TOPOLOGICAL LOGARITHMIC STRUCTURES

John Rognes

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§1. The fraction field of topological K-theory

1.1. The p-adic integers and the p-adic numbers. Fix a prime p, work in a p-complete setting for simplicity, and consider Quillen's localization sequence

$$K(\mathbb{Z}/p) \xrightarrow{i_*} K(\mathbb{Z}_p) \xrightarrow{j^*} K(\mathbb{Q}_p)\,.$$

It is a cofiber sequence of spectra, where i_* is the transfer associated to $i\colon \mathbb{Z}_p \to \mathbb{Z}_p/p = \mathbb{Z}/p$ and j^* is the natural map associated to $j\colon \mathbb{Z}_p \to \mathbb{Z}_p[p^{-1}] = \mathbb{Q}_p$. We use algebro-geometric conventions for the variance of these maps.

The proven Lichtenbaum–Quillen conjectures concern the étale descent properties of p-adically complete algebraic K-theory, here denoted $K(A)_p$. The main case is that of Galois descent: If $A \to B$ is a G-Galois extension, then the induced map $K(A)_p \to K(B)_p^{hG}$ is an equivalence in sufficiently high degrees.

This is more interesting in the case of the fraction field $\mathbb{Q}_p$ than in the case of its valuation ring $\mathbb{Z}_p$, simply because there are many more Galois extensions of the former than of the latter. For example, the p-th cyclotomic extension $\mathbb{Q}_p \to \mathbb{Q}_p(\zeta_p)$ is Δ-Galois, where $\Delta = (\mathbb{Z}/p)^\times \cong \mathbb{Z}/(p-1)$, whereas the map of valuation rings $\mathbb{Z}_p \to \mathbb{Z}_p[\zeta_p]$ is not Galois, since it is ramified at (p). The canonical map $h\colon \mathbb{Z}_p[\zeta_p] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p] \to \prod_\Delta \mathbb{Z}_p[\zeta_p]$ has cokernel a nontrivial finite $\mathbb{Z}_p$-module.

In the limit, we compare $K(\mathbb{Q}_p)_p$ with continuous homotopy fixed points of $K(\overline{\mathbb{Q}}_p)_p$, and in the algebraically closed case we have Suslin's theorem saying that $K(\overline{\mathbb{Q}}_p)_p \simeq ku_p$. Here ku_p denotes p-adically completed connective topological K-theory, with $\pi_* ku_p = \mathbb{Z}_p[u]$.

The local field $\mathbb{Q}_p$ also has the exceptional feature that its Galois cohomology satisfies a form of Poincaré duality, known as Tate–Poitou arithmetic duality. The mod p Galois cohomology is related to the mod p algebraic K-theory by an Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H^{-s}_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(t/2)) \Longrightarrow K_{s+t}(\mathbb{Q}_p; \mathbb{Z}/p)$$

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(for $2s+t \geq 0$), and the cup product

$$H^r_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(i)) \otimes H^{2-r}_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(1-i)) \xrightarrow{\cup} H^2_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(1)) \cong [p]\operatorname{Br}(\mathbb{Q}_p) \cong \mathbb{Z}/p$$

(where $\operatorname{Br}(F)$ is the Brauer group of central simple F-algebras up to Morita equivalence) is a perfect pairing.

1.2. The connective and the periodic p-adic Adams summands. There is a similar localization sequence

$$K(\mathbb{Z}_p) \xrightarrow{\pi_*} K(\ell_p) \xrightarrow{\rho^*} K(L_p)$$

established by Andrew Blumberg and Mike Mandell. Here ℓ_p denotes the Adams summand of ku_p, with $\pi_*\ell_p = \mathbb{Z}_p[v_1]$, π_* is the transfer associated to the zero-th Postnikov section $\pi\colon \ell_p \to \ell_p/v_1 = H\mathbb{Z}_p$, and ρ^* is the natural map associated to the localization map $\rho\colon \ell_p \to \ell_p[v_1^{-1}] = L_p$.

Also in this case there are more Galois extensions for the periodic commutative S-algebras than for their connective covers. Letting $KU_p = ku_p[u^{-1}]$ denote p-adically complete periodic topological K-theory, then $L_p \to KU_p$ is a Δ-Galois extension, with Δ as above, but $\ell_p \to ku_p$ is not Galois. The canonical map $h\colon ku_p \wedge_{\ell_p} ku_p \to \prod_\Delta ku_p$ has cofiber a nontrivial ku_p-module with finite Postnikov tower. In other words, $\ell_p \to ku_p$ is ramified at (u) over (v_1).

Still, it is hard to find many G-Galois extensions of L_p. Andy Baker and Birgit Richter have recently shown that the largest (connected) one is KU_p^{nr}, with $\pi_*KU_p^{nr} = \mathbb{W}\overline{\mathbb{F}}_p[u, u^{-1}]$, where $\mathbb{W}$ denotes the Witt ring functor.

For p odd, it is impossible to adjoin a p-th root of unity to L_p to realize $L_p[\zeta_p]$ as a commutative S-algebra, as can be shown by a θ-ring computation. ((Here is the argument I learned from Vigleik Angeltveit: If $L_p[\zeta_p]$ is E_∞ and $K(1)$-local, there are operations $\psi = \psi^p$ and θ with $\psi(x) = x^p + p\theta(x)$ and ψ a ring operation. Can write $p = q^{p-1}$ in $\mathbb{Z}_p[\zeta_p]$, with q a unit times the uniformizer $\pi = 1 - \zeta_p$, but then $p = \psi(p) = \psi(q)^{p-1} = (q^p + p\theta(q))^{p-1} = (p(q + \theta(q)))^{p-1}$ is divisible by p^2.))

Attempts to realize other coefficient extensions that ramify at p effectively flounder when one tries to apply the Goerss–Hopkins obstruction theory, since the obstruction groups are André–Quillen cohomology groups related to Kähler differentials, which are far from zero in this case.

So there does not seem to be an interesting algebraic closure $\overline{KU_p}$ in commutative S-algebras, whose algebraic K-theory is any simpler than that of KU_p itself, in contrast to Suslin's theorem.

If we assume that there is a motivic cohomology theory for these commutative S-algebras, then we also expect a spectral sequence relating it to algebraic K-theory. This time we prefer to work with $V(1)$-coefficients, where

$$V(1) = S/(p, v_1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$$

is the Smith–Toda complex, and $V(1)_*X = \pi_*(V(1) \wedge X)$. The Atiyah–Hirzebruch spectral sequence may then take the form

$$E^2_{s,t} = H^{-s}_{mot}(\ell_p; \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t}K_{s+t}(\ell_p)$$

(for $2s+t \geq 0$), and similarly for L_p in place of ℓ_p.

By computing with topological cyclic homology, Christian Ausoni and the author determined the abutment of this expected spectral sequence for $p \geq 5$, in which case its E^2-term has a more-or-less obvious form. There is a v_2-self map of degree $(2p^2-2)$ acting on $V(1)$, so v_2 acts on the spectral sequence by a shift of (p^2-1) in $t/2$, as well as on the abutment. The remarkable outcome of the computations is that the algebraic K-theory abutment is v_2-periodic, i.e., it is v_2-torsion free. ((Slight caveat here, caused by the very finite coefficients!))

