ARITHMETIC OF SOME BRAVE NEW RINGS

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§1. Brave new rings.

Each discrete ring R gives rise to a multiplicative cohomology theory

$$X \mapsto H^*(X; R)$$

given by singular cohomology with coefficients in R, and the cup product. There are other such multiplicative cohomology theories, for example, topological K-theory

$$X \mapsto KU^*(X)$$

given by complex vector bundles over X, and their tensor product.

A **brave new ring** A is a structured ring spectrum, in the sense of algebraic topology, that represents such a multiplicative cohomology theory

$$X \mapsto A^*(X) .$$

In particular, A consists of a sequence of spaces

$$A = \{n \mapsto A_n\} ,$$

with additional structure, that represent the graded terms in the multiplicative cohomology theory, so that

$$A^n(X) = [X, A_n]$$

(homotopy classes of maps from X to A_n) in each degree n.

A satisfactory description of the additional structure satisfied by this sequence of spaces $\{n \mapsto A_n\}$ was only published in 1997. Several solutions exist, and go under the names S-algebras (Elmendorf–Kriz–Mandell–May, 1997), symmetric ring spectra (Hovey–Shipley–Smith, 1999), orthogonal ring spectra (May–Mandell–Shipley–Smith, 2001), functors with smash product (Bökstedt, ca. 1986), etc. They rigidify earlier notions of A_∞- and E_∞ ring spectra.

For example, singular cohomology with R-coefficients is represented by the Eilenberg–Mac Lane spectrum

$$HR = \{n \mapsto K(R, n)\} ,$$

with n-th space the Eilenberg–Mac Lane complex $K(R, n)$. Thus discrete rings are special cases of brave new rings.

Topological K-theory provides a non-algebraic example. As a brave new ring it is represented by the complex K-theory spectrum

$$KU = \begin{cases} \mathbb{Z} \times BU & n \text{ even} \\ U & n \text{ odd} \end{cases},$$

with n-th space the group completed union of infinite Grassmannians $\mathbb{Z} \times BU$ for n even, and its loop space U for n odd.

The most fundamental example of a brave new ring is the **sphere spectrum**

$$S = \{n \mapsto S^n\},$$

with n-th space S^n, the n-sphere. It represents stable cohomotopy.

§2. Module categories and K-theory.

The modern description of structured spectra lets us talk about the category

$$\mathcal{M}_A = \{A\text{-modules}\}$$

of modules over a brave new ring A.

For example, the category

$$\mathcal{M}_{HR}$$

of HR-modules is very similar to the category of chain complexes over a ring R.

For another example, the category

$$\mathcal{M}_S \simeq \{\text{spectra}\}$$

of modules over the sphere spectrum S is equivalent to the category of structured spectra, so we may refer to a spectrum as an S-module. This is analogous to viewing abelian groups as $\mathbb{Z}$-modules.

There is also a pairing of S-modules, called the smash product

$$M \wedge_S N,$$

analogous to the tensor product of abelian groups.

A structured ring spectrum A comes with a product map

$$A \wedge_S A \to A$$

that makes it an S-algebra, in the same way as a discrete ring is a $\mathbb{Z}$-algebra. Thus brave new ring, structured ring spectrum and S-algebra are synonyms.

Algebraic K-theory focuses on a subcategory

$$h\mathcal{C}_A \subset \mathcal{M}_A$$

of **finite cell A-modules**, and their homotopy equivalences, analogous to the category of finite cell complexes inside of topological spaces. More precisely, we idempotently complete this category by including all retracts, but we shall suppress this point.

For example,

$$h\mathcal{C}_{HR}$$

is very similar to the category of bounded complexes of finitely generated free R-modules, and their quasi-isomorphisms.

§3. Algebraic K-theory as arithmetic.

The algebraic K-theory of A is a spectrum $K(A)$ generated by the category $h\mathcal{C}_A$. Its 0-th space $K(A)_0$ is a group completion of the classifying space $Bh\mathcal{C}_A$ of this category, as realized by a map

$$Bh\mathcal{C}_A \to K(A)_0 \,.$$

For a discrete ring R, $K(HR) \simeq K(R)$ is Quillen's algebraic K-theory. When $R = \mathcal{O}_F$ is the ring of integers in a number field,

$$\pi_0 K(\mathcal{O}_F) = \mathbb{Z} \oplus \mathrm{Cl}(F)\,,$$
$$\pi_1 K(\mathcal{O}_F) = \mathcal{O}_F^\times$$

and in general $\pi_i K(\mathcal{O}_F)$ carries much arithmetic information about F.

