WALDHAUSEN'S STABLE PARAMETRIZED H-COBORDISM THEOREM

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June 2002

These are notes for the author's lecture at the 2002 Barcelona Conference on Algebraic Topology.

1. Smale's h-cobordism theorem.

Let M be a closed connected n-dimensional manifold. A *cobordism* on M is a compact $(n+1)$-dimensional manifold W containing M as a component of the boundary, $\partial W \cong M \sqcup M'$. It is called an *$h$-cobordism* if the inclusions $M \subset W$ and $M' \subset W$ are homotopy equivalences.

Let R be any ring. Its first algebraic K-group

$$K_1(R) = GL(R)/[GL(R), GL(R)]$$

is the abelianization of the infinite general linear group over R. For any group π, with integral group ring $R = \mathbb{Z}[\pi]$, let $(\pm\pi)$ denote the image of the composite homomorphism

$$\{\pm 1\} \times \pi \subset GL_1(\mathbb{Z}[\pi]) \subset GL(\mathbb{Z}[\pi]) \to K_1(\mathbb{Z}[\pi])\,.$$

The *Whitehead group* of π is the quotient group

$$Wh_1(\pi) = K_1(\mathbb{Z}[\pi])/(\pm\pi)\,.$$

In the simply-connected case $\pi = 1$, $Wh(1) = 0$ is the trivial group.

The h-cobordism theorem (Smale, 1962). *Let M be a closed connected n-dimensional manifold, with fundamental group $\pi = \pi_1(M)$. Suppose that $n \geq 5$. Then there is a one-to-one correspondence between (a) the isomorphism classes of h-cobordisms on M, and (b) the elements of the Whitehead group $Wh_1(\pi)$.*

In particular the h-cobordisms that correspond to the zero element of $Wh_1(\pi)$ are isomorphic to the *trivial* h-cobordism on M, namely the cylinder $W = M \times I$.

The h-cobordism theorem directly implies the Poincaré conjecture in dimensions ≥ 6 (and in dimension 5 with some more work), and is fundamental to the classification of high-dimensional manifolds by surgery theory.

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2. Automorphisms of manifolds.

Ideally, one would like a classification of manifolds up to *isotopy*, so that a fiber bundle $E \downarrow B$ with manifold fibers would be classified by a map from B to the "space of all manifolds". The latter space has the homotopy type of

$$\coprod_{[M]} B\,\mathrm{Aut}(M)$$

where $[M]$ ranges over the set of isomorphism classes of all manifolds, and $\mathrm{Aut}(M)$ is the *automorphism space* of M. The latter can be defined as the simplicial group with q-simplices the automorphisms $g\colon M \times \Delta^q \to M \times \Delta^q$ that commute with projection to Δ^q. So a 0-simplex of $\mathrm{Aut}(M)$ is an automorphism $M \to M$, while a 1-simplex of $\mathrm{Aut}(M)$ with end-points g_0 and g_1 is an isotopy from g_0 to g_1.

Instead, surgery theory classifies manifolds up to the weaker notion of *concordance*. It basically identifies the homotopy type of the *block automorphism space* $\widetilde{\mathrm{Aut}}(M)$, which is defined as the simplicial group with q-simplices the automorphisms $g\colon M \times \Delta^q \to M \times \Delta^q$ that satisfy $g(M \times F) = M \times F$ for each face $F \subset \Delta^q$. So a 0-simplex of $\widetilde{\mathrm{Aut}}(M)$ is again an automorphism $M \to M$, but a 1-simplex of $\widetilde{\mathrm{Aut}}(M)$ with end-points g_0 and g_1 is just a concordance from g_0 to g_1. The classifying space $B\widetilde{\mathrm{Aut}}(M)$ classifies *block bundles* $E \downarrow B$ with manifold fiber M, rather than fiber bundles.

The difference between the block automorphism space offered by surgery theory and the desired automorphism space is measured by the left hand term in the following fiber sequence:

$$\widetilde{\mathrm{Aut}}(M)/\,\mathrm{Aut}(M) \to B\,\mathrm{Aut}(M) \to B\widetilde{\mathrm{Aut}}(M)\,.$$

Hatcher constructed a spectral sequence

$$E^1_{p,q} = \pi_q H(M \times I^p) \Longrightarrow \pi_{p+q}(\widetilde{\mathrm{Aut}}(M)/\,\mathrm{Aut}(M))$$

with $p \geq 0$, $q \geq 1$, abutting to this measure of the failure of surgery theory to precisely classify manifolds.