In this case, once the motivic cohomology is made periodic in the t-grading (and christened Galois cohomology), there is no difference between the ℓ_p- and the L_p-case. Furthermore, there is an apparent perfect cup product pairing

$$H^r_{Gal}(L_p; \mathbb{F}_{p^2}(i)) \otimes H^{3-r}_{Gal}(L_p; \mathbb{F}_{p^2}(d-i)) \xrightarrow{\cup} H^3_{Gal}(L_p; \mathbb{F}_{p^2}(d)) \cong \mathbb{Z}/p\,,$$

where $d = p^2+p$. However, since in this case the target group of the pairing depends on the prime p (in the twist d), there is little hope of an intrinsic understanding of this group, like the identification of $H^2_{Gal}(F; \mathbb{G}_m)$ with the Brauer group $\mathrm{Br}(F)$ mentioned above.

1.3. The fraction field of the p-adic Adams summand. It appears that we should not view L_p as a (fraction) field in commutative S-algebras, due to the possibility of ramification over (p). So we seek some form of localization of L_p away from p, which we will formally denote as

$$f\!f L_p = p^{-1} L_p\,.$$

By this we do not mean the usual algebraic Bousfield localization $L_p[p^{-1}] = L\mathbb{Q}_p$, with $\pi_* L\mathbb{Q}_p = \mathbb{Q}_p[v_1, v_1^{-1}]$, since this is an $H\mathbb{Q}_p$-algebra, so $K(L\mathbb{Q}_p)$ is a $K(\mathbb{Q}_p)$-module, and $V(1)_*K(\mathbb{Q}_p)$ is finite, so $V(1)_*K(L\mathbb{Q}_p)$ will only consist of v_2-torsion. This completely loses the connection with the v_2-periodicity that we think is the essential feature in this story. We are looking for a milder, less drastic, localization.

Instead we can show that $\ell_p/p = \ell/p = k(1)$ admits the structure of a non-commutative ℓ_p-algebra, so $L_p/p = L/p = K(1)$ is an L_p-algebra, and we can form the following commutative 3×3-diagram of horizontal and vertical cofiber sequences:

$$\begin{array}{ccccc}
K(\mathbb{Z}/p) & \xrightarrow{i_*} & K(\mathbb{Z}_p) & \xrightarrow{j^*} & K(\mathbb{Q}_p) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
K(\ell/p) & \xrightarrow{i_*} & K(\ell_p) & \xrightarrow{j^*} & K(p^{-1}\ell_p) \\
\downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*} \\
K(L/p) & \xrightarrow{i_*} & K(L_p) & \xrightarrow{j^*} & K(f\!f L_p)
\end{array}$$

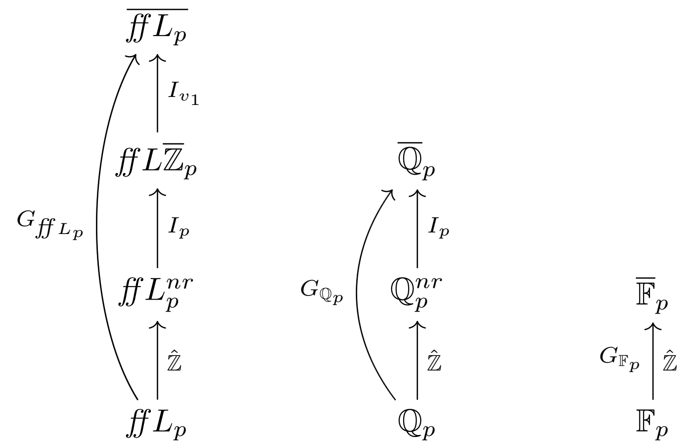
For now, the terms $K(p^{-1}\ell_p)$ and $K(f\!f L_p)$ are just notation for the homotopy cofiber spectra that fit in the diagram. Hence we wish that $\mathrm{Spec}(f\!f L_p)$ is a kind of complement of $\mathrm{Spnc}(L/p)$ in $\mathrm{Spec}(L_p)$, and similarly for $p^{-1}\ell_p$. It is unpleasant that the terms $K(\ell/p)$ and $K(L/p)$ should depend on the choice of ℓ_p-algebra structure, of which there are uncountably many, but this should not be a devastating problem.

Further topological cyclic homology calculations by Ausoni and Rognes determine $V(1)_*K(\ell/p)$, and thereby also $V(1)_*K(p^{-1}\ell_p)$ and $V(1)_*K(f\!f L_p)$. These remain v_2-periodic, and now the expected motivic and Galois cohomology groups of $\mathit{ff}L_p$ are better behaved than those of L_p, in that there is again an apparent perfect cup product pairing

$$H^r_{Gal}(\mathit{ff}L_p; \mathbb{F}_{p^2}(i)) \otimes H^{3-r}_{Gal}(\mathit{ff}L_p; \mathbb{F}_{p^2}(2-i)) \xrightarrow{\cup} H^3_{Gal}(\mathit{ff}L_p; \mathbb{F}_{p^2}(2)) \cong \mathbb{Z}/p\,,$$

where the bigrading of the target group is now independent of p. This is exactly the kind of arithmetic duality as is seen in the Galois cohomology of "higher" local fields, like $\mathbb{F}_p((s,t))$ and $\mathbb{Q}_p((t))$, for which the local class field theory is expressed in terms of Milnor K-theory. The calculations therefore suggest that the fraction field $\mathit{ff}L_p$, and its analogously defined Δ-Galois extension $\mathit{ff}KU_p$, are brave new 2-dimensional "higher" local fields, invisible to classical algebra.

The valuation ring of $\mathit{ff}L_p$ is then $p^{-1}\ell_p$, with residue field $\mathbb{Q}_p = p^{-1}\mathbb{Z}_p$. We expect that there are Galois extensions



and group extensions

$$I_{v_1} \to G_{\mathit{ff}L_p} \to G_{\mathbb{Q}_p} \quad \text{and} \quad I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p}\,,$$

where $G_{\mathbb{F}_p}$ has p-cohomological dimension 1, $G_{\mathbb{Q}_p}$ has p-cohomological dimension 2, and the hypothetical $G_{\mathit{ff}L_p}$ has p-cohomological dimension 3.

§2. Topological logarithmic structures

We propose that a structured ring spectrum version of logarithmic geometry may provide an algebro-geometric context in which the fraction fields $\mathit{ff}L_p$, $\mathit{ff}KU_p$, and some or all of their Galois extensions, will exist.

Let A be a commutative S-algebra. We may view A as an E_∞ ring spectrum, by which we will really mean an $\mathcal{L}$-spectrum, where $\mathcal{L}$ is the linear isometries operad. The underlying space $\Omega^\infty A$ is then an E_∞ ring space. With respect to the additive structure, $\Omega^\infty A_\oplus$ is a grouplike E_∞ space, but this will be of little concern to us. With respect to the multiplicative structure, $\Omega^\infty A_\otimes$ is a pointed E_∞ space (= E_∞ space with zero), by which we mean an $\mathcal{L}_0$-space, i.e., an $\mathcal{L}_+$-space in pointed spaces. In such spaces, the operadic neutral element 1 is almost always different from the base point 0.