The algebraic K-theory of other brave new rings A is also of significant interest, outside of the field of K-theory. A major motivation for the development of the theory of structured ring spectra was the need to have proper foundations for the definitions above, with applications like the following in mind.

For a manifold X, with loop group ΩX, there is a spherical group ring $A = S[\Omega X]$, which is a brave new ring that refines the discrete integral group ring $\mathbb{Z}[\pi_1(X)]$. By Waldhausen's stable parametrized h-cobordism theorem (ca. 1982, writeup in progress at MPI), there is a very close relation between the algebraic K-theory of $S[\Omega X]$

$$K(S[\Omega X]) \sim B\,\mathrm{Aut}(X)$$

and the moduli space of manifolds isomorphic to X, or equivalently, the automorphism group of X, in a range of homotopy groups that grows to infinity with the dimension of X. The theory applies for both smooth and topological manifolds and automorphisms. By the geodesic flow methods of Farrell and Jones, it suffices to consider the cases $X = *$ (a point) and $X = S^1$ (a circle) to deal with all non-positively curved X.

For a different application, there is a relation (Baas–Dundas–Rognes, 2004) between the algebraic K-theory of topological K-theory

$$K(KU) \sim \mathcal{E}\ell\ell \sim CFT\,,$$

elliptic cohomology and conformal field theory. A 2-vector bundle $\mathcal{E}$ over a spacetime X represents an element in $K(KU)^0(X)$, and the holonomy of parallel transport in $\mathcal{E}$ around a string γ in X defines a "number" in KU, i.e., a virtual vector space, which can be interpreted as the state space of γ.

These theories are thus arithmetic aspects of $S[\Omega X]$ and KU, respectively.

§4. The chromatic tower.

We seek a conceptual description of $K(S)$, corresponding to a highly-connected manifold X (like a point, a disc or a sphere) in Waldhausen's theorem, following an approach outlined by him in 1984.

Fixing a rational prime p, we can for simplicity focus on p-local brave new rings. The short filtration of rings

$$\mathbb{Z}_{(p)} \subset \mathbb{Q}$$

corresponds, at the level of module categories, to the localizations away from the subcategories

$$\{0\} \subset \{p\text{-torsion}\}$$

of finitely generated $\mathbb{Z}_{(p)}$-modules, or bounded chain complexes of such.

This unfolds into an infinite tower of brave new rings

$$S_{(p)} \to \cdots \to L_n S \to L_{n-1} S \to \cdots \to L_0 S = H\mathbb{Q}$$

where

$$L_n S = L_{E(n)} S$$

is the (implicitly p-local) n-th **chromatic localization** of the sphere spectrum, which focuses on the n-th kind of periodicity phenomena in stable homotopy theory, i.e., v_n-periodicity, which in turn is related to height n formal groups in characteristic p.

At the level of module categories, these correspond to localizations away from the subcategories

$$\{*\} \subset \cdots \subset \mathcal{T}_n \subset \mathcal{T}_{n-1} \subset \cdots \subset \mathcal{T}_0 = \{p\text{-torsion}\}$$

of $h\mathcal{C}_{S_{(p)}}$, where

$$\mathcal{T}_n = \{\text{type} > n\}$$

is the thick subcategory of p-local finite cell spectra Z of type $> n$. This means that the first $(n+1)$ Morava K-theories

$$K(0)_*(Z), \ldots, K(n)_*(Z)$$

of Z all vanish. By the Devinatz–Hopkins–Smith nilpotence theorem (1988) these are all the thick subcategories of $h\mathcal{C}_{S_{(p)}}$, so the chromatic tower above is a canonical object to study.

In terms of brave new algebraic geometry, we think of $\operatorname{Spec} S_{(p)}$ as being filtered by an increasing sequence of Zariski open subspaces $\operatorname{Spec} L_n S$.

