mathematics, University of Oslo, Norway
E-mail address: rognes@math.uio.no