The homotopy invertible components of $\Omega^\infty A_\otimes$, denoted $GL_1(A)$, constitute a grouplike E_∞-space, or $\mathcal{L}$-space, and the inclusion $\iota\colon GL_1(A) \to \Omega^\infty A_\otimes$ is a map of E_∞ spaces. It extends to a map $\iota_0\colon GL_1(A)_+ \to \Omega^\infty A_\otimes$ of pointed E_∞-spaces.

Definition (prelog and log S-algebras). (i) Let A be a commutative S-algebra. The following definition works best if A is connective: A (pointed) prelog structure (M, α) on A is a pointed E_∞ space M, and a map

$$\alpha\colon M \to \Omega^\infty A_\otimes$$

of pointed E_∞ spaces. We usually write (A, M) for A with this prelog structure, and call it a prelog S-algebra.

(ii) Consider the pullback square

$$\begin{array}{ccc} \alpha^{-1}GL_1(A) & \xrightarrow{\tilde{\alpha}} & GL_1(A) \\ \big\downarrow & & \big\downarrow \iota \\ M & \xrightarrow{\alpha} & \Omega^\infty A_\otimes \end{array}$$

in (unpointed) E_∞ spaces. We say that (M, α) is a (pointed) log structure on A, and that (A, M) is a log S-algebra, if the restricted map $\tilde{\alpha}\colon \alpha^{-1}GL_1(A) \to GL_1(A)$ is a weak equivalence.

(iii) If $M = N_+$, where N is an E_∞ space and zero is the added base point, the restricted E_∞ map $\alpha'\colon N \to \Omega^\infty A_\otimes$ defines an unpointed prelog structure (N, α') on A, making (A, N) an unpointed prelog S-algebra. It is an unpointed log S-algebra if $\tilde{\alpha}'\colon \alpha'^{-1}GL_1(A) \to GL_1(A)$ is a weak equivalence.

Remarks. (i) Note that ι is a fibration, so the pullback square is a homotopy pullback. It is less clear how pullback along ι_0 makes homotopy theoretic sense.

(ii) The emphasis of logarithmic geometry will be on the part of M that does not map to $GL_1(B)$. Hence the log condition is a normalization condition, saying that nothing exceptional happens over the homotopy units: the part of M sitting over $GL_1(B)$ is equivalent to $GL_1(B)$. It may also be viewed as a fibrancy condition.

(iii) For non-connective A, this definition is too restrictive, since a space may act by an equivalence on a non-connective spectrum, without acting homotopy invertibly on its infinite loop space.

(iv) One might alternatively work with diagram spectra, say by letting A be a connective commutative symmetric ring spectrum. Then $\mathbf{n} \mapsto \Omega^n A_n$ defines a commutative monoid $\Omega^\infty A_\otimes$ in pointed I-spaces, where I is the category of finite sets and injective functions, and a prelog structure is a map $\alpha\colon M \to \Omega^\infty A_\otimes$ in pointed I-spaces. Then $GL_1(A)$ should be interpreted as a commutative monoid in pointed I-spaces, following Christian Schlichtkrull.

(v) It is possible to do some work with non-commutative S-algebras and A_∞ spaces with zero, but we will not do so in these notes.

Examples. (i) In classical algebra, only unpointed log structures are considered, but one might just as well have worked with pointed log structures, since the zero will always be isolated in the discrete topology. If R is a commutative ring, $(R, \cdot)$ its multiplicative monoid, N some other commutative monoid, and $\alpha'\colon N \to (R, \cdot)$ a monoid homomorphism, then clearly the induced map $N_+ \to \Omega^\infty HR_\otimes$ defines a prelog structure on the Eilenberg–Mac Lane ring spectrum HR.

(ii) In topology, we have good reasons to consider pointed log structures. For example, to approximate the localization $KU = ku[u^{-1}]$ by a log structure on ku, we think of KU as obtained from ku by inverting the structure maps $\Sigma^2 ku \to ku$. These are induced by the S-module map $\bar{u}\colon \Sigma^\infty S^2 \to ku$ representing the generator $u \in \pi_2 ku$, and the product on ku. In adjoint terms, we have the base-point preserving map $u\colon S^2 \to \Omega^\infty ku \simeq BU \times \mathbb{Z}$, which extends to a pointed E_∞ map

$$\beta\colon L_0 S^2 \to \Omega^\infty ku_\otimes\,,$$

(β for Bott) where

$$L_0 S^2 = \bigvee_{j\ge 0} \mathcal{L}(j) \ltimes_{\Sigma_j} S^{2j}$$

is the free $\mathcal{L}_0$-space (= pointed E_∞ space) on S^2.

((In May's more recent notation, this is the free $\mathcal{L}_+$-space $L_+(S^2)$ on S^2, in pointed spaces, which equals the free $\mathcal{L}$-space $L(S^2 \sqcup \{1\})$ on $S^0 \vee S^2 = S^2 \sqcup \{1\}$, in spaces under $S^0 = \{0,1\}$.))

The associated log S-algebra $(ku, (L_0 S^2)^a)$ is the topological log-geometric approximation to KU. Later calculations with topological André–Quillen homology show that thinking of u as a free (unpointed) map, and working with the free $\mathcal{L}$-space $L(S^2 \sqcup \{0,1\})$, does not yield a satisfactory theory. There is a similar prelog structure

$$\alpha\colon L_0 S^q \to \Omega^\infty \ell_\otimes$$

(α for Adams) on the p-local Adams summand, where $q = 2p-2$.

Definition (logification). (i) To each prelog structure (M,α) on A there is an associated log structure (M^a, α^a) on A, where M^a is the homotopy pushout of the diagram

$$\begin{array}{ccc} (\alpha^{-1}GL_1(A))_+ & \xrightarrow{\tilde{\alpha}_+} & GL_1(A)_+ \\ \downarrow & & \downarrow \iota_0 \\ M & \longrightarrow & M^a \end{array}$$

in pointed E_∞ spaces, and $\alpha^a\colon M^a \to \Omega^\infty A_\otimes$ is the induced map. We write $(A,M)^a$ for the log S-algebra associated to (A,M), to emphasize that M^a depends on A.

(ii) When $M = N_+$ has an isolated zero, $M^a = (N^a)_+$, where N^a is the homotopy pushout of the diagram

$$\begin{array}{ccc} \alpha^{-1}GL_1(A) & \xrightarrow{\tilde{\alpha}'} & GL_1(A) \\ \downarrow & & \downarrow \iota \\ N & \longrightarrow & N^a \end{array}$$

in E_∞ spaces.

Remark. In the context of commutative monoids in pointed I-spaces the coproduct is the smash product, so when $\alpha^{-1}GL_1(A) = 1$ we get

$$M^a = GL_1(A) \ltimes M$$

in the pointed case, and

$$N^a = GL_1(A) \times N$$

in the unpointed case.

Definition (trivial log structures). The trivial (pointed) prelog structure on A is given by $M = 1_+ = \{0, 1\}$. The associated log structure $(A, 1)^a$ equals $(A, GL_1(A)_+)$, and is called the trivial log structure.