In the discrete case, the algebraic K-theory of $K(\mathbb{Z}_{(p)})$ is understood in terms of the algebraic K-theory of fields, by the lower cofiber sequence of spectra in the diagram

$$\begin{array}{ccccc}
K(\mathbb{F}_p) & \longrightarrow & K(\mathbb{Z}_p) & \longrightarrow & K(\mathbb{Q}_p) \\
& \searrow^{i_*} & \uparrow & & \uparrow \\
& & K(\mathbb{Z}_{(p)}) & \xrightarrow{j^*} & K(\mathbb{Q})
\end{array}$$

Here i_* is the transfer/direct image map, that considers a finitely generated $\mathbb{F}_p$-module as a p-torsion $\mathbb{Z}_{(p)}$-module, which goes to zero upon tensoring with $\mathbb{Q}$. By Quillen's dévissage theorem, this map i_* identifies $K(\mathbb{F}_p)$ with the homotopy fiber of j^*.

In the brave new ring case, there is a similar diagram

$$\begin{array}{ccccc}
K^{sm}(\hat{L}_n S) & \longrightarrow & K(\hat{L}_n S) & \longrightarrow & K(L_{n-1}\hat{L}_n S) \\
& \searrow^{i_*} & \uparrow & & \uparrow \\
& & K(L_n S) & \xrightarrow{j^*} & K(L_{n-1} S)
\end{array}$$

for each $n \geq 1$, where

$$\hat{L}_n S = L_{K(n)} S$$

is the $K(n)$-localization of S. Here $K^{sm}(\hat{L}_n S)$ is the algebraic K-theory of the full subcategory

$$h\mathcal{C}^{sm}_{\hat{L}_n S} \subset h\mathcal{C}_{\hat{L}_n S}$$

of (categorically) **small** $\hat{L}_n S$-modules. The lower part of the diagram is essentially a cofiber sequence of spectra, except that j^* may not be surjective on π_0. We do not claim that the upper part of this diagram is a cofiber sequence.

We think of $\operatorname{Spec} \hat{L}_n S$ as a completion of $\operatorname{Spec} L_n S$ away from $\operatorname{Spec} L_{n-1} S$, as $\operatorname{Spec} \mathbb{Z}_p$ is the completion of $\operatorname{Spec} \mathbb{Z}_{(p)}$ away away from $\operatorname{Spec} \mathbb{Q}$. The small $\hat{L}_n S$-modules live over this local neighborhood, and are supported away from the preimage of $\operatorname{Spec} L_{n-1} S$.

§5. Galois extensions.

The algebraic K-theory of fields is quite well understood by the principle of Galois descent, which underlies the now proven Lichtenbaum–Quillen conjectures. To each G-Galois extension $F \to E$ there is a map

$$K(F) \to K(E)^{hG}$$

to the G-homotopy fixed points for the induced G-action on E, and this map is close to a homotopy equivalence, say, with finite coefficients and in sufficiently high degrees. The limiting case, with $E = \bar{F}$ the separable closure and $G = G_F$ the absolute Galois group, is particularly interesting. Taking a common point of view, the presence of this group action separates an arithmetic theory from a geometric theory.

In the case of $K(\mathbb{F}_p)$, all of this is well understood. In the case of $K(\mathbb{Q})$, we understand $K(\bar{\mathbb{Q}})$ after completion at a prime p, by a theorem of Suslin, and our fairly detailed knowledge of $K(\mathbb{Q})$ feeds back to give information about $G_{\mathbb{Q}}$.

There is a theory of Galois extensions of brave new rings (Rognes, MPIM–Bonn preprint 2005–79). A map $A \to B$ of commutative brave new rings is a G-Galois extension if the finite group G acts on B over A and two natural maps

$$A \to B^{hG}$$

$$B \wedge_A B \to \prod_G B$$

are homotopy equivalences. It is natural to conjecture that the Galois descent principle for algebraic K-theory extends, i.e., that the map

$$K(A) \to K(B)^{hG}$$

becomes a homotopy equivalence in sufficiently high degrees, after introducing finite coefficients of sufficiently high type.