Here $H(M)$ is the *space of h-cobordisms* on M. Technically, each h-cobordism W may be taken to be a codimension zero submanifold of the cylinder $M \times I$, containing M as $M \times 0$ at one end. The space is so defined that a map $B \to H(M)$ corresponds to a fiber bundle $E \downarrow B$ of h-cobordisms on M, contained in the trivial bundle $B \times M \times I \downarrow B$ with $B \times M \downarrow B$ at one end.

For the analysis of both the space $B\,\mathrm{Aut}(M)$ of manifolds isomorphic to M, and the group $\mathrm{Aut}(M)$ of automorphisms of M, it is thus essential to also identify the higher homotopy groups of this h-cobordism space. More precisely, we wish to understand the full homotopy type of the h-cobordism space $H(M)$.

3. Higher simple homotopy theory.

Presently, this is understood in a stable range of homotopy groups that grows to infinity as the dimension of M grows. This was the aim of Hatcher's 1975 paper "Higher simple homotopy theory", but the proofs offered there are not satisfactory. Instead, such understanding is now achieved by combining Igusa's stability theorem (1988) with Waldhausen's stable parametrized h-cobordism theorem (ca. 1982), as we now explain.

There are stabilization maps $H(M) \to H(M \times I)$. Passing to the colimit, we define the *stable h-cobordism space*

$$\mathcal{H}(M) = \operatorname*{hocolim}_p H(M \times I^p)\,.$$

Igusa's stability theorem asserts that the natural map $H(M) \to \mathcal{H}(M)$ is at least k-connected if $n \geq \max\{2k+7, 3k+4\}$, i.e., it is roughly $(n/3)$-connected for n large. Work of Goodwillie and his student Meng concerns the fiber of this natural map in a meta-stable range, but this falls outside our present scope.

Smale's h-cobordism theorem identified $\pi_0 H(M) \cong \pi_0 \mathcal{H}(M)$ with the Whitehead group, which is algebraically constructed using algebraic K-theory. Similarly, Waldhausen's stable parametrized h-cobordism theorem identifies the homotopy type of $\mathcal{H}(M)$ with a looped Whitehead space, which is algebraically defined in terms of the algebraic K-theory of spaces.

4. Waldhausen's stable parametrized h-cobordism theorem.

Let $A(X)$ be Waldhausen's algebraic K-theory of any space X. It is defined as the algebraic K-theory of the category $\mathcal{R}(X)$ of finite *retractive spaces* over X, i.e., spaces Y containing X as a retract such that up to homotopy Y can be built from X by attaching finitely many cells. The algebraic K-theory is formed with respect to homotopy equivalences, so that retractive spaces Y and Y' represent points in $A(X)$ and a homotopy equivalence $Y \to Y'$ represents a path connecting these points in $A(X)$.

At this point, the smooth category (DIFF) behaves differently from the piecewise linear (PL) and the topological (TOP) category. (I think this joke is due to Hatcher.) The piecewise linear Whitehead space $Wh^{PL}(X)$ is homotopy equivalent to the topological Whitehead space, and sits in a fiber sequence

$$h(X; A(*)) \to A(X) \to Wh^{PL}(X)$$

where $h(X; A(*)) = \Omega^\infty(X_+ \wedge \mathbf{A}(*))$ is the homology theory associated to the spectrum $\mathbf{A}(*)$ with underlying (infinite loop) space $A(*)$, and the left hand map is the assembly map. The smooth Whitehead space $Wh^{DIFF}(X)$ is different, and sits in a split fiber sequence

$$Q(X_+) \to A(X) \to Wh^{DIFF}(X)$$

where $Q(X_+) = \Omega^\infty \Sigma^\infty (X_+)$.

The stable parametrized h-cobordism theorem (Waldhausen, 1982). *Let M be a compact manifold. The stable h-cobordism space is homotopy equivalent with the looped Whitehead space*

$$\mathcal{H}(M) \simeq \Omega Wh(M)\,.$$

This theorem provides the fundamental connection between geometric topology on one side, via the space $H(M)$ of h-cobordisms on a manifold and the related space $\mathrm{Aut}(M)$ of automorphisms of that manifold, and algebraic K-theory on the other side, via Waldhausen's algebraic K-theory space $A(X)$ and the fiber sequences above. Waldhausen's space $A(X)$ is in turn closely related to Quillen's algebraic