Definition (prelog and log maps). (i) A map $(A, M) \to (B, N)$ of prelog S-algebras is a pair $(f, f^\flat)$, where $f \colon A \to B$ is a map of commutative S-algebras, and $f^\flat \colon M \to N$ is a map of pointed E_∞ spaces, such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & \Omega^\infty A_\otimes \\ {\scriptstyle f^\flat}\downarrow & & \downarrow{\scriptstyle \Omega^\infty f} \\ N & \xrightarrow{\beta} & \Omega^\infty B_\otimes \end{array}$$

of pointed E_∞ spaces is commutative. Equivalently, the adjoint diagram

$$\begin{array}{ccc} \Sigma^\infty M & \xrightarrow{\bar\alpha} & A \\ {\scriptstyle \Sigma^\infty f^\flat}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma^\infty N & \xrightarrow{\bar\beta} & B \end{array}$$

of commutative S-algebras is required to commute.

(ii) A map of log S-algebras is a map of the underlying prelog S-algebras. The log S-algebras generate a reflective full subcategory $\mathcal{L}og_S$ of the prelog S-algebras, with the logification functor $(A, M) \mapsto (A, M)^a$ as the reflector, i.e., the left adjoint to the forgetful functor.

Example. There is a map $(f, f^\flat) \colon (\ell, L_0 S^q) \to (ku_{(p)}, L_0 S^2)$ of prelog S-algebras, where $f \colon \ell \to ku_{(p)}$ is the usual map and $f^\flat \colon L_0 S^q \to L_0 S^2$ is the pointed E_∞-map that extends the pointed map

$$S^q \subset \mathcal{L}(p{-}1) \ltimes_{\Sigma_{p-1}} S^{2(p-1)} \subset L_0 S^2 \,.$$

There is also an induced map of associated log S-algebras.

Remark. Even if $\ell \to ku_{(p)}$ is not étale as a map of commutative S-algebras, we may expect that $(\ell, L_0 S^q) \to (ku_{(p)}, L_0 S^2)$ is étale as a map of (pre)log S-algebras. This would fit with the classical expectations, since $\ell \to ku_{(p)}$ appears to be tamely ramified at $(v_1) = (u)^{p-1}$. An alternative guess is that the prelog structures with $M = W \sqcup \{1\}$ and $N = BU_{(p)} \sqcup \{1\}$, where $\Omega^\infty \ell \simeq W \times \mathbb{Z}_{(p)}$, will have this property.

Definition (localizations). (i) Let (A, M) be a (pre)log S-algebra, and assume $M = N_+$ has an isolated zero. Let $\Gamma N \simeq \Omega B N$ be the (multiplicative) group completion of N, and let

$$A[N^{-1}] = S[\Gamma N] \wedge_{S[N]} A$$

where $S[N] = \Sigma^\infty N_+ = \Sigma^\infty M$, and similarly for $S[\Gamma N]$. Then there are prelog maps

$$(A, 1_+) \to (A, N_+) \to (A[N^{-1}], N_+)$$

with logifications

$$(A, 1_+)^a \to (A, N_+)^a \to (A[N^{-1}], 1_+)^a \,,$$

since $N \to \Omega^\infty A_\otimes \to \Omega^\infty A[N^{-1}]_\otimes$ maps into $GL_1(A[N^{-1}])$. Hence there are maps of affine topological log schemes

$$U = \mathrm{Spec}(A[N^{-1}]) \to \Lambda = \mathrm{Spec}(A, N_+) \to X = \mathrm{Spec}(A)$$

exhibiting $\Lambda = \mathrm{Spec}(A, N_+)$ as an intermediate "compactification" of the Zariski open U in X. The latter topological schemes are viewed as topological log schemes by equipping them with the trivial log structures. For the purposes of differential algebraic geometry, including TAQ, THH and TC, we propose to work with the log scheme Λ in place of the open subscheme U.

(ii) On the other hand, if $M = L_0T$ has an isolated unit, for some zero-pointed space T, then the natural map

$$\mathcal{L}(j) \ltimes_{\Sigma_j} T^{\wedge j} \wedge T \to \mathcal{L}(j+1) \ltimes_{\Sigma_{j+1}} T^{\wedge (j+1)}$$

induces maps $M \wedge T \to M$ and $\sigma \colon \Sigma^\infty M \wedge T \to \Sigma^\infty M$. If $T = S^d$ for some $d \ge 0$, then we can let $T^{-1}\Sigma^\infty M$ be the homotopy colimit of the infinite sequence of maps

$$\Sigma^\infty M \xrightarrow{\Sigma^{-d}\sigma} \Sigma^{-d}\Sigma^\infty M \xrightarrow{\Sigma^{-2d}\sigma} \Sigma^{-2d}\Sigma^\infty M \to \dots \,.$$

For $d > 0$, this will usually not be a connective spectrum. ((More generally, it might make sense to let $T^{-1}\Sigma^\infty M$ be the homotopy colimit of the adjoint maps

$$\Sigma^\infty M \xrightarrow{\tilde\sigma} F(T, \Sigma^\infty M) \xrightarrow{\tilde\sigma} F(T \wedge T, \Sigma^\infty M) \to \dots \,,$$

but this is less clear.)) By the theory of Bousfield localizations in the category of $\Sigma^\infty M$-modules [EKMM, VIII.2.2], we can take $T^{-1}\Sigma^\infty M$ to be a commutative $\Sigma^\infty M$-algebra. Then we let

$$T^{-1}A = T^{-1}\Sigma^\infty M \wedge_{\Sigma^\infty M} A \,.$$

There are prelog maps

$$(A, 1_+) \to (A, L_0T) \to (T^{-1}A, L_0T) \,,$$

where $T^{-1}A$ might not be connective. Still, for $T = S^d$ we will have that the multiplication map

$$\Sigma^\infty T \wedge T^{-1}A \xrightarrow{\bar\tau \wedge 1} A \wedge T^{-1}A \xrightarrow{\mu} T^{-1}A$$

is an equivalence, so the space T generating M acts by an equivalence on $T^{-1}A$. Here $\tau \colon T \to \Omega^\infty A_\infty$ is the restriction of α over $T \subset M$, and $\bar\tau$ is its left adjoint. We view this as saying that the non-connective prelog S-algebras $(T^{-1}A, L_0T)$ and $(T^{-1}A, 1_+)$ should be log equivalent. (This example shows that our definition of a log structure is not the right one for non-connective S-algebras.) With this interpretation, we again have maps

$$U = \mathrm{Spec}(T^{-1}A) \to \Lambda = \mathrm{Spec}(A, M) \to X = \mathrm{Spec}(A)$$

that exhibit Λ as intermediate between U and X.

Example. For $A = ku$, $T = S^2$, $M = L_0 S^2$ and β the Bott prelog structure, we get $T^{-1}A = KU$.

Remark. Following Martin Olsson, we can embed topological logarithmic geometry into the topological algebraic geometry of stacks, by taking each topological logarithmic scheme Λ (Zariski or étale locally modeled on $\mathrm{Spec}(A, M)$) to a moduli stack of topological logarithmic schemes Y over Λ, with morphisms the "exact" maps $Y \to Y'$ over Λ. Then notions like smooth, unramified and étale morphisms of topological logarithmic schemes should become special cases of the same notions for topological stacks. Clark Barwick understands this point of view well.