Conjecture. *Let $A \to B$ be a $K(n)$-local G-Galois extension, let F be a type $(n+1)$ finite cell spectrum, and let $T = v_{n+1}^{-1} F$ be the mapping telescope of any v_{n+1} self-map of F. Then*

$$T \wedge K(A) \to T \wedge K(B)^{hG}$$

is a homotopy equivalence.

§6. The Lubin–Tate spectra.

By the Morava change-of-rings theorem and the Goerss–Hopkins–Miller obstruction theory, there is a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension

$$\hat{L}_n S \xrightarrow{\mathbb{G}_n} E_n$$

with $\mathbb{G}_n$ the (extended) Morava stabilizer group of automorphisms of the height n Honda formal group law, and E_n an even periodic form of $E(n)$, with

$$\pi_0 E_n = \mathbb{W}\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]]$$

the complete local ring supporting the universal Lubin–Tate deformation of the Honda formal group law. The commutative S-algebra E_n is a Landweber exact complex oriented spectrum.

By Hovey–Strickland (1999), a finite cell $\hat{L}_n S$-module M is small if and only if the smash product

$$E_n \wedge_{\hat{L}_n S} M \simeq \hat{L}_n(E_n \wedge M)$$

is **degreewise finite**, in the sense that each homotopy group (usually denoted $E^\vee_{n*}(M)$) is finite. We obtain the diagram

$$\begin{array}{ccc}
K(K_n) & & \\
\downarrow{\scriptstyle\sim} & \searrow^{i_*} & \\
K^{df}(E_n) & \longrightarrow & K(E_n) \\
\uparrow{\scriptstyle \mathbb{G}_n} & & \uparrow{\scriptstyle \mathbb{G}_n} \\
K^{sm}(\hat{L}_n S) & \longrightarrow & K(\hat{L}_n S)
\end{array}$$

where $K^{df}(E_n)$ is the algebraic K-theory of the full subcategory

$$h\mathcal{C}^{df}_{E_n} \subset h\mathcal{C}_{E_n}$$

of degreewise finite E_n-modules. There is a map $i \colon E_n \to K_n$ of brave new rings, where K_n is a (non-commutative!) even periodic form of $K(n)$, with $\pi_0 K_n = \mathbb{F}_{p^n}$. Composing the module action with i defines a functor

$$h\mathcal{C}_{K_n} \to h\mathcal{C}^{df}_{E_n}$$

and an induced map in algebraic K-theory.

Conjecturally then, $K^{sm}(\hat{L}_n S) \to K^{df}(E_n)^{h\mathbb{G}_n}$ is close to a homotopy equivalence, by a form of Galois descent, and $K(K_n) \to K^{df}(E_n)$ is a homotopy equivalence, by a form of dévissage.

We think of $\operatorname{Spec} E_n$ as a regular covering space, or étale cover, of $\operatorname{Spec} \hat{L}_n S$, and the degreewise finite E_n-modules live over this covering space, supported away from the preimage of $\operatorname{Spec} L_{n-1} S$. These are then part of an étale topology on $\operatorname{Spec} S_{(p)}$.

More generally, there is a sequence of categories

$$h\mathcal{C}^{df}_{E_n} \subset \cdots \subset h\mathcal{C}^{I_k\text{-tors}}_{E_n} \subset \cdots \subset h\mathcal{C}_{E_n}$$