K-theory of the group ring $R = \mathbb{Z}[\pi]$ for $\pi = \pi_1(X)$, in that Dundas proved that there is a homotopy cartesian square

$$\begin{array}{ccc} A(X) & \longrightarrow & TC(X) \\ \downarrow & & \downarrow \\ K(R) & \longrightarrow & TC(R) \end{array}$$

where TC denotes the topological cyclic homology theory of Bökstedt, Hsiang and Madsen, in the integral form defined by Goodwillie. Note that $\pi_0 \Omega Wh(M) = \pi_1 Wh(M) = Wh_1(\pi)$ for $\pi = \pi_1(M)$, so the stable form of Smale's h-cobordism theorem is recovered by applying π_0 to that of Waldhausen. The stability theorem of Igusa for stable (smooth) concordances = pseudoisotopies also recovers the unstable form of Smale's theorem, but with a less precise dimension range.

The topological and smooth cases of the stable parametrized h-cobordism theorem can be reduced to the piecewise linear case by *triangulation theory* and *smoothing theory*, respectively. These assert that the homotopy fiber of $H^{PL}(M) \to H^{TOP}(M)$ is the space of sections in a suitable bundle over M, and similarly for the fiber of $H^{DIFF}(M) \to H^{PL}(M)$, and these section spaces give homology theories in M, which can be controlled. Granting this, we may hereafter restrict attention to the PL case.

5. Retractive simplicial sets.

Let X be a simplicial set, and recall the category $\mathcal{R}(X)$ of finite retractive simplicial sets Y over X. It has a full subcategory $\mathcal{R}^h(X)$ with objects the retractive simplicial sets for which the inclusion $X \to Y$ is a weak homotopy equivalence.

A map $Y \to Y'$ of simplicial sets is a weak homotopy equivalence if and only if its geometric realization $|Y| \to |Y'|$ is a homotopy equivalence. We say that $Y \to Y'$ is a *simple map* if the geometric realization $|Y| \to |Y'|$ satisfies the stronger condition that every point inverse is contractible.

Simple maps form a category, i.e., are closed under composition, and simple maps are indeed weak homotopy equivalences. To prove this, one proceeds by characterizing simple maps of finite simplicial sets as the *hereditary* weak equivalences over open subsets, which obviously form a category.

(In the PL category, simple maps can be arbitrarily well approximated by homeomorphisms. In a sense, the space of simple maps is a closure of the space of homeomorphisms.)

Let $h\mathcal{R}(X)$ be the subcategory of $\mathcal{R}(X)$ with morphisms the weak homotopy equivalences $Y \to Y'$, and let $s\mathcal{R}(X)$ be the (smaller) subcategory with morphisms the simple maps $Y \to Y'$. Likewise define subcategories $h\mathcal{R}^h(X)$ and $s\mathcal{R}^h(X)$ of $\mathcal{R}^h(X)$.

These are categories with cofibrations and weak equivalences, hence are susceptible to Waldhausen's $S_\bullet$-construction, which yields a simplicial category that in each simplicial degree has objects given by suitable diagrams in the original category.

Let Δ^q denote the simplicial q-simplex, and let X^{Δ^q} be the simplicial set of maps $\Delta^q \to X$. Then $[q] \mapsto X^{\Delta^q}$ is a (bi-)simplicial set denoted $X^{\Delta^\bullet}$.

Theorem (Wa85, 3.3.1). *The square*

$$\begin{array}{ccc} sS_\bullet \mathcal{R}^h(X^{\Delta^\bullet}) & \longrightarrow & sS_\bullet \mathcal{R}(X^{\Delta^\bullet}) \\ \downarrow & & \downarrow \\ hS_\bullet \mathcal{R}^h(X^{\Delta^\bullet}) & \longrightarrow & hS_\bullet \mathcal{R}(X^{\Delta^\bullet}) \end{array}$$

is homotopy cartesian, and the term on the lower left is contractible (it has an initial object). The other terms are as follows,

$$|sS_\bullet \mathcal{R}^h(X^{\Delta^\bullet})| \simeq Wh^{PL}(X)\,,$$
$$\Omega|sS_\bullet \mathcal{R}(X^{\Delta^\bullet})| \simeq h(X; A(*))\,,$$
$$\Omega|hS_\bullet \mathcal{R}(X^{\Delta^\bullet})| \simeq A(X)\,,$$

and each of the homotopy equivalences can be described by a natural chain of maps.

This provides the fiber sequence relating $Wh^{PL}(X)$ to $A(X)$. The simplicial direction provided by $X^{\Delta^\bullet}$ is necessary to make the term in the upper right corner a homotopy functor.