§3. Logarithmic TAQ and THH

We would like to define a cyclotomic spectrum $\mathrm{THH}(A, M)$, for (A, M) a log S-algebra, such that there is a cofiber sequence

$$\mathrm{THH}(\ell/p) \xrightarrow{i_*} \mathrm{THH}(\ell_p) \xrightarrow{j^*} \mathrm{THH}(\ell_p, M)$$

of cyclotomic spectra, for a suitable log structure $\alpha\colon M \to (\Omega^\infty \ell_p)_\otimes$. Then we would get a cofiber sequence

$$\mathrm{TC}(\ell/p; p) \xrightarrow{i_*} \mathrm{TC}(\ell_p; p) \xrightarrow{j^*} \mathrm{TC}(\ell_p, M; p)\,,$$

and $\mathrm{trc}\colon K(p^{-1}\ell_p) \to \mathrm{TC}(\ell_p, M; p)$ will be a p-adic equivalence in positive degrees. So for the purpose of K-theory calculations with topological cyclic homology, the log S-algebra (ℓ_p, M) would be a suitable interpretation for $p^{-1}\ell_p$.

Thinking of topological Hochschild homology as generalized differential forms, we expect logarithmic THH to be a generalization of differential forms with logarithmic poles. For a discrete (unpointed) log ring (R, N), the log differential forms $\Omega^*_{(R,N)}$ are generated by 1-forms dr and $d\log x$, for $r \in R$ and $x \in N$, with $\alpha(x) \cdot d\log x = d\alpha(x)$.

3.1. Logarithmic topological André–Quillen homology. It is a little clearer how to deal with Kähler differentials, i.e., 1-forms, than general Kähler forms, so we begin by defining logarithmic topological André–Quillen homology, $\mathrm{TAQ}(A, M)$, for a log S-algebra (A, M).

Definition (derivations). Let K be an A-module, and consider the square-zero extension $pr\colon A \vee K \to A$, with section $in\colon A \to A \vee K$. The space of derivations of A with values in K is the mapping space of dashed lifts

$$\mathrm{Der}(A, K) = \mathcal{C}_S \left\{ \begin{array}{ccc} & & A \vee K \\ & \overset{d}{\nearrow} & \downarrow{\scriptstyle pr} \\ A & \xrightarrow[=]{} & A \end{array} \right\}$$

in the (model) category of commutative S-algebras. (This equals $(\mathcal{C}_S/A)(A, A \vee K)$, but we find the diagram notation easier to understand.)

Topological André–Quillen homology is constructed so as to corepresent derivations, in the category of A-modules.

Lemma.
$$\mathcal{M}_A(\mathrm{TAQ}(A), K) \simeq \mathrm{Der}(A, K)\,.$$

We shall define the logarithmic topological André–Quillen homology $\mathrm{TAQ}(A, M)$ as an A-module, corepresenting a space of log derivations.

Remark. Steffen Sagave has thought of how to give an improved definition of $\mathrm{TAQ}(A, M)$ as a log A-module. We will not pursue this here.

Definition (inverse image log structures). Let (A, M) be a log S-algebra and K an A-module. We define the inverse image log structure in^*M on $A \vee K$, along $in\colon A \to A \vee K$, to be the log structure associated to the composite map

$$M \xrightarrow{\alpha} \Omega^\infty A_\otimes \xrightarrow{\Omega^\infty in} \Omega^\infty(A \vee K)_\otimes\,.$$

Note that there are homotopy pushout squares

$$\begin{array}{ccccc}
1_+ & \longrightarrow & GL_1(A)_+ & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle in^\flat} \\
(1+\Omega^\infty K)_{\otimes +} & \longrightarrow & GL_1(A \vee K)_+ & \longrightarrow & in^*M
\end{array}$$

in pointed E_∞ spaces, so that

$$in^*M \simeq (1+\Omega^\infty K)_\otimes \ltimes M\,.$$

Here $(1+\Omega^\infty K)_\otimes \simeq \Omega^\infty K_\oplus$, since K is a square-zero ideal. There are log maps

$$(A, M) \xrightarrow{(in, in^\flat)} (A \vee K, in^*M) \xrightarrow{(pr, pr^\flat)} (A, M)$$

where $pr^\flat$ collapses $(1+\Omega^\infty K)_\otimes$ to a point.

Definition (log derivations). The space of log derivations of (A, M) with values in K is the mapping space of dashed lifts

$$\mathrm{Der}((A, M), K) = \mathcal{L}og_S \left\{ \begin{array}{ccc}
 & & (A \vee K, in^*M) \\
 & \overset{(d, d^\flat)}{\nearrow} & \downarrow{\scriptstyle (pr, pr^\flat)} \\
(A, M) & \xrightarrow[=]{} & (A, M)
\end{array} \right\}$$

in the (model/infinity) category of log S-algebras.

Lemma. *There is a homotopy pullback square*

$$\begin{array}{ccc}
\mathrm{Der}((A, M), K) & \longrightarrow & \mathrm{Der}(A, K) \\
\downarrow & & \downarrow{\scriptstyle \phi^*} \\
\mathrm{Der}^\flat(M, K) & \xrightarrow{\psi^*} & \mathrm{Der}(\Sigma^\infty M, K)
\end{array}$$

where

$$\mathrm{Der}^\flat(M,K) = \mathcal{L}_0 \left\{ \begin{array}{ccc} & & in^*M \\ & \overset{d^\flat}{\nearrow} & \downarrow{\scriptstyle pr^\flat} \\ M & \xrightarrow[=]{} & M \end{array} \right\} .$$

Sketch proof. The condition for $(d, d^\flat)$ to be a log morphism over (A, M) is that the two composites

$$M \xrightarrow{\alpha} \Omega^\infty A_\otimes \xrightarrow{\Omega^\infty d} \Omega^\infty(A \vee K)_\otimes$$

and

$$M \xrightarrow{d^\flat} in^*M \xrightarrow{in^*\alpha} \Omega^\infty(A \vee K)_\otimes$$

agree as maps over $\Omega^\infty A_\otimes$, which by adjunction is equivalent to them agreeing in the mapping space of dashed lifts

$$\mathcal{C}_S \left\{ \begin{array}{ccc} & & A \vee K \\ & \nearrow & \downarrow{\scriptstyle pr} \\ \Sigma^\infty M & \xrightarrow[\bar\alpha]{} & A \end{array} \right\} .$$

This is naturally equivalent to $\mathrm{Der}(\Sigma^\infty M, K)$. □

Lemma. *(i) When $M = N_+$ has an isolated zero, $in^*N_+ \simeq (\Omega^\infty K_\oplus \times N)_+$, so*

$$\mathrm{Der}^\flat(M,K) \simeq \mathcal{L}(N, \Omega^\infty K) \simeq \mathcal{M}_A(A \wedge n, K)$$

where $n = B^\infty N$ is the connective spectrum generated by the E_∞ space N.