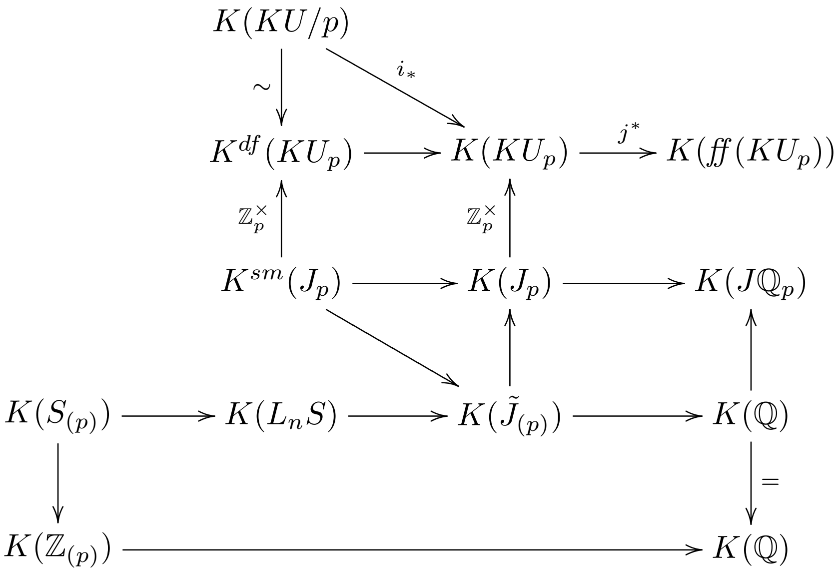
for $0 \le k \le n$, where $I_k = (p, u_1, \dots, u_{k-1})$ is an invariant ideal in $\pi_0 E_n$, and $h\mathcal{C}^{I_k\text{-tors}}_{E_n}$ is the full subcategory of finite cell E_n-modules whose homotopy groups are I_k-torsion. For $k = 0$, this equals $h\mathcal{C}_{E_n}$ and for $k = n$ it equals $h\mathcal{C}^{df}_{E_n}$. There is a map $E_n \to E_n/I_k$ that makes E_n/I_k a finite cell E_n-module, a functor

$$h\mathcal{C}_{E_n/I_k} \to h\mathcal{C}^{I_k\text{-tors}}_{E_n},$$

and an induced map in algebraic K-theory. Optimistically, each of these maps may be homotopy equivalences.

§7. Topological K-theory.

In the first non-algebraic case, $n = 1$, we have $L_0 S = H\mathbb{Q}$, $L_1 S = \tilde{J}_{(p)}$, $\hat{L}_1 S = J_p$, the p-adic image-of-J spectrum, $E_1 = KU_p$, p-adic topological K-theory, $\mathbb{G}_1 = \mathbb{Z}_p^\times$ with $k \in \mathbb{Z}_p^\times$ acting on KU_p by the Adams operation ψ^k, and $K_1 = KU/p$, mod p topological K-theory.



To go from $K(\mathbb{Q})$ to $K(S_{(p)})$, the first step is to $K(\tilde{J}_{(p)})$, and the change is $K^{sm}(J_p)$, which we expect to obtain by Galois descent from $K^{df}(KU_p)$, which we in turn expect is homotopy equivalent to $K(KU/p)$.

We define the algebraic K-theory $K(\mathit{ff}(KU_p))$ of the **fraction field** $\mathit{ff}(KU_p)$ of p-adic topological K-theory to be the mapping cone of i_*, so as to have the i_*-j^*-cofiber sequence above. This is a "motivic" definition, for now. No definition of $\mathit{ff}(KU_p)$ as a brave new ring is yet known.

We view $h\mathcal{C}^{df}_{KU_p}$ as a category of coherent sheaves over $\operatorname{Spec} KU_p$ that are supported over a closed (non-commutative) subobject $\operatorname{Spec} KU/p$. The result of cutting out this subobject is to be $\operatorname{Spec} \mathit{ff}(KU_p)$.

There is a cofiber sequence

$$K(\mathbb{Z}) \xrightarrow{\pi_*} K(ku) \xrightarrow{\rho^*} K(KU)$$

due to Blumberg–Mandell, where $\rho\colon ku \to KU$ is the connective cover of KU, and $\pi\colon ku \to H\mathbb{Z}$ is the 0-th Postnikov section. The same proof will yield similar cofiber sequences for KU_p and KU/p. These have the advantage that the algebraic K-theory spectra

$$K(ku_p), K(ku/p), K(\mathbb{Z}_p), K(\mathbb{Z}/p)$$

can be calculated by means of the cyclotomic trace map to the **topological cyclic homology** of Bökstedt–Hsiang–Madsen (1993). This is an invariant derived from topological Hochschild homology, rather like crystalline cohomology in the algebraic case, and for connective p-complete brave new rings, like ku_p, ku/p, $\mathbb{Z}_p$ and $\mathbb{Z}/p$, the cyclotomic trace map is very close to a homotopy equivalence.

§8. Lichtenbaum–Quillen periodicity.