Let X be a simplicial set again, and let $\mathcal{C}(X)$ be the category of simplicial sets Y under X that can be built from X by attaching finitely many simplices. (So $X \to Y$ is a cofibration and Y/X is a finite simplicial set.) Let $\mathcal{C}^h(X)$ be the full subcategory of objects for which the inclusion $X \to Y$ is a weak homotopy equivalence, and let $s\mathcal{C}^h(X)$ be the further subcategory with morphisms the simple maps $Y \to Y'$ (under X).

It can be shown that the functor $X \mapsto s\mathcal{C}^h(X)$ respects weak homotopy equivalences. This proceeds by proving that $s\mathcal{C}^h(-)$ respects filling horns and sequential colimits, hence also respects fibrant replacement. Likewise $s\mathcal{C}^h(-)$ takes simplicial homotopy equivalences to weak homotopy equivalences, which completes the argument.

Theorem (Wa85, 3.1.1 and 3.1.7). *There is a natural chain of homotopy equivalences*

$$|s\mathcal{C}^h(X)| \simeq \Omega|sS_\bullet \mathcal{R}^h(X^{\Delta^\bullet})|\,.$$

Hence there is a homotopy equivalence $|s\mathcal{C}^h(X)| \simeq \Omega Wh^{PL}(X)$, and it remains to prove that $\mathcal{H}(M) \simeq |s\mathcal{C}^h(M)|$ when M is a compact PL manifold. Later we will need:

Proposition (W-J-R, 5.7). *Let X be a finite simplicial set. There is a homotopy fiber sequence*

$$s\mathcal{C}^h(X) \to s\mathcal{C}(\varnothing) \to h\mathcal{C}(\varnothing)\,.$$

6. Non-singular simplicial sets.

A simplicial set X is called *non-singular* if for every non-degenerate simplex x the corresponding representing map $\bar{x} \colon \Delta^q \to X$ is an embedding, i.e., the geometric realization $|\bar{x}| \colon |\Delta^q| \to |X|$ is an embedding.

The geometric realization of a non-singular simplicial set X has a canonical piecewise linear structure, for which the embedding of each non-degenerate simplex is linear. This is not the case for general simplicial sets, which is related to the fact that pushouts do not generally exist in the category of polyhedra and PL maps. They do however exist for pushout diagrams where both arrows are injective, which ensures that for non-singular X the geometric realization $|X|$ is a polyhedron in a natural way.

In order to relate $s\mathcal{C}^h(M)$ to PL h-cobordisms on M, which in particular are polyhedra, it is therefore necessary to improve on the simplicial sets occurring in $s\mathcal{C}^h(M)$ to become non-singular.

Once again, let X be a simplicial set, and let $\mathcal{D}(X) \subset \mathcal{C}(X)$ be the full subcategory of non-singular simplicial sets Y under X that are finite relative to X. (X must be non-singular for this category to be non-empty.) Let $\mathcal{D}^h(X)$ and $s\mathcal{D}(X)$ be the usual subcategories, with objects Y such that the inclusion $X \to Y$ is a weak homotopy equivalence, and morphisms $Y \to Y'$ that are simple maps, respectively, and let $s\mathcal{D}^h(X)$ be the intersection of these two subcategories.

Proposition (W-J-R, 5.8). *Let X be non-singular. The inclusion*

$$s\mathcal{D}^h(X) \to s\mathcal{C}^h(X)$$

is a homotopy equivalence.

The proof involves an *improvement functor* I from the category of finite simplicial sets to itself, such that each $I(X)$ is non-singular and comes equipped with a natural simple map $I(X) \to X$. The functor I preserves simple maps and cofibrations, hence can be used to produce a deformation retraction of $s\mathcal{C}^h(X)$ down to $s\mathcal{D}^h(X)$.

Next it is necessary to reintroduce an extra simplicial direction. We assume X is finite.

Let $\widetilde{\mathcal{D}}_q(X)$ be the category whose objects are the commutative diagrams of non-singular finite simplicial sets

$$\begin{array}{ccc} X \times \Delta^q & \longrightarrow & Z \\ & {\scriptstyle pr_2} \searrow \quad \swarrow & \\ & \Delta^q & \end{array}$$

where $X \times \Delta^q \to Z$ is a cofibration and $Z \to \Delta^q$ is a Serre fibration, i.e., its geometric realization $|Z| \to |\Delta^q|$ has the homotopy lifting property for polyhedra. The morphisms in $\widetilde{\mathcal{D}}_q(X)$ are the maps $Z \to Z'$ over Δ^q that restrict to the identity map on $X \times \Delta^q$.