(ii) When $M = L_0T$ has an isolated unit, and $T = S^d$ ((or more generally, $T = \Sigma V$ is a suspension)), then

$$\mathrm{Der}^\flat(M,K) \simeq \mathcal{T} \left\{ \begin{array}{ccc} & & \Omega^\infty K_\oplus \ltimes T \\ & \nearrow & \downarrow{\scriptstyle pr} \\ T & \xrightarrow[=]{} & T \end{array} \right\} \simeq \Omega^\infty K_\oplus \simeq \mathcal{M}_A(A, K)$$

where $\mathcal{T}$ denotes the category of pointed spaces.

Remark. In part (ii), the homotopy invariance of the mapping space seems troublesome, since *pr* is not a fibration. The second equivalence assumes that we are talking about strict lifts, not lifts in an equivalent fibration. Since T is a suspension, the space of lifts then deformation retracts to the space of constant lifts.

Definition (log TAQ). (i) When $M = N_+$ has an isolated zero, we define $\mathrm{TAQ}(A, M)$ as the homotopy pushout

$$\begin{array}{ccc} A \wedge_{S[N]} \mathrm{TAQ}(S[N]) & \xrightarrow{\psi} & A \wedge n \\ {\scriptstyle \phi}\downarrow & & \downarrow \\ \mathrm{TAQ}(A) & \longrightarrow & \mathrm{TAQ}(A, M) \end{array}$$

of A-modules, where $n = B^\infty N$, ϕ is induced by $\bar\alpha\colon \Sigma^\infty M \to A$ (corepresenting ϕ^*), and ψ corepresents ψ^*.

(ii) When $M = L_0 T$ has an isolated unit, and $T = S^d$, we define $\mathrm{TAQ}(A, M)$ as the homotopy pushout

$$\begin{array}{ccc} A \wedge_{\Sigma^\infty M} \mathrm{TAQ}(\Sigma^\infty M) & \xrightarrow{\psi} & A \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ \mathrm{TAQ}(A) & \longrightarrow & \mathrm{TAQ}(A, M) \end{array}$$

of A-modules.

Remarks. (i) By a slight extension of a theorem of Basterra–Mandell, $\mathrm{TAQ}(S[N]) \simeq S[N]\wedge n$, so $A\wedge_{S[N]}\mathrm{TAQ}(S[N]) \simeq A\wedge n$, but ψ is typically only an equivalence when (A, N_+) is a trivial log structure, so in general $\mathrm{TAQ}(A) \not\simeq \mathrm{TAQ}(A, N_+)$. Informally, ψ takes $1 \wedge d\alpha(x)$ to $\alpha(x) \wedge d\log x$, for $x \in N$.

(ii) Since $\Sigma^\infty L_0 T = L_0 \Sigma^\infty T \cong L(S \vee \Sigma^\infty T)$ is the free commutative S-algebra on $S \vee \Sigma^\infty T$, it is easy to see that $\mathrm{TAQ}(\Sigma^\infty L_0 T) \simeq \Sigma^\infty L_0 T \wedge T$, so $A \wedge_{\Sigma^\infty M} \mathrm{TAQ}(\Sigma^\infty M) \simeq A \wedge T$.

Proposition. *Let (A, M) be a log S-algebra, with $M = N_+$ or $M = L_0 T$, and let K be an A-module. Then*

$$\mathcal{M}_A(\mathrm{TAQ}(A, M), K) \simeq \mathrm{Der}((A, M), K)\,.$$

Hence $\mathrm{TAQ}(A, M)$ corepresents log derivations, and deserves the name "log differentials". We can usually make the maps ψ more explicit:

Proposition. *(i) When $M = N_+$ has an isolated zero, and $n = B^\infty N$ is a reduced Thom spectrum $Th(\xi)/S$ with Thom diagonal $\theta\colon n \to S[N] \wedge n$, then*

$$\psi\colon A \wedge n \simeq A \wedge_{S[N]} \mathrm{TAQ}(S[N]) \to A \wedge n$$

is the composite

$$A \wedge n \xrightarrow{1\wedge\theta} A \wedge S[N] \wedge n \xrightarrow{1\wedge\bar\alpha\wedge 1} A \wedge A \wedge n \xrightarrow{\mu\wedge 1} A \wedge n\,.$$

(ii) When $M = L_0 T$ has an isolated unit, and $T = S^d$,

$$\psi\colon A \wedge T \simeq A \wedge_{\Sigma^\infty M} \mathrm{TAQ}(\Sigma^\infty M) \to A$$

is the composite

$$A \wedge T \cong A \wedge \Sigma^\infty T \xrightarrow{1\wedge\bar\tau} A \wedge A \xrightarrow{\mu} A\,.$$

((The first result assumes that the universal derivation

$$d^u\colon S[N] \to S[N] \vee \mathrm{TAQ}(S[N]) \simeq S[N] \vee (S[N] \wedge n)$$

has second component

$$S[N] \xrightarrow{\pi} \Sigma^\infty N \xrightarrow{\theta} S[N] \wedge \Sigma^\infty N \xrightarrow{1\wedge\bar\kappa} S[N] \wedge n$$

where π collapses 1_+ in N_+, θ is induced by the half-smash diagonal $\Delta\colon N \to N_+ \wedge N$, and $\bar\kappa$ is left adjoint to the group completion map $\kappa\colon N \to \Omega^\infty n$. Check this!))

Example. If $N = LX$ is the free E_∞ space on a unit-pointed space X, then $n = \Sigma^\infty X = S[X]/S$, and ψ is the composite

$$A \wedge X \xrightarrow{1\wedge\Delta} A \wedge X_+ \wedge X \cong A \wedge S[X] \wedge X \xrightarrow{1\wedge\bar{\alpha}\wedge 1} A \wedge A \wedge X \xrightarrow{\mu\wedge 1} A \wedge X\,.$$

Here $\bar{\alpha}$ really means the restriction of $\bar{\alpha}\colon S[N] \to A$ to $S[X] \subset S[N]$.

Example. If $A = \ell_p$ and $M = L_0 S^q$, with $q = 2p-2$, we get a cofiber sequence

$$\mathrm{TAQ}(\ell_p) \to \mathrm{TAQ}(\ell_p, M) \xrightarrow{res} H\mathbb{Z}_p$$

where $H\mathbb{Z}_p$ is the homotopy cofiber of $v_1\colon \ell_p \wedge S^q \to \ell_p$. Similarly, for $B = ku_p$ and $N = L_0 S^2$ we get a cofiber sequence

$$\mathrm{TAQ}(ku_p) \to \mathrm{TAQ}(ku_p, N) \xrightarrow{res} H\mathbb{Z}_p$$

where $H\mathbb{Z}_p$ is the homotopy cofiber of $u\colon ku_p \wedge S^2 \to ku_p$. There is a transitivity sequence, also for log TAQ, giving a cofiber sequence

$$\Sigma^2 ku_p/\Sigma^q ku_p \xrightarrow{\partial} \mathrm{TAQ}(ku_p/\ell_p) \to \mathrm{TAQ}((ku_p, N)/(\ell_p, M))$$

of relative differentials. A chase of definitions and identifications shows that the left hand term is the homotopy cofiber of the map

$$\Sigma^q ku_p \xrightarrow{(p-1)u^{p-2}} \Sigma^2 ku_p\,,$$

i.e., the different of $(v_1) = (u^{p-1})$. So

$$\Sigma^2 ku_p/\Sigma^q ku_p \simeq \Sigma^2 H\mathbb{Z}_p \vee \dots \vee \Sigma^{2p-4} H\mathbb{Z}_p$$

and $(f, f^\flat)\colon (\ell_p, M) \to (ku_p, N)$ is (formally) log étale if and only if ∂ is an equivalence. ((Is it?))