Such calculations have been made with Christian Ausoni (Uni-Bonn, writeup in progress at MPI) in the last year. Assume $p \geq 5$ and let

$$V(1) = S/(p, v_1) = \mathrm{cone}(v_1\colon S^{2p-2}/p \to S^0/p)$$

be a type 2 finite cell spectrum representing homotopy with mod (p, v_1) coefficients, i.e.,

$$V(1)_*X = \pi_*(V(1) \wedge X)\,.$$

It admits a v_2-self map of degree $2(p^2-1)$, so $V(1)_*X$ is naturally a module over $\mathbb{Z}/p[v_2]$. There is a long exact sequence

$$\dots \to \pi_{i-2p+2}(X; \mathbb{Z}/p) \xrightarrow{v_1} \pi_i(X; \mathbb{Z}/p) \to V(1)_i(X) \to \dots$$

where $\pi_i(X; \mathbb{Z}/p)$ denotes mod p homotopy.

Theorem (Ausoni–Rognes, 2005). *Each of*

$$V(1)_*K(KU/p), \quad V(1)_*K(KU_p), \quad V(1)_*K(\mathit{ff}(KU_p))$$

is a finitely generated free $\mathbb{Z}/p[v_2]$-module on explicit generators, of rank

$$2(p-1)(p^2-p+4), \quad 2(p-1)(2p+2), \quad 2(p-1)(p^2+3)$$

respectively.

We view this purely calculational result as a verification of a generalization to brave new rings of the Lichtenbaum–Quillen conjecture, to the effect that

$$\pi_*(K(F); \mathbb{Z}/p)$$

is a finitely generated free $\mathbb{Z}/p[v_1]$-module, for a local or global number field F, whose v_1-localization

$$v_1^{-1}\pi_*(K(F); \mathbb{Z}/p) \cong \pi_*(K^{\acute{e}t}(F); \mathbb{Z}/p)$$

agrees with the étale K-theory of F, and similarly for the ring $\mathcal{O}_F$ of integers in F. For example, $\pi_*(K(\mathbb{Q}_p); \mathbb{Z}/p)$ is a finitely generated free $\mathbb{Z}/p[v_1]$-module of rank $(p+3)$.

§9. Beilinson–Lichtenbaum motivic truncation.

A more precise form of the Lichtenbaum–Quillen conjecture, again for a field F, is offered by the motivic version of the Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H^{-s}_{mot}(F; \mathbb{Z}(t/2)) \Longrightarrow \pi_{s+t} K(F)$$

in its mod p form, and the Beilinson–Lichtenbaum conjecture that

$$H^r_{mot}(F; \mathbb{Z}/p(i)) \cong \begin{cases} H^r_{ét}(F; \mathbb{Z}/p(i)) & \text{for } r \le i \\ 0 & \text{for } r > i \end{cases}$$

for $\mathrm{char}(F) \neq p$, where in the field case

$$H^r_{ét}(F; \mathbb{Z}/p(i)) \cong H^r_{Gal}(F; \mu_p^{\otimes i}) \cong H^r_{cont}(G_F; \mu_p^{\otimes i})$$

is the continuous group cohomology of the absolute Galois group, and

$$H^r_{ét}(F; \mathbb{Z}/p(*)) \cong v_1^{-1} H^r_{mot}(F; \mathbb{Z}/p(*))$$

where $v_1 \in H^0_{mot}(F; \mathbb{Z}/p(p-1))$. This conjecture is now a theorem, by the work of Voevodsky, Rost, Suslin and others.

No definition of motivic cohomology for brave new rings is yet known to me, but if we assume its existence, together with an analogue of the Atiyah–Hirzebruch spectral sequence with mod (p, v_1) coefficients, then we can run the spectral sequence backwards to get a prediction for the motivic cohomology of

$$H^r_{mot}(\mathit{ff}(KU_p); \mathbb{F}_{p^2}(i)) \,,$$

assuming that $\mathit{ff}(KU_p)$ is a "brave new field". Here $\mathbb{F}_{p^2}(i) = \pi_{2i} K_2 = V(1)_{2i} E_2$.