Let $\mathcal{D}_q(X) \subset \widetilde{\mathcal{D}}_q(X)$ be the full subcategory of objects such that the pair of polyhedra $(|Z|, |X \times \Delta^q|)$ is a locally trivial pair over $|\Delta^q|$.

We obtain simplicial categories $\widetilde{\mathcal{D}}_\bullet(X)$ and $\mathcal{D}_\bullet(X)$, with simplicial subcategories $s\widetilde{\mathcal{D}}^h_\bullet(X)$ and $s\mathcal{D}^h_\bullet(X)$, as usual.

Proposition (W-J-R, 5.9). *The natural maps*

$$s\mathcal{D}^h(X) \to s\mathcal{D}^h_\bullet(X) \quad \textit{and} \quad s\mathcal{D}^h_\bullet(X) \to s\widetilde{\mathcal{D}}^h_\bullet(X)$$

are both homotopy equivalences.

The proof involves subdividing the Serre fibration $p\colon Z \to \Delta^q$, and forming the preimage $\mathrm{Sd}(p)^{-1}(\beta)$ of the barycenter $\beta \in \Delta^q$, viewed as a vertex of the subdivision $\mathrm{Sd}(\Delta^q)$. There is a natural simple map

$$\mathrm{Sd}(p)^{-1}(\beta) \times \Delta^q \to Z$$

over Δ^q, which must be suitably adapted to provide the two homotopy inverses.

7. Polyhedra.

A *finite triangulation* of a compact space X is a homeomorphism $h\colon |T| \to X$ from the geometric realization of a finite non-singular simplicial set T. A *compact polyhedron* is a compact space together with a preferred class of finite triangulations related to each other by piecewise linear isomorphism.

Let X be a compact polyhedron, and let $\widetilde{\mathcal{E}}_q(X)$ be the category whose objects are the commutative diagrams of polyhedra

$$\begin{array}{ccc} X \times |\Delta^q| & \longrightarrow & Z \\ & {\scriptstyle pr_2}\searrow \quad \swarrow & \\ & |\Delta^q| & \end{array}$$

where $X \times |\Delta^q| \to Z$ is injective and $Z \to |\Delta^q|$ is a Serre fibration. The morphisms in $\widetilde{\mathcal{E}}_q(X)$ are the maps $Z \to Z'$ over $|\Delta^q|$ that restrict to the identity map on $X \times |\Delta^q|$.

Let $\mathcal{E}_q(X)$ be the full subcategory of $\widetilde{\mathcal{E}}_q(X)$ whose objects have the property that the pair of polyhedra $(Z, X \times |\Delta^q|)$ is a locally trivial pair over $|\Delta^q|$.

We obtain simplicial categories $\mathcal{E}_\bullet(X)$ and $\widetilde{\mathcal{E}}_\bullet(X)$, with simplicial subcategories $s\mathcal{E}^h_\bullet(X)$ and $s\widetilde{\mathcal{E}}^h_\bullet(X)$, as usual. Geometric realization of non-singular simplicial sets provides functors $j\colon s\mathcal{D}^h_\bullet(X) \to s\mathcal{E}^h_\bullet(|X|)$, and $j\colon s\widetilde{\mathcal{D}}^h_\bullet(X) \to s\widetilde{\mathcal{E}}^h_\bullet(|X|)$.

Proposition (W-J-R, 6.1). *Let X be a finite non-singular simplicial set. The natural maps*

$$\begin{array}{ccc} s\mathcal{D}^h_\bullet(X) & \longrightarrow & s\mathcal{E}^h_\bullet(|X|) \\ \downarrow & & \downarrow \\ s\widetilde{\mathcal{D}}^h_\bullet(X) & \longrightarrow & s\widetilde{\mathcal{E}}^h_\bullet(|X|) \end{array}$$

are all homotopy equivalences.

The proof involves triangulating a chain of simple maps of polyhedra

$$|X| \times L \subset P_0 \to \cdots \to P_n$$

parametrized over a suitable polyhedron L, relative to a fixed triangulation on the subpolyhedron $|X| \times L$ common to all the polyhedra in the chain. This requires, in particular, that in each case the subpolyhedron $|X| \times L$ is *collared*, i.e., comes equipped with a suitable neighborhood of the form $|X| \times L \times [0, \epsilon]$ for some $\epsilon > 0$. No functorial homotopy inverse is constructed in this case, but the relative homotopy groups of the maps j are shown to vanish.