3.2. Logarithmic topological Hochschild homology. We do not have as clear a characterization of what we should mean by $\mathrm{THH}(A, M)$. For a commutative S-algebra A and an A-module K, thought of as a symmetric A-bimodule, the space of (associative) S-algebra lifts

$$\mathcal{A}_S\left\{\begin{array}{ccc} & & A \vee K \\ & \nearrow & \big\downarrow pr \\ A & \xrightarrow[=]{} & A \end{array}\right\}$$

is corepresented by the A-bimodule I_A given by the homotopy fiber of $\mu\colon A^e = A \wedge A \to A$, and there is a cofiber sequence

$$A \wedge_{A^e} I_A \to A \xrightarrow{\zeta} \mathrm{THH}(A)\,.$$

Furthermore, $\mathrm{THH}(A)$ is a commutative A-algebra, and there is a Quillen spectral sequence

$$\mathrm{Symm}^*_A\, \Sigma\,\mathrm{TAQ}(A) \Longrightarrow \mathrm{THH}(A)$$

starting with the extended powers in A-modules of $\Sigma\,\mathrm{TAQ}(A)$.

Let

$$\mathcal{A}\,\mathrm{Der}^\flat(M, K) = \mathcal{A}_0\left\{\begin{array}{ccc} & & in^* M \\ & \nearrow & \big\downarrow pr \\ M & \xrightarrow[=]{} & M \end{array}\right\}$$

be the space of pointed A_∞ lifts.

Lemma. *(i) When $M = N_+$ has an isolated zero,*

$$\mathcal{A}\,\mathrm{Der}^\flat(M, K) \simeq \mathcal{T}(BN, \Omega^\infty \Sigma K) \simeq \mathcal{M}_{A^e}(A^e \wedge \Sigma^{-1}\Sigma^\infty BN, K)\,,$$

where $BN = \Omega^\infty \Sigma n$.

(ii) When $M = L_0 T$ has an isolated unit, with $T = S^d$,

$$\mathcal{A}\,\mathrm{Der}^\flat(M, K) \simeq \Omega^\infty K_\oplus \simeq \mathcal{M}_{A^e}(A^e, K)\,.$$

This leads to the corepresenting diagrams of A-bimodules

$$\begin{array}{ccc} A^e \wedge_{\Sigma^\infty M^e} I_{\Sigma^\infty M} & \xrightarrow{\psi} & A^e \wedge \Sigma^{-1}\Sigma^\infty BN \\ \downarrow{\scriptstyle\phi} & & \\ I_A & & \end{array}$$

for $M = N_+$, and

$$\begin{array}{ccc} A^e \wedge_{\Sigma^\infty M^e} I_{\Sigma^\infty M} & \xrightarrow{\psi} & A^e \\ \downarrow{\scriptstyle\phi} & & \\ I_A & & \end{array}$$

for $M = L_0 T$, $T = S^d$. The cofiber sequence above and the commutative A-algebra structure on $THH(A)$ then suggest the following construction.

Definition (log THH). (i) When $M = N_+$ has an isolated zero, we define $\mathrm{THH}(A, M)$ as the homotopy pushout

$$\begin{array}{ccc} A \wedge_{S[N]} \mathrm{THH}(S[N]) & \xrightarrow{\psi} & A \wedge BN_+ \\ \downarrow{\scriptstyle\phi} & & \downarrow \\ \mathrm{THH}(A) & \longrightarrow & \mathrm{THH}(A, M) \end{array}$$

in commutative A-algebras, where ϕ is induced by $\bar{\alpha}\colon S[N] \to A$.

(ii) When $M = L_0 T$ has an isolated unit, with $T = S^d$, we define $\mathrm{THH}(A, M)$ as the homotopy pushout

$$\begin{array}{ccc} A \wedge_{\Sigma^\infty M} \mathrm{THH}(\Sigma^\infty M) & \xrightarrow{\psi} & A \wedge S^1_+ \\ \downarrow{\scriptstyle\phi} & & \downarrow \\ \mathrm{THH}(A) & \longrightarrow & \mathrm{THH}(A, M) \end{array}$$

in commutative A-algebras, where ϕ is induced by $\bar{\alpha}\colon \Sigma^\infty M \to A$.

Remarks. (i) $\mathrm{THH}(S[N]) \simeq S[B^{cy}N]$ where $B^{cy}N$ is the cyclic bar construction on N, formed with respect to the cartesian product in spaces.

(ii) $\mathrm{THH}(\Sigma^\infty M) \simeq \Sigma^\infty B^{cy}_\wedge M$, were $B^{cy}_\wedge M$ is the cyclic bar construction on M, formed with respect to the smash product in pointed spaces.

(iii) In order to proceed to a definition of log TC, we would need to endow $\mathrm{THH}(A, M)$ with cyclotomic structure. This is presently not apparent from this definition. The algebraic condition $d(d\log x) = 0$ in the logarithmic de Rham complex suggests that the circle action on $A \wedge BN_+$ (or $A \wedge S^1_+$) is trivial (up to homotopy), which in turn suggests that also the cyclic structure on this part is essentially trivial.

Example. Let $A = H\mathbb{Z}_p$ and $M = N_+$ where $N = \langle p \rangle = \{p^j \mid j \geq 0\}$. Then $BN \simeq S^1$, while

$$B^{cy}N \simeq \{1\} \sqcup \coprod_{j=1}^{\infty} S^1$$

consists of a free 0-N-cell and a free 1-N-cell, and the definition simplifies to a homotopy pushout square

$$\begin{array}{ccc} H\mathbb{Z}_p \wedge S^1_+ & \xrightarrow{\psi} & H\mathbb{Z}_p \wedge S^1_+ \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ \mathrm{THH}(\mathbb{Z}_p) & \longrightarrow & \mathrm{THH}(\mathbb{Z}_p, \langle p \rangle_+) \end{array}$$

of commutative $H\mathbb{Z}_p$-algebras. Here ψ is induced by the degree p map $S^1 \to S^1$, and ϕ is induced by the inclusion $\zeta \colon H\mathbb{Z}_p \to \mathrm{THH}(\mathbb{Z}_p)$ and the circle action on the target.