Pre-theorem.

$$H^*_{mot}(\mathit{ff}(KU_p); \mathbb{F}_{p^2}(*)) \cong \mathbb{Z}/p[b] \otimes \Phi_{*,*}$$

Here the "higher Bott element" b is a $(p-1)$-st root of v_2, in bidegree $(r, i) = (0, p+1)$, and $\Phi_{*,*}$ is an explicit finite bigraded $\mathbb{Z}/p$-algebra of rank $2(p^2+3)$ concentrated in the fundamental domain of bidegrees

$$0 \le r \le 3, \quad r \le i < r + p + 1 \,.$$

In particular, $\mathit{ff}(KU_p)$ has p-cohomological dimension 3.

Inverting $v_2 = b^{p-1}$, we must then have

$$H^*_{ét}(\mathit{ff}(KU_p); \mathbb{F}_{p^2}(*)) \cong \mathbb{Z}/p[b, b^{-1}] \otimes \Phi_{*,*}$$

and we can observe that, indeed,

$$H^r_{mot}(\mathit{ff}(KU_p); \mathbb{F}_{p^2}(i)) \cong \begin{cases} H^r_{ét}(\mathit{ff}(KU_p); \mathbb{F}_{p^2}(i)) & \text{for } r \le i \\ 0 & \text{for } r > i. \end{cases}$$

We view this calculational conclusion as verifying a generalization of the Beilinson–Lichtenbaum conjecture for the fraction field of p-adic topological K-theory. In particular, it strongly supports the idea that the symbol $K(\mathit{ff}(KU_p))$ is really the algebraic K-theory of a field-like object, and that the presumptions made in its construction are justified.

If a similar argument is made to predict the motivic and étale cohomology of KU/p and KU_p, then the analogue of the Beilinson–Lichtenbaum conjecture does not hold. This is perhaps to be expected, since KU/p is most like a non-commutative field of characteristic p, and KU_p is more like a valuation ring than a field. On the other hand, it also illustrates that something special is going on in the algebraic K-theory and (hypothetical) motivic cohomology of $\mathit{ff}(KU_p)$, something that is not shared by most brave new rings.

§10. Tate–Poitou arithmetic duality.

The Galois cohomology of local and global number fields has an additional special property, known as Tate–Poitou **arithmetic duality**.

In the local number field case, $\mathbb{Q}_p \subset F$, the cup-product pairing

$$H^r_{ét}(F;\mathbb{Z}/p(i)) \otimes H^{2-r}_{ét}(F;\mathbb{Z}/p(1-i)) \xrightarrow{\cup} H^2_{ét}(F;\mathbb{Z}/p(1)) \cong \mathrm{Br}(F)[p] \xrightarrow[\cong]{inv} \mathbb{Z}/p$$

is a perfect pairing, so induces an isomorphism

$$H^{2-r}_{ét}(F;\mathbb{Z}/p(1-i)) \cong H^r_{ét}(F;\mathbb{Z}/p(i))^* .$$

For higher local fields, say a d-local field F_d for $d \geq 0$ (with $\mathbb{Q}_p \subset F_1$), there is a similar perfect cup-product pairing landing in

$$H^{d+1}_{ét}(F_d;\mathbb{Z}/p(d)) \cong \mathbb{Z}/p$$

(Deninger–Wingberg, 1986).

Similarly, there is an isomorphism

$$H^3_{ét}(\mathit{ff}(KU_p);\mathbb{F}_{p^2}(2)) \cong \mathbb{Z}/p$$

and the expected cup-product pairing

$$H^r_{ét}(\mathit{ff}(KU_p);\mathbb{F}_{p^2}(i)) \otimes H^{3-r}_{ét}(\mathit{ff}(KU_p);\mathbb{F}_{p^2}(2-i)) \xrightarrow{\cup} H^3_{ét}(\mathit{ff}(KU_p);\mathbb{F}_{p^2}(2)) \cong \mathbb{Z}/p$$

is visibly perfect.

Pre-theorem. *The finite $\mathbb{Z}/p$-algebra $\Phi_{*,*}$ above is a Poincaré algebra.*

In this sense, the fraction field of p-adic topological K-theory behaves as a brave new 2-local field. In particular, it has the arithmetic characteristics of such a higher local field. Conceivably, the fraction field $\mathit{ff}(E_n)$ of the n-th Lubin–Tate spectrum may behave like a brave new $(n+1)$-local field, and the fraction field $\mathit{ff}(KU)$ of integral topological K-theory may have global arithmetic behavior.

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