8. Piecewise linear manifolds.

A PL manifold M admits a *tangent microbundle* τ_M, which is the germ of PL neighborhoods of the diagonal $\Delta(M) \subset M \times M$. A *stable framing* of M is an equivalence class of isomorphisms $\tau_M \oplus \epsilon^k \cong \epsilon^{k+n}$ (for k sufficiently large). For stably framed PL manifolds M and N, a *stably framed map* $f\colon M \to N$ is one such that the composite map of bundles over M

$$\epsilon^{n+k} \cong \tau_M \oplus \epsilon^k \xrightarrow{f_* \oplus 1} f^*\tau_N \oplus \epsilon^k \cong \epsilon^{n+k}$$

is the identity.

Let $\mathcal{M}^n_q$ denote the category of (locally trivial) PL bundles of compact, stably framed, n-dimensional manifolds, with base space $|\Delta^q|$. The morphisms are the stably framed isomorphisms of such stably framed PL bundles, hence this category is a groupoid. For varying q these assemble to a simplicial groupoid $\mathcal{M}^n_\bullet$. Product with the unit interval defines a stabilization map $\mathcal{M}^n_\bullet \to \mathcal{M}^{n+1}_\bullet$.

Let $h\mathcal{M}^n_q$ denote the category with the same objects, and with stably framed homotopy equivalences of PL bundles as morphisms. Again there is a simplicial category $h\mathcal{M}^n_\bullet$ and a stabilization map $h\mathcal{M}^n_\bullet \to h\mathcal{M}^{n+1}_\bullet$.

Proposition. *Let X be a PL manifold. There is a vertical map of horizontal homotopy fiber sequences*

$$\begin{array}{ccccc} \mathcal{H}(X) & \longrightarrow & \operatorname{hocolim}_n \mathcal{M}^n_\bullet & \longrightarrow & \operatorname{hocolim}_n h\mathcal{M}^n_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ s\mathcal{E}^h_\bullet(X) & \longrightarrow & s\mathcal{E}_\bullet(\varnothing) & \longrightarrow & h\mathcal{E}_\bullet(\varnothing)\,. \end{array}$$

The vertical map $\operatorname{hocolim}_n h\mathcal{M}^n_\bullet \to h\mathcal{E}_\bullet(\varnothing)$ is a homotopy equivalence by general position. Hence to prove that $\mathcal{H}(X) \simeq s\mathcal{E}^h_\bullet(X)$ it suffices to prove the following.

Theorem (W-J-R, II.1.2). *The stabilized map* $\operatorname{hocolim}_n \mathcal{M}^n_\bullet \to s\mathcal{E}_\bullet(\varnothing)$ *is a weak homotopy equivalence.*

This is an application of Quillen's theorem A. Let K be a polyhedron. Let $\mathcal{S}^n_0(K)$ be the set of compact, stably framed, n-dimensional PL manifolds M together with a PL simple map $f\colon M \to K$. More generally, let $\mathcal{S}^n_q(K)$ be the set of PL fiber bundles of such objects, fibered over Δ^q. These assemble to a simplicial set $\mathcal{S}^n_\bullet(K)$. It suffices to show that $\operatorname{hocolim}_n \mathcal{S}^n_\bullet(K) \simeq *$ for any K.

9. Thickenings.

In the case $K = *$, $\mathcal{S}^n_\bullet(*)$ is a space of weak homotopy n-balls, which have *homology* $(n-1)$-spheres as boundary. This space is typically not even connected. To avoid this pathology, let

$$\mathcal{T}^n_\bullet(K) \subset \mathcal{S}^n_\bullet(K)$$

be the simplicial subset of *thickenings* of K, where we additionally require that the restricted map $f|\colon \partial M \to K$ has 1-connected point inverses. Then stably

$$\operatorname*{hocolim}_n \mathcal{T}^n_\bullet(K) \xrightarrow{\simeq} \operatorname*{hocolim}_n \mathcal{S}^n_\bullet(K)$$

and the stable parametrized h-cobordism theorem finally follows from the key input:

Proposition. *The space $\mathcal{T}^n_\bullet(K)$ is $(n-2k-6)$-connected, where k is the dimension of K.*

Its proof depends on the *principle of global transversality in patches*, which was invented by Hatcher for the purpose of proving a theorem on 3-manifolds.

This completes our outline of Waldhausen's work.

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