The resulting Künneth spectral sequence in mod p homotopy is interesting. We have $(S/p)_*(H\mathbb{Z}_p \wedge S^1_+) = E(dp)$ at the upper left, $(S/p)_*(H\mathbb{Z}_p \wedge S^1_+) = E(d \log p)$ at the upper right, and $(S/p)_* \mathrm{THH}(\mathbb{Z}_p) = E(\lambda_1) \otimes P(\mu_1)$. Here dp and $d \log p$ have degree 1, λ_1 has degree $(2p-1)$, and μ_1 has degree $2p$. The map ψ takes dp to $p \cdot d \log p$, which is zero mod p. So

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}^{E(dp)}_{**}(E(d \log p), E(\lambda_1) \otimes P(\mu_1)) \\ &\cong E(d \log p, \lambda_1) \otimes P(\mu_1) \otimes \Gamma(\kappa_0) \end{aligned}$$

where $\kappa_0 = [dp]$ has bidegree $(1,1)$. There is a differential

$$d^p(\gamma_p \kappa_0) = \lambda_1$$

leaving

$$E^{p+1}_{**} = E(d \log p) \otimes P(\mu_1) \otimes P_p(\kappa_0) = E^{\infty}_{**} \,,$$

and a multiplicative extension $\kappa_0^p = \mu_1$ in the abutment, so

$$(S/p)_* \mathrm{THH}(\mathbb{Z}_p, \langle p \rangle_+) = E(d \log p) \otimes P(\kappa_0)$$

with $d \log p$ in degree 1 and κ_0. It follows that there is an equivalence

$$\mathrm{THH}(\mathbb{Z}_p, \langle p \rangle_+) \simeq \mathrm{THH}(\mathbb{Z}_p | \mathbb{Q}_p) \,,$$

where $\mathrm{THH}(\mathbb{Z}_p | \mathbb{Q}_p)$ is Hesselholt–Madsen's relative THH, defined for valuation rings in p-local fields, so as to sit in the desired cofiber sequence

$$\mathrm{THH}(\mathbb{Z}/p) \xrightarrow{i_*} \mathrm{THH}(\mathbb{Z}_p) \xrightarrow{j^*} \mathrm{THH}(\mathbb{Z}_p | \mathbb{Q}_p)$$

of cyclotomic spectra. Their construction does not obviously generalize to non-classical rings, unlike ours, which appears to work for a large class of commutative S-algebras. This compatibility computation suggests that both theories are correct.

pre-Example. Let $A = \ell_p$ and $M = N_+$, with a pre-log structure $\alpha\colon N_+ \to \Omega^\infty(\ell_p)_\otimes$. In order for (ℓ_p, M) to model $p^{-1}\ell_p$, we expect to have $\pi_0 N \cong \langle p \rangle$, so perhaps

$$N = \coprod_{j \ge 0} \mathcal{L}(j) \times_{\Sigma_j} U^j$$

for a connected, unpointed space U, with α mapping U to the component $W_p \times \{p\}$ of $\Omega^\infty \ell_p$. We get a homotopy pushout square

$$\begin{array}{ccc} \ell \wedge_{S[N]} S[B^{cy}N] & \xrightarrow{\psi} & \ell \wedge BN_+ \\ \phi \downarrow & & \downarrow \\ \mathrm{THH}(\ell_p) & \longrightarrow & \mathrm{THH}(\ell_p, N_+) \end{array}$$

of commutative ℓ_p-algebras. We desire an equivalence

$$\mathrm{THH}(\ell_p, N_+) \simeq \mathrm{THH}(p^{-1}\ell_p)\,,$$

where the right hand side is defined to sit in a cofiber sequence

$$\mathrm{THH}(\ell/p) \xrightarrow{i_*} \mathrm{THH}(\ell_p) \xrightarrow{j^*} \mathrm{THH}(p^{-1}\ell_p)\,.$$

Computations of McClure–Staffeldt and Ausoni–Rognes show that in mod p and v_1 homotopy, which we write as $V(1)$-homology,

$$V(1)_* \,\mathrm{THH}(\ell_p) = E(\lambda_1, \lambda_2) \otimes P(\mu_2)$$

and

$$V(1)_* \,\mathrm{THH}(p^{-1}\ell_p) = E(d\log p, \lambda_2) \otimes P_p(\kappa_0) \otimes P(\mu_2)\,.$$

Also note that

$$V(1)_*(\ell \wedge_{S[N]} S[B^{cy}N]) = H_*(N \backslash B^{cy}N)$$

if $B^{cy}N$ is a free N-cell complex, and

$$V(1)_*(\ell \wedge BN_+) = H_*(BN)\,,$$

where H_* denotes mod p homology. Hence we would like to find an E_∞ space N, giving a pre-log structure on ℓ_p, such that the spectral sequence

$$E^2_{**} = \mathrm{Tor}^{H_*(N\backslash B^{cy}N)}_{**}(H_*(BN), E(\lambda_1, \lambda_2) \otimes P(\mu_2))$$

converges to

$$E(d\log p, \lambda_2) \otimes P_p(\kappa_0) \otimes P(\mu_2)\,.$$

If $N = \langle p \rangle$ admitted an E_∞ map to $\Omega^\infty(\ell_p)_\otimes$, then it would work fine, but nontrivial Dyer–Lashof operations on $[p] \in H_0(\ell_p)$ may pose an obstruction to this. ((Check this. Lemma II.2.8 in Cohen–Lada–May seems to say that $Q^s[p] = 0$ in spectrum homology, for positive s.)) One guess is that a somewhat larger N may contain extra terms in $H_*(N\backslash B^{cy}N)$ and $H_*(BN)$, that cancel in Tor.

pre-Example. Let $A = \ell_p$ and $M = L_0 S^q$, with the Adams prelog structure $\alpha\colon M \to \Omega^\infty(\ell_p)_\otimes$. We get a homotopy pushout square

$$\begin{array}{ccc} \ell_p \wedge_{\Sigma^\infty M} \Sigma^\infty B^{cy}_\wedge M & \xrightarrow{\psi} & \ell_p \wedge S^1_+ \\ \phi \downarrow & & \downarrow \\ \mathrm{THH}(\ell_p) & \longrightarrow & \mathrm{THH}(\ell_p, M) \end{array}$$

of commutative ℓ_p-algebras. In order to have a cofiber sequence

$$\mathrm{THH}(\mathbb{Z}_p) \xrightarrow{\pi_*} \mathrm{THH}(\ell_p) \xrightarrow{\rho^*} \mathrm{THH}(\ell_p, M)$$

we should have

$$V(1)_* \,\mathrm{THH}(\mathbb{Z}_p) = E(\lambda_1, \lambda_2) \otimes P(\mu_2)$$

(as before) and

$$V(1)_* \,\mathrm{THH}(\ell_p, M) = E(d \log v_1, \lambda_1) \otimes P(\kappa_1)$$

with $d \log v_1$ in degree 1 and κ_1 in degree $2p$. Clearly $V(1)_*(\ell_p \wedge S^1) = E(d \log v_1)$, so if

$$V(1)_*(\ell_p \wedge_{\Sigma^\infty M} \Sigma^\infty B^{cy}_\wedge M) \cong H_*(M \backslash B^{cy}_\wedge M) \cong E(dv_1)$$

with dv_1 in degree $(2p-1)$, then we can indeed have such a cofiber sequence. This requires that $\kappa_1 = [dv_1]$ has bidegree $(1, 2p-1)$, there is a differential $d^p(\gamma_p \kappa_1) = \lambda_2$ and a multiplicative extension $\kappa_1^p = \mu_2$, quite like in the case of $\mathrm{THH}(\mathbb{Z}_p, \langle p \rangle_+)$.

We have not yet analyzed the $M = L_0 S^q$-action on $B^{cy}_\wedge M$, in this case.

Department of Mathematics, University of Oslo, Norway
E-mail address: rognes@math.uio.no